T. Liggett Mathematics 131BH – Final Exam Solutions March 17, 2010

(15) 1. For which real values of x does the series

$$\sum_{n=1}^{\infty} \frac{(nx)^n}{n!}$$

converge? Explain.

By Stirling's formula, $n! \sim n^n e^{-n}\sqrt{2\pi n}$, so

$$\frac{(nx)^n}{n!} \sim \frac{(xe)^n}{\sqrt{2\pi n}}.$$

It follows that the series converges for $|x| < e^{-1}$, and diverges for $|x| > e^{-1}$ and for $x = e^{-1}$. The series converges for $x = -e^{-1}$ by the alternating series test. To check this, one needs to know that

$$a_n = \frac{(ne^{-1})^n}{n!}$$

is decreasing to zero. The inequality $a_n \geq a_{n+1}$ is equivalent to $n \log(1+\frac{1}{n}) \leq 1$. This follows from $\log(1+x) \leq x$ for $x \geq 0$. (Note: $a_n \sim 1/\sqrt{n}$ does not imply that $\sum(-1)^n a_n$ converges. For an example in which it diverges, take $a_n = 1/\sqrt{n}$ for n odd and $a_n = (1/\sqrt{n}) + (1/n)$ for n even.)

(15) 2. Use power series to compute

$$\lim_{x\to 0} \frac{a^x - 1}{x}$$

for $a > 0$.

For $x \neq 0$, write

$$\frac{a^x - 1}{x} = \frac{e^{x \log a} - 1}{x} = \log a \sum_{n=1}^{\infty} \frac{(x \log a)^{n-1}}{n!}.$$

Therefore, the limit is $\log a$.

(20) 3. Suppose f is a real-valued function on $[-1, 1]$ with a continuous derivative. Prove that there exist polynomials $p_n(x)$ so that $p_n \to f$ and $p_n' \to f'$ uniformly on $[-1, 1]$. (You may use the Weierstrass Theorem.)

Choose polynomials q_n so that $q_n \to f'$ uniformly on $[-1, 1]$, and let

$$p_n(x) = f(-1) + \int_{-1}^{x} f'(t)dt.$$

(20) 4. Suppose f is Riemann integrable on compact subsets of $[0, \infty)$.

(a) Show that $\lim_{x\to\infty} f(x) = 0$ implies

$$\lim_{t\downarrow 0} t \int_0^\infty e^{-tx} f(x)dx = 0.$$

Given $\epsilon > 0$, choose M so that $|f(x)| < \epsilon$ for $x \geq M$. Then

$$\left| t \int_0^\infty e^{-tx} f(x)dx \right| \leq \epsilon \int_M^\infty te^{-tx}dx + ||f||_\infty \int_0^M te^{-tx}dx = \epsilon e^{-Mt} + ||f||_\infty \left[1 - e^{-Mt}\right].$$

Therefore,

$$\limsup_{t\downarrow 0} \left| t \int_0^\infty e^{-tx} f(x)dx \right| \leq \epsilon.$$

Since ϵ is arbitrary, the limit is 0.

(b) Is the converse to the statement in part (a) true? If so, prove it; if not give a counterexample.

The converse is not true. For a counterexample, take $f(x) = (-1)^n$ on $[n, n+1)$. Then

$$t \int_0^\infty e^{-tx} f(x)dx = \sum_{n=0}^\infty (-1)^n \int_n^{n+1} te^{-tx}dx = (1-e^{-t}) \sum_{n=0}^\infty (-e^{-t})^n = \frac{1-e^{-t}}{1+e^{-t}}.$$

(15) 5. Change the order of summation, with appropriate justification, to compute

$$\sum_{k=0}^\infty \sum_{l=0}^\infty \binom{k+l}{k} x^k y^l \quad \text{for } |x| + |y| < 1.$$

If $x, y \geq 0$, the following interchange is justified, whether or not the series converges:

$$\sum_{k=0}^\infty \sum_{l=0}^\infty \binom{k+l}{k} x^k y^l = \sum_{n=0}^\infty \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = \sum_{n=0}^\infty (x+y)^n.$$

Therefore,

$$\sum_{k=0}^\infty \sum_{l=0}^\infty \binom{k+l}{k} |x|^k |y|^l = \frac{1}{1-|x|-|y|} < \infty \quad \text{for } |x| + |y| < 1.$$

It follows that the interchange in justified whenever $|x| + |y| < 1$, and this gives

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k+l}{k}x^k y^l = \frac{1}{1-x-y}$$

in this case.

(15) 6. Suppose that f_n is continuous on E for each n and $f_n \to f$ uniformly on E. Prove that f is continuous on E.

Begin by writing

$$|f(y) - f(x)| \le |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|.$$

To prove continuity at $x \in E$, given $\epsilon > 0$, choose n so that $|f_n(z) - f(z)| < \epsilon$ for all $z \in E$, and then $\delta > 0$ so that $d(y, x) < \delta$ implies $|f_n(y) - f_n(x)| < \epsilon$ for that n. Then $d(y, x) < \delta$ implies $|f(y) - f(x)| < 3\epsilon$.

(20) 7. Suppose that for each n, f_n is continuous on $[0, 1]$ and satisfies $|f_n(x)| \le 1$ for $0 \le x \le 1$. Define

$$g_n(x) = \int_0^x f_n(t)dt.$$

(a) Show that there is a sequence n_k and a continuous function g on $[0, 1]$ so that $g_{n_k} \to g$ uniformly on $[0, 1]$.

Since $|g_n(y) - g_n(x)| \le |y - x|$, and $|g_n(x)| \le 1$ for all n, x, y, the family $\{g_n\}$ is uniformly bounded and equicontinuous. The result follows from Theorem 7.25.

(b) Is the function g in part (a) necessarily differentiable on $[0, 1]$? Explain.

No. For a counterexample, let

$$g_n(x) = \begin{cases} |x - \frac{1}{2}| & \text{if } |x - \frac{1}{2}| \ge \frac{1}{n}; \\ \frac{n}{2}(x - \frac{1}{2})^2 + \frac{1}{2n} & \text{if } |x - \frac{1}{2}| \le \frac{1}{n}, \end{cases}$$

and $f_n = g_n'$.

(20) 8. Suppose f is continuous on $[0, 1]$, and let $||f||_p = \left[\int_0^1 |f(x)|^p dx\right]^{1/p}$ for $p \ge 1$ and $||f||_\infty = \max_{0\le x\le 1} |f(x)|$.

(a) Show that $||f||_p \le ||f||_\infty$ for each p.

$$||f||_p \le \big[\int_0^1 ||f||_\infty^p dx\big]^{1/p} = ||f||_\infty.$$

(b) Show that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

Without loss of generality, we may assume that $||f||_\infty > 0$. Take $M \in (0, ||f||_\infty$, and then an interval $(a, b) \subset [0, 1]$ (with $b > a$) so that $|f(x)| \ge M$ on (a, b). Then

$$||f||_p \ge \big[\int_a^b |f(x)|^p dx\big]^{1/p} \ge M(b-a)^{1/p}.$$

It follows that

$$\liminf_{p\to\infty} ||f||_p \ge M.$$

Now take M close to $||f||_\infty$, and use part (a).

(30) 9. In each case, decide whether the statement is true or false. If true, prove it; if false, give a counterexample, or otherwise show it is false.

(a) If f_n and f are Riemann integrable on $[0, 1]$ and $f_n \to f$ uniformly on $[0, 1]$, then $\int_0^1 f_n dx \to \int_0^1 f dx$ as $n \to \infty$.

True:

$$\left| \int_0^1 f_n dx - \int_0^1 f dx \right| \le ||f_n - f||_\infty \to 0.$$

(b) If $\sum_n a_n e^{inx}$ is the Fourier series for the function

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi; \\ x - 2\pi & \text{for } \pi < x \le 2\pi, \end{cases}$$

then $\sum_n |a_n| < \infty$.

False. If it were true, then $\sum_n a_n e^{inx}$ would converge uniformly to a continuous function g. But $g = f$ for $x \ne \pi$, since f is differentiable there. This is a contradiction, since f has a jump discontinuity at π.

(c) Every bounded function on $[0, 1]$ is Riemann integrable.

False. Counterexample: the indicator of the rationals.

(d) If f in a continuous complex-valued function on $[0,1]$, then there exists a $t \in [0,1]$ so that

$$\int_0^1 f(x)dx = f(t).$$

False. Counterexample: $f(x) = e^{2\pi i x}$.

(30) 10. In each case, decide whether the statement is true or false. If true, prove it; if false, give a counterexample, or otherwise show it is false.

(a) If the power series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 1, then $\sum_{n=0}^{\infty} c_n$ converges.

False. Counterexample: $c_n = 1/n$.

(b) The space $C[0,1]$ of continuous functions on $[0,1]$ with the norm $||\cdot||_2$ is complete.

False. Take $f_n(x) = \min\{(2x)^n, 1\}$ and $f = 1_{[1/2,1]}$. Then $f_n \to f$, so $\{f_n\}$ is Cauchy. If $f_n \to g$ for some continuous g, then $||f-g||_2 = 0$. Since $f-g$ is continuous except at $\frac{1}{2}$, $f = g$ except at $\frac{1}{2}$. This is a contradiction.

(c)

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{1}{n^3+n^4x^2} = -2x\sum_{n=1}^{\infty}\frac{1}{n^2(1+nx^2)^2}.$$

True. The series on the right converges uniformly on compact sets, so the statement follows from Theorem 7.17, applied to the partial sums.

(d) If f is nonnegative and continuous on $[0,\infty)$, and satisfies $\lim_{x\to\infty} f(x) = 0$, then

$$\int_0^{\infty} f(x)dx < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} f(n) < \infty.$$

False. Take $f(x) = (x-n)(1-x+n)/(n+1)$ for $n \le x \le n+1$.