(15) 1. (a) Prove the following part of the ratio test: If $a_n \neq 0$ for each n and

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1,$$

then $\sum_n a_n$ converges absolutely.

Proof. Choose β so that

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \beta < 1,$$

and then N so that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq \beta$$

for $n \geq N$. Then $|a_n| \leq \beta^{n-N}|a_N|$ for $n \geq N$. Now use the comparison test – compare with the geometric series $\sum_n \beta^n$. □

(b) Show by example that the statement in (a) is false if lim sup is replaced by lim inf.

Example. Take $a_{2n} = 1, a_{2n+1} = 2$. Then the lim inf and lim sup are $\frac{1}{2}$ and 2 respectively, and the series diverges since the summands to not tend to 0.

(20) 2. Suppose $f : X \to Y$ is continuous. In each case, decide whether the statement is true or false. If true, prove it; if false give a counterexample with $X = Y = R^1$.

(a) If $K \subset X$ is compact, then $f(K)$ is compact.

True; see Theorem 4.14.

(b) If $K \subset Y$ is compact, then $f^{-1}(K)$ is compact.

False; take $f(x) \equiv 0$ and $K = \{0\}$.

(c) If $E \subset X$ is connected, then $f(E)$ is connected.

True; see Theorem 4.22.

(d) If $E \subset Y$ is connected, then $f^{-1}(E)$ is connected.

False; take $f(x) = x^2$ and $E = [1, 4]$.

(28) 3. In each case, say whether the statement is true or false. Briefly explain your answer.

(a) If $\{a_n, n \geq 1\}$ is decreasing and $\sum a_n$ converges, then there exists a constant C so that $a_n \leq C/n$.

True. Since the series converges, $a_n \downarrow 0$. By the monotonicity,

$$na_n \le \sum_{k=1}^{n} a_k \le \sum_{k=1}^{\infty} a_k < \infty.$$

(b) Let X be $C[0,1]$, the metric space of all continuous functions on $[0,1]$, with $d(f,g) = \max_{0\le t\le 1} |f(t) - g(t)|$. Then $\{f \in X : d(f,0) \le 1\}$ is compact.

False, since $f_n(t) = t^n$ is a sequence in the unit ball that does not have a convergent subsequence.

(c) If f is continuous on $(0,1)$, it is uniformly continuous on $(0,1)$.

False; take $f(x) = 1/x$.

(d) $Q \cap [0,1]$ is compact.

False; take $x \in [0,1]\backslash Q$, and $x_n \in Q$ so that $x_n \to x$. This sequence has no convergent subsequence in $Q \cap [0,1]$.

(e) If $\sum_n a_n$ converges and $\{b_n\}$ is bounded, then $\sum_n a_n b_n$ converges.

False; $\sum_n (-1)^n/n$ converges, but $\sum_n (-1)^n(-1)^n/n = \sum_n 1/n$ does not.

(f) If a_n and b_n are real and $\sum_n (a_n^2 + b_n^2) < \infty$, then $\sum_n a_n b_n$ converges.

True. This follows from the comparison test, since $2|a_n b_n| \le a_n^2 + b_n^2$. Alternatively, use the Schwarz inequality.

(g) If $\sum_n |a_{n+1} - a_n| < \infty$, then $\lim_{n\to\infty} a_n$ exists.

True. The sequence a_n is Cauchy, since for $m < n$,

$$|a_n - a_m| \le \sum_{k=m}^{\infty} |a_{k+1} - a_k|,$$

which tends to 0 as $m \to \infty$.

(15) 4. A family $\mathcal{F}$ of functions is said to be uniformly equicontinuous if

$$\forall\ \epsilon > 0\ \exists\ \delta > 0 \ni d(x,y) < \delta, f \in \mathcal{F} \Rightarrow d(f(x), f(y)) < \epsilon. \tag{1}$$

(a) Suppose $g : R^2 \to R^1$, and define $f_\theta(x) = g(x\cos\theta, x\sin\theta)$ for $0 \le \theta \le 2\pi$. Prove that if $\{f_\theta, 0 \le \theta \le 2\pi\}$ is uniformly equicontinuous, then g is continuous at the origin.

Proof. Given $\epsilon > 0$, let $\delta > 0$ be the value provided in (1) for this family. Then $|x| < \delta$ implies that

$$|g(x\cos\theta, x\sin\theta) - g(0,0)| < \epsilon.$$

Writing $(u,v) \in R^2$ in polar coordinates gives the result. □

(b) Show by example that the statement in (a) is false if it is only assumed that each f_θ is continuous.

Example:

$$g(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then

$$f_\theta(x) = \frac{x\cos^2\theta\sin\theta}{x^2\cos^2\theta + \sin^2\theta},$$

and $g(x,x^2) = \frac{1}{2}$ for $x \neq 0$.

(10) 5. Prove that if $\lim_{n\to\infty} a_n = a$, then

$$\lim_{x\uparrow 1}(1-x)\sum_{n=0}^{\infty} x^n a_n = a.$$

Proof. This is equivalent to

$$\lim_{x\uparrow 1}(1-x)\sum_{n=0}^{\infty} x^n (a_n - a) = 0.$$

Given $\epsilon > 0$, choose N so that $n \geq N \Rightarrow |a_n - a| < \epsilon$. Then

$$\left|(1-x)\sum_{n=0}^{\infty} x^n(a_n - a)\right| \leq \left|(1-x)\sum_{n=0}^{N-1} x^n(a_n - a)\right| + \epsilon x^N,$$

so

$$\limsup_{x\uparrow 1}\left|(1-x)\sum_{n=0}^{\infty} x^n(a_n - a)\right| \leq \epsilon.$$

Since ϵ is arbitrary, the result follows. □

(12) 6. (a) Suppose that $F, K \subset X$, $F \cap K = \emptyset$, F is closed and K is compact. Show that $\inf\{d(x,y) : x \in F, y \in K\} > 0$.

Proof. Suppose that $\inf\{d(x,y) : x \in F, y \in K\} = 0$. Then there are sequences x_n in F and y_n in K so that $d(x_n, y_n) \to 0$. Since K is compact, we can pass to a subsequence so that $y_{n_k} \to y$ for some $y \in K$. By the triangle inequality, $x_{n_k} \to y$ as well. Since F is closed, $y \in F$. Therefore $F \cap K \neq \emptyset$, which is a contradiction. □

(b) Show by example that the statement in (a) is not correct if K is only assumed to be closed, rather than compact.

Example. Take $F = \{n \in Z : n \geq 2\}$ and $K = \{n + \frac{1}{n} : n \in Z, n \geq 2\}$.

(15) 7. Suppose $a < c < b$, f in continuous on (a,b), and f is differentiable on $(a,b)\backslash\{c\}$. Show that if $\lim_{x \to c} f'(x)$ exists, then f is differentiable at c also.

Proof. By the mean value theorem, if $t \in (a,b)\backslash\{c\}$, there is a d strictly between t and c so that

$$\frac{f(t) - f(c)}{t - c} = f'(d).$$

As $t \to c$, the corresponding $d \to c$. Therefore,

$$f'(c) = \lim_{t \to c} \frac{f(t) - f(c)}{t - c} = \lim_{x \to c} f'(x).$$

□

(10) 8. Suppose f is a nonnegative function on R^1 such that for some M,

$$\sum_{x \in F} f(x) \leq M$$

for all finite $F \subset R^1$. Show that $\{x : f(x) > 0\}$ is at most countable.

Proof. For each positive integer n,

$$\#\left\{x : f(x) \geq \frac{1}{n}\right\} \leq \frac{M}{n},$$

so $\{x : f(x) > 0\}$ is the union of countably many finite sets. □

(10) 9. Is Q, the set of rational numbers, connected? Prove your answer.

Proof. It is not; $A = Q \cap (-\infty, \sqrt{2})$ and $B = Q \cap (\sqrt{2}, \infty)$ are separated sets. □

(15) 10. (a) Define $f : R^1 \to R^1$ is differentiable at x.

(b) Prove that if f is differentiable at x, then it is continuous at x.

(c) Prove that if f and g are differentiable at x, then so is their product fg.

See Definition 5.1 and Theorems 5.2 and 5.3.

(10) 11. Suppose that f is strictly positive and continuous on $[0, \infty)$, and that $\lim_{x\to\infty} f(x) = 1$. Show that there is an $\epsilon > 0$ so that $f(x) \geq \epsilon$ for all $x \geq 0$.

Proof. Choose N so that $f(x) \geq \frac{1}{2}$ for $x \geq N$. f achieves its minimum $\alpha > 0$ on $[0, N]$ by compactness and continuity. Let ϵ be the smaller of $\frac{1}{2}$ and α. □

(10) 12. Suppose that $f : [0, 1] \to R^1$ is continuous and satisfies $f(0) = f(1) = 0$ and $f'(0) = f'(1) = 1$. (f may not be differentiable on $(0, 1)$.) Show that there is an $x \in (0, 1)$ so that $f(x) = 0$.

Proof. Since $f'(0) = f'(1) = 1$, there exist $0 < x < y < 1$ so that $f(x) > 0$ and $f(y) < 0$. By the intermediate value theorem, there is a $x < z < y$ so that $f(z) = 0$. □

(10) 13. Show that the sequence x_n defined by $x_1 = 1$ and

$$x_{n+1} = x_n + \frac{1}{x_n^2}, \quad n \geq 1$$

is unbounded.

Proof. Since

$$x_{n+1} - x_n = \frac{1}{x_n^2} \geq 0,$$

the sequence is increasing. If it were bounded, it would have to converge, say to x. But then

$$x = x + \frac{1}{x^2},$$

which is impossible. □

(20) 14. Suppose $a_n \downarrow 0$ and $\sum_n a_n = \infty$.

(a) Determine exactly for which complex z's the series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges absolutely.

Since a_n is bounded, the radius of convergence $R \geq 1$, and since $\sum_n a_n$ diverges, $R \leq 1$. Moreover, $\sum_n a_n |z|^n = \infty$ if $|z| = 1$. Therefore, $R = 1$, and the series converges absolutely iff $|z| < 1$.

(b) Determine exactly for which complex z's the series

$$\sum_{n=0}^{\infty} a_n z^n \tag{2}$$

converges.

The series diverges if $|z| > 1$ and if $z = 1$. Therefore, we need to consider only z so that $|z| = 1$ and $z \neq 1$. In this case, write

$$S_n = \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z},$$

which is bounded in n. Use summation by parts to write

$$\sum_{n=0}^{N} a_n z^n = a_N S_N + \sum_{n=0}^{N-1} (a_n - a_{n+1}) S_n.$$

The first term on the right tends to 0 as $N \to \infty$. The second term converges, since

$$\sum_{n=0}^{\infty} |(a_n - a_{n+1}) S_n| \leq M \sum_{n=0}^{\infty} (a_n - a_{n+1}) = M a_0 < \infty,$$

where M is a bound on $\{S_n\}$. Therefore, the series in (2) converges if $|z| = 1, z \neq 1$.