Weighted Estimates for Singular Integrals

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1 Weighted norm inequalities for Calderón-Zygmund operators without doubling conditions

after X. Tolsa [15]
A summary written by Jonas Azzam

Abstract

For an arbitrary Borel measure μ satisfying $\mu(B(x,r)) \leq Cr$ that *may be nondoubling*, we characterize those weights w such that all $L^2(\mu)$-bounded CZOs are also weak-type (p,p) bounded with respect to $wd\mu$. These in turn coincides with the class of weights for which weighted inequalities hold for a certain maximal operaters, and also satisfy a certain Sawyer-type condition.

1.1 Introduction

In a little over a decade, several results from classical harmonic analysis concerning Calderon-Zygmund operators (CZOs) have been generalized to the setting where the underlying measure is nondoubling. Much of this has been motivated by Painlevé's problem and the Cauchy integral operator. In David's work on Vitushkin's conjecture in [2], for example, it was necessary to prove a $T(b)$-theorem for CZOs on sets whose associated measures were non-doubling, contrasting to the classical case where the underlying measure is simply Lebesgue measure on Euclidean space.

Another example is Tolsa's work on extending the definitions of BMO and Hardy spaces on Euclidean space to the nondoubling scenario (see [12]). In this paper, for a measure μ of the form (1) below, he constructs a Banach space of functions RBMO=RBMO(μ) with the property that L^2 bounded CZOs are bounded from $L^\infty \to$ RBMO, functions in RBMO satisfy a John-Nirenberg inequality, and that RBMO coincides with the usual BMO space if μ is Lebesgue measure. Other works include (but are not limited to) [5, 6, 7, 13, 14, 16].

The main difficulty in these results is that the most of classical Calderon-Zygmund theory depends on the doubling property of Lebesgue measure. Without the doubling condition, essential tools such as the Vitali covering lemma are difficult to use.

In the paper being considered in this summary [15], Tolsa attempts to generalize some of the theory of weighted inequalities for singular integrals to the nondoubling setting.

We say that a function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a *(n-dimensional) Calderon-Zygmund kernel* if

1. $|k(x,y)| \lesssim \frac{1}{|x-y|^n}$ for $x \neq y$, and
2. there is some $\gamma \in (0,1]$ such that for $|x-x'| \leq \frac{1}{2}|x-y|$,

$$|k(x,y) - k(x',y)| + |k(y,x) + k(y,x')| \lesssim \frac{|x-x'|}{|x-y|}^{n+\gamma}.$$

In this case, we say the associated singular integral operator T is in $CZO(\gamma)$. We also let T_ε denote the truncated singular integral operator and define $T_* f(x) = \sup_{\varepsilon>0} |Tf(x)|$.

For a fixed $0 < n \leq d$, let μ be *any* Borel measure satisfying

$$\mu(B(x,r)) \leq r^n \qquad \text{for all } r > 0, x \in \text{supp}\mu. \tag{1}$$

We consider the problem of classifying weights $w : \mathbb{R}^d \to (0,\infty)$ such that all $L^2(\mu)$-bounded n-dimensional Calderon-Zygmund T are also $L^p(w)$ bounded, that is,

$$\int |Tf|^p w d\mu \lesssim \int |f|^p w d\mu \qquad \text{for all } f \in L^p(w). \tag{2}$$

The case when $n = d$ and μ is Lebesgue measure is a result of Coifman and Fefferman, see [1]. In particular, (2) holds if and only if w in the class of A_p-weights. This in turn coincides with the set of weights for which the Hardy-Littlewood maximal function is $L^p(w)$-bounded by [4]. (Some references for definitions and basic results of A_p weights include [3] and [11], although we will not need them here).

Loosely speaking, Tolsa establishes that, for a general nondoubling measure μ as above, the class of weights for which (2) holds is equal to the class of weights for which a certain maximal function is bounded on $L^p(w)$. In addition, these weights are characterized by certain Sawyer-type conditions. We state these results precisely in Theorems 7 and 8 below.

1.2 Notation and Preliminaries

Here we overview some of the tools developed by Tosla in [12, 14, 13].

1.2.1 Cubes

We assume that $\mu(Q) \leq \ell(Q)^n$, where Q is any cube in $\mathbb{R}^d$ with sides parallel to the axes and $\ell(Q)$ denotes the side-length of Q. For such a cube, we will denote its center by z_Q. We will also call $\{x\}$ for $x \in \mathbb{R}^d$ and $\mathbb{R}^d$ cubes (with side-lengths 0 and ∞ respectively). Note that the cube $\mathbb{R}^d$ is considered to be concentric with all cubes. To distinguish between these classes, we will call all Q with $0 < \ell(Q) < \infty$ *transit cubes.*

We define a "distance" on cubes as follows: for $Q \subseteq R$ two cubes, let Q_R be the smallest cube concentric with Q containing R. Define

$$\delta(Q, R) = \int_{Q_R \setminus Q} \frac{1}{|x - z_Q|^n} d\mu(x).$$

Note that $\delta(x, Q)$ and $\delta(\mathbb{R}^d, Q)$ make sense, and can be infinite.

Lemma 1. *Let $P \subseteq Q \subseteq R$ be cubes. Then the following hold:*

1. *If $\ell(Q) \sim \ell(R)$, then $\delta(Q, R) \sim 1$. In particular, $\ell(Q) \leq t\ell(R)$ implies $\delta(Q, R) \lesssim 2^n t^n$.*

2. *$\delta(Q, R) \lesssim 1 + \log \frac{\ell(R)}{\ell(Q)}$.*

3. *$\delta(P, R) = \delta(P, Q) + \delta(Q, R) \pm \varepsilon_0$.*

From hereon, $a = b \pm \varepsilon_0$ denotes that $a = b + \varepsilon$ where $|\varepsilon| \leq \varepsilon_0$.

One can think of δ as a way of measuring the difference in scale between cubes. In the case that μ is Lebesgue measure, notice that for any cube Q we have $\delta(Q, 2Q) \sim 1$. This doesn't necessarily hold for general measures μ of the form we are considering.

For the analysis ahead, we need a lattice of cubes that act as a substitute for the usual dyadic grid in $\mathbb{R}^d$. In the Lebesgue measure case, the δ-distance between concentric cubes of consecutive scales (say they are sizes 2^k and 2^{k+1}) is a positive constant, and translates of the same cube have the same scale. We would like a lattice of cubes that behaves in a similar fashion.

Lemma 2. *Let A be a large constant. There is a family of cubes $\{Q_{x,k}\}_{x\in supp\mu}^{k\in\mathbb{Z}}$ such that the following hold:*

1. *For each $x \in supp\mu$, the cubes $Q_{x,k} \subseteq Q_{x,k-1}$ are decreasing concentric cubes such that $\bigcap_k Q_{x,k} = \{x\}$ and $\bigcup_k Q_{x,k} = \mathbb{R}^d$.*
2. *$\delta(Q_{x,k}, Q_{x,j}) \le (j-k)A \pm \varepsilon$ if $j > k$.*
3. *$\delta(Q_{x,k}, Q_{x,j}) = (j-k)A \pm \varepsilon$ if $j > k$ and $Q_{x,k}, Q_{x,j}$ are transit cubes.*
4. *$2Q_{x,k} \cap 2Q_{y,k} \neq \emptyset$ implies that $2Q_{x,k} \subseteq Q_{y,k-1}$ and $\ell(Q_{x,k}) \le \frac{1}{100}\ell(Q_{y,k-1})$.*
5. *There is $\eta > 0$ such that If $m \ge 1$ and $2Q_{x,k+m} \cap 2Q_{y,k} \neq \emptyset$, then $\ell(Q_{x,k}) \le 2^{-\eta A m}\ell(Q_{y,k})$.*
6. *$\ell(Q_{x,k})$ is Lipschitz in $x \in supp\mu$ uniformly in k. Here, all constants depend on d, ε_0, and n and are independent of A.*

Remark 3. *Condition (2) is necessary if either $\delta(x, Q)$ or $\delta(Q, \mathbb{R}^d)$ are finite. In the former case, there is a constant K_x such that $k \ge K_x$ implies $Q_{x,k} = \{x\}$, and we call $\{x\}$ a stopping cube. In the latter case, there are constants K_x' and K_x'' such that for any $x \in supp\mu$ and $k \ge K$ we have $Q_{x,k}$ we call $\mathbb{R}^d$ an initial cube,*

The following definition gives us a way of determining the "scale" of an arbitrary cube not part of our lattice

Definition 4. *Let $\mathcal{D}_k = \{Q_{x,k} : x \in supp\mu\}$, $\mathcal{D} = \bigcup \mathcal{D}_k$. We say $Q \in \mathcal{AD}_k$ if there is $Q_{x,k}$ such that*

$$\ell(Q_{x,k}) \le \frac{100}{99}\inf\{\ell(Q_{y,j}) : Q \subseteq Q_{y,j}\}.$$

1.2.2 The averaging functions s_k and the maximal operator N

The following lemma constructs functions $s_k(x, \cdot)$ that are intended to play the role of a scaled bump function centered at x, that is, a compactly supported function with some smoothness whose mass is concentrated near the cube $Q_{x,k}$ and whose mass is roughly constant as we shift in position and scale.

Lemma 5. *If A is large enough, then for $x \in supp\mu$ and $k \in \mathbb{Z}$ there are radially decreasing functions $s_k(x, \cdot)$ supported in $2Q_{x,k-1}$ such that:*

1. $s_k(x, y) \sim \frac{1}{A\ell(Q_{x,k})}$ *for* $y \in Q_{x,k}$.
2. $s_k(x, y) = \frac{1}{A|x-y|^n}$ *for* $y \in Q_{x,k-1} \backslash Q_{x,k}$, *(so in particular,* $s_k(x, y) \leq \frac{1}{A|x-y|^n}$ *for all* $y \in \mathbb{R}^d$*).*
3. $\nabla_y s_k(x, y) \lesssim A^{-1} \min\left(\frac{1}{\ell(Q_{x,k})^{n+1}}, \frac{1}{|x-y|^{n+1}}\right)$ *for all* $y \in \mathbb{R}^d$.
4. *The* $s_k(x, \cdot)$ *are normalized in the sense that* $\int s_k(x, y) d\mu(y) \in (\frac{9}{10}, \frac{10}{9})$.

By the previous lemma and the properties of the cubes $Q_{x,k}$, we have the following lemma about the $s_k(x, \cdot)$ and their conjugates:

Lemma 6.

1. *If* $y \in supp\mu$, *then* $supps_k(\cdot, y) \in Q_{y,k-2}$.
2. *If* $Q \in \mathcal{AD}_k$ *and* $y \in Q \cap supp\mu$, *then* $supps_{k+m}(y, \cdot) \subseteq \frac{10}{9}Q$ *for* $m \geq 3$ *and* $s_{k+m}(\cdot, y) \subseteq \frac{10}{9}Q$ *for* $m \geq 4$.
3. *For all* $k \in \mathbb{Z}$ *and* $y \in supp\mu$, *the conjugates are normalized in the sense that* $\int s_k(x, y) d\mu(x) \in (\frac{9}{10}, \frac{10}{9})$.
4. *The kernels are almost symmetric, in the sense that*
$$s_k(x, y) \lesssim s_{k-1}(y, x) + s_k(y, x) + s_{k+1}(y, x).$$
5. $\nabla_x s_k(x, y) \lesssim \frac{1}{A|x-y|^{n+1}}$.

In particular, the operator $S_k f(x) = \int s_k(x, y) f(y) d\mu(y)$ *is a CZO, with associated constants uniform in* k.

All items except for (5) may be shown by previous lemmas, whereas item (5) depends on the actual construction of the $s_k(x, \cdot)$ (although the key is that $\ell(Q_{x,k})$ is Lipschitz in x).

In the classical case, most of these technicalities are avoided since we may let $s_k(x, y) = \varphi_{\ell(Q_{x,k})}(x - y)$, where φ is some bump function adapted to the unit cube centered at zero, and $\varphi_{\ell(Q)}$ is the same function scaled by $\ell(Q)$.

Now that we have an appropriate averaging function, we may now define the maximal operator N. For $f \in L^1_{loc}(\mu)$ simply define

$$Nf(x) = \sup_k S_k f(x).$$

It is not difficult to show that $Nf \lesssim M^c f$ where M^c is the usual centered Hardy-Littlewood maximal function. However, the reverse inequality does not hold in general.

1.3 Statement of Results

We are now in the position to state the main results. For a measurable function w, let $w(Q) = \int_Q w d\mu$.

Theorem 7. *Let $p \in [1, \infty)$ and $\gamma \in (0, 1]$. Let $w > 0$ be a μ measurable function, and if $p \neq 1$ let $\sigma = w^{-p'/p}$. Then the following are equivalent:*

1. *All $T \in CZO(\gamma)$ are weak-type (p, p) bounded with respect to $wd\mu$.*
2. *For all $T \in CZO(\gamma)$, T_* is weak-type (p, p) bounded with respect to $wd\mu$.*
3. *N is weak-type (p, p) bounded with respect to $wd\mu$.*
4. *The S_k are uniformly weak-type (p, p) bounded with respect to $wd\mu$.*
5. *If $p > 1$, then the following holds uniformly in $k \in \mathbb{Z}$ and $Q \subseteq \mathbb{R}^d$:*

$$\int |S_k^*(w\chi_Q)|^{p'} \sigma d\mu \lesssim w(Q). \tag{3}$$

Theorem 8. *With the same conditions as in the previous theorem, the same theorem holds with the words "weak-type (p, p) bounded" replaced with "L^p bounded" and in addition to* (3) *in (5), the following inequality holds:*

$$\int |S_k(\sigma\chi_Q)|^p w d\mu \lesssim \sigma(Q). \tag{4}$$

Weights satisfying the conditions in the first or second theorem are called weak-Z_p and Z_p-weights respectively.

Items (1) and (2) of these theorems are generalizations of the A_p-weight characterizations via singular integrals and maximal functions. In addition, also note the Sawyer-type conditions (3) and (4) on the averaging operators. In fact, the structures of the proofs of the two theorems are based upon those in [9] and [10] respectively.

Remark 9. *In [8], it is shown that for μ an arbitrary Borel measure, if $w \in A_p$ (where A_p is defined with respect to μ in the usual sense), then the centered Hardy-Littlewood maximal operator M^c is $L^p(w)$ bounded. Since $Nf(x) \lesssim Mf(x)$, by the above results, A_p is not necessarily equivalent to Z_p. However, there are examples of weights that don't satisfy a reverse Hölder inequality, and thus $A_p \subsetneq Z_p$.*

We will focus entirely on the proof of Theorem 7 in the case that there are no stopping or initial cubes (so all cubes are transition cubes).

1.4 Outline of proof of Theorem 7

1.4.1 The simple implications: (2) $\Rightarrow$ (1) $\Rightarrow$ (4) $\Rightarrow$ (5)

Most of the implications in Theorem 7 follow easily. For example, (2) implies (1) trivially.

If we assume (1), then each individual S_k is weakly bounded, so it remains to show that they are uniformly weakly-bounded. This follows from the fact that the S_k are CZO with uniform constants. Suppose they were not uniformly bounded, so we may find a sequence k_j such that $||S_{k_j}||_{L^p(w)\to L^{p,\infty}(w)} \geq j^2$. Let $T = \sum \frac{1}{j^2} S_{k_j}$, which is also CZO (here, we are using the fact that the CZO constants for the S_k are all uniformly bounded) and thus also bounded by (1). By the positivity of the S_{k_j}, we have for any $j_0 \in \mathbb{N}$,

$$||T||_{L^p(w)\to L^{p,\infty}(w)} \geq \sum_{j=1}^{j_0} \frac{||S_{k_j}||_{L^p(w)\to L^{p,\infty}(w)}}{j^2} \geq j_0,$$

which contradicts the boundedness of T. Thus, (1) implies (4).

The implication (4) $\Rightarrow$ (5) is proven almost verbetum as its analogue in [9]. Since S_k are uniformly weak-type (with respect to $wd\mu$), so are their duals. Let $f \in L^p(\sigma d\mu)$. Then by the uniform weak-boundedness of the S_k^*,

$$\begin{aligned}\int S_k(w\chi_Q)f\sigma d\mu = \int_Q S_k^*(f\sigma)wd\mu = \int_0^\infty w\{x \in Q : S_k^*(f\sigma) > \lambda\}d\lambda \\ \leq \int_0^\infty \min\{\lambda^{-p}||S_k^* f||_{L^{p,\infty}(w)}, w(Q)\}d\lambda \lesssim ||f||_{L^{p'}(\sigma)} w(Q)^{\frac{1}{p'}}\end{aligned}$$

and the implication follows.

1.4.2 The implication (3) $\Rightarrow$ (2)

The implication (3) $\Rightarrow$ (2) resembles the traditional proof of weak weighted inequality for singular integrals. Namely, we first establish a good-λ inequality of the form

$$w\{x : T_* f(x) > (1+\varepsilon)\lambda, Nf(x) \leq \delta\lambda\} \leq (1-\eta)w\{x : T_* f(x) > \lambda\} \quad (5)$$

where $\eta > 0$ is some fixed constant, $\varepsilon, \lambda > 0$, and $\delta = \delta(\varepsilon) > 0$.

One way to establish this in the traditional case (that is, when μ is Lebesgue measure and N is instead the uncentered maximal function) is, more or less, to first prove the inequality for $w \equiv 1$ and then the A_∞ condition on w to get the inequality. However, we do not have this condition when working with a general measure μ. Instead, we have the following weaker form of the A_∞ condition:

Definition 10. *We say that a weight w is in Z_∞ if there is $\tau > 0$ such that for any $Q \in \mathcal{AD}_k$ and $A \subseteq \mathbb{R}^d$,*

$$S_{k+3}(\chi_A)(x) \geq \frac{1}{4}\chi_Q \quad \Rightarrow \quad w(A \cap 2Q) \geq \tau w(Q). \quad (6)$$

If (3) holds in the Theorem, it is not hard to show that $w \in Z_\infty$. With this fact in hand, the proof runs in a similar fashion to the traditional proof, although a bit more technical given our framework and the lack of the full A_∞ condition.

1.4.3 The implication (5) $\Rightarrow$ (3)

The model for this part is Sawyer's theorem on two-weight weak-type inequalities for fractional integral operators (see [9]). We will give a rough sketch of the proof of this implication.

Let $T = N + \beta M_R$, where M is the radial uncentered Hardy-Littlewood maximal function

$$Mf(x) = \sup_{B \ni x} \frac{1}{r(B)^n} \int_B f d\mu,$$

and β is some large fixed constant. It is not difficult to show that $M_R f(x) \lesssim Nf(x)$, so it suffices to prove T is weakly bounded, and in particular, we do so by proving that, for some $\eta \in (0,1)$, the following good-λ inequality holds for all $\varepsilon, \lambda > 0$

$$w(\Omega_{(1+\varepsilon)\lambda}) \leq \eta w(\Omega_\lambda) + \frac{C_\varepsilon}{\lambda^p} \int |f|^p w d\mu, \tag{7}$$

where $\Omega_\lambda = \{Tf > \lambda\}$.

The reason for introducing this intermediate operator is that it is more convenient to work with due to the following lemma:

Lemma 11. *For $\varepsilon > 0$ there are $\beta > 0$ and $m \in \mathbb{N}$ large enough such that the following holds: if $\mathcal{W} = \{Q_j\}$ is a Whitney decomposition of Ω_λ, and $U_m(Q)$ is the union of all Whitney cubes that are linked to $3Q$ by a chain of no more than m adjacent cubes in $\mathcal{W}$, then for any $Q \in \mathcal{W}$ and $x \in Q$,*

$$T(f\chi_{\mathbb{R}^d \setminus U_m(Q)} \leq (1 + \frac{\varepsilon}{2})\lambda. \tag{8}$$

Since $\Omega_{(1+\varepsilon)\lambda} \subseteq \Omega_\lambda$, it is covered by the collection $\mathcal{W}$, so if $Q \in \mathcal{W}$ and $x \in Q \cap \Omega_{(1+\varepsilon)\lambda}$, by the lemma we have the estimate

$$T(f\chi_{U_m(Q)}) \geq \frac{\lambda\varepsilon}{2} \chi_{Q \cap \Omega_{(1+\varepsilon)\lambda}}. \tag{9}$$

Loosely speaking, since T is bounded pointwise by a constant times N, this says that for each $x \in Q \cap \Omega_{(1+\varepsilon)\lambda}$ there is a k with $S_k(f\chi_{U_m(Q)})$ large. If we knew that there was one (or a finite sum of) such $S_k(f\chi_{U_m(Q)})$ where this was true on a large "good" subset of $Q \cap \Omega_{(1+\varepsilon)\lambda}$, then we could estimate the $wd\mu$-measure of this subset by integrating $S_k(f\chi_{U_m(Q)})$ over Q, which would put us in a position to use the Sawyer-type estimate (3). This will give us a decent estimate on the size of our "good" set, and the theorem will follow so long as we verify that the "bad" complement of this set has small measure.

More precisely, for each $Q \in \mathcal{W}$, let h_Q be such that $Q \in \mathcal{AD}_h$. For a fixed $\delta > 0$ small and n_1 a large integer, we define

$$G_\lambda = \bigcup_{Q \in \mathcal{W}} \{x \in Q \cap \Omega_{(1+\varepsilon)\lambda} : S_k(f\chi_{U_m(Q)}) \geq \delta\lambda, \quad k = h_Q - n_1, ..., h_Q + 5\},$$

and let $B_\lambda = \Omega_{(1+\varepsilon)\lambda} \backslash G_\lambda$.

For n_1 large enough and δ small enough, we may ensure that

$$w(B_\lambda) \leq \eta_1 w(\Omega_\lambda) \tag{10}$$

for some $\eta_1 \in (0, 1)$. To do this, we prove that

$$S_{h_Q+3}(\chi_{\mathbb{R}^d \backslash B_\lambda}) \geq \frac{1}{4}\chi_Q,$$

then apply the Z_∞ condition to get the estimate $w(2Q\backslash B_\lambda) \geq \tau w(Q)$, and then sum over all $Q \in \mathcal{W}$ (recalling that the Whitney decomposition has bounded overlap).

Finally, with n_1 and δ fixed, for each $Q \in \mathcal{W}$ we estimate $w(G_\lambda \cap Q)$ using the Sawyer-type estimates. Using (3) and the definition of G_λ, it is not difficult to show

$$w(Q \cap G_\lambda) \lesssim w(Q)^{\frac{1}{p'}} \left(\int_{U_i} fwd\mu\right)^{\frac{1}{p}} \leq \theta w(Q) + C_{\theta,p}\frac{1}{\lambda^p}\int_{U_i} |f|^p wd\mu,$$

where we just used the inequality $a^{\frac{1}{p'}} b^{\frac{1}{p}} \leq \theta a + \theta^{-\frac{p'}{p}} b$. Fixing θ small enough, and using the facts that $G_\lambda \cup B_\lambda = \Omega_{(1+\varepsilon)\lambda}$ and that the U_i have bounded overlap, we add this to (10) to obtain (9).

For the case $p = 1$, a slight adjustment of the arguments above gives (4) $\Rightarrow$ (3) in the theorem.

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2 Astala's Conjecture on Distortion of Hausdorff Measures under Quasiconformal Maps in the Plane

after Michael T. Lacey, Eric T. Sawyer, and Ignacio Uriarte-Tuero[4]
A summary written by Oleksandra Beznosova

Abstract

We outline the proof of the Astala's conjecture on distortion of Hausdorff measures under quasiconformal maps in the plane.

2.1 Introduction

The notion of a quasiconformal mapping, but not the name, was introduced by H. Grötzsch in 1928. In 1935 this notion reappeared in the work of Lavrentiev in relation to the partial differential equations. In 1936 Ahlfors included a reference to the quasiconformal case to the theory of covering surfaces. From then on the notion became generally known and widely used.

An orientation-preserving homeomorphism $\phi : \Omega \to \Omega'$ between planar domains Ω, $\Omega' \subset \mathbb{C}$ is called *K-quasiconformal* if it belongs to the Sobolev space $W^{1,2}_{loc}(\Omega)$ and satisfies the *distortion inequality*

$$\max_{\alpha} |\partial_\alpha \phi| \ \leq K \min_{\alpha} |\partial_\alpha \phi| \quad a.e. \ in \ \Omega.$$

We will focus on the properties of quasiconformal maps with respect to the Hausdorff measure on the complex plane.

Given set $E \subset \mathbb{C}$, $0 \leq s \leq 2$, and $0 < \delta \leq \infty$, first define

$$\mathcal{H}^s_\delta(E) := \inf \left\{ \sum_{i=1}^{\infty} diam(B_i)^s \quad : \quad E \subset \bigcup_{i=1}^{\infty} B_i \ , \ diam(B_i) \leq \delta \right\},$$

where $B_i \subset \mathbb{C}$ is a set, and $diam(B_i)$ stands for its diameter. Clearly, $\mathcal{H}^s_\delta(E)$ decreases as δ increases. The quantity $\mathcal{H}^s_\infty(E)$ is usually called the *Hausdorff content* of the set E.

We define the s-dimensional Hausdorff measure to be:

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

Note that $\mathcal{H}^s = 0 \quad \Leftrightarrow \quad \mathcal{H}^s_\infty = 0$.

It is well known (see Ahlfors [1]) that the image under a quasiconformal mapping of a set of zero Lebesgue measure also has the Lebesgue measure zero. Similar is true for the sets of Hausdorff dimension zero, but they need not preserve Hausdorff dimension bigger than zero. In 1956 Mori showed that K-quasiconformal mappings are locally Hölder continuous with exponent $\frac{1}{K}$, and this exponent is the best possible. In 1957 Bojarski showed that quasiconformal mappings also distort the area by a power that only depends on K. It was conjectured by Gehring and Reich in 1966 and proven Astala in 1994 [2] that optimal exponent in area distortion is the same, $\frac{1}{K}$. As a corollary to the area distortion result, Astala showed the theorem below, which proves the special case of a conjecture (for $n = 2$) of Iwanec and Martin.

Theorem 1. ***(Astala)****Let $\phi : \Omega \to \Omega'$ be K-quasiconformal and suppose that $E \subset \Omega$ is compact . Then*

$$\frac{1}{K}\left(\frac{1}{dim(E)} - \frac{1}{2}\right) \leq \frac{1}{dim(\phi E)} - \frac{1}{2} \leq K\left(\frac{1}{dim(E)} - \frac{1}{2}\right). \tag{1}$$

Moreover, these bounds are optimal since equality may occur in either estimate.

This is a sharp theorem for Hausdorff dimensions, we will state and prove its refinement to Hausdorff measures.

Theorem 2. ***(Main Theorem) Lacey, Sawyer, and Uriarte-Tuero***

If ϕ is a planar K-quasiconformal mapping, $0 \leq t \leq 2$ and $t' = \frac{2Kt}{2+(K-1)t}$, then we have implication below for all compact sets $E \subset \mathbb{C}$.

$$\mathcal{H}^t(E) = 0 \quad \Rightarrow \quad \mathcal{H}^{t'}(\phi E) = 0 \tag{2}$$

First notice that the left side of (1) follows from (2) in a straight-forward way. In order to see that (2) implies the right side of (1), we first observe that the inverse of a K-quasiconformal mapping is K-quasiconformal, apply main theorem to the inverse mapping and rewrite (2) as $\mathcal{H}^{t'}(F) > 0 \quad \Rightarrow \mathcal{H}^t(\phi F) > 0$, which implies the right side of (1).

2.2 Sketch of the Proof of Main Theorem

1. We will first reduce the Main theorem to the case of K-quasiconformal mappings of bounded dilatation K. Lemma 3 is a restatement of the main theorem for such mappings.

2. Bound on the dilatation constant of the quasiconformal mapping allows us to apply Stoilow's factorization, split it into the 'conformal inside' part and 'conformal outside' part and consider those two parts separately. Moreover, 'conformal inside' part is handled in [3], which reduces the proof of the main theorem to the 'conformal outside' mapping.

3. The 'conformal outside' part is the main focus of the paper. In order to handle this, we will introduce families $\mathcal{P}$ of the dyadic cubes that obey a t-packing condition and corresponding measures $w_{t,\mathcal{P}}$. Following Astala's approach, statement then reduces to the boundedness of the Beurling operator on $L^p(w_{t,\mathcal{P}})$ spaces, which does not follow from the A_p theory.

Step 1. Let us first restate the Main Theorem for the case of quasiconformal mappings of bounded dilatation.

Lemma 3. *Let $0 < t < 2$. Then there is a small constant $0 < k_0 < 1$ ($k_0 = k_0(t)$ is a decreasing function of t) so that the following holds.*

Let $\phi : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping with $\frac{K-1}{K+1} \leq k_0$. Then we have the following implication for all compact subsets $E \subset \mathbb{C}$.

$$\mathcal{H}^t(E) = 0 \quad \Rightarrow \quad \mathcal{H}^{t'}(\phi E) = 0,$$

where $t' = \frac{2Kt}{2+(K-1)t}$.

The Main Theorem follows from the Lemma 3 by standard decomposition argument, that can be found in [1] or [5]. We will skip this proof. What follows is the proof of Lemma 3

Step 2. We consider a K-quasiconformal mapping ϕ with $\frac{K-1}{K+1} \leq k_0$ and a compact set $E \subset \mathbb{C}$. Following [3], we normalize mapping ϕ using properties of the Möbius transformation which allows us to assume that E is a compact subset of $(\frac{1}{32}, \frac{1}{16})^2 \subset \frac{1}{8}\mathbb{D} \subset \mathbb{C}$.

Using Stoilow's factorization and Koebe's distortion theorem (see [6]), one can show that without loss of generality, we can further assume that ϕ is a principal mapping (i.e. they are K-quasiconformaal mappings that are conformal outside $\bar{\mathbb{D}}$ and normalized by $\phi(z) - z = O\left(\frac{1}{|z|}\right)$ *as* $z \to \infty$).

Step 3. On this step we will go over the covering arguments.

Hausdorff measure produced by dyadic cubes is equivalent measure to the one produced by balls or arbitrary sets. By the dyadic cubes we will mean $[2^{-k}m_1, 2^{-k}(m_1+1)] \times [2^{-k}m_2, 2^{-k}(m_2+1)]$, $k \in \mathbb{N}$, $m_1, m_2 \in \mathbb{Z}$.

Let $\mathcal{P} = \{P_j\}_{j=1}^N$ be a finite collection of disjoint dyadic cubes in the plane. Let $0 < t < 2$. We define the t-Carleson packing norm of $\mathcal{P}$ as follows:

$$||\mathcal{P}||_{t-pack} := \sup_Q \left[\ell(Q)^{-t} \sum_{P\in\mathcal{P}, P\in Q} \ell(P)^t \right]^{\frac{1}{t}},$$

where supremum is taken over all dyadic cubes Q and $\ell(Q)$ is the side-length of the cube Q.

We also define the measure $w_{t,\mathcal{P}}$ associated with $\mathcal{P}$ by

$$w_{t,\mathcal{P}} := \sum_j l(P_j)^{t-2} \chi_{P_j}(x).$$

Note that the measure $w_{t,\mathcal{P}}$ behaves as a t-dimensional measure, i.e. for any cube Q (dyadic or not) with sides parallel to the coordinate axes

$$w_{t,\mathcal{P}} \leq 16||\mathcal{P}||^t_{t-pack} \ell(Q)^t.$$

The following proposition allows us to approximate E by a finite union of cubes and case $m = 2$ is one of the key tools in the proof of the Main Theorem. We will state it without a proof.

Proposition 4. *Let $m \geq 0$ be an integer. Then there is a positive constant C such that, for any compact set $E \subset (0,1)^2 \subset \mathbb{C}$, $0 < t < 2$, and $\epsilon > 0$, there is a finite collection of closed dyadic cubes $\mathcal{P} = \{P_j\}_{j=1}^N$ such that*

(a) $2^m P_i \cap 2^m P_j = \emptyset$ *for* $i \neq j$.

(b) $E \subset \bigcup_j 3 \cdot 2^m P_j$.

(c) $||\mathcal{P}||_{t-pack} \leq 1$.

(d) $\sum_j \ell(P_j)^t \leq C(\mathcal{H}^t_\infty(E) + \epsilon)$.

Consider $\epsilon > 0$ and use Proposition 4, with $m = 2$, to obtain a collection of cubes $\mathcal{P} = \{P_i\}$, satisfying the conclusions of Proposition 4 with respect to the compact set E. Denote $\Omega = \bigcup_i P_i$.

Following [2], decompose $\phi = g \circ f$, where both g and f are principal K-quasiconformal mappings, f is conformal outside Ω, and g is conformal in $f(\Omega) \cup \mathbb{C} \setminus \mathbb{D}$. The 'conformal inside' part, g, has been addressed in [3], see Theorem 5. The 'conformal outside' mapping f is handled by the following Lemma 6.

Theorem 5. *Let $g : \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping. Let $\{S_j\}_{j=1}^N$ be a finite family of pairwise disjoint quasi-disks in $\mathbb{D}$, such that $S_j = f(D_j)$ for a single K-quasiconformal map f and for discs (or cubes)*

D_j, and assume that g is conformal in $\Omega = \bigcup_j S_j$. Then for any $t \in (0, 2]$ and $t' = \frac{2Kt}{2+(K-1)t}$, we have

$$\left(\sum_{j=1}^{N} diam(g(S_j))^{t'}\right)^{\frac{1}{t'}} \leq C(K) \left(\sum_{j=1}^{N} diam(g(S_j))^{t}\right)^{\frac{1}{tK}}.$$

Lemma 6. *Let $0 < t < 2$. There is a positive constant ϵ_0 (which is a decreasing function of t) so that the following holds.*

Let $\mathcal{P} = \{P_j\}_{j=1}^N$ be a finite collection of dyadic cubes which satisfies the t-Carleson packing condition $||\mathcal{P}||_{t-pack} \leq C$. Assume further that cubes $3P_j$ are pairwise disjoint. Let $E = \bar{P} = \bigcup_j P_j$ and let $f : \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping which is conformal outside the compact set E, with $\frac{K-1}{K+1} < \epsilon_0$.

Then, there is a constant $C(K, t)$, which for fixed K is a decreasing function of t, such that

$$\sum_{j=1}^{N} diam(fP_j)^t \leq C(K, t) \sum_{j=1}^{N} \ell(P_j)^t.$$

Proof of this Lemma heavily relies on the weighted norm inequalities for Beurling operator on $L^2(w_{t,\mathcal{P}})$. They do not follow from the standard A_p theory and require a careful proof. The proof uses combinatorial properties of measures $w_{t,\mathcal{P}}$ and can be extended to the class of Calderón-Zygmund operators.

Since dilatation of ϕ is at most ϵ_0, the dilatation of f satisfies the same bound, so that Lemma 6 applies to it.

By quasi-symmetry, Theorem 5 and Lemma 6, one can show that

$$\begin{aligned}
\mathcal{H}^{t'}_\infty(\phi E) &\leq \mathcal{H}^{t'}_\infty\left(\phi\left(\bigcup_i 12 \cdot P_i\right)\right) \leq \sum_i diam(\phi(12 \cdot P_i))^{t'} \\
&\leq C(K) \sum_i diam(\phi(P_i))^{t'} \leq C(K) \left(\sum_i diam(f(P_i))^t\right)^{\frac{t'}{tK}} \\
&\leq C(K, t) \left(\sum_i \ell(P_i)^t\right)^{\frac{t'}{tK}} \leq C(K, t) \left(\mathcal{H}^t_\infty(E) + \epsilon\right)^{\frac{t'}{tK}} \leq C(K, t)\, \epsilon^{\frac{t'}{tK}}.
\end{aligned}$$

This, up to Proposition 4 and Lemma 6 Completes the proof of the Main Theorem.

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3 The Bellman functions and two-weight inequalities for Haar multipliers

after F. Nazarov, S. Treil, and A. Volberg [1]
A summary written by Nicholas Boros

Abstract

We outline a proof of the dyadic two weight Carleson imbedding theorem and a proof of the bilinear imbedding theorem, which is a dyadic version of a Sawyer type theorem.

3.1 Introduction

Let u, v be weights that are locally integrable, $\phi \in L^2(\mathbb{R}, \mathbb{R})$ and T_0 an operator that will act on ϕ. We will find certain classes of T_0 such that

$$\int_{\mathbb{R}} |T_0\phi|^2 v \leq C \int_{\mathbb{R}} |\phi|^2 u, \forall \phi.$$

To help answer this question we first give a proof of the two-weight dyadic Carleson imbedding theorem in Section 3.2. In Section 3.3 we prove the bilinear imbedding theorem, which is a partial answer to the question and can help find more such operators T_0 that answer the question. This result is the dyadic version of Verbitsky's result [3], which is a generalization of Sawyer's original result [2], for which T_0 is an integral operator with positive kernel. The Bellman function technique will be used to prove both, which requires no background except basic analysis and linear algebra. In Section 3.4 we use the two theorems to give more classes of T_0 for which the above estimate is true. For notation we use $\langle \phi \rangle_I$ to denote $\frac{1}{|I|}\int_I \phi dx$, where dx is the Lebesgue measure and $I \in \mathcal{D}$ denotes I as a dyadic subinterval of $\mathbb{R}$. Also, we use the notation I^- to denote the left half of I and I^+ for the right half.

3.2 Two-weight dyadic Carleson imbedding theorem

Theorem 1. *Let w be any weight and $\{\alpha\}_{J\in\mathcal{D}}$ be a sequence of non-negative numbers. Then $\sum_{J\in\mathcal{D}} \alpha_J \langle \phi\sqrt{w}\rangle_J^2 \leq C\|\phi\|_{L^2}^2$ for all $\phi \in L^2$ is equivalent to (SC) $\frac{1}{|I|}\sum_{J\subset I} \alpha_J \langle w\rangle_J^2 \leq C\langle w\rangle_I, \forall I \in \mathcal{D}$.*

Proof. ($\Rightarrow$) Use $\phi = \sqrt{w}\chi_I$ and take the sum over intervals $J \subset I$.

($\Leftarrow$) Reductions: (1) We can assume that $\phi \geq 0$, since the result can then be easily extended using the fact that for any $x, y \in \mathbb{R}$, $(x-y)^2 \lesssim x^2 + y^2$. (2) We can assume that $C = 1$ in (SC) because it can be made so by scaling the weight. (3) We only need to prove the result summing over the dyadic intervals inside of some $I_k \in \mathcal{D}$.

We now define the Bellman function, $\mathcal{B}$, which we can use to prove our result. Let $w_I := \langle w\rangle_I$, $f_I := \langle \phi\sqrt{w}\rangle_I$, $F_I := \langle \phi^2\rangle_I$, $M_I := \frac{1}{|I|}\sum_{J\subset I, J\in\mathcal{D}} \alpha_J \langle w\rangle_J^2$. For $I \in \mathcal{D}$ define $\mathcal{B}(F_I, f_I, w_I, M_I) :=$

$$\sup_{\phi, w, \{\alpha_J\}_{J\in\mathcal{D}}} \left\{ \frac{1}{|I|} \sum_{J\subset I, J\in\mathcal{D}} \alpha_J \langle \phi\sqrt{w}\rangle_J^2 : \sum_{J\subset K} \alpha_J \langle w\rangle_J^2 \leq \langle w\rangle_K, \forall K \in \mathcal{D} \right\}$$

on the domain $\Omega = \{(F, f, w, M) \in \mathbb{R}^4 : f^2 \leq Fw, M \leq w; F, f, w, M \geq 0\}$. The first inequality in Ω is because of Cauchy-Schwarz and the second is due to (SC). Note that $\mathcal{B}$ is independent of the initial choice of $I \in \mathcal{D}$.

Bellman function properties:

1) For all $(F_I, f_I, w_I, M_I) \in \Omega$ we have $0 \leq \mathcal{B}(F_I, f_I, w_I, M_I) \leq CF_I$.

2) (Weak Concavity) For all $(F_{I^\pm}, f_{I^\pm}, w_{I^\pm}, M_{I^\pm}), (F_I, f_I, w_I, M_I) \in \Omega$ satisfying $F_I = \frac{F_{I^-}+F_{I^+}}{2}$, $f_I = \frac{f_{I^-}+f_{I^+}}{2}$, $w = \frac{w_{I^-}+w_{I^+}}{2}$, $M_I = \frac{M_{I^-}+M_{I^+}}{2} + \frac{1}{|I|}\alpha_I w_J^2$ the following estimate holds:

$$\mathcal{B}(F_I, f_I, w_I, M_I) \geq \frac{1}{2}\Big(\mathcal{B}(F_{I^-}, f_{I^-}, w_{I^-}, M_{I^-}) + \mathcal{B}(F_{I^+}, f_{I^+}, w_{I^+}, M_{I^+})\Big) + \frac{\alpha_I f_I^2}{|I|}.$$

The upper bound of Property 1 is what we need to finish proving the theorem. When setting up a Bellman function we almost always have or require a weak concavity/convexity property. The proof of Property 2 is just a matter of plugging in the variables and taking supremums. Assuming that such a Bellman function, $\mathcal{B}$, exists we can now finish the proof of the theorem.

Property 2 can be rewritten as

$$\alpha_J f_J^2 \leq |J|\mathcal{B}(F_J, \ldots, M_J) - |J^-|\mathcal{B}(F_{J^-}, \ldots, M_{J^-}) - |J^+|\mathcal{B}(F_{J^+}, \ldots, M_{J^+}),$$

for all $J \in \mathcal{D}$, to show that

$$\sum_{J\subset I, J\in\mathcal{D}, |J|>2^{-k}|I|} \alpha_J f_J^2 \leq |I|\mathcal{B}(F_I, \ldots, M_I) - \sum_{J\subset I, J\in\mathcal{D}, |J|=2^{-k}|I|} |J|\mathcal{B}(F_J, \ldots, M_J)$$
$$\leq |I|\mathcal{B}(F_I, f_I, w_I, M_I) \leq C|I|F_I.$$

The desired estimate follows by taking the limit as k approaches infinity.

Note that the following properties $(2')$ $\mathcal{B}$ is concave (i.e. $d^2\mathcal{B} \geq 0$) and $(2'')$ $\mathcal{B}_M \geq \frac{f^2}{w^2}$ imply Property 2 of the Bellman function. So we present the function $B(F, f, w, M) = 4\Big(F - \frac{f^2}{w+M}\Big)$, which satisfies Properties $1, 2'$ and $2''$, thus we are finished with the proof. $\square$

3.3 Bilinear imbedding theorem

Lemma 2. *Let u, v be weights, $\{\alpha_J\}_{J\in\mathcal{D}} \subset \mathbb{R}_+$, $T_0\phi := \sum_{J\in\mathcal{D}} \alpha_J \langle\phi\rangle_J \chi_J$, and the truncation of T_0 at $I \in \mathcal{D}$ be given by $T_0^{(I)}\phi := \sum_{J\subset I, J\in\mathcal{D}} \alpha_J\langle\phi\rangle_J\chi_J$. Denote $\mathcal{D}_1' = \Big\{J \in \mathcal{D} : \frac{\langle\phi\sqrt{w}\rangle_J^2}{\langle w\rangle_J} \geq \frac{\langle\theta\sqrt{v}\rangle_J^2}{\langle v\rangle_J}\Big\}$. Then (1') $M_I := \frac{1}{|I|}\int_I |T_0^{(I)}(w\chi_I)|^2 v \leq C\langle w\rangle_I$ for all $I \in \mathcal{D}$ implies $\sum_{J\in\mathcal{D}_1'} \alpha_J\langle\phi\sqrt{w}\rangle_J\langle\theta\sqrt{v}\rangle_J|J| \leq C\|\phi\|_{L^2}^2 + \|\theta\|_{L^2}^2$ for all $\phi, \theta \in L^2$.*
(Dual result) Let $\mathcal{D}_2' = \Big\{J \in \mathcal{D} : \frac{\langle\phi\sqrt{w}\rangle_J^2}{\langle w\rangle_J} \leq \frac{\langle\theta\sqrt{v}\rangle_J^2}{\langle v\rangle_J}\Big\}$. Then (2') $\int_I |T_0^*(v\chi_I)|^2 w \leq C\int_I v$ for all $I \in \mathcal{D}$ implies $\sum_{J\in\mathcal{D}_2'} \alpha_J\langle\phi\sqrt{w}\rangle_J\langle\theta\sqrt{v}\rangle_J|J| \leq C\|\phi\|_{L^2}^2 + \|\theta\|_{L^2}^2$ for all $\phi, \theta \in L^2$.

Proof. We only prove the first part since the dual result follows immediately by replacing u with v, ϕ with θ and T_0 with T_0^*. We will use a Bellman function, with properties very similar to that of the Carleson imbedding theorem, to prove the result. We define the variables as

$$F_J = \langle\phi^2\rangle_J, f_J = \langle\phi\sqrt{w}\rangle_J, w_J = \langle w\rangle_J, G_J = \langle\theta^2\rangle_J, \theta_J = \langle g\sqrt{v}\rangle_J, v_J = \langle v\rangle_J.$$

These variables satisfy $F_J = \frac{1}{2}(F_{J^-} + F_{J^+}), \dots, v_J = \frac{1}{2}(v_{J^-} + v_{J^+})$, but M_J does not. Using the fact that $T_0^{(J^\pm)}w = T_0^{(J)}w - \alpha_J\langle w\rangle_J$ and the trivial estimate $A^2 - (A-a)^2 \geq aA$, for $a \leq a \leq A$, gives

$$\begin{aligned} M_J - \frac{1}{2}(M_{J^+} + M_{J^-}) &\geq \frac{1}{|J|}\int_J \Big[(T_0^{(J)}w)^2 - (T_0^{(J)} - \alpha_J\langle w\rangle_J)^2\Big]v \\ &\geq \alpha_J\langle w\rangle_J \frac{1}{|J|}\int_J (T_0^{(J)}w)v. \end{aligned}$$

Since we now have an estimate including an expression not defined as a variable (or constant), we define N_J as $\frac{1}{|J|}\int_J (T_0^{(J)}w)v$. Note that $N_J - \frac{1}{2}(N_{J^+} +$

$N_{J^-}) = \alpha_J w_J v_J$. So $F_J, f_J, w_J, g_J, v_J, M_J$ and N_J are the variables needed to define the Bellman function. The dynamics of the variables are:

$$F_J = \frac{1}{2}(F_{J^-} + F_{J^+}), \dots, v_J = \frac{1}{2}(v_{J^-} + v_{J^+}), \tag{1}$$

$$M_J \geq \frac{1}{2}(M_{J^-} + M_{J^+}) + \alpha_J w_J N_J, \tag{2}$$

$$N_J = \frac{1}{2}(N_{J^+} + N_{J^-}) + \alpha_J w_J v_J \tag{3}$$

Define the Bellman function, $\mathcal{B}$, as the function satisfying the properties below and with domain, Ω, of non-negative octets $X = (F, f, \dots, N)$ such that $f^2 \leq Fw, g^2 \leq Gv, M \leq w$. The first two inequalities, in Ω, are Cauchy-Schwartz and the last one is just Condition $(1')$.

Bellman function properties:

1) For all $X \in \Omega$ we have $0 \leq \mathcal{B}(X) \leq C(F + G)$

2) (Weak Concavity) For all X satisfying the dynamics (1), (2), (3), the Bellman function satisfies the following estimate for some $\gamma > 0$,

$$\mathcal{B}(X) - \frac{1}{2}\Big(\mathcal{B}(X_-) + \mathcal{B}(X_+)\Big) \geq \begin{cases} \gamma\alpha f g & , \frac{f^2}{w} \geq \frac{g^2}{v} \\ 0 & , \frac{f^2}{w} < \frac{g^2}{v} \end{cases}$$

Or one can equivalently we can write this as

$$\gamma |J| \alpha_J \langle \phi\sqrt{w}\rangle_J \langle \theta\sqrt{v}\rangle_J \leq |J|\mathcal{B}(X_J) - |J^+|\mathcal{B}(X_{J^+}) - |J^-|\mathcal{B}(X_{J^-}).$$

Assuming that such a Bellman function exists, we can now prove the lemma.

$$\begin{aligned} \sum_{J\in\mathcal{D}, J\subset I, |J|>2^{-k}|I|} \alpha_J \langle \phi\sqrt{w}\rangle_J \langle \theta\sqrt{v}\rangle_J |J| &\leq |I|\mathcal{B}(X_I) - \sum_{J\in\mathcal{D}, J\subset I, |I|=2^{-k}|I|} |J|\mathcal{B}(X_J) \\ &\leq |I|\mathcal{B}(X_I). \end{aligned} \tag{4}$$

Letting k approach infinity gives

$$\begin{aligned} \sum_{J\in\mathcal{D}, J\subset I} \alpha_J \langle \phi\sqrt{w}\rangle_J \langle \theta\sqrt{v}\rangle_J |J| &\leq |I|\mathcal{B}(X_I) \leq |I| C(F_I + G_I) \\ &= C\left(\int_I \phi^2 + \int_I \theta^2\right). \end{aligned} \tag{5}$$

Taking the limit as I expands to $\mathbb{R}$ proves the lemma.

Notice how the Bellman function properties here are almost identical to that of the Carleson imbedding theorem (CIT). The lower bound in the weak concavity is what we need to sum over to make the proof work, as before, so it makes sense to require this property. Property 1 is the last piece needed to finish the estimate, as before. It turns out that we can use the Bellman function from the CIT as an initial start and then build to build upon to get the Bellman function for this lemma. For more details refer to [1]. $\square$

Theorem 3. *(Bilinear Imbedding Theorem) Let* u, v *be two arbitrary weights,* $\{\alpha_J\}_{J\in\mathcal{D}} \subset \mathbb{R}_+$, $T_0\phi := \sum_{J\in\mathcal{D}} \alpha_J \langle\phi\rangle_J \chi_J$, $T := M_{\sqrt{v}} T_0 M_{\sqrt{w}}$ *and the truncation of* T_0 *at* $I \in \mathcal{D}$ *be given by* $T_0^{(I)}\phi := \sum_{J\subset I, J\in\mathcal{D}} \alpha_J\langle\phi\rangle_J\chi_J$. *We will refer to the following inequalities below: (1)* $\int_I |T_0(w\chi_I)|^2 v \leq C\int_I w, \forall I \in \mathcal{D}$, *(2)*$\int_I |T_0^*(v\chi_I)|^2 w \leq C\int_I v, \forall I \in \mathcal{D}$.
(**A**) *Inequalities (1) and (2) imply* $(T\phi,\theta)_{L^2} = \sum_{J\in\mathcal{D}} \alpha_J\langle\phi\sqrt{w}\rangle_J\langle\theta\sqrt{v}\rangle_J|J| \leq C\|\phi\|_{L^2}\|\theta\|_{L^2}$ *for all* $\phi, \theta \in L^2$.
(**B**) *Let* $\mathcal{D}_1 = \left\{ J \in \mathcal{D} : \frac{\langle f\sqrt{w}\rangle_J^2}{\langle f^2\rangle_J\langle w\rangle_J} \geq \frac{\langle g\sqrt{v}\rangle_J^2}{\langle g^2\rangle_J\langle v\rangle_J} \right\}$. *Then (1) is equivalent to* $\sum_{J\in\mathcal{D}_1} \alpha_J\langle f\sqrt{w}\rangle_J\langle g\sqrt{v}\rangle_J|J| \leq C\|f\|_{L^2}\|g\|_{L^2}$ *for all* $f, g \in L^2$.
(Dual result) Let $\mathcal{D}_2 = \left\{ J \in \mathcal{D} : \frac{\langle\phi\sqrt{w}\rangle_J^2}{\langle\phi^2\rangle_J\langle w\rangle_J} \leq \frac{\langle\theta\sqrt{v}\rangle_J^2}{\langle\theta^2\rangle_J\langle v\rangle_J} \right\}$. *Then (2) is equivalent to* $\sum_{J\in\mathcal{D}_2} \alpha_J\langle\phi\sqrt{w}\rangle_J\langle\theta\sqrt{v}\rangle_J|J| \leq C\|\phi\|_{L^2}\|\theta\|_{L^2}$ *for all* $\phi, \theta \in L^2$.

Proof. First we will prove part (A). By the Lemma, we obtain

$$\begin{aligned}\sum_{J\in\mathcal{D}} \alpha_J\langle\phi\sqrt{w}\rangle_J\langle\theta\sqrt{v}\rangle_J|J| &\leq \left(\sum_{J\in\mathcal{D}_1'} + \sum_{J\in\mathcal{D}_2'}\right)\alpha_J\langle\phi\sqrt{w}\rangle_J\langle\theta\sqrt{v}\rangle_J|J| \\ &\leq C\left(\|\phi\|_{L^2}^2 + \|\theta\|_{L^2}^2\right).\end{aligned}$$

For $t > 0$ we replace ϕ by $t\phi$ and θ by $t^{-1}\theta$ to get

$$\sum_{J\in\mathcal{D}} \alpha_J\langle\phi\sqrt{w}\rangle_J\langle\theta\sqrt{v}\rangle_J|J| \leq C\left(t^2\|\phi\|_{L^2}^2 + t^{-2}\|\theta\|_{L^2}^2\right).$$

Taking the infimum over all $t \in (0, \infty)$ gives

$$\sum_{J\in\mathcal{D}} \alpha_J\langle\phi\sqrt{w}\rangle_J\langle\theta\sqrt{v}\rangle_J|J| \leq 2C\left(\|\phi\|_{L^2}\|\theta\|_{L^2}\right),$$

which completes the proof of part (A). Note that the Lemma gives a Young-type inequality rather than the Hölder-type because it was easier to find a Bellman function in that case.

(B) We only need to prove the first part because the duality will follow from that by interchanging v with w, T_0 with T_0^* and ϕ with θ. Suppose that

$$\sum_{J\in\mathcal{D}_\infty} \alpha_J \langle \phi\sqrt{w}\rangle_J \langle \theta\sqrt{v}\rangle_J |J| \le C\|\phi\|_{L^2}\|\theta\|_{L^2}, \forall \phi, \theta \in L^2.$$

Fix $I \in \mathcal{D}$ and consider $\phi = \sqrt{w}\chi_I$. For $\theta \in L^2$ supported on I if $J \subset I$ such that $J \in \mathcal{D}$ then $\frac{\langle \phi\sqrt{w}\rangle_J^2}{\langle \phi^2\rangle_J \langle w\rangle_J} \ge \frac{\langle \theta\sqrt{v}\rangle_J^2}{\langle \theta^2\rangle_J \langle v\rangle_J}$ by Cauchy-Schwarz. So $(T\phi, \theta)_{L^2} =$

$$\sum_{J\in\mathcal{D}, J\subset I} \alpha_J \langle \phi\sqrt{w}\rangle_J \langle \theta\sqrt{v}\rangle_J |J| = \sum_{J\in\mathcal{D}_1, J\subset I} \alpha_J \langle \phi\sqrt{w}\rangle_J \langle \theta\sqrt{v}\rangle_J |J| \le C\|\phi\|_{L^2}\|\theta\|_{L^2}.$$

Taking the supremum over all such θ with $\|\theta\|_{L^2} \le 1$ gives $\int_I |T\phi|^2 \le C\|\phi\|_{L^2}^2$.

Conversely we can bootstrap, from the Bellman function found in the Lemma, to prove the result. From [1], the Bellman function, $\mathcal{B}$, satisfies

1) For all $X \in \Omega$ we have $0 \le \mathcal{B}(X) \le 2(F+G)$

2) (Weak Concavity) For all X satisfying the dynamics (1), (2),(3) the Bellman function satisfies

$$\mathcal{B}(X) - \frac{1}{2}\Big(\mathcal{B}(X_- + X_-) + \mathcal{B}(X_+)\Big) \ge \begin{cases} \frac{1}{16}\alpha f g & , \frac{f^2}{w} \ge \frac{g^2}{v} \\ 0 & , \frac{f^2}{w} < \frac{g^2}{v}. \end{cases}$$

Consider $\widetilde{\mathcal{B}}^{(\tau)}(X) := \tau F + 14\tau^{-1}G + \mathcal{B}(\tau F, \sqrt{\tau} f, w, \tau^{-1}G, \sqrt{\tau^{-1}} g, v, M, N)$, where in Bellman function $\mathcal{B}$, we have multiplied f by $\sqrt{\tau}$ and g by $\sqrt{\tau^{-1}}$. Observe that

$$\tau F + 14\tau^{-1}G \le \widetilde{\mathcal{B}}^{(\tau)}(X) \le 3\tau F + 16\tau^{-1}G$$

Define $\widetilde{\mathcal{B}}(X) := \inf_{\tau>0} \widetilde{\mathcal{B}}^{(\tau)}(X)$. Choosing $\tau = 2\sqrt{G/F}$ implies that $\widetilde{\mathcal{B}}^{(\tau)}(X) \le 6\sqrt{FG} + 8\sqrt{FG}$. Therefore, $\widetilde{\mathcal{B}}(X) \le 14\sqrt{FG}$. Let $\tau_* = \tau_*(X)$ be where the infimum is attained in $\widetilde{\mathcal{B}}(X)$. Then $14\tau_*^{-1}G \le \widetilde{\mathcal{B}}^{(\tau_*)}(X) \le 14\sqrt{FG} \Rightarrow \tau_* \ge \sqrt{G/F}$. Now we have the estimate

$$\widetilde{\mathcal{B}}(X) - \frac{1}{2}\Big(\widetilde{\mathcal{B}}(X^+) + \widetilde{\mathcal{B}}(X^-)\Big) \ge \widetilde{\mathcal{B}}^{(\tau_*)}(X) - \frac{1}{2}\Big(\widetilde{\mathcal{B}}^{(\tau_*)}(X^+) + \widetilde{\mathcal{B}}^{(\tau_*)}(X^-)\Big) \ge \frac{\alpha f g}{16},$$

when $\frac{\tau_* f^2}{w} \ge \frac{\tau_*^{-1} g^2}{v}$. Since $\tau_* \ge \sqrt{G/F}$ and $\frac{f^2}{Fw} \ge \frac{g^2}{Gv}$ imply $\frac{\tau_* f^2}{w} \ge \frac{\tau_*^{-1} g^2}{v}$ then the estimate is also true when $\frac{f^2}{Fw} \ge \frac{g^2}{Gv}$. Since zero is always a lower bound, then

$$\widetilde{\mathcal{B}}(X) - \frac{1}{2}\Big(\widetilde{\mathcal{B}}(X_- + X_-) + \widetilde{\mathcal{B}}(X_+)\Big) \ge \begin{cases} \frac{1}{16}\alpha f g & , \frac{f^2}{Fw} \ge \frac{g^2}{Gv} \\ 0 & , \frac{f^2}{Fw} < \frac{g^2}{Gv}, \end{cases}$$

which can be rewritten as $\frac{1}{16}|J|\alpha_J\langle\phi\sqrt{w}\rangle_J\langle\theta\sqrt{v}\rangle_J \le |J|\mathcal{B}(X_J) - |J^+|\mathcal{B}(X_{J^+}) - |J^-|\mathcal{B}(X_{J^-})$. Using this together with the estimate $\widetilde{\mathcal{B}}^{(\tau)}(X) \le 14\sqrt{FG}$ we can now finish the proof just as in (4) and (5). $\square$

3.4 Applications

We define the Haar functions for $J \in \mathcal{D}$ as $h_J := \chi_{J^+} - \chi_{J^-}$ and denote $\Delta_J\phi := \frac{1}{|I|}\int_{\mathbb{R}} \phi h_I$. Also, define $T = M_{\sqrt{v}} T_0 M_{\sqrt{w}}$ for weights u, v. The following results are proven in full detail in [1]. The first is an application of the dyadic two weight Carleson imbedding theorem and disbalanced Haar functions. The second is an application of the bilinear imbedding theorem and dyadic two weight Carleson imbedding theorem.

Theorem 4. *(Sawyer type theorem for the square function operator) Let* $T_0\phi := \left(\sum_{J\in\mathcal{D}} \alpha_J[\Delta_J\phi]^2\chi_J\right)^{\frac{1}{2}}$ *for some* $\{\alpha_J\}_{J\in\mathcal{D}}$*. Then the operator* T *is bounded in* L^2 *if, and only if, there exists* $C > 0$ *such that* $\int_{\mathbb{R}} [T_0(w\chi_I)]^2 v \le C\int_I w$ *for all* $I \in \mathcal{D}$.

Theorem 5. *(Sawyer type theorem for Haar Multipliers) Let* $\boldsymbol{\alpha} = \{\alpha_J\}_{J\in\mathcal{D}} \subset \mathbb{R}_+$, $\boldsymbol{\sigma} = \{\sigma_J\}_{J\in\mathcal{D}} \subset \{\pm 1\}$ *and* $T_0(\boldsymbol{\sigma\alpha})\phi := \sum_{J\in\mathcal{D}} \sigma_J\alpha_J[\Delta_J]\phi h_J$*. Then* $\sup_{\boldsymbol{\sigma}} \|T(\boldsymbol{\sigma\alpha})\|_{L^2\to L^2} < \infty$ *if, and only if, there exists a* $C > 0$ *such that* $\int_{\mathbb{R}} [T_0(\boldsymbol{\alpha\sigma})(w\chi_I)]^2 v \le C\int_I w$ *and* $\int_{\mathbb{R}} [T_0^*(\boldsymbol{\alpha\sigma})(v\chi_I)]^2 w \le C\int_I v$ *for all* $I \in \mathcal{D}$ *and all sign sequences* $\boldsymbol{\sigma}$.

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4 The sharp bound for the Hilbert transform on weighted Lebesgue spaces

after S. Petermichl [4]
A summary written by Daewon Chung

Abstract

We summarize the proof in [4] of the linear estimate for the Hilbert transform on the weighted Lebesgue space $L^2(w)$.

4.1 Introduction

We say w is a weight if it is locally integrable and positive almost everywhere. In 1973, Hunt, Muckenhoupt and Wheeden proved that the A_p-condition:

$$w \in A_p \text{ iff } [w]_{A_p} := \sup_I \langle w \rangle_I \langle w^{-1/(p-1)} \rangle_I^{p-1} < \infty$$

characterizes the boundedness of the Hilbert transform on $L^p(w)$. Here the notation $\langle \cdot \rangle_I$ stands for the average over the interval I and the supremum is taken over all intervals I. A year later, Coifman and Fefferman extended this result to a larger class of convolution singular integrals with standard kernels. Recently, many authors have been interested in finding the sharp bounds for the operator norms in terms of the A_p-characteristic $[w]_{A_p}$ of the weight. That is, one looks for a function $\phi(x)$, sharp in terms of its growth, such that

$$\|Tf\|_{L^p(w)} \leq C\phi([w]_{A_p})\|f\|_{L^p(w)} .$$

Most recently it has been solved for the general Calderón-Zygmund operators in [5]. However, understanding this problem in the case of the Hilbert transform will be a very important step. In this summary, we are interested in obtaining sharp weight inequalities for the Hilbert transform which is defined as follows:

$$Hf(x) = p.v.\frac{1}{\pi}\int \frac{f(y)}{x-y}\,dy .$$

We now provide the main result of [4].

Theorem 1. *There exists a constant C such that, for all weights $w \in A_p$,*

$$\|Hf\|_{L^p(w)} \leq C[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}$$

and this is sharp for $1 < p < \infty$.

4.2 Preliminaries

4.2.1 Notations

We write $\langle f\rangle_{I,w}$ for the weighted average, $\langle f\rangle_{I,w} := \frac{1}{w(I)}\int_I f(x)dx$, where $w(I) = \int_I w$. The notation $\langle\cdot,\cdot\rangle$ stands for the standard inner product in L^2 and $\langle\cdot,\cdot\rangle_w$ denotes the weighted inner product. Intervals of the form $[k2^{-j},(k+1)2^{-j})$ for integers j,k are called dyadic intervals. Let us denoted $\mathcal{D}$ the collection of all dyadic intervals. For any interval $I\in\mathcal{D}$, there is a Haar function defined by

$$h_I(x) = \frac{1}{\sqrt{|I|}}(\chi_{I_+}(x) - \chi_{I_-}(x))\,,$$

where $I_\pm$ denote the left and right halves of I . We also consider the different grids of dyadic intervals parameterized by α, r, defined by $D^{\alpha,r} = \{\alpha + rI : I\in\mathcal{D}\}$, for $\alpha\in\mathbb{R}$ and positive r . For each grid $\mathcal{D}^{\alpha,r}$ of dyadic intervals, there are corresponding Haar functions $h_I^{\alpha,r}$, $I\in\mathcal{D}^{\alpha,r}$ that are an orthonormal system in L^2. Let us introduce a proper orthonormal system for $L^2(w)$ with respect to the weighted inner product, the weighted Haar system, defined by

$$h_I^w := \frac{1}{w(I)^{1/2}}\left[\frac{w(I_-)^{1/2}}{w(I_+)^{1/2}}\chi_{I_+} - \frac{w(I_+)^{1/2}}{w(I_-)^{1/2}}\chi_{I_-}\right].$$

Then, every function $f\in L^2(w)$ can be written as

$$f = \sum_{I\in\mathcal{D}}\langle f, h_I^w\rangle_w h_I^w\,,$$

where the sum converges a.e. in $L^2(w)$. Moreover, $\|f\|^2_{L^2(w)} = \sum_{I\in\mathcal{D}}\langle f, h_I^w\rangle_w^2$. Since h_I^w is a constant on J for $I\supsetneq J$, we denote this constant by $h_I^w(J)$. We can then write the weighted averages

$$\langle f\rangle_{I,w} = \sum_{J\supsetneq I}\langle g, h_I^w\rangle_w h_J^w(I)\,.$$

We also use the notation $\Delta_I w = \frac{1}{2}(\langle w\rangle_{I_+} - \langle w\rangle_{I_-})$.

4.2.2 Theorems and Lemmas

To prove Theorem 1 we need several theorems and lemmas. One can find the proof[1] in the indicated references. Another main result in [4] is a two

[1] In fact, all of them were proved by the Bellman function argument.

weighted bilinear embedding theorem, which was proved by a Bellman function argument.

Theorem 2. *[Bilinear Embedding Theorem] Let w and v be weights so that $\langle w\rangle_I\langle v\rangle_I \le Q$ for all intervals I and let $\{\alpha_I\}$ be a non-negative sequence so that the three estimates below hold for all J*

$$\sum_{I\subseteq J}\frac{\alpha_I}{\langle w\rangle_I} \le Qv(J)\,,\quad \sum_{I\subseteq J}\frac{\alpha_I}{\langle v\rangle_I} \le Qw(J)\,,\ \text{and}\ \sum_{I\subseteq J}\alpha_I \le Q|J|\,.$$

Then there is C such that for all $f\in L^2(w)$ and $g\in L^2(v)$

$$\sum_{I\in\mathcal{D}}\alpha_I\langle f\rangle_{I,w}\langle g\rangle_{I,v} \le CQ\|f\|_{L^2(w)}\|g\|_{L^2(w)}\,.$$

We will also need some inequalities for weights.

Lemma 3. *[6] There exist a constant C such that for all weight $w\in A_2$ and dyadic interval $I\in\mathcal{D}$,*

$$\sum_{I\subseteq J}\frac{|\Delta_I w|^2|\,I|}{\langle w\rangle_I} \le C[w]_{A_2}w(J)\,.$$

Lemma 4. *[4] For all dyadic intervals J and all weights w,*

$$\sum_{I\subseteq J}\frac{|\Delta_I w|^2|\,I|}{\langle w\rangle_I^3} \le w^{-1}(J)\,.$$

Lemma 5. *[1] If $w\in A_2$ then there exists a constant $C>0$ such that for all dyadic intervals J*

$$\sum_{I\subseteq J}\frac{|\Delta_I w|^2\langle w^{-1}\rangle_I|\,I|}{\langle w\rangle_I} \le C[w]_{A_2}|J|\,.$$

Lemma 6. *[2] If $w\in A_2$ then there exists a constant $C>0$ such that for all dyadic intervals J*

$$\sum_{I\subseteq J}\frac{\big(|\Delta_{I_+}w|+|\Delta_{I_-}w|\big)^2\langle w^{-1}\rangle_I|\,I|}{\langle w\rangle_I} \le C[w]_{A_2}|J|\,.$$

4.3 The main argument

In [3], S. Petermichl introduced the dyadic shift operator and proved that the kernel of the Hilbert transform can be written as a well chosen averages of certain dyadic shift operators which are defined by

$$S^{\alpha,r} f = \sum_{I \in \mathcal{D}^{\alpha,r}} \langle f, h_I \rangle (h_{I_-} - h_{I_+}) \,.$$

This allows us to use the operator $S^{\alpha,r}$ instead of H to prove Theorem 1. In the following arguments there is no dependence on the choice of the grid, so we omit the indices α and r. Also, it suffices to consider the operator $Sf = \sum_{I\in\mathcal{D}} \langle f, h_I\rangle h_{I_-}$, and we will get the estimate

$$\|Sf\|_{L^2(w)} \leq C[w]_{A_2} \|f\|_{L^2(w)} \,. \tag{1}$$

Then the sharp extrapolation yields Theorem 1. We are going to show the inequality (1) by duality, that is for any positive functions $f \in L^2(w^{-1})$ and $g \in L^2(w)$

$$|\langle S(w^{-1}f), g\rangle_w| \leq C[w]_{A_2} \|f\|_{L^2(w^{-1})} \|g\|_{L^2(w)} \,. \tag{2}$$

Expanding f and g in the weighted Haar systems for $L^2(w^{-1})$ and $L^2(w)$, respectively, we have

$$|\langle S(w^{-1}f, g\rangle_w| = \left| \sum_{I,J} \langle f, h_I^{w^{-1}}\rangle_{w^{-1}} \langle g, h_J^w\rangle_w \langle S(w^{-1}h_I^{w^{-1}}), h_J^w\rangle_w \right| \tag{3}$$

Since $\langle S(w^{-1}h_I^{w^{-1}}), h_I^w\rangle = \sum_L \langle h_L, h_I^{w^{-1}}\rangle_{w^{-1}} \langle h_{L_-}, h_I^{w^{-1}}\rangle_I$ can be non-zero only for $L \subseteq I$ and $L_- \subseteq J$. Also, there exist such intervals L only for $I \subseteq J$ or $\hat{J} \subseteq I$, where $\hat{J}$ stands for the parent of J. Thus, we can decompose the right hand of (3) into two parts, *paraproducts*: $\sum_{\hat{J}\subsetneq I}$, $\sum_{I\subsetneq J}$ and *diagonal part*: $\sum_{I=J}$, $\sum_{I=\hat{J}}$. The following Lemma lies at the heart of the matter for the estimate for both parts.

Lemma 7. *For all intervals I and weight $w \in A_2$, there exists a constant C such that*

$$\left\| \sum_{L\subseteq I} \langle w^{-1}, h_L\rangle h_{L_-} \right\|_{L^2(w)} \leq C[w]_{A_2} w^{-1}(I)^{1/2} \,. \tag{4}$$

Once we have Lemma 7, the desired estimates for the two parts are very straightforward. See [4] for more detail. We will emphasize the proof of Lemma 7.

Proof of Lemma 7. We are going to show Lemma 7 by duality, that is for any positive function $f \in L^2(w)$ there exists a constant C such that

$$\left|\left\langle \sum_{L\subseteq I} \langle w^{-1}, h_L\rangle h_{L_-}, f \right\rangle_w\right| \le C[w]_{A_2} w^{-1}(I)^{1/2} \|f\|_{L^2(w)} \,. \tag{5}$$

Expanding f in the weighted Haar system in $L^2(w)$, we have

$$\left|\left\langle \sum_{L\subseteq I} \langle w^{-1}, h_L\rangle h_{L_-}, f \right\rangle_w\right| = \left| \sum_{L\subseteq I} \langle w^{-1}, h_L\rangle \sum_{J\in\mathcal{D}} \langle f, h_J^w\rangle_w \langle h_{L_-}, h_J^w\rangle_w \right| \tag{6}$$

Since $\langle h_{L_-}, h_J^w\rangle_w$ can be non-zero only for $L_- \subseteq J$, we can split the sum in (6) into three sums, $J = L_-$, $J = L$, and $J \supsetneq L$ .The sums for $J = L_-$ and $J = L$ can be estimated by

$$\begin{aligned}
&\sum_{L\subseteq I} \left|\langle w^{-1}, h_L\rangle \langle f, h_{L_-}^w\rangle_w \langle h_{L_-}, h_{L_-}^w\rangle_w\right| + \sum_{L\subseteq I} \left|\langle w^{-1}, h_L\rangle \langle f, h_L^w\rangle_w \langle h_{L_-}, h_L^w\rangle_w\right| \\
&\le \left(\sum_{L\subseteq I} \langle f, h_{L_-}^w\rangle_w^2 \right)^{1/2} \left(\sum_{L\subseteq I} \langle w^{-1}, h_L\rangle^2 \langle h_{L_-}, h_{L_-}^w\rangle_w^2 \right)^{1/2} \\
&\qquad + \left(\sum_{L\subseteq I} \langle f, h_L^w\rangle_w^2 \right)^{1/2} \left(\sum_{L\subseteq I} \langle w^{-1}, h_L\rangle^2 \langle h_{L_-}, h_L^w\rangle_w^2 \right)^{1/2} \\
&\le 2\sqrt{2} \|f\|_{L^2(w)} \left(\sum_{L\subseteq I} \langle w^{-1}, h_L\rangle^2 \langle w\rangle_L \right)^{1/2} \le C \|f\|_{L^2(w)} [w]_{A_2} w^{-1}(I)^{1/2} \,.
\end{aligned}$$

The last inequality is due to Lemma 3. Since $\langle \chi_I\rangle_{L,w^{-1}} = 1$ for all $L \subseteq I$, we have for the sum $J \supsetneq L$,

$$\left| \sum_{L\subseteq I} \langle w^{-1}, h_L\rangle \sum_{J:J\supsetneq L} \langle f, h_J^w\rangle_w \langle h_{L_-}, h_J^w\rangle_w \right| \le \sum_{L\subseteq I} |\langle w^{-1}, h_L\rangle|\,|\langle w, h_{L_-}\rangle| \langle f\rangle_{L,w} \langle \chi_I\rangle_{L,w^{-1}}$$

By Theorem 2, our desired estimate for this sum holds, provided the following three embedding conditions hold:

$$\sum_{I\subseteq J} \frac{|\langle w^{-1}, h_I\rangle|\,|\langle w, h_{I_-}\rangle|}{\langle w\rangle_I} \le C[w]_{A_2} w^{-1}(J) \tag{7}$$

$$\sum_{I\subseteq J} \frac{|\langle w^{-1}, h_I\rangle|\,|\langle w, h_{I_-}\rangle|}{\langle w^{-1}\rangle_I} \le C[w]_{A_2} w(J) \tag{8}$$

$$\sum_{I\subseteq J} |\langle w^{-1}, h_I\rangle|\,|\langle w, h_{I_-}\rangle| \le C[w]_{A2} |J| \tag{9}$$

For the embedding condition (7), one can check by using Cauchy-Schwarz inequality, Lemma 4, and Lemma 5. One also uses Cauchy-Schwarz inequality, Lemma 3 and Lemma 4 for the embedding condition (8). Finally, for the embedding condition (9), we have

$$\begin{aligned}
&\sum_{I\subseteq J}|\langle w^{-1}, h_I\rangle|\,|\langle w, h_{I_-}\rangle| \\
&\quad\le C\Big(\sum_{I\subseteq J}\frac{|\Delta_I w^{-1}|^2\langle w\rangle_I|\,I|}{\langle w^{-1}\rangle_I}\Big)^{1/2}\Big(\sum_{I\subseteq J}\frac{|\Delta_{I_-} w|^2\langle w^{-1}\rangle_I|\,I|}{\langle w\rangle_I}\Big)^{1/2} \\
&\quad\le C[w]_{A_2}|J|\,. \qquad (10)
\end{aligned}$$

Here inequality (10) uses Lemma 5 and Lemma 6. Note that the proof of the embedding condition (9) was directly deduced by the single Bellman function in [4].

$\square$

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5 A characterization of a two-weight norm inequality for maximal operators

after E.T. Sawyer [1]
A summary written by Francesco Di Plinio

Abstract

We outline the proof of Sawyer's characterization of the two-weight norm inequality for the Hardy-Littlewood maximal operator.

5.1 The main result and some perspective

Notation. Hereafter, if ω is a positive Borel measure on $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is measurable, we use the notation $\omega(E) = \int_E \mathrm{d}\omega$. If $\mathrm{d}\omega(x) = w(x)\mathrm{d}x$ for some positive locally integrable w, we abuse the above notation by writing $w(E) = \int_E w(x)\mathrm{d}x$ in place of $\omega(E)$ and $L^p(w, E)$ in place of $L^p(\omega, E)$.

Let $\mathcal{Q}$ be the collection of cubes in $\mathbb{R}^n$ and let us denote with

$$Mf(x) = \sup_{x \in Q \in \mathcal{Q}} \frac{1}{|Q|} \int_Q |f| \mathrm{d}y$$

the standard Hardy-Littlewood maximal operator. The main result of [1] is a necessary and sufficient condition for the two-weight boundedness of M. That is, if v and w are two weights (nonnegative functions) on $\mathbb{R}^n$ and $1 < p \leq q \leq \infty$, $p < \infty$,

$$\|Mf\|_{L^q(w)} \leq C\|f\|_{L^p(v)} \tag{1}$$

if and only if

$$\left\| M\big(v_{[p]} \mathbf{1}_Q\big) \right\|_{L^q(w,Q)} \leq C\big(v_{[p]}(Q)\big)^{\frac{1}{p}} < \infty \qquad \forall Q \in \mathcal{Q}, \tag{2}$$

where $v_{[p]}(x) := (v(x))^{1-p'}$.

Ten years before Sawyer's paper appeared, Muckenhoupt [3] proved that the weak type inequality ($1 < p < \infty$)

$$w(\{x : Mf(x) > \lambda\}) \leq C\lambda^{-p}\|f\|_{L^p(v)} \tag{3}$$

holds if and only if

$$\sup_{Q\in\mathcal{Q}}\left(\frac{w(Q)}{|Q|}\right)\left(\frac{v_{[p]}(Q)}{|Q|}\right)^{p-1}<\infty. \tag{4}$$

If $w=v$, the quantity on the lhs of (4) is the A_p-characteristic of w; in this case, Muckenhoupt showed that (4) is actually equivalent to the strong-type version of (3) when $p=q$, i.e. M is bounded on $L^p(w)$ if and only w has finite A_p-characteristic. However, (4) is not in general sufficient for (1) to hold in the two-weight case.

Sawyer's result appears to be the first complete characterization of pair of weights for which (1) holds true. Although condition (2) (with $w=v, p=q$) is not obviously equivalent to (4), the ideas in Sawyer's proof of the equivalence between (2) and (1) inspired a very elementary proof of Muckenhoupt's result by M. Christ and R. Fefferman [2].

5.2 Weighted norm inequalities for fractional maximal operators

The equivalence (1) $\Longleftrightarrow$ (2) is obtained as a particular case of a more general result about the weighted fractional maximal operator

$$M_{\mu,\alpha}f(x)=\sup_{x\in Q\in\mathcal{Q}}\mu(Q)^{\frac{\alpha}{n}-1}\int_Q|f|\mathrm{d}\mu.$$

where μ is a positive locally finite Borel measure and $0\le\alpha<n$.

Theorem 1. *Assume that the measure μ is a doubling measure. Then, for every pair (ν,ω) of positive Borel measures and for $1<p\le q\le\infty$, $p<\infty$,*

$$\|M_{\mu,\alpha}f\|_{L^q(\omega)}\le C\|f\|_{L^p(\nu)} \tag{5}$$

if and only if $\mu\ll\nu$ and, setting $u=\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$,

$$\left\|M_{\mu,\alpha}\left[u^{p'-1}\mathbf{1}_Q\right]\right\|_{L^q(\omega,Q)}\le C\left\|u^{p'-1}\right\|_{L^p(\nu,Q)}<\infty\quad\forall Q\in\mathcal{Q}. \tag{6}$$

In turn, Theorem 1 is a consequence of the correspondent result for the dyadic version of $M_{\mu,\alpha}$, namely

$$M^{\Delta(t)}_{\mu,\alpha}f(x)=\sup_{x\in Q\,:\,Q-t\in\mathcal{Q}^\Delta}\mu(Q)^{\frac{\alpha}{n}-1}\int_Q|f|\mathrm{d}\mu.$$

where $t\in\mathbb{R}^n$ and $\mathcal{Q}^\Delta$ is the collection of dyadic cubes of $\mathbb{R}^n$.

Theorem 2. *For every $t \in \mathbb{R}^n$ and every pair (ν, ω) of positive Borel measure, for $1 < p \le q \le \infty$, $p < \infty$,*

$$\|M_{\mu,\alpha}^{\Delta(t)} f\|_{L^q(\omega)} \le C\|f\|_{L^p(\nu)} \tag{7}$$

if and only if $\mu \ll \nu$ and, setting $u = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}$,

$$\Big\| M_{\mu,\alpha}^{\Delta(t)} \big[u^{p'-1}\mathbf{1}_Q\big]\Big\|_{L^q(\omega,Q)} \le C\big\|u^{p'-1}\big\|_{L^p(\nu,Q)} < \infty \tag{8}$$

holds for every Q such that $Q - t \in \mathcal{Q}^\Delta$.

Observe that Theorem 2 does not require μ to be a doubling measure, while Theorem 1 does. In fact, the doubling condition is only needed in the averaging argument used to deduce Theorem 1 from Theorem 2, which we now sketch.

Sketch of proof of (6) $\implies$ (5). By a limiting argument in k, we can prove the correspondent statement with the collection $\mathcal{Q}$ replaced by the collection $\mathcal{Q}_k$ of cubes of $\mathbb{R}^n$ with sidelength at most 2^k in the definition of $M_{\mu,\alpha}$. Next, (6) is clearly stronger than (8), so that we can apply Theorem 2 to get the bound

$$\|M_{\mu,\alpha}^{\Delta(t)}\|_{L^p(\nu)\to L^q(\omega)} \le C, \qquad \forall t \in \mathbb{R}^n.$$

We then finish the proof of (5) by showing the pointwise bound

$$M_{\mu,\alpha} f(x) \le C \int_{[-2^{k+2},2^{k+2}]^n} M_{\mu,\alpha}^{\Delta(t)} f(x)\, \frac{\mathrm{d}t}{2^{n(k+3)}}.$$

To do this, let $Q \in \mathcal{Q}_k$ be a cube with $x \in Q$, $\ell(Q) \approx 2^j$, $j \le k$ and

$$\mu(Q)^{\frac{\alpha}{n}-1} \int_Q |f| \mathrm{d}\mu \gtrsim M_{\mu,\alpha} f(x).$$

Let Ω be the set of $t \in [-2^{k+2}, 2^{k+2}]^n$ such that there exists a cube $Q_t \supset Q$ with $\ell(Q_t) = 2^{j+1}$ and $Q_t - t \in \mathcal{Q}^\Delta$. For $t \in \Omega$, we have $Q_t \subset 7Q$, so that using that the measure μ is doubling, $\mu(Q_t) \le C\mu(Q)$, and therefore

$$M_{\mu,\alpha}^{\Delta(t)} f(x) \ge \mu(Q_t)^{\frac{\alpha}{n}-1} \int_{Q_t} |f|\,\mathrm{d}\mu \gtrsim \mu(Q)^{\frac{\alpha}{n}-1} \int_Q |f|\,\mathrm{d}\mu \gtrsim M_{\mu,\alpha} f(x).$$

Noting that Ω occupies at least one fourth of $[-2^{k+2}, 2^{k+2}]^n$ and then averaging concludes the proof of the pointwise bound. $\square$

5.3 Proof of the dyadic case.

We sketch the proof of the sufficiency of (8) in Theorem 2. It is enough to prove the case $t = 0$ since the statements are invariant under translations and reflections of $\mathbb{R}^n$. For the rest of the section, $M_{\mu,\alpha}$ stands for $M_{\mu,\alpha}^{\Delta(0)}$.

For a fixed $k \in \mathbb{Z}$, let $\mathcal{J}_k$ be the collection of Calderon-Zygmund cubes corresponding to the level set $E_k = \{M_{a,\mu} f > 2^k\}$. Then, for $Q \in \mathcal{J}_k$, by Hölder,

$$\begin{aligned} \mu(Q)^{1-\frac{\alpha}{n}} &\le 2^{-k} \int_Q |f|\, u \mathrm{d}\nu \le 2^{-k} \|f\|_{L^p(\nu,Q)} \|u\|_{L^{p'}(\nu,Q)} \\ &\le 2^{-k} \|f\|_{L^p(\nu,Q)} \sigma(Q)^{\frac{1}{p'}}, \end{aligned} \tag{9}$$

having set $\mathrm{d}\sigma = \left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right)^{p'} \mathrm{d}\nu = u^{p'-1}\mathrm{d}\mu$. Note that $\sigma(Q)$ is finite (by (8)) and positive (since $\mu(Q) > 0$), and by definition of σ,

$$M_{\mu,\alpha}\big(u^{p'-1}\mathbf{1}_Q\big) \ge \mu(Q)^{\frac{\alpha}{n}-1}\sigma(Q) \qquad \text{on } Q. \tag{10}$$

At this point, the case $q = \infty$ follows by using assumption (8) and disjointness of the cubes of $\mathcal{J}_k$. The case $p \le q < \infty$ requires extra work. Assumption (8) followed by (10) yields

$$\omega(Q)\mu(Q)^{\frac{q\alpha}{n}-q}\sigma(Q)^q \le \left\| M_{\mu,\alpha}^{\Delta(t)}\big[u^{p'-1}\mathbf{1}_Q\big] \right\|_{L^q(\omega,Q)}^q \lesssim \|u^{p'-1}\|_{L^p(\nu,Q)}^q = \sigma(Q)^{\frac{q}{p}}.$$

We use the last inequality and (9) to get that $\omega(Q) \le 2^{-kq}\|f\|_{L^p(\nu,Q)}^q$, for $Q \in \mathcal{J}_k$. Therefore, by disjointness of $Q \in \mathcal{J}_k$ and using $q \ge p$, we obtain

$$\omega(E_k) = \sum_{\mathcal{J}_k} \omega(Q) \lesssim 2^{-kq}\|f\|_{L^p(\nu)}^q.$$

Furthermore, if we set $Q^\star = Q \cap (E_{k+1})^c$ for $Q \in \mathcal{J}_k$, we can estimate

$$\begin{aligned} \|M_{\mu,\alpha} f\|_{L^q(\omega)}^q &\lesssim \sum_k 2^{kq} w(E_{k+1} - E_k) \le \sum_k \sum_{Q\in\mathcal{J}_k} w(Q^\star) \Big(\mu(Q)^{\frac{\alpha}{n}-1} \textstyle\int_Q |f|\mathrm{d}\mu\Big)^q \\ &= \sum_k \sum_{Q\in\mathcal{J}_k} \gamma(Q) \Big(\sigma(Q)^{-1} \textstyle\int_Q |f| u^{1-p'}\mathrm{d}\sigma\Big)^q \end{aligned} \tag{11}$$

where, for $Q \in \mathcal{J}_k$

$$\gamma(Q) := \omega(Q^\star)\Big(\mu(Q)^{\frac{\alpha}{n}-1}\textstyle\int_Q u^{p'-1}\mathrm{d}\mu\Big)^q \le \displaystyle\int_{Q^\star} \Big(M_{\alpha,\mu}\big[u^{p'-1}\mathbf{1}_Q\big]\Big)^q \mathrm{d}\omega.$$

Define now the measure space $\Omega = \{(k, Q) : k \in \mathbb{Z}, Q \in \mathcal{J}_k\}$, with measure assigning mass $\gamma(Q)$ to (k, Q). The operator

$$g \mapsto Tg, \qquad [Tg]_{(k,Q)} = \frac{1}{\sigma(Q)} \int_Q |g| \, \mathrm{d}\sigma,$$

is clearly of strong type (∞, ∞). We claim that it is also weak type $(1, q/p)$, so that we get strong type (p, q) by Marcienkiewicz interpolation theorem. Assuming that the claim is true, we get from (11) that

$$\begin{aligned} \|M_{\mu,\alpha} f\|_{L^q(\omega)}^q &\lesssim \sum_k \sum_{Q \in \mathcal{J}_k} \gamma(Q) \Big(\big[T\big(|f| u^{1-p'}\big)\big]_{(k,Q)} \Big)^q = \big\|T\big(|f| u^{1-p'}\big)\big\|_{L^q(\Omega)}^q \\ &\lesssim \big\||f| u^{1-p'}\big\|_{L^p(\sigma)}^q = \|f\|_{L^p(\nu)}^q, \end{aligned}$$

i.e. (7) follows.

Let us conclude by sketching the proof of the claim. Let $\lambda > 0$ and $\mathcal{I}_\lambda$ be the Calderon-Zygmund cubes corresponding to the level set

$$\{(k, Q) : Q \in \mathcal{J}_k, \textstyle\int_Q |g| \, \mathrm{d}\sigma > \lambda \sigma(Q)\}.$$

Then, using (8) to go from the first to the second line, and recalling $q \geq p$,

$$\begin{aligned} |\{|Tg| > \lambda\}| &\lesssim \sum_{I \in \mathcal{I}_\lambda} \sum_{(k,Q):Q \subset I} \gamma(Q) \leq \sum_{I \in \mathcal{I}_\lambda} \sum_{(k,Q):Q \subset I} \int_{Q^\star} \Big(M_{\alpha,\mu}\big[u^{p'-1} \mathbf{1}_Q\big] \Big)^q \mathrm{d}\omega \\ &\lesssim \sum_{I \in \mathcal{I}_\lambda} \sigma(I)^{\frac{q}{p}} \leq \Bigg(\sum_{I \in \mathcal{I}_\lambda} \sigma(I) \Bigg)^{\frac{q}{p}} \lesssim \Bigg(\lambda^{-1} \sum_{I \in \mathcal{I}_\lambda} \int_I |g| \Bigg)^{\frac{q}{p}} \leq \frac{\|g\|_{L^1(\sigma)}^{\frac{q}{p}}}{\lambda^{\frac{q}{p}}}. \end{aligned}$$

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6 A two weight inequality for the Hilbert transform assuming an energy hypothesis

after M.T. Lacey, E.T. Sawyer and I. Uriarte-Tuero [1]
A summary written by Kabe Moen

Abstract

We explore the two weight problem for the Hilbert transform, summarizing [1]. The authors introduce a range of side conditions, known as Energy conditions. At one endpoint the Energy conditions are necessary for the two weight inequality and at the other endpoint they become the Pivotal conditions of Nazarov, Treil, and Volberg [3]. Assuming the Energy conditions hold, the two weight boundedness of the Hilbert transform is shown to be equivalent to two testing conditions and a Poisson A_2 condition. The authors also provide an example showing that the Pivotal conditions are not necessary for the two weight inequality.

6.1 Introduction

Let H be the Hilbert transform. The two weight problem for H is to find necessary and sufficient conditions on a pair of non-negative functions (u, v) so that H maps $L^2(v)$ into $L^2(u)$, i.e.

$$\int_{\mathbb{R}} |Hf(x)|^2 u(x)\, dx \le C \int_{\mathbb{R}} |f(x)|^2 v(x)\, dx.$$

Notice that by setting $\sigma = v^{-1}$, an equivalent formulation is for the operator $H(\,\cdot\,\sigma)$ being bounded from $L^2(\sigma)$ to $L^2(u)$. We examine inequalities of the form

$$\|H(f\sigma)\|_{L^2(\omega)} \le C\|f\|_{L^2(\sigma)} \tag{1}$$

where ω and σ are nonnegative locally finite Borel measures, i.e. weights. We refer to (1) as the two weight inequality.

Motivated by the one weight theory, a natural condition for the weights to satisfy would be the two weight version of the A_2 condition:

$$\sup_I \frac{1}{|I|}\int_I d\omega \cdot \frac{1}{|I|}\int_I d\sigma < \infty. \tag{2}$$

However this condition is not sufficient. In fact, it is even not sufficient for the weak-type inequality

$$\omega(\{x : |H(f\sigma)(x)| > t\}) \leq \frac{C}{t^2} \int_{\mathbb{R}} |f(x)|^2 \, d\sigma,$$

see [2]. This is in stark contrast to the Hardy-Littlewood maximal operator, M. The operator M is bounded from $L^2(v)$ to $L^{2,\infty}(u)$ if and only if (u, v) satisfies (2) with $d\omega = u \, dx$ and $d\sigma = v^{-1} \, dx$.

One way to strengthen the A_2 condition is to consider Poisson A_2:

$$\begin{aligned} \mathsf{P}(I, \mu) &\equiv \int_{\mathbb{R}} \frac{|I|}{(|I| + \operatorname{dist}(x, I))^2} \, d\mu & (3) \\ \mathcal{A}_2^2 &= \sup_I \mathsf{P}(I, \omega) \cdot \mathsf{P}(I, \sigma). & (4) \end{aligned}$$

When $\mathcal{A}_2 < \infty$ we say the pair (ω, σ) satisfies the Poisson A_2 condition.

The $T1$ theorem of David and Journé coupled with Sawyer's work on two weight inequalities for positive operators provides motivation for additional conditions. These conditions are referred to as the "testing conditions",

$$\begin{aligned} \mathcal{H}^2 &= \sup_I \frac{\int_I |H(\chi_I \sigma)|^2 \, d\omega}{\sigma(I)} < \infty & (5) \\ (\mathcal{H}^*)^2 &= \sup_I \frac{\int_I |H(\chi_I \omega)|^2 \, d\sigma}{\omega(I)} < \infty. & (6) \end{aligned}$$

It is conjectured that the two weight inequality (1) is equivalent to (ω, σ) satisfying the Poisson A_2 condition and the testing conditions (5) and (6). This is analogous to the $T1$ Theorem with the testing conditions playing the role of $T1, T^*1 \in BMO$ and the Poisson A_2 condition playing the role of weak boundedness. The approach of Nazarov, Treil, and Volberg [3] proves this conjecture subject to two additional conditions, the *Pivotal Conditions*

$$\sum_{r=1}^{\infty} \omega(I_r) \mathsf{P}(I_r, \chi_I \sigma)^2 \leq C \sigma(I) \quad \text{and} \quad \sum_{r=1}^{\infty} \sigma(I_r) \mathsf{P}(I_r, \chi_I \omega)^2 \leq C \omega(I)$$

where these inequalities holds for all intervals I and pairwise disjoint partitions of $I = \bigcup_r I_r$.

One of the many novelties introduced in [1] is the *Energy Functional*, E. Let

$$\mathbb{E}_I^{\omega} f = \frac{1}{\omega(I)} \int_I f \, d\omega$$

and

$$\mathsf{E}(I,\omega)^2 = \mathbb{E}_I^{\omega(dx)}\mathbb{E}_I^{\omega(dy)}\left(\frac{|x-y|}{|I|}\right)^2 = \frac{1}{\omega(I)^2}\iint_{I\times I}\frac{|x-y|^2}{|I|^2}\,d\omega(x)d\omega(y).$$

Notice that $\mathsf{E}(I,\omega) \leq 1$ and if $\omega = \delta_a$, the point mass for $a \in I$, then $\mathsf{E}(I,\delta_a) = 0$. A refinement of the Pivotal Conditions is the following, given $0 \leq \epsilon \leq 2$ let

$$\mathcal{E}_\epsilon = \sup_I \sup_{I=\cup_r I_r} \frac{1}{\sigma(I)}\sum_r \omega(I_r)\mathsf{E}(I_r,\omega)^\epsilon \mathsf{P}(I_r,\chi_I\sigma)^2 \tag{7}$$

$$\mathcal{E}_\epsilon^* = \sup_I \sup_{I=\cup_r I_r} \frac{1}{\omega(I)}\sum_r \sigma(I_r)\mathsf{E}(I_r,\sigma)^\epsilon \mathsf{P}(I_r,\chi_I\omega)^2. \tag{8}$$

Notice that when $\mathcal{E}_0$ and $\mathcal{E}_0^*$ are finite the weights (ω,σ) satisfy the Pivotal Conditions above. Since $\mathsf{E}(J,\omega) \leq 1$ we have $\mathcal{E}_\epsilon \leq \mathcal{E}_\delta$ for $0 \leq \delta < \epsilon \leq 2$. When $\mathcal{E}_\epsilon$ and $\mathcal{E}_\epsilon^*$ are finite we say that the *Energy Conditions* are satisfied.

We are now ready to state the main theorem.

Theorem 1. *Let ω and σ be weights with no common point masses, i.e., $\omega(\{x\})\sigma(\{x\}) = 0$ for all $x \in \mathbb{R}$. Suppose that for some $0 \leq \epsilon < 2$, $\mathcal{E}_\epsilon$ and $\mathcal{E}_\epsilon^*$ are finite. Then the two weight inequality* (1) *holds if and only if $\mathcal{A}_2 + \mathcal{H} + \mathcal{H}^* < \infty$. Moreover,*

$$\|H(\,\cdot\,\sigma)\| \lesssim \mathcal{E}_\epsilon + \mathcal{E}_\epsilon^* + \mathcal{A}_2 + \mathcal{H} + \mathcal{H}^*.$$

One of the remarkable aspects of the Energy Conditions is they are necessary for the two weight inequality when $\epsilon = 2$. Specifically, we have the following theorem.

Theorem 2. *The following inequalities hold*

$$\mathcal{E}_2 \lesssim \mathcal{A}_2 + \mathcal{H}, \quad \textit{and} \quad \mathcal{E}_2^* \lesssim \mathcal{A}_2 + \mathcal{H}^*.$$

If one could show that the Energy Conditions are also sufficient when $\epsilon = 2$ then one would have a complete characterization of the two weight inequality for the Hilbert transform. Alternatively one might attempt to show that the Pivotal Conditions are necessary, thus providing a characterization of the two inequality. However, we construct an example of a pair of weights that satisfy the two weight inequality but not the Pivotal Conditions.

Theorem 3. *There exists a pair of weights* (ω, σ) *and* ϵ_0, $0 < \epsilon_0 < 2$ *such that* $\mathcal{E}_\epsilon + \mathcal{E}^*_\epsilon < \infty$, *for* $\epsilon_0 \le \epsilon < 2$, *and* $\mathcal{A}_2 + \mathcal{H} + \mathcal{H}^* < \infty$, *but* $\mathcal{E}^*_0 = \infty$. *In particular, the Pivotal Conditions are not necessary for the two weight inequality* (1).

The rest of this summary is devoted to the analysis of the two weight inequality with the added Energy Conditions. In Section 6.2, we motivate the definition of the Energy Conditions. In Section 6.3 we provide a glimpse into the proof of Theorem 1. Section 6.4 is devoted to showing the Pivotal Conditions are not necessary, the content of Theorem 3.

6.2 Motivation for the Energy Conditions

We now motivate the formulation of the Energy Condition in terms of a Calderón-Zygmund decomposition type argument. Suppose b is a function supported on an interval I that has cancellation with respect to the measure $d\sigma$,

$$\int_I b \, d\sigma = 0.$$

Consider estimating

$$\int_{\mathbb{R}\setminus 2I} |H(b\sigma)| d\omega = \int_{\mathbb{R}\setminus 2I} \left| \int_I \frac{b(y)}{x-y} d\sigma(y) \right| d\omega(x).$$

The usual move is to use the cancellation of b to add the quantity

$$\int_I \frac{b(y)}{c_I - x} \, d\sigma(y) = \frac{1}{c_I - x} \int_I b \, d\sigma = 0$$

where c_I is the center of the interval I. However, a more precise estimate that takes into account the weight σ is to add the quantity

$$\mathbb{E}_I^{d\sigma(z)} \left(\frac{1}{z-x} \right) \cdot \int_I b \, d\sigma = \frac{1}{\sigma(I)} \int_I \frac{1}{z-x} \, d\sigma(z) \cdot \int_I b \, d\sigma = 0.$$

Then we have,

$$\begin{aligned} \int_{\mathbb{R}\setminus 2I} |H(b\sigma)| d\omega &= \int_{\mathbb{R}\setminus 2I} \left| \int_I b(y) \cdot \mathbb{E}_I^{d\sigma(z)} \left(\frac{z-y}{|I|} \frac{|I|}{(x-y)(z-x)} \right) d\sigma(y) \right| d\omega(x) \\ &\le 4 \int_I |b(y)| \int_{\mathbb{R}\setminus 2I} \mathbb{E}_I^{d\sigma(z)} \left(\frac{|z-y|}{|I|} \right) \frac{|I|}{(|I| + \mathrm{dist}(x, I))^2} \, d\omega(x) d\sigma(y) \\ &\le 4 \|b\|_{L^2(\sigma)} \cdot \omega(I)^{\frac{1}{2}} \cdot \mathsf{E}(I, \sigma) \cdot \mathsf{P}(I, \chi_{\mathbb{R}\setminus 2I}\omega). \end{aligned}$$

6.3 Proof of Theorem 1

In this section we give a (very) brief sketch of the proof of Theorem 1. See also the summary of the paper [3] by M.C. Reguera, which uses similar techniques. We wish to estimate $\langle H(f\sigma), g\rangle_\omega$, for "nice" functions $f \in L^2(\sigma)$ and $g \in L^2(\omega)$ with norm one. Expanding in Haar bases adapted to the measures ω and σ we have

$$\begin{aligned}\langle H(f\sigma), g\rangle_\omega &= \sum_{I,J\in\mathcal{D}} \langle f, h_I^\sigma\rangle_\sigma \langle H(\sigma h_I^\sigma), h_J^\omega\rangle_\omega \langle g, h_I^\omega\rangle_\omega \\ &= \sum_{I,J\in\mathcal{D}} \langle H(\sigma\Delta_I^\sigma f), \Delta_J^\omega g\rangle_\omega\end{aligned}$$

where $\langle\cdot,\cdot\rangle_\mu$ denotes the pairing with respect to the measure μ, $\mathcal{D}$ is a dyadic grid, and

$$\Delta_I^\mu f = \langle f, h_I^\mu\rangle_\mu \cdot h_I^\mu.$$

The different measures σ and ω may not interact well with this decomposition. By shifting the grids and using a 'good-bad' decomposition of the functions f and g one can make the reduction to sum over certain 'good' intervals (I, J). This is a minor point in the proof and we do not discuss it further.

We decompose the sum into several layers which each contain several pieces. We will use the notation $\mathcal{C}_1^1, \mathcal{C}_2^1, \mathcal{C}_1^2, \ldots$, for our estimates, where the superscript indicates the level of the decomposition and the subscript indicates the piece within that level. We will also use $\mathcal{I}_1^1, \mathcal{I}_2^1, \mathcal{I}_1^2, \ldots$, to indicate the index of summation of the decomposition. We begin by spitting the sum in (9)

$$\begin{aligned}\langle Tf, g\rangle_\omega &= \sum_{(I,J)\in\mathcal{I}_1^1} \langle H(\sigma\Delta_I^\sigma f), \Delta_J^\omega g\rangle_\omega + \sum_{(I,J)\in\mathcal{I}_2^1} \langle H(\sigma\Delta_I^\sigma f), \Delta_J^\omega g\rangle_\omega \\ &= \mathcal{C}_1^1 + \mathcal{C}_2^1\end{aligned}$$

where the sums in $\mathcal{C}_1^1$ and $\mathcal{C}_2^1$ are over the index sets

$$\mathcal{I}_1^1 = \{(I, J) : |J| \le |I|\} \quad \text{and} \quad \mathcal{I}_2^1 = \{(I, J) : |I| > |J|\}.$$

The estimate $\mathcal{C}_2^1$ is similar to $\mathcal{C}_1^1$ using duality: $\langle H(f\sigma), g\rangle_\omega = -\langle H(g\omega), f\rangle_\sigma$. The term $\mathcal{C}_1^1$ is split into three pieces,

$$\mathcal{C}_1^1 = \mathcal{C}_1^2 + \mathcal{C}_2^2 + \mathcal{C}_3^2,$$

with corresponding index sets

$$\begin{aligned}
\mathcal{I}_1^2 &= \{(I,J)\in\mathcal{I}_1^1 : 2^{-r}|I|\le|J|\le|I|, \mathrm{dist}(I,J)\le|I|\},\\
\mathcal{I}_2^2 &= \{(I,J)\in\mathcal{I}_1^1 : |J|\le|I|, \mathrm{dist}(I,J)>|I|\},\\
\mathcal{I}_3^2 &= \{(I,J)\in\mathcal{I}_1^1 : |J|<2^{-r}|I|, \mathrm{dist}(I,J)\le|I|\}
\end{aligned}$$

where r is a sufficiently large integer depending on the good-bad decomposition of f and g. Roughly speaking, the terms in $\mathcal{C}_1^2$ are 'diagonal short-range', the terms in $\mathcal{C}_2^2$ are 'long-range', and the terms in $\mathcal{C}_3^2$ are 'short-range'. The testing conditions and Poisson A_2 are used to estimate $\mathcal{C}_1^2$ and $\mathcal{C}_2^2$, in particular $\mathcal{C}_1^2 \lesssim \mathcal{A}_2$ and $\mathcal{C}_2^2 \lesssim \mathcal{H}$.

The terms in $\mathcal{C}_3^2$ are decomposed further, $\mathcal{C}_3^2 = \mathcal{C}_1^3 + \mathcal{C}_2^3$. The substantial term is $\mathcal{C}_2^3$, as $\mathcal{C}_1^3 \lesssim \mathcal{E}_\epsilon$. Estimating this term is the crux of the proof, and we provide a brief glimpse into the machinery that is used. We have

$$\mathcal{C}_2^3 = \sum_{(I,J)\in\mathcal{I}_2^3} \langle H(\sigma\Delta_I^\sigma f), \Delta_J^\omega g\rangle_\omega$$

where

$$\mathcal{I}_2^3 = \{(I,J) : |J| < 2^{-r}|I|, J\subset I\}.$$

For $(I,J)\in\mathcal{I}_2^3$ since $J\subset I$, let I_J denote the child that contains J and $\hat{I}$ be some ancestor of I_J. Then

$$\begin{aligned}
\langle H(\sigma\Delta_I^\sigma f), \Delta_J^\omega g\rangle_\omega &= \langle H(\sigma\chi_{I\setminus I_J}\Delta_I^\sigma f), \Delta_J^\omega g\rangle_\omega + \langle H(\sigma\chi_{I_J}\Delta_I^\sigma f), \Delta_J^\omega g\rangle_\omega\\
&= \langle H(\sigma\chi_{I\setminus I_J}\Delta_I^\sigma f), \Delta_J^\omega g\rangle_\omega + \mathbb{E}_{I_J}^\sigma \Delta_I^\sigma f \cdot \langle H(\sigma\chi_{\hat{I}}), \Delta_J^\omega g\rangle_\omega \qquad (9)\\
&\quad - \mathbb{E}_{I_J}^\sigma \Delta_I^\sigma f \cdot \langle H(\sigma\chi_{\hat{I}\setminus I_J}), \Delta_J^\omega g\rangle_\omega. \qquad (10)
\end{aligned}$$

The point is that $\Delta_I^\sigma f$ is constant on I_J and equals its σ-average. Thus $\mathcal{C}_2^3$ can be split into three sums corresponding to the three terms in (9) and (10). The terms a referred to as, respectively, 'neighboring terms', 'paraproduct terms', and 'stopping terms'. The neighboring terms are bounded by $\mathcal{A}_2$ and the stopping terms are bounded by $\mathcal{E}_\epsilon$. The paraproduct is the essential term. It is decomposed further using a Corona decomposition modified with the Energy Condition. The Corona decomposition yields Carleson measure estimates for the paraproducts.

6.4 Counterexample for the Pivotal Conditions

Let C denote the Cantor set, $\mathsf{C} = \bigcap_n E_n$ where $E_n = \bigcup_{k=1}^{2^n} I_k^n$ and $|I_k^n| = 3^{-k}$. The Cantor measure is the unique probabiltiy measure supported on C with the property that

$$\omega(I_k^n) = 2^{-n}, \qquad n \geq 0,\ 1 \leq k \leq 2^n.$$

Let G_k^n be the removed open middle third of I_k^n and

$$\sigma = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \left(\frac{2}{9}\right)^n \delta_{c_k^n}.$$

where $c_k^n \in G_k^n$ is to be chosen. The usual choice of c_k^n would be the center of G_k^n. However the following construction exploits the cancelation of $H\omega$. Notice that since ω is supported in C, $H\omega$ is monotonically decreasing on G_k^n from ∞ at the left endpoint to $-\infty$ at the right endpoint. Therefore $H\omega$ has a unique zero, say $H\omega(c_k^n) = 0$. The pair of weights (ω, σ) satisfies the Poisson A_2 condition and the testing conditions (5) and (6).

The weights (ω, σ) also verify $\mathcal{E}_\epsilon < \infty$ for $0 < \epsilon \leq 2$ and $\mathcal{E}_\epsilon^* < \infty$ for $\epsilon_0 \leq \epsilon \leq 2$ with

$$\epsilon_0 = \frac{\ln 4}{\ln(9/2)} \approx 0.92$$

but $\mathcal{E}_0^* = \infty$. This shows that (ω, σ) satisfy the two weight inequality (1) but not the Pivotal conditions.

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7 A sharp estimate on the norm of the martingale transform

after J. Wittwer [7]
A summary written by Jean Moraes

Abstract

In this paper, following [7], we show that the bound for the martingale transform in $L^2(w)$ is linear in the A_2 Muckenhoupt characteristic constant of w, for $w \in A_2$.

7.1 Introduction

It was known since the 70s that singular integrals operators in $L^2(w)$ are bounded for $w \in A_2$, this result is due to Hunt, Muckenhaupt and Wheeden. However until the 90s nothing was known about how the bounds of these operators depend on the A_2 characteristic $[w]_{A_2}$ of the weights. Nowadays this dependence is known to be linear for many operators: Hilbert transform, Ahlfors-Beurling transform, Riesz transforms, martingale transforms; in fact Tuomas Hytönen just posted in the arXiv what is claimed to be the proof that a CZ-singular integral operators must obey a linear bound in $L^2(w)$ with respect to $[w]_{A_2}$. In [7], Wittwer proved that the bound for the martingales transform is linear. This was one of the first family of operators to be shown a sharp linear dependence in $[w]_{A_2}$.

7.2 Main Theorem

In what follows, $h_I(x)$ will denote the normalized Haar function for the dyadic interval I, i.e., $h_I(x) = \frac{\chi_{I_-}(x) - \chi_{I_+}(x)}{\sqrt{|I|}}$, where I_-,I_+ denotes the left and the right children of I respectively. The weight w will be dyadic A_2 weight on $[0,1]$. Also, w will be normalized to have $\int_0^1 w(x)dx = 1$. Let, $\langle f \rangle_I = \frac{1}{|I|}\int_I f(x)dx$ for I a dyadic interval. Note if we define $P(I) = \langle w \rangle_I \langle w^{-1} \rangle_I$, then $[w]_{A_2} = \sup_{I \in D} P(I)$.
The weights w and w^{-1} can be decomposed as $w(x) = \prod\limits_{I \in D([0,1])} (1 + c_I h_I(x))$ and $w^{-1}(x) = \prod\limits_{I \in D([0,1])} (1 + d_I h_I(x))$ where $c_I = \frac{\langle w \rangle_{I_-} - \langle w \rangle_{I_+}}{2\langle w \rangle_I}\sqrt{|I|}$ and $d_I =$

$\frac{\langle w^{-1}\rangle_{I_-} - \langle w^{-1}\rangle_{I_+}}{2\langle w^{-1}\rangle_I}\sqrt{|I|}$. where $D([0,1])$ is the collection of all dyadic intervals in $[0,1]$. Let $r(I)$ be a function from $D([0,1])$ to $\{-1,1\}$, then we define a martingale transform as

$$\big(T_r f\big)(x) = \sum_{I\in D([0,1])} r(I)\langle f, h_I\rangle h_I(x)$$

Theorem 1. *For all $w \in A_2$ and all $f \in L^2(w)$, there exists a constant c, independent of r, such that*

$$||T_r f||_{L^2(w)} \le c[w]_{A_2}||f||_{L^2(w)}.$$

7.3 Preliminaries

In order to prove this theorem we will need some lemmas and theorems:

Theorem 2 (**Weighted Carleson Embedding Theorem**). *Let $\alpha_I \ge 0$. Then*

$$\sum_{I\in D([0,1])} \langle f w^{1/2}\rangle_I^2 \alpha_I \le 4c||f||_{L^2}^2, \quad \forall f \in L^2(dx) \quad \textit{iff}$$

$$\frac{1}{|J|}\sum_{I\subset J}\langle w\rangle_I^2\alpha_I \le c\langle w\rangle_J, \quad \forall J \in D[0,1].$$

Theorem 3. *Let $\alpha_I \ge 0$. If*

$$\frac{1}{|J|}\int_J\Big(\sum_{I\subset J}\alpha_I\langle w\rangle_I\chi_I(x)\Big)^2 w^{-1}(x)dx \le c_1\langle w\rangle_J, \quad \forall J\in D \quad \textit{and} \qquad (1)$$

$$\frac{1}{|J|}\int_J\Big(\sum_{I\subset J}\alpha_I\langle w^{-1}\rangle_I\chi_I(x)\Big)^2 w(x)dx \le c_1\langle w^{-1}\rangle_J, \quad \forall J\in D \qquad (2)$$

$$\textit{Then} \quad \sum_{I\in D}\alpha_I\langle f w^{1/2}\rangle_I\langle g w^{-1/2}\rangle_I|I| \le c_2||f||_{L^2}||g||_{L^2} \quad \forall f,g \in L^2(dx), \quad (3)$$

where $c_2 \le c\sqrt{c_1}$ and c is an constant.

The proof of Theorem 2 can be found in [5], section 2, and Theorem 3 can be found in [7].

Theorem 4. *Let Sf be the dyadic square function of f. Then for all $w \in A_2$*

$$||Sf||_{L^2(w)} \leq c[w]_{A_2}||f||_{L^2(w)}$$

and this estimate is sharp, where $Sf(x) = \Big(\sum_{I \in D} \frac{|\langle f, h_I\rangle|^2}{|I|}\chi_I(x)\Big)^{1/2}$

This result was first proved in [4], Wittwer used a different method to also prove this in [7].

Corollary 5. *If $w \in A_2$, then* $\dfrac{1}{|J|}\sum_{I \subset J}\langle w^{-1}\rangle_I^2 d_I^2 \langle w\rangle_I \leq c[w]_{A_2}^2 \langle w^{-1}\rangle_J$

Proof. Just note that $\frac{1}{|J|}\sum_{I\subset J}\langle w^{-1}\rangle_I^2 d_I^2\langle w\rangle_I \leq ||S(w^{-1}\chi_J||^2_{L^2(w)}$ which by Theorem 4 is bounded by $c[w]^2_{A_2}||w^{-1}\chi_J||^2_{L^2(w)}$. □

The next two lemmas are proved in [7], we will give the idea of the proof of Lemma 7.

Lemma 6. *If $w \in A_2$, then* $\dfrac{1}{|J|}\sum_{I \subset J}\dfrac{c_I}{\sqrt{|I|}}\dfrac{d_I}{\sqrt{|I|}}P(I)|I| \leq C[w]_{A_2}$.

Lemma 7. *If $w \in A_2$, then* $\dfrac{1}{|J|}\sum_{I \subset J}\dfrac{c_I}{\sqrt{|I|}}\dfrac{d_I}{\sqrt{|I|}}\langle w\rangle_I|I| \leq C[w]_{A_2}\langle w\rangle_J$.

Proof. This lemma is proved by the method of Bellman functions. Consider $B(x,y) = x\big(\frac{-4A}{xy} - \frac{xy}{4A} + 4A + 1\big)$, this function has the following properties:

- $0 \leq B(x,y) \leq 5Ax$ on $D_1 = \{x, y > 0; 1 \leq xy \leq A\}$.

- $\begin{bmatrix} -B_{xx} & -B_{xy} - \frac{1}{y} \\ -B_{xy} - \frac{1}{y} & -B_{yy} \end{bmatrix}$ and $\begin{bmatrix} -B_{xx} & -B_{xy} + \frac{1}{y} \\ -B_{xy} + \frac{1}{y} & B_{yy} \end{bmatrix}$ are positive semidefinite on $D_2 = \{x, y > 0; 1 \leq xy \leq 2A\}$.

These properties imply that:

$$B(x,y) - \frac{B(x_-,y_-) + B(x_+,y_+)}{2} \geq C\big|(x_- - x_+)(y_- - y_+)\big|$$

where $x = \frac{x_- + x_+}{2}$, $y = \frac{y_- + y_+}{2}$ and $(x,y), (x_-,y_-), (x_+,y_+) \in D_1$. In order to get the desired estimate, we just have to run the usual type of Bellman function argument using $x = \langle w\rangle_J$, $y = \langle w^{-1}\rangle_J$, $x_- = \langle w\rangle_{J_-}$, $x_+ = \langle w\rangle_{J_+}$, $y_- = \langle w^{-1}\rangle_{J_-}$, $y_+ = \langle w^{-1}\rangle_{J_+}$ and $A = [w]_{A_2}$. □

7.4 Proof of Theorem 1

Let us estimate the norm of the martingale transform by duality,

$$||T_r f||_{L^2(w)} = \sup_{||g||_{L^2(w^{-1})}=1} \int \sum_{I,J\in D} r(I)\langle f, h_I\rangle h_I(x)\langle g, h_J\rangle h_J(x)dx$$

Using the fact that $\{h_I(x)\}_I$ is an orthonormal basis in $L^2(dx)$ we can collapse the double sum in just one sum,

$$||T_r f||_{L^2(w)} = \sup_{||g||_{L^2(w^{-1})}=1} \sum_{I\in D} r(I)\langle f, h_I\rangle\langle g, h_J\rangle \tag{4}$$

Now, note that $||f||_{L^2(w)} = ||fw^{1/2}||_{L^2(dx)}$ and $||g||_{L^2(w^{-1})} = ||gw^{-1/2}||_{L^2(dx)}$, so we can replace f by $fw^{1/2}$ and g by $gw^{-1/2}$ in (4), and then write

$$||T_r||_{L^2(w)\to L^2(w)} = \sup_{||f||_{L^2(dx)}=1} \sup_{||g||_{L^2(dx)}=1} \sum_{I\in D} r(I)\langle fw^{-1/2}, h_I\rangle\langle gw^{1/2}, h_I\rangle \tag{5}$$

Let's consider

$$h_I^w(x) = \frac{h_I(x) + \gamma_I^w \chi_I(x)}{\delta_I^w},$$

where $\gamma_I^w = \frac{-c_I}{|I|}$ and $\delta_I^w = \sqrt{\langle w\rangle_I(1 - c_I^2/|I|)} = \sqrt{\frac{\langle w\rangle_{I_-}\langle w\rangle_{I_+}}{w_I}}$, for these choices $\{h_I^w\}_I$ is a normalized and orthogonal system in $L^2(w)$ with respect to the weighted inner product $\langle f, g\rangle = \int fgwdx$.

Substituting $h_I(x) = \delta_I^w h_I^w(x) - \gamma_I^w \chi_I(x)$ and $h_I(x) = \delta_I^{w^{-1}} h_I^{w^{-1}}(x) - \gamma_I^{w^{-1}} \chi_I(x)$ in (5) we can rewrite $\sum_{I\in D} r(I)\langle fw^{-1/2}, h_I\rangle\langle gw^{1/2}, h_I\rangle$ as $\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, where

$$\Gamma_1 = \sum_{I\in D[0,1]} r(I)\langle fw^{-1/2}, h_I^{w^{-1}}\rangle\delta_I^{w^{-1}}\langle gw^{1/2}, h_I^w\rangle\delta_I^w \tag{6}$$

$$\Gamma_2 = -\sum_{I\in D[0,1]} r(I)\langle fw^{-1/2}, \chi_I\rangle\gamma_I^{w^{-1}}\langle gw^{1/2}, h_I^w\rangle\delta_I^w \tag{7}$$

$$\Gamma_3 = -\sum_{I\in D[0,1]} r(I)\langle fw^{-1/2}, h_I^{w^{-1}}\rangle\delta_I^{w^{-1}}\langle gw^{1/2}, \chi_I\rangle\gamma_I^w \tag{8}$$

$$\Gamma_4 = \sum_{I\in D[0,1]} r(I)\langle fw^{-1/2}, \chi_I\rangle\gamma_I^{w^{-1}}\langle gw^{1/2}, \chi_I\rangle\gamma_I^w \tag{9}$$

We will estimate each sum separately in absolute value.

7.4.1 Sum Γ_1

$$|\Gamma_1| \leq \sum_{I\in D[0,1]} \left|\langle fw^{-1/2}, h_I^{w^{-1}}\rangle\langle gw^{1/2}, h_I^w\rangle\right| \sqrt{\frac{\langle w\rangle_{I_+}\langle w\rangle_{I_-}\langle w^{-1}\rangle_{I_+}\langle w^{-1}\rangle_{I_-}}{\langle w\rangle_I\langle w^{-1}\rangle_I}}$$

By geometric-arithmetic ineq. $\langle w\rangle_{I_+}\langle w\rangle_{I_-} \leq \langle w\rangle_I{}^2$ and $\langle w^{-1}\rangle_{I_+}\langle w^{-1}\rangle_{I_-} \leq \langle w^{-1}\rangle_I{}^2$ then

$$\begin{aligned} |\Gamma_1| &\leq \sum_{I\in D([0,1])} \left|\langle fw^{-1/2}, h_I^{w^{-1}}\rangle\langle gw^{1/2}, h_I^w\rangle\right| \sqrt{P(I)} \\ &\leq [w]_{A_2}^{1/2} \sum_{I\in D([0,1])} \left|\langle fw^{1/2}, h_I^{w^{-1}}\rangle_{w^{-1}}\langle gw^{-1/2}, h_I^w\rangle_w\right| \qquad (10) \end{aligned}$$

Since $\{h_I^w\}_{I\in D}$ and $\{h^{w^{-1}}\}_{I\in D}$ are orthonormal in $L^2(w)$ and $L^2(w^{-1})$ respectively, then we can use Cauchy-Schwarz and the Bessel inequality in (10) and get that $|\Gamma_1| \leq [w]_{A_2}^{1/2}||fw^{1/2}||_{L^2(w^{-1})}||gw^{-1/2}||_{L^2(w)} = [w]_{A_2}^{1/2}||f||_{L^2(dx)}||g||_{L^2(dx)}$

7.4.2 Sums Γ_2 and Γ_3

Note that the sum Γ_2 and the sum Γ_3 are similar, so its enough to estimate just one of them, the other follow by the same argument.

$$\begin{aligned} |\Gamma_2| &\leq \sum_{I\in D([0,1])} \left|\langle fw^{-1/2}, \chi_I\rangle\gamma_I^{w^{-1}}\langle gw^{1/2}, h_I^w\rangle\delta_I^w\right| \\ &= \sum_{I\in D([0,1])} \left|\langle fw^{-1/2}\rangle_I|I|\frac{d_I}{|I|}\langle gw^{1/2}, h_I^w\rangle\sqrt{\langle w\rangle_I(1-c_I^2/|I|)}\right| \qquad (11) \\ &\leq \left(\sum_{I\in D[0,1]} \left|\langle fw^{-1/2}\rangle_I^2|d_I|^2\langle w\rangle_I\right)^{1/2}\left(\sum_{I\in D([0,1])}\langle gw^{1/2}, h_I^w\rangle^2\right)^{1/2} \qquad (12) \end{aligned}$$

Above we used the fact that $(1-c_I^2/|I|) \leq 1$ in (11) and then applied Cauchy-Schwarz inequality in order to get (12). Arguing as we did in the sum of Γ_1 we can get the following estimative by Bessel's inequality

$$\left(\sum_{I\in D([0,1])}\langle gw^{1/2}, h_I^w\rangle^2\right)^{1/2} \leq ||g||_{L^2(dx)}.$$

Corollary 5 and Theorem 2, for w^{-1} instead of w, imply that

$$\left(\sum_{I\in D([0,1])} |\langle fw^{-1/2}\rangle_I|^2 |d_I|^2 \langle w\rangle_I\right)^{1/2} \leq c[w]_{A_2}||f||_{L^2(dx)}.$$

Therefore $|\Gamma_2| + |\Gamma_3| \leq 2c[w]_{A_2}||f||_{L^2(dx)}||g||_{L^2(dx)}$.

7.4.3 Sums Γ_4

$$|\Gamma_4| \leq \sum_{I\in D([0,1])} \left|\langle fw^{-1/2}\rangle_I |I| \langle gw^{1/2}\rangle_I |I| \frac{c_I d_I}{|I|^2}\right| \tag{13}$$

We will use Theorem 3 with $\alpha_I = \frac{c_I d_I}{|I|}$ to estimate the sum above. In order to to use theorem 2 we have to check 1 and 2, since $[w]_{A_2} = [w^{-1}]_{A_2}$ it is enough to check just one of these conditions.

$$\frac{1}{|J|}\int_J \left(\sum_{I\subset J} \alpha_I \langle w\rangle_I \chi_I(x)\right)^2 w^{-1}(x)dx = \tag{14}$$

$$= \frac{1}{|J|}\left(\sum_{I\subset J} \alpha_I^2 \langle w\rangle_I^2 \langle w^{-1}\rangle_I |I| + 2\sum_{I,K,K\subset I\subset J} \alpha_I \alpha_K \langle w\rangle_I \langle w\rangle_K \langle w^{-1}\rangle_K |K|\right)$$

Using the fact that $\alpha_I = \frac{c_I d_I}{|I|}$ and $\left|\frac{d_I}{\sqrt{|I|}}\right| \leq 1$, then the first sum above is bounded by $\frac{1}{|J|}\sum_{I\subset J}\left(\frac{c_I}{\sqrt{|I|}}\right)^2 \langle w\rangle_I |I| P(I)$ which is bounded by $c[w]_{A_2}^2 \langle w\rangle_J$ by corollary 5. The last step is to estimate the second sum, again use the definition of α_I and α_K to rewrite it as

$$\frac{2}{|J|}\sum_{I\subset J} \frac{|c_I|}{\sqrt{|I|}} \frac{|d_I|}{\sqrt{|I|}} \langle w\rangle_I \sum_{K\subset I} \frac{|c_K|}{\sqrt{|K|}} \frac{|d_K|}{\sqrt{|K|}} P(K)|K| \tag{15}$$

By Lemma 6, $\sum_{K\subset I} \frac{|c_K|}{\sqrt{|K|}} \frac{|d_K|}{\sqrt{|K|}} P(K)|K| \leq C[w]_{A_2}|I|$ and then (15) is bounded by $C[w]_{A_2}\frac{2}{|J|}\sum_{I\subset J} \frac{|c_I|}{\sqrt{|I|}} \frac{|d_I|}{\sqrt{|I|}} \langle w\rangle_I |I|$ which is bounded by Lemma 7 by $C[w]_{A_2}\langle w\rangle_J$. Therefore (13) is bounded by $C[w]_{A_2}^2 \langle w\rangle_J$ which allows us to use Theorem 2 and conclude that $|\Gamma_4| \leq c[w]_{A_2}||f||_{L^2(dx)}||g||_{L^2(dx)}$.

Therefore we have that all Γ's are bounded by $c[w]_{A_2}||f||_{L^2(dx)}||g||_{L^2(dx)}$, with constant $c > 0$ independent of r, which implies

$$||T_r f||_{L^2(w)} \leq c[w]_{A_2}||f||_{L^2(w)}.$$

□

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8 Sharp A_2 inequality for Haar shift operators

after M. Lacey, S. Petermichl and M. C. Reguera [2]
A summary written by Diogo Oliveira e Silva

Abstract

The authors of [2] prove linear growth in the A_2 characteristic for weighted L^2 inequalities involving Haar shift operators. We describe how two new ingredients of the proof, a two weight $T1$ theorem and a corona decomposition of the weight, come into play in the proof.

8.1 Introduction

It is a classical result [5] that the Hilbert transform H is bounded on $L^p(w)$ if and only if $w \in A_p$. Specializing to $p = 2$, the weight w satisfies

$$[w]_{A_2} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w\Big)\Big(\frac{1}{|Q|}\int_Q w^{-1}\Big) < \infty$$

if and only if we have an inequality

$$\|Hf\|_{L^2(w)} \leq C\|f\|_{L^2(w)},$$

where the constant C depends on the A_2-constant $[w]_{A_2}$. Determining the exact dependent of C on $[w]_{A_2}$ is a difficult problem with nontrivial consequences. In [4] the following optimal bound is proved:

$$\|Hf\|_{L^2(w)} \lesssim [w]_{A_2}\|f\|_{L^2(w)}. \tag{1}$$

Later work showed that if one replaces H by a Riesz transform R_j or the Beurling-Ahlfors transform B, inequality (1) still holds. For applications of these results to the theory of elliptic PDE, see [4].

Earlier proofs made use of Haar shift operators (which are entirely natural in this context since H, R_j and B are all obtained by appropriate averaging of Haar shifts) together with Bellman function techniques. The present paper [2] still deals with Haar shift operators, but instead of Bellman functions uses a deep two weight $T1$ theorem from [3], together with an appropriate corona decomposition of the weight w to verify the relevant Carleson measure estimates.

8.2 Haar shift operators

We will denote the family of all dyadic cubes in $\mathbb{R}^d$ by $\mathcal{Q}$, and all cubes will from now onwards be assumed dyadic. For $Q \in \mathcal{Q}$ and $m \in \mathbb{N}$, we let $Q^{(m)}$ be the m-fold parent of Q. By a Haar function h_Q on a cube Q we mean any function supported on Q which is constant on dyadic subcubes of Q and which satisfies the following conditions:

(i) $\int_Q h_Q = 0$; (cancellation)

(ii) $\|h_Q\|_\infty \leq |Q|^{-1/2}$. (size)

In particular, Haar functions are L^2 normalized. They play an essential role in the following definition:

Definition 1. *We say that T is a Haar shift operator of index $\tau \in \mathbb{N}_0$ on $\mathbb{R}^d$ if $Tf = \sum_{Q \in \mathcal{Q}} \langle f, g_Q \rangle \gamma_Q$, where*

$$g_Q, \gamma_Q \in span\{h_{Q'} : Q' \subset Q, 2^{-\tau d}|Q| \leq |Q'|\}, \text{ and} \tag{2}$$

$$\|g_Q\|_\infty, \|\gamma_Q\|_\infty \leq |Q|^{-1/2}. \tag{3}$$

Let us turn to some illustrative examples of Haar shift operators.

Example 2. *(Index 0) Given a bounded sequence $\alpha = (\alpha_Q)_{Q \in \mathcal{Q}}$, consider the Haar multiplier T^α given by*

$$T^\alpha f = \sum_{Q \in \mathcal{Q}} \alpha_Q \langle f, h_Q \rangle h_Q.$$

If $\alpha \subset \{-1, 1\}^{\mathbb{N}}$, then T^α is called the martingale transform, also known as the dyadic Hilbert transform.

Example 3. *(Index 1) For a one-dimensional example, consider the Haar shift S given by*

$$Sf = \sum_{I \in \mathcal{Q}} \langle f, h_I \rangle (h_{I_r} - h_{I_l}),$$

where I_r and I_l denote the right and left halves of the interval $I \subset \mathbb{R}$, respectively.

The purpose of conditions (2) and (3) is to ensure that Haar shift operators are Calderón-Zygmund operators. That is the content of the following proposition:

Proposition 4. *Let T be a Haar shift operator of index τ on $\mathbb{R}^d$. Then T is bounded on $L^2(dx)$ with norm $\lesssim \tau$, and T maps $L^1(dx)$ into $L^{1,\infty}(dx)$ with norm $\lesssim 2^{\tau d}$.*

We mention some elements that go into the proof of proposition 4. The cancellation condition (2) allows us to use the lemma of Cotlar, Knapp and Stein, which together with the size condition (3) implies $\|T\|_{L^2(dx)} \lesssim \tau$. The weak (1,1) bound is obtained by the usual Calderón-Zygmund decomposition, where the analysis of the "bad" part is simplified by noting that

$$x \notin Q^{(\tau)} \Rightarrow T(1_Q b)(x) = 0.$$

8.3 Main result and tools

The main result of [2] is the following:

Theorem 5. *Let T be a Haar shift operator of index τ on $\mathbb{R}^d$, and let w be an A_2 weight. Then*

$$\|T\|_{L^2(w)\to L^2(w)} \lesssim_{d,\tau} [w]_{A_2}. \tag{4}$$

We will sketch the proof given in [2] in the next section. For now we explore two essential ingredients which go into the proof of theorem 5.

8.3.1 A two weight $T1$ theorem

The paper [3] gives an elegant characterization of some two weight inequalities in L^2. Its main result implies the following theorem:

Theorem 6. *Let T be a Haar shift operator of index τ on $\mathbb{R}^d$, and let σ and μ be two positive measures. The inequality*

$$\|T(\sigma f)\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\sigma)}$$

holds if and only if there exist constants $C_1, C_2, C_3 < \infty$ such that for all cubes Q, Q', Q'' with $Q', Q'' \subset Q$ and $2^{-\tau d}|Q| \le |Q'|, |Q''|$,

$$\Big| \int_{Q''} T(\sigma 1_{Q'}) d\mu \Big| \le C_1 \sigma(Q')^{1/2} \mu(Q'')^{1/2}, \tag{5}$$

$$\|T(\sigma 1_Q)\|_{L^2(Q,\mu)} \le C_2 \sigma(Q)^{1/2} \text{ and} \tag{6}$$

$$\|T^*(\mu 1_Q)\|_{L^2(Q,\sigma)} \le C_3 \mu(Q)^{1/2}.$$

As a corollary of the proof, one gets that $\|T(\sigma\cdot)\|_{L^2(\sigma)\to L^2(\mu)} \lesssim C_1 + C_2 + C_3$.

8.3.2 The corona decomposition

The following somewhat involved definition will play an essential role in the proof of the main estimate:

Definition 7. *Let $\mathcal{Q}' \subset \mathcal{Q}$ be any bounded collection of cubes, and let μ be a positive measure. Let $\mathcal{L} \subset \mathcal{Q}'$. We call $(\mathcal{Q}'(L))_{L\in\mathcal{L}}$ a corona decomposition of $\mathcal{Q}'$ with respect to μ if the following conditions hold:*

(i) For every $Q \in \mathcal{Q}'$, there exists $L \in \mathcal{L}$ such that $Q \subset L$. Let $\lambda(Q) \in \mathcal{L}$ denote the minimal cube which contains Q, and set $\mathcal{Q}'(L) := \{Q \in \mathcal{Q}' : \lambda(Q) = L\}$. Then

$$\frac{\mu(Q)}{|Q|} \leq 4\frac{\mu(\lambda(Q))}{|\lambda(Q)|};$$

(ii) If $L, L' \in \mathcal{L}$ are such that $L' \subsetneq L$, then

$$4\frac{\mu(L)}{|L|} < \frac{\mu(L')}{|L'|}.$$

Observe that the collections $\mathcal{Q}'(L)$ partition $\mathcal{Q}'$. A construction of the corona decomposition is accomplished via the following stopping-time argument from [1]. Let $\mathcal{L}_0$ consist of all $Q \in \mathcal{Q}'$ which are maximal for set inclusion. Recursively, $\mathcal{L}_{m+1}$ shall consist of all Q in the set

$$\bigcup_{L\in\mathcal{L}_m} \left\{Q \in \mathcal{Q}' : Q \subset L \text{ and } \frac{\mu(Q)}{|Q|} > 4\frac{\mu(L)}{|L|}\right\}$$

for which Q is maximal for set inclusion, and $\mathcal{Q}'(L)$ is the collection of all $Q \in \mathcal{Q}'$ such that $Q \subset L$ and $Q \not\subset L'$ for any $L' \in \mathcal{L} := \bigcup_{m\geq 0} \mathcal{L}_m$ with $L' \subsetneq L$.

A straightforward consequence of the construction is

$$\Big|\bigcup_{\mathcal{L}\ni L'\subsetneq L} L'\Big| \leq \frac{1}{4}|L|, \ L \in \mathcal{L}. \tag{7}$$

To what extent does (7) still hold if we replace Lebesgue measure by the weight w? The following lemma gives a partial answer to this question:

Lemma 8. *Let $\mathcal{L}$ be associated with the corona decomposition of an A_2 weight w. For any cube Q we have*

$$\sum_{\mathcal{L}\ni L\subset Q} w(L) \leq \frac{16}{9}[w]_{A_2} w(Q). \tag{8}$$

8.4 Idea of the proof of theorem 5

In order to be able to apply theorem 6, we start by noting that inequality (4) is equivalent to the following two weight version:

$$\|T(fw)\|_{L^2(w^{-1})} \lesssim [w]_{A_2} \|f\|_{L^2(w)}. \tag{9}$$

To verify (9), it is enough to show that the following conditions hold for all cubes Q, R of comparable size (*i.e.* such that $2^{-d\tau}|Q| \le |R| \le 2^{d\tau}|Q|$):

$$|\langle T(w1_Q), w^{-1}1_R\rangle| \lesssim [w]_{A_2} w(Q)^{1/2} w^{-1}(R)^{1/2}; \tag{10}$$

$$\int_Q |T(w1_Q)|^2 w^{-1} dx \lesssim [w]_{A_2}^2 w(Q). \tag{11}$$

The "weak boundedness" inequality (10) can be derived from the "T1" condition (11), and so we concentrate on verifying the latter. "Large scales" are easy to handle, and so we will limit ourselves to showing that

$$\Big\| \sum_{Q:Q\subset Q_0} \langle w, g_Q\rangle \gamma_Q \Big\|_{L^2(w^{-1})} \lesssim [w]_{A_2} w(Q_0)^{1/2}. \tag{12}$$

For cubes Q_0 and collections of cubes $\mathcal{Q}'$, we define two quantities:

$$H(Q_0, \mathcal{Q}') := \sum_{\mathcal{Q}' \ni Q \subset Q_0} \langle w, g_Q\rangle \gamma_Q;$$

$$H(\mathcal{Q}') := \sup_{Q_0 \in \mathcal{Q}'} \frac{\|H(Q_0, \mathcal{Q}')\|_{L^2(w^{-1})}}{w(Q_0)^{1/2}}.$$

We pave the way to the corona decomposition by introducing the sets

$$\mathcal{Q}_n := \Big\{ Q \in \mathcal{Q} : 2^{n-1} < \frac{w(Q)}{|Q|} \frac{w^{-1}(Q)}{|Q|} \le 2^n \Big\},$$

which essentially fix the local A_2 characteristic. Our goal is of course to show that $H(\mathcal{Q}) \lesssim [w]_{A_2}$. For that purpose, it is enough to show that $H(\mathcal{Q}_n) \lesssim 2^{n/2}[w]_{A_2}^{1/2}$.

Fix $Q_0 \in \mathcal{Q}_n$ which tests the supremum in the definition of $H(\mathcal{Q}_n)$. Let

$$\mathcal{P}_n := \{Q \in \mathcal{Q}_n : Q \subset Q_0\},$$

and consider the corona decomposition $(\mathcal{P}_n(L))_{L\in\mathcal{L}_n}$ of $\mathcal{P}_n$ with respect to the measure w (accomplished via the construction outlined in the previous section). Note that $\mathcal{L}_n \subset \mathcal{P}_n \subset \mathcal{Q}_n$.

We are seeking to prove

$$\|H(Q_0, \mathcal{Q}_n)\|^2_{L^2(w^{-1})} \lesssim 2^n [w]_{A_2} w(Q_0). \tag{13}$$

Since $H(Q_0, \mathcal{Q}_n) = \sum_{L\in\mathcal{L}_n} H(L, \mathcal{P}_n(L))$, the following lemma will be useful in the proof of (13).

Lemma 9. *The following uniform distributional estimates hold for $L \in \mathcal{L}_n$:*

$$\Big|\Big\{x \in L : |H(L, \mathcal{P}_n(L))(x)| > Kt\frac{w(L)}{|L|}\Big\}\Big| \lesssim e^{-t}|L|; \tag{14}$$

$$w^{-1}\Big(\Big\{x \in L : |H(L, \mathcal{P}_n(L))(x)| > Kt\frac{w(L)}{|L|}\Big\}\Big) \lesssim e^{-t}w^{-1}(L). \tag{15}$$

The proof of lemma 9 involves a further dyadic decomposition, and uses proposition 4 and a version of John-Nirenberg inequality. I will omit the details and present them at the summer school.

Letting $H_n(L) := |H(L, \mathcal{P}_n(L))|$, we have that

$$\begin{aligned}
\|H(Q_0, \mathcal{Q}_n)\|^2_{L^2(w^{-1})} \leq & \Big\| \sum_{L\in\mathcal{L}_n} H_n(L)\Big\|^2_{L^2(w^{-1})} \\
&= \sum_{L\in\mathcal{L}_n} \|H_n(L)\|^2_{L^2(w^{-1})} + 2\sum_{L\in\mathcal{L}_n}\sum_{\mathcal{L}_n\ni L'\subsetneq L}\int H_n(L)H_n(L')w^{-1} \\
&=: I + II.
\end{aligned}$$

We estimate I (estimating II is similar but slightly more involved):

$$\begin{aligned}
\|H_n(L)\|^2_{L^2(w^{-1})} \lesssim & \Big(\frac{w(L)}{|L|}\Big)^2 w^{-1}(L) && \text{(by (15))} \\
=& w(L)\Big(\frac{w(L)}{|L|}\frac{w^{-1}(L)}{|L|}\Big) \\
\leq & 2^n w(L). && \text{(since } L \in \mathcal{L}_n \subset \mathcal{Q}_n\text{)}
\end{aligned}$$

The desired result now follows from lemma 8:

$$I \lesssim 2^n \sum_{L\in\mathcal{L}_n} w(L) \lesssim 2^n [w]_{A_2} w(Q_0).$$

The author is indebted to Maria Carmen Reguera for a very useful discussion of some parts of [2] and to Michael Christ for valuable recommendations on the exposition.

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9 Heating of The Ahlfors-Beurling operator: Weakly quasiregular maps on the plane are quasiregular

after S. Petermichl and A. Volberg [8]
A summary written by Nikolaos Pattakos

Abstract

We outline a proof of a sharp weighted estimate of the Ahlfors-Beurling operator and then we establish borderline regularity for solutions of the Beltrami equation $f_z - \mu f_{\overline{z}}$ on the plane, where μ is a bounded measurable function, $\|\mu\|_\infty = k < 1$.

9.1 Introduction and notation

We are interested in the following Ahlfors-Beurling operator (dA denotes area Lebesgue measure on $\mathcal{C}$):

$$T\phi(z) = \frac{1}{\pi}\int\int \frac{\phi(\zeta)dA(\zeta)}{(z-\zeta)^2}$$

understood as a Calderón-Zygmund operator. For any function f on the plane, it's heat extension is given by the formula

$$f(y,t) = \frac{1}{\pi t}\int\int_{\mathcal{R}^2} f(x)exp\Big(-\frac{|x-y|^2}{t}\Big)dx_1dx_2, (y,t) \in \mathcal{R}^3_+$$

This is just the convolution of the function f with the fundamental solution of the heat equation $k(x,t) = \frac{1}{t\pi}exp(-(|x|^2)/t))$. We use the same notation for the function and for it's extension. Now we can define the heat A_p characteristic of a weight w as:

$$Q^{heat}_{w,p} = \sup_{x\in\mathcal{R}^2, t>0}\int\int w(x-y)k(y,t)dy_1dy_2 \cdot \Big(\int\int w^{-\frac{1}{(p-1)}}(x-y)k(y,t)dy_1dy_2\Big)^{p-1}.$$

The weights with finite $Q^{heat}_{w,p}$ are called A_p weights. There is an extensive theory of A_p weights (see, e.g.,[2], [10]). The usual definition i.e. weights with finite

$$Q_{w,p}^{class} = \sup_{B(x,R)} \Big(\frac{1}{|B(x,R)|}\int_{B(x,R)} wdA\Big)\Big(\frac{1}{|B(x,R)|}\int_{B(x,R)|} w^{(-\frac{1}{p-1})}dA\Big)^{p-1}$$

where the supremum is taken over all disks on the plane, differs from this one but it actually describes the same class of weights. In the following μ is a measurable function with $\|\mu\|_{L^\infty} = k < 1$. When we write $F = F^\mu$ we mean that F solves the Beltrami equation $F_z - \mu F_{\overline{z}} = 0$, where $F_z = \frac{1}{2}(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y})$ and $F_{\overline{z}} = \frac{1}{2}(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y})$. Solutions of this equation are called q-weakly quasiregular if they belong to the space $W^{1,q}_{loc}$ where $1 \leq q$. It is known that if $q > 1+k$ then a q-weakly quasiregular map is actually in $W^{1,2+\epsilon}_{loc}$ for a certain positive ϵ. In particular it is quasiregular i.e. in the space $W^{1,2}_{loc}$. In [2] one can find an easy example that shows this is no longer true for $q < 1+k$. Thus, the remaining question is about the critical exponent $q = 1+k$. In the same paper [2] the authors suggest the following problem: Prove that the operators $I - \mu T$ and $I - T\mu$ have dense range in $L^{1+\frac{1}{k}}(\mathcal{C})$ and are injective in $L^{1+k}(\mathcal{C})$. All of these questions are going to be answered just by proving the sharp estimate for the Ahlfors-Beurling operator.

9.2 The sharp weighted estimate for the Ahlfors-Beurling operator

Theorem 1. *For any A_p weight w, $p \geq 2$, we have*

$$\|T\|_{L^p(wdA)\to L^p(wdA)} \leq CQ^{heat}_{w,p}.$$

In order to prove this we fix w to be an arbitrary positive function on the plane and we assume that $p = 2$. The general case then follows from this one. The operator T is given in the Fourier domain (ξ_1, ξ_2) by the multiplier

$$\frac{\overline{\zeta}}{\zeta} = \frac{\overline{\zeta}^2}{\|\zeta\|^2} = \frac{(\xi_1 - i\xi_2)^2}{\xi_1^2+\xi_2^2} = \frac{\xi_1^2}{\xi_1^2+\xi_2^2} - \frac{\xi_2^2}{\xi_1^2+\xi_2^2} - 2i\frac{\xi_1\xi_2}{\xi_1^2+\xi_2^2}.$$

Thus T can be written as $T = R_1^2 - R_2^2 - 2iR_1R_2$, where R_1, R_2 are the Riesz tranforms on the plane. Another way of writing T is:

$$T = m_1 - im_2$$

where m_1, m_2 are multiplier operators and they are connected (as functions, not as multiplier operators) by

$$m_2 = m_2 \circ \rho,$$

where ρ is a $\frac{\pi}{4}$ rotation of the plane. So the multiplier operators are related by

$$m_2 = U_\rho m_1 U_\rho^{-1},$$

where U_ρ is an operator of ρ-rotation in the (x_1, x_2) plane. But for any operator K,

$$\|U_\rho K U_\rho^{-1}\|_{L^2(wdA)\to L^2(wdA)} = \|K\|_{L^2(w\circ\rho^{-1}dA)\to L^2(w\circ\rho^{-1}dA)}.$$

Combining this with the fact that $Q_{w,2}^{heat} = Q_{w\circ\rho^{-1},2}^{heat}$ for any rotation, we conclude that we need the desired estimate only for $m_1 = R_1^2 - R_2^2$. Actually we show that

$$\|R_i^2\|_{L^2(wdA)\to L^2(wdA)} \le CQ_{w,2}^{heat}, i = 1, 2.$$

Lemma 2. *Let $\phi, \psi \in C_c^\infty$. Then the integral $\int\int\int_{\mathcal{R}_+^3} \frac{\partial\phi}{\partial x_1}\frac{\partial\psi}{\partial x_1} dx_1 dx_2 dt$ converges absolutely and*

$$\int\int_{\mathcal{R}^2} R_1^2\phi \cdot \psi dx_1 dx_2 = -2\int\int\int_{\mathcal{R}_+^3} \frac{\partial\phi}{\partial x_1}\frac{\partial\psi}{\partial x_1} dx_1 dx_2 dt$$

Our goal is to estimate the right-hand side of this equality and we do that by using the following theorem:

Theorem 3. *For any $\phi, \psi \in C_c^\infty$ and any positive function w on the plane, we have*

$$\int\int\int_{\mathcal{R}_+^3} \Big|\frac{\partial\phi}{\partial x_1}\Big|\Big|\frac{\partial\psi}{\partial x_1}\Big| dx_1 dx_2 dt \le AQ_{w,2}^{heat}\Big(\int\int_{\mathcal{R}^2} |\phi|^2 w dx_1 dx_2 + \int\int_{\mathcal{R}^2} |\psi|^2 \frac{1}{w} dx_1 dx_2\Big)$$

where A is an absolute constant.

This immediately implies that

$$\int\int\int_{\mathcal{R}_+^3} \Big|\frac{\partial\phi}{\partial x_1}\Big|\Big|\frac{\partial\psi}{\partial x_1}\Big| dx_1 dx_2 dt \le 2AQ_{w,2}^{heat}\|\phi\|_{L^2(w)}\|\psi\|_{L^2(w^{-1})}$$

by substituting ϕ/t, $t\psi$ into the equality of the theorem and minimizing over t. But this is exactly what we want for the operator R_1^2.

9.3 The boarderline regularity for solutions of the Beltrami equation on the plane

In [2] we can find the following theorems:

Theorem 4. *Let f be a K-quasiconformal map where $K = \frac{1+k}{1-k}$. Consider $w = |f_z \circ f^{-1}|^{p-2}$ for $p \in (1+k, 1+\frac{1}{k})$. Then $\|(I-\mu T)^{-1}\|_{L^p \to L^p}, \|(I - T\mu)^{-1}\|_{L^p \to L^p}$ are bounded by $c(k)\|T\|_{L^p(wdA) \to L^p(wdA)}$.*

Theorem 5. *Consider $f = f^\mu$, $w = |f_z \circ f^{-1}|^{p-2}$, $p \in [2, 1+\frac{1}{k})$. If*

$$\|T\|_{L^p(wdA) \to L^p(wdA)} \leq \frac{C}{1 + (\frac{1}{k}) - p},$$

then $I - \mu T$ and $I - T\mu$ have dense ranges in $L^{1+\frac{1}{k}}(\mathcal{C})$ and are injective in $L^{1+k}(\mathcal{C})$.

Theorem 6. *Let f, w be as in theorem 5, and $p \in [2, 1+\frac{1}{k})$. Then*

$$Q^{heat}_{w,p} \leq \frac{c}{1 + \frac{1}{k} - p}.$$

Now it is clear that that we have exactly what we want at the critical exponents. The fact that weakly quasiregular maps are quasiregular follows from the injectivity of $I - \mu T$ at the critical exponent.

Theorem 7. *Let $\|\mu\|_{L^\infty} = k < 1$. Then $(1+k)$-quasiregular maps are also quasiregular.*

To prove this choose a function $\phi \in C_c^\infty$ and set $G = F\phi$ where F is a solution of the Beltrami equation in $W^{1,1+k}_{loc}$. Then

$$G_{\overline{z}} - \mu G_z = (\phi_{\overline{z}} - \mu\phi_z)F.$$

So, G is a Cauchy transform of the compactly supported function $\psi = G_{\overline{z}}$. Then the above equality can be rewritten as $((\phi_{\overline{z}} - \mu\phi_z)F = F_0)$

$$(I - \mu T)\psi = F_0 \in L^2(\mathcal{C}).$$

In fact $F \in W^{1,1}_{loc}$, so by Sobolev's theorem $(\phi_{\overline{z}} - \mu\phi_z)F \in L^{\frac{2+2k}{1-k}}(\mathcal{C})$ and $(\phi_{\overline{z}} - \mu\phi_z)F$ has compact support. So it is in $L^{2+\epsilon}(\mathcal{C})$. Looking at the last equality we can see that $\mu = 0$ outside of *supp*ϕ and actually we can find an obvious solution as a Neumann series. It converges in $L^{2+\epsilon}(\mathcal{C})$ (we use the fact that $\|T\|$ is close to 1 when p is close to 2) and has its support in *supp*ϕ. So it is in $L^{1+k}(\mathcal{C})$ as well. Let us call it ψ_0. Now we have two solutions of the second equation in $L^{1+k}(\mathcal{C})$-ψ, ψ_0. However we have already proved the injectivity of $I - \mu T$ in $L^{1+k}(\mathcal{C})$. Therefore, $\psi = \psi_0$. Thus, $G_{\overline{z}} = \psi \in L^{2+\epsilon}(\mathcal{C})$. Therefore, the compactly supported function G is in $W^{1,2+\epsilon}(\mathcal{C})$. This means that F is in $W^{1,2+\epsilon}_{loc}$, so it is quasiregular and in particular continuous, open and so on.

We should remark the following: It is obvious that there is a positive absolute constant a, such that $aQ_{w,p}^{class} \leq Q_{w,p}^{heat}$. The opposite inequality is easy to prove too. There is a positive absolute constant b such that $Q_{w,p}^{heat} \leq bQ_{w,p}^{class}$ (this is due to F. Nazarov).

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10 Two weight estimates for Calderón-Zygmund operators and corona decomposition for non-doubling measures.

After F. Nazarov, S. Treil and A. Volberg [3]
A summary written by Maria Carmen Reguera

Abstract

We summarize the characterization of two weighted estimates for Calderón-Zygmund operators asumming an extra property on the weights: the pivotal conditions. This result was given by F. Nazarov, S. Treil and A. Volberg in [3].

10.1 Introduction

Let w, ν be non-negative locally integrable functions on the real line $\mathbb{R}$. And let H be the Hilbert transform,

$$Hf(x) := p.v. \int \frac{1}{x-y} f(y)dy$$

We want to characterize boundedness of H from $L^2(w)$ to $L^2(\nu)$. A more suitable and commonly used reformulation of this problem is to find conditions to prove boundedness of H_μ from $L^2(\mu)$ to $L^2(\nu)$, where $H_\mu(f) := H(\mu f)$ and $\mu = w^{-1}$, the dual measure of w.

If $w = \nu$ and $w > 0$ a.e., the problem was solved in early seventies by R. Hunt, B. Muckenhoupt and R. Wheeden, they proved that $H : L^2(w) \mapsto L^2(w)$ if and only if the weight w is in $\mathcal{A}_2$, i.e.,

$$\|w\|_{\mathcal{A}_2} := \sup_{I \subset \mathbb{R}} \langle w \rangle_I \langle w^{-1} \rangle_I < \infty \,, \tag{1}$$

where $\langle w \rangle_I = \frac{1}{|I|} \int_I w dx$.

In the 2-weighted case, it seemed natural to provide characteristics in the spirit of (1). But even replacing the 2-weighted version of (1) by the stronger condition

$$\|\mu, \nu\|_{\mathcal{P}\mathcal{A}_2} = \sup_{z \in \mathcal{C}_+} P_\mu(z) P_\nu(z) \leq C_p \,, \tag{2}$$

where

$$P_\mu(z) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{\Im z}{(\Re z - t)^2 + (\Im z)^2} \, dt \,,$$

we will have a necessary (see [2]) but not sufficient condition (work of F. Nazarov). In turn, the conditions that paper [2] and [3] present are the ones that E. Sawyer used to characterize boundedness of positive operators in the 2-weighted setting. The heuristic is the same as the one used by G. David and J.L. Journé for the $T1$ theorem:

"*The operator is bounded between two weighted L^2 spaces if and only if it is bounded on a family of simple test functions.*"

Proving boundedness of singular integrals in the two weighted setting is a very difficult task. On one hand, one has to deal with the singularity of the kernel, on the other hand, the degeneracy of the two measures μ, ν. To avoid these difficulties the main result of [3] assumes an extra condition on the weights μ and ν, the "pivotal conditions". We present here the instance of the Hilbert transform, for a general Calderón-Zygmund operator a reformulated pivotal condition is needed, see [3].

Definition 1. *Given an interval I and any measure $d\mu$ on the real line, we write*

$$P_I(d\mu) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{|I|}{|I|^2 + (c(I) - t)^2} \, d\mu(t)\,.$$

This is the Poisson integral at the point whose real part is the center of the interval, and imaginary part is the length of the interval.

Let $I \in \mathcal{D}_\mu$. Let $\{I_\alpha\}$ be a finite family of disjoint subintervals of I belonging to the same lattice. The following is the so-called pivotal property:

$$\sum_\alpha [P_{I_\alpha}(1_{I \setminus I_\alpha} d\mu)]^2 \nu(I_\alpha) \leq P\, \mu(I)\,. \tag{3}$$

We are now ready to state the main theorem of [2].

Theorem 2. *Let μ, ν be arbitrary positive measures. Let us assume the measures μ, ν satisfy extra conditions: the pivotal property* (3) *and the corresponding "dual", where μ and ν interchange roles. Then H_μ is bounded from $L^2(\mu)$ to $L^2(\nu)$ if and only if the following inequalities are satisfied*

$$\|1_I H_\mu(1_I)\|_{L^2(\nu)} \leq C_\chi \nu(I), \ \ \forall I \subset \mathbb{R}\,, \tag{4}$$

$$\|1_I H_\nu(1_I)\|_{L^2(\mu)} \leq C_\chi \mu(I), \ \ \forall I \subset \mathbb{R}\,, \tag{5}$$

$$\|\mu, \nu\|_{\mathcal{PA}_2} = \sup_{z \in \mathcal{C}_+} P_\mu(z) P_\nu(z) \leq C_p\,. \tag{6}$$

The pivotal condition is not necessary (see [1]), nevertheless there are interesting cases where the pivotal condition is satisfied. For instance, if the measures μ and ν are doubling and their support is the whole $\mathbb{R}^n$ and the operators are either the Hilbert transform or the Riesz transforms. It is also true when we assume boundedness of M_μ from $L^2(\mu)$ to $L^2(\nu)$ and M_ν from $L^2(\nu)$ to $L^2(\mu)$, where M is the Hardy-Littlewood maximal operator. This assumption is very natural if we consider that this is the case when we have just one measure.

10.2 Proof of Main Theorem: Initial considerations

In this summary we will only consider sufficiency.
In what follows, $|\cdot|$ stands for Lebesgue measure, 1_E is the characteristic function on the set E. Let $f \in L^2(\mu)$, $g \in L^2(\nu)$ be two test functions with compact support contained in I_0^μ and I_0^ν respectively and with zero mean. Let $\mathcal{D}^\mu$, $\mathcal{D}^\nu$ be two dyadic lattices of $\mathbb{R}$. Let us consider the weighted Haar functions h_I^μ, defined to be supported in I, have constant value on each of the dyadic halves of I, with mean zero and $\|h_I^\mu\|_{L^2(\mu)} = 1$. Similarly, we define h_J^ν with $J \in \mathcal{D}^\nu$. We also introduce the operators $\Delta_I^\mu(f) := (f, h_I^\mu)_\mu h_I^\mu, \quad I \in \mathcal{D}^\mu,\ I \subset I_0^\mu$. Δ_J^ν is defined in an analogous manner.

It is easy to see that we can decompose our linear form as

$$(H_\mu f, g)_\nu = \sum_{I \in \mathcal{D}^\mu, J \in \mathcal{D}^\nu} (H_\mu \Delta_I^\mu f, \Delta_J^\nu g)_\nu \,. \tag{7}$$

We will make one more reduction, for that purpose we will use the "good-bad" decomposition from the work of F. Nazarov, S. Treil and A. Volberg on non-homegenous spaces. We will not enter in the precise definition of good intervals, essentially I is good if there is no J so that I can be close to its boundary or its center. For more details we refer the reader to any of the papers in the bibliography.

Our problem is reduced to estimate

$$(H_\mu f, g)_\nu = \sum_{\substack{I \in \mathcal{D}^\mu, J \in \mathcal{D}^\nu \\ I, J \text{ are good}}} (H_\mu \Delta_I^\mu f, \Delta_J^\nu g)_\nu \,. \tag{8}$$

Let $\mathcal{D}^\mu, \mathcal{D}^\nu$ be fixed. Let $\mathcal{F} := \{(I,J) : I, J \text{ are good}\}$. Let r be a fixed number that comes from the definition of good intervals. We can decompose the set $\mathcal{F}$ as $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_2^* \cup \mathcal{F}_3 \cup \mathcal{F}_3^* \cup \mathcal{F}_4 \cup \mathcal{F}_4^*$, where $\mathcal{F}_1 := \{(I,J) : \ 2^{-r}|J| \le |I| \le 2^r|J|, \mathrm{dist}(I,J) \le \max(|I|,|J|)\}$, $\mathcal{F}_2 := \{(I,J) : \ 2^{-r}|J| \le |I| \le |J|, \mathrm{dist}(I,J) \ge |J|\}$, $\mathcal{F}_3 := \{(I,J) : \ |I| < 2^{-r}|J|, I \cap J = \emptyset\}$, and $\mathcal{F}_4 := \{(I,J) : \ |J| < 2^{-r}|I|, J \subset I\}$. We are omitting the definition of the sets $\mathcal{F}_i^*$, they are defined and treated in a symmetric fashion. Boundedness when we restrict to the set $\mathcal{F}_1$ is obtained through the testing conditions (4) and (5). Boundedness on the sets $\mathcal{F}_2$ and $\mathcal{F}_3$ are obtained using (2). Finally, in the next section we consider the sum in $\mathcal{F}_4$. This is the term that most clearly has the paraproduct structure in it.

10.3 Corona decomposition

We are left to analyze the following piece

$$\tau := \sum_{(I,J)\in\mathcal{F}_4} (H_\mu \Delta_I^\mu f, \Delta_J^\nu g)_\nu \,. \tag{9}$$

We are going to split τ into three terms. Let I_J denote the half of I, which contains J. And I_n is the other half. Let $\widehat{I}$ denote an arbitrary super interval of I_J in the same lattice: $\widehat{I} \in \mathcal{D}^\mu$. We write

$$(H_\mu \Delta_I^\mu f, \Delta_J^\nu g)_\nu = (H_\mu(1_{I_n}\Delta_I^\mu f), \Delta_J^\nu g)_\nu + (H_\mu(1_{I_J}\Delta_I^\mu f), \Delta_J^\nu g)_\nu =$$

$$(H_\mu(1_{I_n}\Delta_I^\mu f), \Delta_J^\nu g)_\nu + \langle \Delta_I^\mu f\rangle_{\mu, I_J} (H_\mu(1_{\hat{I}}), \Delta_J^\nu g)_\nu - \langle \Delta_I^\mu f\rangle_{\mu, I_J} (H_\mu(1_{\hat{I}\setminus I_J}), \Delta_J^\nu g)_\nu \,.$$

We will name them according to the notation in [2]: the first one is "the neighbor term", the second one is "the difficult term" and the third one is "the stopping term". The proof of boundedness of the neighbor term follows the line of reasoning of the $\mathcal{F}_2$ and $\mathcal{F}_3$ terms and we refer the reader to section 7.1 in [2]. The proof of the paraproduct and stopping terms will require the following reorganization of the intervals. We first define the collection of stopping intervals.

Definition 3. *(Stopping criteria) Given any interval $\hat{I}$, set $\mathcal{S}(\hat{I})$ to be the maximal $\mathcal{D}^\mu$ strict subintervals $I \subsetneq \hat{I}$ such that*

$$\left[P_I(1_{\hat{I}\setminus I} d\mu)\right]^2 \nu(I) \geq B\mu(I), \tag{10}$$

for B a fixed constant such that $B > 2P$. We start with I_0^μ and define $\mathcal{S}$ to be the set of all stopping intervals.

We now define the associated Corona Decomposition.

Definition 4. *(Corona decomposition) For $S \in \mathcal{S}$, we set $\mathcal{P}(S)$ to be all the pairs of intervals (I, J) such that*

1. *$I \in \mathcal{D}^\mu$, $J \in \mathcal{D}^\nu$, $J \subset I$, and $|J| < 2^{-r}|I|$.*
2. *S is the $\mathcal{S}$-parent of I_J, the child of I that contains J.*

Let $\mathcal{C}^\mu(S)$ to be all those $I \in \mathcal{D}^\mu$ such that S is a minimal member of $\mathcal{S}$ that contains a $\mathcal{D}^\mu$-child of I. $\mathcal{C}^\nu(S)$ is the set of all those $J \in \mathcal{D}^\nu$ such that S is the smallest member of $\mathcal{S}$ that contains J and satisfies $2^r|J| < |S|$.

Notice that the Corona decomposition allow us to organize the set of pair of intervals $\mathcal{F}_4 = \bigcup_{S\in\mathcal{S}} \mathcal{P}(S)$.

We now go back to analyze the stopping term:

$$T := \sum_{S\in\mathcal{S}} \sum_{(I,J)\in\mathcal{P}(S)} |\langle \Delta_I^\mu f\rangle_{\mu,I_J}||(H_\mu(1_{S\setminus I_J}), \Delta_J^\nu g)_\nu| \,.$$

The proof follows from the estimate below and the crucial fact that I_J is not a stopping interval and the stopping inequality (10) is reversed. This is the place were we need the precise definition of stopping interval.

$$|\langle \Delta_I^\mu f\rangle_{\mu,I_J}||(H_\mu(1_{\hat{I}\setminus I_J}), \Delta_J^\nu g)_\nu| \le A\left(\frac{|J|}{|I|}\right)^{1/2}\left(\frac{\nu(J)}{\mu(I_J)}\right)^{1/2} P_{I_J}(1_{\hat{I}\setminus I_J}\, d\mu)\|\Delta_J^\nu g\|_\nu \|\Delta_I^\mu f\|_\mu \,.$$

10.3.1 The difficult term: the paraproduct

In this section $r(S,S')$ stands for the distance in the tree formed by the collection $\mathcal{S}$. $\pi_{\mathcal{S}}(S)$ and $\pi(S)$ will denote the parent of S in the collection $\mathcal{S}$ and the collection $\mathcal{D}^\mu$ respectively. And P will denote the orthogonal projection onto the span of Haar fuctions h_J^ν with $J \in \mathcal{C}^\nu(S)$. We now write the paraproduct term as

$$\sum_{S\in\mathcal{S}} \sum_{(I,J)\in\mathcal{P}(S)} \langle \Delta_I^\mu f\rangle_{\mu,I_J} \cdot \langle H_\mu(\mathbf{1}_S), \Delta_J^\nu g\rangle_\nu \,.$$

There is a subtle point that should be mentioned, different clusters $\mathcal{P}(S)$ and $\mathcal{P}(S')$ could have common $J's$, what prevents from having orthogonality among the clusters. Taking that into account, we can decompose the operator in the following two pieces.

$$\Theta_1 := \sum_{S\in\mathcal{S}} \sum_{\substack{(I,J)\in\mathcal{P}(S)\\ J\in\mathcal{C}^\nu(S)}} \langle \Delta_I^\mu f\rangle_{\mu,I_J} \cdot \langle H_\mu(\mathbf{1}_S), \Delta_J^\nu g\rangle_\nu \,, \tag{11}$$

$$\Theta_2 := \sum_{S\in\mathcal{S}\setminus\{I^0\}} \sum_{t\ge 1} \sum_{\substack{(I,J)\in\mathcal{P}(\tilde{S})\\ J\in\mathcal{C}^\nu(S),\ S\subset\tilde{S},\ r(S,\tilde{S})=t}} \langle \Delta_I^\mu f\rangle_{\mu,I_J} \cdot \left\langle H_\mu(\mathbf{1}_{\tilde{S}}), \Delta_J^\nu g\right\rangle_\nu \,. \tag{12}$$

(11) is the main term and is a paraproduct. The following lemma states the Carleson estimate needed in order to appeal to Carleson's embedding theorem.

The fact that I_J is not a stopping interval and we can reverse inequality (10) is crucial in the proof.

Lemma 5. *Let $l(J)$ denotes the smallest children of $I \in \mathcal{C}^\mu(S)$ containing J. Let $\{\beta_I\}$ be the sequence*

$$\beta_I = \sum_{J \in \mathcal{C}^\nu(S)\,:\, l(J)=I} \left|\langle H_\mu(\mathbf{1}_S), \Delta^\nu_J g\rangle_\nu\right|^2 , \quad \textit{then}$$

$$\sum_{I \in \mathcal{C}^\mu(S), I \subset K} \beta_I \leq C(B + C_\chi)\mu(K), \qquad K \in \mathcal{D}^\mu.$$

(12) is the error term. When $r(S, \tilde{S}) = t$ we get extra decay from the pivotal condition, what allows to overcome the lack of orthogonality. The study of this term can be reduced to the study of the following two paraproducts.

$$\Xi^1 f := \sum_{S \in \mathcal{S}} \sum_{t \geq 1,\ S \subset \tilde{S}\, r(S,\tilde{S})=t} \langle f\rangle_{\mu,\pi(\tilde{S}))} \mathbb{P}_S(H_\mu 1_{\pi_{\mathcal{S}}(\tilde{S})\setminus\tilde{S}})\,. \tag{13}$$

$$\Xi^2 f := \sum_{S \in \mathcal{S}} \langle f\rangle_{\mu,\pi(S)} \mathbb{P}_S(H_\mu 1_{\pi_{\mathcal{S}}(S)})\,, \tag{14}$$

Boundedness of Ξ_1 is a consequence of Carleson embedding theorem. An improvement with geometrical decay on t of the Carleson measure estimate will be obtained. This is the term where the difficulty of not having doubling measures appears, what saves the argument is the use of the pivotal condition (3).

Theorem 6. *Let $\alpha_t(\tilde{S})$ be the sequence*

$$\alpha_t(\tilde{S}) := \sum_{S : S \subset \tilde{S},\, r(S,\tilde{S})=t} \left\| \mathsf{P}^\nu_S H(\mu \mathbf{1}_{\pi_{\mathcal{S}}(\tilde{S})\setminus\tilde{S}})\right\|^2_{L^2(\nu)} \tag{15}$$

The following Carleson measure estimate holds:

$$\sum_{S \in \mathcal{S}\,:\, \pi(S) \subset K} \alpha_t(S) \lesssim 2^{-t} C(C_\chi, P)\mu(K)\,, \qquad K \in \mathcal{D}^\mu. \tag{16}$$

The second paraproduct Ξ^2 can be handle using similar arguments.

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11 The Bellman Function, The Two-Weight Hilbert Transform, and Embeddings of the Model Spaces K_θ

after F. Nazarov and A. Volberg [1]
A summary written by Alexander Reznikov

Abstract

We study some natural conditions of boundedness of the Hilbert Transform and prove that they are not sufficient.

11.1 Basic Definitions

We consider a unit disc $\mathbb{D} \subset \mathbb{C}$ and a unit circle $\mathbb{T}$. We say that θ is an inner function if it's bounded and analytic in $\mathbb{D}$ and it's boundary values on $\mathbb{T}$ are unimodular, namely, $|\theta(\zeta)| = 1$ for m-a.e. $\zeta \in \mathbb{T}$. Here m is the normalized Lebesgue measure on $\mathbb{T}$.

By H^2 we denote set of analytic functions, who's boundary values lie in $L^2(\mathbb{T}, m)$. Denote

$$\theta H^2 = \{\theta f \colon f \in H^2\},$$
$$K_\theta = H^2 \ominus \theta H^2.$$

From now on we assume $\theta(0) = 0$. We now introduce a new measure on $\mathbb{T}$. Namely, let σ be a measure on $\mathbb{T}$ such that

$$(1 - \theta(\lambda))^{-1} = \int_{\mathbb{T}} \frac{d\sigma(\zeta)}{1 - \lambda\overline{\zeta}}, \quad \forall \lambda \in \mathbb{D}.$$

Note that $\sigma(\mathbb{T}) = 1$. It is a famous construction from Complex Analysis, developed by Clark and Poltoratski. For example, the transformation

$$(U^* f)(\lambda) = (1 - \theta(\lambda)) \int_{\mathbb{T}} \frac{f(\zeta) d\sigma(\zeta)}{1 - \lambda\overline{\zeta}}$$

is unitary from $L^2(\sigma)$ to K_θ. There is a natural question: when U^* is bounded from $L^2(\sigma)$ to $L^2(\mu)$? We can study only such μ that supports of μ and σ are disjoint, and θ is well-defined μ-a.e.

Finally, we introduce a Hilbert Transform

$$H_\sigma f = \int_{\mathbb{T}} \frac{f d\sigma(\zeta)}{1 - \lambda\overline{\zeta}}, \quad f \in L^2(\sigma).$$

11.2 Questions, which are answered in the paper

We now move on and state some connected questions, which are studied in the paper.

11.2.1 Two-weighted Hilbert Transform

We introduce an operator

$$H_u f = \int \frac{f u dx}{x-y}$$

and ask if H_u is bounded as an operator $L^2(u) \to L^2(v)$.

Remark 1. *If $u = v$ then H_u is bounded if and only if $u \in A_2$.*

By $P_u(z)$ we denote the Harmonic extention of u from $\mathbb{R}$ to $\mathbb{C}_+$. Here is the conjecture.

Conjecture 2. *Assume that*

$$P_u(z)P_v(z) \leqslant C \forall z \in \mathbb{C}_+ \tag{1}$$

$$\|H_u \chi_I\|^2_{L^2(v)} \leqslant Cu(I), \quad \forall \textit{ interval } I. \tag{2}$$

Then H_u is bounded.

We prove

Theorem 3. *Conjecture 2 is false.*

11.2.2 Hilbert Transform H_σ

We return now to our K_θ. We denote

$$P_\mu(\lambda) = \int\limits_{Cl\mathbb{D}} \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2} d\mu(z).$$

P_μ is the Poisson extension of μ. We state the conjecture.

Conjecture 4. *Assume that*

$$P_\mu(z)P_\sigma(z) \leqslant A_2 \forall z \in \mathbb{D} \tag{3}$$

$$\|H_\sigma 1\|^2_{L^2(\mu)} \leqslant A_1. \tag{4}$$

Then $H_\sigma \colon L^2(\sigma) \to L^2(\mu)$ is bounded.

This is also false, and here is what will be proven.

Theorem 5. *There exists a measure σ, which is a finite linear combination of delta-measures, and a measure μ such that:*

$$\mathrm{supp}\sigma \cap \mathrm{supp}\mu = \emptyset, \tag{5}$$
$$\sigma(\mathbb{T}) = 1, \tag{6}$$

and (3) *and* (4) *are true. On the other hand,*

$$\|H_\mu 1\|_{L^2(\sigma)} \geqslant C,$$

where C is as large as we want.

11.3 The Dyadic model and Bellman approach

In this part of the paper authors study some discrete analog of the Hilbert Transform. To state the main result, we need some more notation.

We fix an interval $[0,1]$ and let $\mathcal{D}$ be a dyadic lattice for this interval. By $\langle f\rangle_J$ we denote the average of f over an interval $J \subset I$:

$$\langle f\rangle_J = \frac{1}{|J|}\int_J f(t)dt.$$

If we average over the whole $[0,1]$, we drop the subindex and write $\langle f\rangle$. Next, let

$$\Delta_I f = \langle f\rangle_{I_-} - \langle f\rangle_{I_+},$$

where $I_\pm$ is right and left half of I. Let now $\mathcal{D}_m = \{I \in \mathcal{D},\ |I| = 2^{-m}\}$. Our goal is to construct two functions U and V such that

1. U and V are constants on every $I \in \mathcal{D}_m$;
2. $\langle U\rangle_I \langle V\rangle_I \leqslant A_0$ for every $I \in \mathcal{D}$;
3. $\sum_{I\in\mathcal{D}} (\Delta_I U)^2 \langle V\rangle_I |I| \leqslant \langle U\rangle$;
4. $\sum_{I\in\mathcal{D}} (\Delta_I V)^2 \langle U\rangle_I |I| \geqslant C\langle V\rangle$.

We also guarantee that

$$\frac{\langle U\rangle_{I_+}}{\langle U\rangle_{I_-}},\ \frac{\langle V\rangle_{I_+}}{\langle V\rangle_{I_-}} \in (1-\tau, 1+\tau), \quad I \in \mathcal{D}.$$

Here C is as big as we want and τ is as small as we want.

Sketch of the proof. Fix a dyadic interval J. For every point $x = (x_1, x_2, x_3)$ such that $x_3 \leqslant x_1$, $x_1 x_2 \leqslant 1$ we consider a family

$$\mathcal{F}_x = \Big\{(U,V)\colon \langle U\rangle_J = x_1,\ \langle V\rangle_J = x_2\ \frac{1}{|J|}\sum_{I\in\mathcal{D}, I\subset J}(\Delta_I U)^2\langle V\rangle_I |I| = x_3,$$

$$\langle U\rangle_I \langle V\rangle_I \leqslant 1,\ \frac{1}{|I|}\sum_{\ell\subset I, \ell\in\mathcal{D}}(\Delta_\ell U)^2\langle V\rangle_\ell |\ell| \leqslant \langle U\rangle_I,\ I\in\mathcal{D}, I\subset J\Big\}.$$

On the set $\Omega = \{x = (x_1, x_2, x_3)\colon 0 \leqslant x_{1,2,3}, x_1 x_2 \leqslant 1, x_3 \leqslant x_1\}$ we define a function

$$B(x) = \sup\left\{\frac{1}{|J|}\sum_{I\in\mathcal{D}, I\subset J}(\Delta_I V)^2\langle U\rangle_I |I|\colon (U,V)\in\mathcal{F}_x\right\}.$$

Lemma 6. *If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is such that $x, x\pm\alpha, (x_1, x_2, x_3 + \alpha_1^2 x_2) \in \Omega$ then*

$$\frac{B(x+\alpha) + B(x-\alpha)}{2} + \alpha_2^2 x_1 \leqslant B(x_1, x_2, x_3 + \alpha_1^2 x_2).$$

Assume we have proved this lemma. Then, by homogeneity, we get

$$B(x) = x_2 \xi(x_1 x_2, x_2 x_3).$$

Lemma 7. *ξ is unbounded.*

This lemma gives us that

$$\forall C\ \exists x\colon B(x) \geqslant C x_2,$$

which is exactly what we need. $\square$

11.4 References

In addition to what is said, we want to cite few papers for further reading. The Bellman Function was considered in many papers by Nazarov, Treil, Vasyunin and Volberg. For example, [2], [3], [6].

We hardly recommend the work [4], which is the first work, where Bellman approach was used in Harmonic Analysis.

We also refer to a paper [5] because it describes the "jump" from dyadic model to the Hilbert Transform.

Reader can find much more references and information on the web-site http://sashavolberg.files.wordpress.com/2010/05/vvbell05_24_10.pdf

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12 A_1 bounds for Calderón-Zygmund operators related to a problem of Muckenhoupt and Weeden

after Andrei K. Lerner, Sheldy Ombrosi, and Carlos Péres [1]
A summary written by Prabath Silva

Abstract

We summarize the paper [1] here. Also we mention some recent improvements of the results from [2].

12.1 Introduction

Let T be a Calderón-Zygmund singular integral operator and w a weight (i.e., $w \geq 0$ and $w \in L^1_{loc}(\mathbb{R}^n)$). The Muckenhoupt and Weeden conjecture says

$$||Tf||_{L^{1,\infty}(w)} \leq c||f||_{L^1(Mw)}; \tag{1}$$

here M is the Hardy-Littlewood maximal operator. It is well known that (1) is true if we replace T by M. In this paper we have improvements towards the weak Muckenhoupt and Weeden conjecture,

$$||Tf||_{L^{1,\infty}(w)} \leq c||w||_{A_1}||f||_{L^1(w)}. \tag{2}$$

First we state the main result in [3] by the same authors which contains a result towards the weak Muckenhoupt and Weeden conjecture.

Theorem 1. *Let T be a Calderón-Zygmund operator. Then*

$$||Tf||_{L^p(w)} \leq c\nu_p||w||_{A_1}||f||_{L^p(w)} \quad (1 < p < \infty) \tag{3}$$

and

$$||Tf||_{L^{1,\infty}(w)} \leq c\phi(||w||_{A_1})||f||_{L^1(w)}; \tag{4}$$

here $\nu_p = \frac{p^2}{p-1}log(e + \frac{1}{p-1})$ and $\phi(t) = t(1 + log^+t)(1 + log^+log^+t)$.

The main theorem in paper [1] is an improvement of the above theorem.

Theorem 2. *Let T be a Calderón-Zygmund operator. Then*

$$||Tf||_{L^p(w)} \leq cpp'||w||_{A_1}||f||_{L^p(w)} \quad (1 < p < \infty) \tag{5}$$

and

$$||Tf||_{L^{1,\infty}(w)} \leq c||w||_{A_1}(1 + log||w||_{A_1})||f||_{L^1(w)}, \tag{6}$$

where $c = c(n, T)$.

The improvement of $\nu_p = pp'$ in here is optimal. As a corollary to the main theorem we get

Corollary 3. *Let $1 < p < \infty$ and let T be a Calderón-Zygmund operator. Also, let $w \in A_p$; then*

$$||Tf||_{L^{p,\infty}(w)} \leq c||w||_{A_p}(1 + log||w||_{A_p})||f||_{L^p(w)}, \tag{7}$$

By a duality argument we get the following Sawer-type testing condition,

Corollary 4. *Let $1 < p < \infty$ and let T be a Calderón-Zygmund operator. Also, let $w \in A_p$; then for any measurable set E,*

$$||T(\sigma\chi_E)||_{L^p(w)} \leq c||w||_{A_p}^{\frac{1}{p-1}}(1 + log||w||_{A_p})\sigma(E)^{1/p}. \tag{8}$$

A recent result in [2] solved the problem of sharp weighted estimates for general Calderón-Zygmund operators.

Theorem 5. *Let $1 < p < \infty$ and let T be a Calderón-Zygmund operator. Also let $w \in A_p$, then*

$$||Tf||_{L^p(w)} \leq c||w||_{A_p}^{max(1,1/(p-1))}||f||_{L^p(w)}, \tag{9}$$

Note that this result give improvement for (5) and (7) for $p > 2$ in terms of the $||w||_{A_p}$. Also note that (2) gives better estimates when $p > 2$ in term of $||w||_{A_p}$ for $f = \sigma\chi_E$ than in (9).

12.2 Proof of the main theorem

The new ingredient to the proof of the theorem is the following lemma, the rest of the proof is similar to the proof in [3].

Lemma 6. *Let T be a Calderón-Zygmund operator. There exists a constant $c = c(n, T)$ such that for any weight w and for any $p, r \geq 1$,*

$$\left\| \frac{Tf}{M_r w} \right\|_{L^p(M_r w)} \leq cp \left\| \frac{Mf}{M_r w} \right\|_{L^p(M_r w)}, \tag{10}$$

where $M_r w = M(w^r)^{1/r}$.

Once we have this lemma the proof of (5) follows form the duality argument and equation (6) follows from Calderón-Zygmund decomposition.

To prove the lemma we need the following two lemmas.

Lemma 7. *Let T be a Calderón-Zygmund operator and let $w \in A_p$, $p \geq 1$. Then there is a constant $c = c(n, p, T)$ such that*

$$||Tf||_{L^1(w)} \leq c||w||_{A_p}||Mf||_{L^1(w)}. \tag{11}$$

Proof of this lemma follows from a sharp good-λ inequality from [4] relating T^*, the maximal truncation of T, with M and a sharp A_∞ condition from [5] for A_p weights.

Lemma 8. *Let $1 < s < \infty$, and let v be a weight. Then there exists a nonnegative sublinear operator R satisfying the following properties:*

1. $h \leq R(h)$
2. $||R(h)||_{L^s(v)} \leq 2||h||_{L^s(v)}$
3. $R(h)v^{1/s} \in A_1$ *with* $||R(h)v^{1/s}||_{A_1} \leq cs'$.

We can take $R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{(||S||_{L^s(v)})^k}$ where $S(h) = \frac{M(hv^{1/s})}{v^{1/s}}$.

Having these two lemmas we can prove Lemma 6. By duality we have

$$\left\|\frac{Tf}{M_r w}\right\|_{L^p(M_r w)} = \sup_{||h||_{L^{p'}(M_r w)}=1} \int_{\mathbb{R}^n} |Tf| h dx$$

Now applying Lemma 8 with $s = p'$ and $v = M_r w$, and using factorization properties of A_p weights with properties from the Lemma 8, we get $||R(h)||_{A_3} \leq cp$.

$$\begin{aligned} \int_{\mathbb{R}^n} |Tf| h dx &\leq \int_{\mathbb{R}^n} |Tf| R(h) dx \leq c||R(h)||_{A_3} \int_{\mathbb{R}^n} M(f) R(h) dx \\ &\leq cp \left\|\frac{Mf}{M_r w}\right\|_{L^p(M_r w)} ||h||_{L^{p'}(M_r w)}. \end{aligned}$$

Here we used (11) and property 2 in Lemma 8.

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13 Two weight inequalities for discrete positive operators

after M. Lacey, E. Sawyer, I. Uriarte-Tuero [1]
A summary written by Michal Tryniecki

Abstract

We characterize two weight inequalities for general positive dyadic operators

$$||T_{\boldsymbol{\tau}}(f\sigma)||_{L^q(w)} \lesssim ||f||_{L^p(\sigma)}, \quad 1 < p \leq q < \infty$$

in terms of Sawyer-type testing conditions.

13.1 Introduction

The main theorem that we are interested in is a generalization of the Embedding Inequality of Sawyer and Nazarov-Treil-Volberg. The inequality was obtained by Nazarov-Treil Volberg [2] as a deep extension of the Theorem of Eric Sawyer [3] on two-weight inequalities. The proof by Nazarov-Treil Volberg uses the Bellman Function approach. We present here the work of Lacey, Sawyer, Uriarte-Tuero which extends the previous results to higher dimensions (new for $d \geq 2$) and works for general case $1 < p \leq q < \infty$ (as opposed to $p = q = 2$).

We start with the presentation of the previous result. Let $\mathcal{Q}$ be a choice of dyadic cubes in $\mathbb{R}^d$. For a cube Q we define the average of f over Q:

$$\mathbb{E}_{\mathrm{Q}} f := |Q|^{-1} \int_{\mathrm{Q}} f dx.$$

By a *weight* we mean a non-negative locally integrable function $w : \mathbb{R}^d \to [0, \infty)$. For such weights and "nice" sets (like dyadic cubes) Q we set:

$$w(\mathrm{Q}) := \int_{\mathrm{Q}} w dx.$$

Theorem 1 (Embedding Inequality of Sawyer and Nazarov-Treil-Volberg). *Let $\{\boldsymbol{\tau}_{\mathrm{Q}} : \mathcal{Q}\}$ be non-negative constants. Let w, σ be weights. Define:*

$$C_1^2 := \sup_R \sigma(R)^{-1} \int [\sum_{\mathrm{Q} \subset R} \tau_{\mathrm{Q}} 1_{\mathrm{Q}} \mathbb{E}_{\mathrm{Q}} \sigma]^2 w,$$

$$C_2^2 := \sup_R w(R)^{-1} \int [\sum_{\mathrm{Q}\subset R} \tau_{\mathrm{Q}} 1_{\mathrm{Q}} \mathbb{E}_{\mathrm{Q}} w]^2 \sigma,$$

$$C_3 := \sup_{||f||_{L^2(\sigma)=1}} \sup_{||g||_{L^2(w)=1}} \sum_{\mathrm{Q}\in\mathcal{Q}} \tau_{\mathrm{Q}} \mathbb{E}_{\mathrm{Q}}(f\sigma)\cdot \mathbb{E}_{\mathrm{Q}}(gw)|\mathrm{Q}|.$$

Then we have the equivalence $C_3 \simeq C_1 + C_2$.

Now we will present the announced extension of this theorem. We will need some more definitions.
Let us set:

$$\mathbb{E}^w_{\mathrm{Q}} f := w(\mathrm{Q})^{-1} \int_{\mathrm{Q}} f w dx.$$

The *dyadic maximal function associated to* w is given by:

$$M_w f(x) := \sup_{\mathrm{Q}\in\mathcal{Q}} 1_{\mathrm{Q}}(x) \mathbb{E}^w_{\mathrm{Q}} |f|.$$

The following fact (proved exactly in the same way as the corresponding statement without weights) and its "linearized" version will be essential in the proof:

Theorem 2. *For* $1 < p < \infty$ *we have:*

$$||M_w f||_{L^p(w)} \lesssim ||f||_{L^p(w)}. \tag{1}$$

For a dyadic cube Q let $Q^{(1)}$ denote its dyadic parent, and define $Q^{(\rho)}$ inductively for $\rho \geq 2$. Fix non-negative constants $\boldsymbol{\tau} = \{\tau_{\mathrm{Q}} : \mathrm{Q} \in \mathcal{Q}\}$ and define linear operator $T_{\boldsymbol{\tau}}$ and its two "localizations" corresponding to a dyadic cube R:

$$Tf := \sum_{\mathrm{Q}\in\mathcal{Q}} \tau_{\mathrm{Q}} \cdot \mathbb{E}_{\mathrm{Q}} f \cdot 1_{\mathrm{Q}},$$

$$T^{in}_R f := \sum_{\substack{\mathrm{Q}\in\mathcal{Q}\\ \mathrm{Q}\subset R}} \tau_{\mathrm{Q}} \cdot \mathbb{E}_{\mathrm{Q}} f \cdot 1_{\mathrm{Q}},$$

$$T^{out}_R f := \sum_{\substack{\mathrm{Q}\in\mathcal{Q}\\ \mathrm{Q}\supset R}} \tau_{\mathrm{Q}} \cdot \mathbb{E}_{\mathrm{Q}} f \cdot 1_{\mathrm{Q}},$$

Notice that $T^{out}_{\boldsymbol{\tau},R} f$ is constant on R and we have the following decomposition: $T_{\boldsymbol{\tau}} f(x) = T^{in}_{\boldsymbol{\tau},R} f(x) + T^{out}_{\boldsymbol{\tau},R^{(1)}} f(x')$, where $x \in R$, $x' \in R^{(1)}$.
We introduce the so called local and global testing conditions. Let $1 < p \leq q < \infty$ and let p', q' be conjugate to p, q. Set:

$$[[\sigma, w]]^{Loc}_{p,q} := \sup_{R\in\mathcal{Q}} w(R)^{-1/q'} ||T^{in}_R (w 1_R)||_{L^{p'}(\sigma)}$$

$$[[\sigma, w]]^{Glo}_{p,q} := \sup_{R\in\mathcal{Q}} w(R)^{-1/q'} ||T^{out}_R(w1_R)||_{L^{p'}(\sigma)}$$

We say that the testing condition (local or global) holds if the above are finite. The main theorem we are interested in characterizes the two weight inequalities in terms of the testing conditions:

Theorem 3. ***Main result.*** *We have the equivalences of norms :*

$$||T(\sigma\cdot)||_{L^p(\sigma)\mapsto L^q(w)} \simeq [[\sigma, w]]^{Loc}_{p,q} + [[w, \sigma]]^{Loc}_{q',p'}, \quad 1 < p \leq q < \infty,$$

$$||T(\sigma\cdot)||_{L^p(\sigma)\mapsto L^q(w)} \simeq [[\sigma, w]]^{Glo}_{p,q} + [[w, \sigma]]^{Glo}_{q',p'}, \quad 1 < p < q < \infty.$$

Note that for $p = q = 2$ we get the Embedding Inequality of Sawyer and Nazarov-Treil-Volberg.

In the next section we will sketch the proof of this result.The first step will be to show that weak-type inequalities are true, namely we prove:

Theorem 4. ***Weak-type inequality.*** *We have the equivalence of norms:*

$$||T(\sigma\cdot)||_{L^p(\sigma)\mapsto L^{q,\infty}(w)} \simeq [[\sigma, w]]^{Loc}_{p,q}, \quad 1 < p \leq q < \infty,$$

$$||T(\sigma\cdot)||_{L^p(\sigma)\mapsto L^{q,\infty}(w)} \simeq [[\sigma, w]]^{Glo}_{p,q}, \quad 1 < p < q < \infty.$$

13.2 Sketch of proof

13.2.1 The weak case

The necessity of the testing conditions is straightforward. If we assume that $\mathfrak{N} = ||T(\sigma\cdot)||_{L^p(\sigma)\mapsto L^{q,\infty}(w)} < \infty$ then by the duality for Lorentz spaces we have:

$$||T(fw)||_{L^{p'}(\sigma)} \leq \mathfrak{N}||f||_{L^{q',1}(w)}.$$

Applying this to $f = 1_Q$ and decomposing T into local and global parts we get both testing conditions.
Now we show that testing conditions imply the weak-type bound for T. Let's start with the local case. We fix smooth, compactly supported $f \in L^p(\sigma)$ and $\lambda > 0$. Fix $Q_0 \in \mathcal{Q}_\lambda$ a cube in the family of maximal dyadic cubes in $\{T(f\sigma) > \lambda\}$ that intersect $\{T(f\sigma) > 2\lambda\}$. It follows from maximality that:

$$\lambda \leq T^{in}_{Q_0}(f\sigma)(x) \quad \text{for } x \in Q_0 \cap \{T(f\sigma)(x) > 2\lambda\}. \tag{2}$$

Now we split the cubes in $\mathcal{Q}_\lambda$ into two groups, the first one $\mathcal{E}$ consisting of cubes for which $w(Q \cap \{T(f\sigma) > 2\lambda\}) < \eta w(Q)$, where $\eta = 2^{-q-1}$.
We estimate:

$$(2\lambda)^q w(T(f\sigma) > 2\lambda) \leq \eta(2\lambda)^q \sum_{Q \in \mathcal{E}} w(Q) + \eta^{-q} \sum_{Q \in \mathcal{Q}_\lambda \setminus \mathcal{E}} [\frac{1}{w(Q)} \int_Q T_Q^{in}(f\sigma) w dx]^q w(q)$$

$$\leq \eta 2^q \lambda^q w(T(f\sigma) > \lambda) + C\eta^{-q}([[\sigma, w]]_{p,q}^{Loc})^q ||f||_{L^p(\sigma)}^q$$

where the first inequality follows from the fact that for cubes in the second group we have (2) and the second one from the self duality of T^{in}. The estimate follows if we take λ so that the left-hand side is close to maximal (it is bounded by the assumptions on f).
Now the global case. Let Q_0 be as before. The idea here will be a comparison to the following maximal function involving both weights in the definition:

$$\overline{M} f(x) := \sup_{Q \subset Q_0} 1_Q(x) \left[w(Q)^{-1} \int_Q f^p \sigma \right]^{1/p}.$$

After some nontrivial calculations we are able to show that for $x \in Q_0 \cap \{T(f\sigma) > 2\lambda\}$ we have:

$$\lambda \lesssim [[\sigma, w]]_{p,q}^{Glo} w(Q_0)^{1/p - 1/q} \overline{M} f(x).$$

Remark: It is this estimate that requires strict inequality $p < q$.
¿From here for each maximal Q_0 we get the following estimate:

$$\lambda^q w(Q_0 \cap \{T(f\sigma) > 2\lambda\}) \lesssim ([[\sigma, w]]_{p,q}^{Glo})^p \lambda^{q-p} w(Q_0)^{1-p/q} \int_{Q_0} f^p \sigma.$$

By summing over all maximal Q_0 and applying Hölder inequality we get:

$$(2\lambda)^q w(T(f\sigma) > 2\lambda) \lesssim ([[\sigma, w]]_{p,q}^{Glo})^p [\lambda^q w(T(f\sigma) > \lambda)]^{1-p/q} \int f^p \sigma.$$

from which the result follows in the same manner as for the local case.

13.2.2 The strong case

The proof of the strong type inequality is based on several ideas. We will use the "linearized" form of the maximal function. Let $\{E(Q) : Q \in \mathcal{Q}\}$ be any selection of measurable disjoint sets $E(Q) \subset Q$ indexed by the dyadic cubes. Define the linear operator:

$$Lf(x) := \sum_{Q \in \mathcal{Q}} 1_{E(Q)}(x) \mathbb{E}_Q^w f.$$

This is a "linearized" form of the maximal function in the sense that the bound (1) is equivalent to the bound $||Lf||_{L^p(w)} \lesssim ||f||_{L^p(w)}$ with implied constant independent of w and the sets $E(Q)$.
We are assuming that

$$\mathfrak{L} = [[\sigma, w]]_{p,q}^{Loc}, \quad \mathfrak{L}_* = [[w, \sigma]]_{q',p'}^{Loc}$$

are finite. By the weak-type case it follows that

$$\sup_{Q \in \mathcal{Q}} w(Q)^{-1/q'} ||T(1_Q w)||_{L^{p'}(\sigma)} \lesssim \mathfrak{L}.$$

We will be working with open sets $\Omega_k = \{T(f\sigma) > 2^k\}$. For a fixed integer $\rho \geq 2$ we construct collections $\mathcal{Q}_k$ of disjoint dyadic cubes that have a Whitney-style properties:

$$\text{disjoint cover}: \quad \Omega_k = \bigcup_{Q \in \mathcal{Q}_k} Q,$$

$$\text{Whitney condition}: \quad Q^{(\rho)} \subset \Omega_k, \ Q^{(\rho+1)} \cap \Omega_k^c \neq \emptyset,$$

$$\text{finite overlap}: \quad \sum_{Q \in \mathcal{Q}_k} 1_{Q^{(\rho)}} \lesssim 1_{\Omega_k},$$

$$\text{crowd control}: \quad \sup_{Q \in \mathcal{Q}_k} \sharp\{Q' \in \mathcal{Q}_k : Q' \cap Q^{(\rho)} \neq \emptyset\} \lesssim 1,$$

$$\text{nested property} \quad Q \in \mathcal{Q}_k, Q' \in \mathcal{Q}_l, Q \subsetneq Q' \Rightarrow k > l.$$

To construct such a decomposition it is enough to take $\mathcal{Q}_k$ to be the maximal dyadic cubes in Ω_k such that the Whitney condition holds. The following easy lemma will be used to decompose our operator:

Lemma 5. ***Maximum Principle*** *For all k and $Q \in \mathcal{Q}_k$ we have:*

$$\max\{T^{out}_{Q^{(\rho)}}(f 1_{Q^{(\rho+1)}}\sigma)(x), T(1_{(Q^{(\rho+1)})} f\sigma)(x)\} \leq 2^k, \quad \text{for } x \in Q.$$

For $Q \in \mathcal{Q}_k$ we define the sets:

$$E_k(Q) = Q \cap (\Omega_{k+4} \setminus \Omega_{k+5}).$$

After decomposing T into the local and global part the maximum principle gives us for $x \in Q$:

$$T^{in}_{Q^{(\rho)}}(1_{Q^{(\rho)}} f\sigma)(x) \geq 2^k,$$

which by self duality of T^{in} leads to

$$2^k w(E_k(Q)) \leq \int_{E_k(Q)} T^{in}_{Q^{(\rho)}}(1_{Q^{(\rho)}} f\sigma) w = \int_{Q^{(\rho)}} f \cdot T^{in}_{Q^{(\rho)}}(1_{E_k(Q)} w)\sigma = \alpha_k(Q) + \beta_k(Q),$$

where

$$\alpha_k(Q) = \int_{Q^{(\rho)} \setminus \Omega_{k+5}} f \cdot T^{in}_{Q^{(\rho)}}(1_{E_k(Q)} w)\sigma, \quad \beta_k(Q) = \int_{Q^{(\rho)} \cap \Omega_{k+5}} f \cdot T^{in}_{Q^{(\rho)}}(1_{E_k(Q)} w)\sigma$$

We split the integral into two parts according to the difficulty of the estimates that will be needed. It is β_k where hard work needs to be done.
Then we estimate:

$$\int |T(f\sigma)|^q w \leq \sum_{k=-\infty}^{\infty} 2^{(k+5)q} w(\Omega_{k+4} \setminus \Omega_{k+5}) = 2^{mq} \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{Q}_k} 2^{kq} w(E_k(Q).$$

We split the last sum into three $S_i, i = 1, 2, 3$ according to the properties of cube $Q \in \mathcal{Q}_k$, namely depending on $0 < \eta < 1$ we have cubes of three types:

$$\mathcal{Q}_k^1 := \{Q \in \mathcal{Q}_k : w(E_x(Q)) \leq \eta w(Q)\},$$

$$\mathcal{Q}_k^2 := \{Q \in \mathcal{Q}_k : w(E_x(Q)) > \eta w(Q), \alpha_k(Q) > \beta_k(Q)\},$$

$$\mathcal{Q}_k^3 := \mathcal{Q} \setminus (\mathcal{Q}_k^1 \cup \mathcal{Q}_k^2).$$

Because of the definition of $\mathcal{Q}_k^1$ "most" of the cubes are of this type. The sum over the cubes of this type is handled almost identically as in the weak case giving $S_1 \lesssim \eta ||T(f\sigma)||^q_{L^q(w)}$. The estimate for the second sum S_2 is also fairly straightforward since $\alpha_k(Q)$ is easy to control. We get $S_2 \lesssim \eta^{-q} \mathfrak{L}^q ||f||^q_{L^p(\sigma)}$. The vast majority of work is done to estimate S_3. The estimate that is finally obtained is $S_3 \lesssim \eta^{-q}[\mathfrak{L}^q + \mathfrak{L}_*^q]||f||^q_{L^p(\sigma)}$. It requires introduction of so called "principal cubes" and some careful analysis of the number of $\mathcal{Q}_k$ that a given cube can be a member of (notice that there is no a priori bound for that number). Once the mentioned estimates are obtained we get:

$$\int |T(f\sigma)|^q w \lesssim \eta ||T(f\sigma)||^q_{L^q(w)} + \eta^{-q}[\mathfrak{L}^q + \mathfrak{L}_*^q] \cdot ||f||^q_{L^p(\sigma)}.$$

That proves the theorem since for η small enough the first term on the right-hand side can be absorbed into the left-hand side.

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14 Two weight inequalities for individual Haar multipliers and other well localized operators

after F. Nazarov, S.Treil, A.Volberg [1]
A summary written by Armen Vagharshakyan

Abstract

The article provides Sawyer type testing conditions for two-weight estimates for 'well localized' operators e.g. Haar multipliers.

14.1 Introduction

The authors provide Sawyer type conditions for two weight estimates for individual Haar multipliers as well as more general operators - the so called well localized operators.
One motivation to study two weight estimates for Haar multipliers is that they were found useful to restore certain singular integral operators: like the Hilbert transform ([2]), Beurling transform ([3]), and one dimensional convolution type Calderon-Zygmund operators with twice differentiable kernels ([4]).

14.2 Statement of Results

In order to phrase the main theorems of the article, we need to introduce some notations.
Let D denote the family of dyadic grids in R^d. For every $R \in D$ denote by X_R the characteristic function of R. Given a measure μ denote by h_R^μ a μ-weighted *Haar function* associated to R, that is: a function supported on Q which is constant on first-order dyadic subcubes of Q and satisfies the following condition

$$\int h_Q^\mu d\mu = 0.$$

If μ is the Lebesque measure, then we just write h_Q instead of h_Q^μ.
The *sidelength* of a dyadic cube Q will be denoted by $l(Q)$.
We introduce the notion of *distance* between dyadic cubes in the following way: let $R, Q \in D$ and $R \subset Q$, then $d(R,Q) = log_2(l(Q)/l(R))$. In the general case, for a pair of dyadic cubes $R, Q \in D$, let $S \in D$ be the smallest dyadic rectangle that contains both R and Q. Then $d(R,Q) = d(R,S) + d(S,Q)$.

Definition. Let T be a linear bounded operator from $L_2(R^d)$ to $L_2(R^d)$. We will call T a *band operator*, if the following condition holds:

$$\langle T(h_Q), h_R \rangle = 0 \quad for \quad d(Q,R) > r.$$

Example. Haar shift operators are band operators.
Now we are ready to phrase the main theorem:

Theorem 1 (band operator). *Let T be a band operator and $T^\star$ be its conjugate. Let $u, v \geq 0$ and $u, v \in L_1^{loc}$.*
Then
$G(f) \mathrel{\hat{=}} v^{1/2}T(u^{1/2}f) \in C(L_2, L_2)$
iff
$\int_{R^n} |T(X_Q u)|^2 v \leq C \int_Q u$ *and*
$\int_{R^n} |T^\star(X_Q v)|^2 u \leq C \int_Q v.$

Note. Formally, the expressions $T(X_Q u)$, $T^\star(X_Q v)$ that appear in the conditions of the theorem are defined only for $u \in L_2^{loc}$. But one can make the statement of the theorem rigorous, if the conditions of the theorem are bound to hold uniformly for a sequence of functions $u_n, v_n \in L_2^{loc}$, which tend to the functions u and v.
Actually, we will consider a more general type of operators - well localized operators, and then the theorem for band operators will imply from that of well localized operators.
Definition. The operator T is called *well localized* for a pair of weights μ,ν if the following conditions hold:

$$\langle T(X_Q), h_R^\nu \rangle_\nu = 0$$

and

$$< T^\star(X_Q), h_R^\mu >_\mu = 0$$

for
1. $l(R) \leq l(Q)$ and $R \not\subset Q^r$, or
2. $l(R) \leq 2^{-r} l(Q)$ and $R \not\subset Q$.

Note. Here $T^\star$ denotes the adjoint of T and R^r is the r'th dyadic parent of R.
Note. In order to define a well localized operator, we don't need the operator T to be bounded from $L_2(\mu)$ to $L_2(\nu)$, rather we only need T to be a linear operator defined on finite linear combinations of characteristic functions of dyadic cubes which maps those linear combinations into $L_2(\nu)$.
Remark. One can prove that if T is a band operator (with respect to functions u and v) then $f \to T(uf)$ is well a localized operator with respect to the measures $d\mu = u$,$d\nu = v$.
Now we phrase the theorem for well localized operators:

Theorem 2 (well localized)**.** *Suppose T is a well localized operator, then $T \in C\left(L_2(\mu), L_2(\nu)\right)$ iff*
$||T(X_Q)||_{L_2(\nu)} \leq C\sqrt{\mu(Q)}$ *and*
$||T^\star(X_Q)||_{L_2(\mu)} \leq C\sqrt{\nu(Q)}$.

14.3 Some Details of the Proof

In order to prove the theorem about well localized operators, we introduce the following paraproduct:

$$\Pi(f) = \sum_R \Delta_R^\nu \left(TX_{R^r}\right) E_{R^r}^\mu(f).$$

Note. Here R^r is the r'th dyadic parent of R, $E_{R^r}^\mu(f)$ is the μ-weighted average of f over R^r and by $\Delta_R^\nu(g)$ we denote the function supported on R, which is constant on every first-order dyadic subcube $P \subset R$, this constant being equal to:

$$\Delta_R^\nu(g)(x) = E_R^\nu(g) - E_P^\nu(g), \ for\ x \in P.$$

One can use the fact that the operator T is well localized to derive the following property, which would relate the operator T to the paraproduct $\Pi(f)$:

$$\left\langle \Pi\left(h_{Q_0}^\mu\right), h_{R_0}^\nu \right\rangle_\nu =$$

$$= \begin{cases} 0 & \text{,otherwise} \\ \left\langle T(h_{Q_0}^\mu), h_{R_0}^\nu \right\rangle_\nu & \text{,if } R_0^r \subsetneqq Q_0 \end{cases}$$

So, speaking very roughly the paraproduct $\Pi(f)$ captures the coefficients in the expansion of $T(f)$ which lie well below the diagonal. One can also introduce an 'adjoint' paraproduct $\Pi^\star(f)$, which would capture the coefficients in the expansion of $T(f)$ which lie well above the diagonal. And then the main step in proving the theorem for well localized operators will be checking the boundedness of the paraproducts $\Pi(f)$ and $\Pi^\star(f)$.
This boundedness will follow from the Carleson embedding theorem. Namely, in order to prove the boundedness of the paraproduct $\Pi(f)$ from $L_2(\mu)$ to $L_2(\nu)$, we would rewrite:

$$\Pi(f) = \sum_Q \left[\sum_{R^r = Q} \Delta_R^\nu \left(TX_Q\right) \right] E_Q^\mu(f),$$

Now we take

$$a_Q = \sum_{R^r = Q} ||\Delta_R^\nu \left(TX_Q\right)||_{L_2(\nu)}^2$$

and check that these numbers satisfy the Carleson embedding theorem:

Theorem 3 (Carleson, see e.g. [5])**.** *For numbers $a_Q \geq 0$ indexed by dyadic cubes we have:*
If $\sum_{Q\subset\widetilde{Q}} a_Q \leq \mu(\widetilde{Q})$ *then* $\sum_Q a_Q \left|E^\mu_Q f\right|^2 \leq C||f||^2_{L_2(\mu)}$.

Remark. Checking the conditions of the Carleson embedding theorem for our choice of coefficients a_Q will use the fact that the operator T is well localized.

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15 Multiparameter operators and sharp weighted inequalities

after R. Fefferman and J. Pipher [5]
A summary written by Daniel Wang

Abstract

We give a summary of [5] by R. Fefferman and Pipher. In this paper, weighted inequalities are obtained for maximal and singular integral operators over Zygmund rectangles. These ideas are then used to prove sharp weighted estimates.

15.1 Weighted inequalities of classical operators

The two main operators considered here are the maximal operator $M_{\mathfrak{z}}$ and singular integral operators of convolution type, both adapted to the geometry generated by the Zygmund rectangles. Weighted inequalities for the (Zygmund) maximal operator and singular integral operators are proved in theorems 1 and 2, respectively.

15.1.1 Main definitions and results

Let $\rho_{s,t}(x,y,z) = (sx, ty, stz)$ for $s,t > 0$, and Q the unit cube in $\mathbb{R}^3$. The Zygmund rectangles, $\mathcal{R}_{\mathfrak{z}}$, are translates of the family $\{\rho_{s,t}(Q)\}_{s,t}$. Associated with these rectangles is the Zygmund maximal operator $M_{\mathfrak{z}}$, given by

$$M_{\mathfrak{z}}(f)(x,y,z) = \sup_{(x,y,z)\in R, R\in\mathcal{R}_{\mathfrak{z}}} \frac{1}{m(R)}\int_R |f|dm,$$

The weights associated with $\mathcal{R}_{\mathfrak{z}}$ are defined as follows: For $p \in (1,\infty)$, with $\frac{1}{p}+\frac{1}{p'} = 1$, a non-negative function ω belongs to $A^p(\mathfrak{z})$ provided

$$\|\omega\|_{A^p(\mathfrak{z})} = \sup_{R\in\mathcal{R}_{\mathfrak{z}}} \left(\frac{1}{m(R)}\int_R \omega dm\right)\left(\frac{1}{m(R)}\int_R \frac{1}{\omega^{p'/p}}dx\right)^{p/p'} < \infty,$$

Theorem 1. *[4] The maximal operator $M_{\mathfrak{z}}$ is bounded on $L^p(\omega)$ for $p \in (1,\infty)$ if and only if $\omega \in A^p(\mathfrak{z})$.*

The next result pertains to singular integrals. For $N \in \mathbb{N}$ sufficiently large, let $\{\psi_{k,j}\}_{k,j\in\mathbb{Z}} \subset C^\infty(\mathbb{R}^3)$ satisfy the following size and smoothness assumption: There exists C, independent of j,k, such that

$$\|\psi_{k,j}\|_{S_N} = \sup_{(x,y,z)\in\mathbb{R}^3} (1+|(x,y,z)|^N) \sum_{\alpha,\beta,\gamma=0}^{N} |\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \psi(x,y,z)| \leq C, \qquad (1)$$

and the cancelation condition: For all $\alpha, \beta \leq N$,

$$\begin{aligned}
&\int_{\mathbb{R}^2} y^\alpha z^\beta \psi_{k,j}(x,y,z)dydz = 0 \text{ for all fixed } x \in \mathbb{R}^1 \\
&\int_{\mathbb{R}^2} x^\alpha y^\beta \psi_{k,j}(x,y,z)dxdy = 0 \text{ for all fixed } z \in \mathbb{R}^1 \qquad (2) \\
&\int_{\mathbb{R}^2} x^\alpha z^\beta \psi_{k,j}(x,y,z)dxdz = 0 \text{ for all fixed } y \in \mathbb{R}^1.
\end{aligned}$$

Theorem 2. *Suppose $Tf = f * K$ is a singular integral, where*

$$K(x,y,z) = \sum_{k,j\in\mathbb{Z}} 2^{-2(k+j)}\psi_{k,j}\left(\frac{x}{2^k}, \frac{y}{2^j}, \frac{z}{2^{k+j}}\right) \qquad (3)$$

and $\{\psi_{k,j}\}_{k,j}$ satisfy the size/smoothness condition (1) and cancelation condition (2). Then T is bounded on $L^p(\omega)$ if $\omega \in A^p(\mathfrak{z})$, $p \in (1,\infty)$.

15.1.2 Outline of theorem 1

Since this was first proved in [4], we will not give too many details. But the key component in proving the sufficiency of $\omega \in A^p(\mathfrak{z})$ is the covering lemma in [3] regarding rectangles with sides parallel to the axes. Using this covering lemma, the operator corresponding to ω, $M_{\mathfrak{z}}^{\omega}(f)(x,y,z) = \sup_{(x,y,z)\in R, R\in\mathcal{R}_{\mathfrak{z}}} \frac{1}{\omega(R)}\int_R |f|\omega dx$ is shown to be bounded on $L^1 \to L^{1,\infty}$. Interpolation and an argument analogous to the classical case (of Muckenhoupt weights) give theorem 1.

15.1.3 Outline of theorem 2

Step 1: We first define the multiplier classes $\mathcal{M}_{\mathfrak{z}}^x$ and $\mathcal{M}_{\mathfrak{z}}^y$. Starting with $\mathcal{M}_{\mathfrak{z}}^x$, we define a 'unit annulus' adapted to this setting:

$$A^x = \{(\xi,\eta,\zeta) \in \mathbb{R}^3 : \frac{1}{2} < |\xi| \leq 1 \text{ and } \frac{1}{2} < |(\eta,\zeta)| \leq 1\},$$

whose dyadic dilates $\{\rho_{2^k,2^j}(A^x)\}_{k,j\in\mathbb{Z}}$ partition $\mathbb{R}^3$. Associated to A^x is a 'set of singularities' S^x, given by $S^x = \{(\xi,\eta,\zeta) : \xi = 0 \text{ or } (\eta,\zeta) = (0,0)\}$. Then $\mathcal{M}_{\mathfrak{z}}^x$ is the collection of all $m \in C^N(\mathbb{R}\backslash S^x)$ such that for all $\alpha,\beta,\gamma \leq N$ and $s,t > 0$, and all $(\xi,\eta,\zeta) \in A^x$, we have the following Mihlin condition:

$$|\partial_\xi^\alpha \partial_\eta^\beta \partial_\zeta^\gamma (m \circ \rho_{s,t})(\xi,\eta,\zeta)| \leq C.$$

The class $\mathcal{M}_{\mathfrak{z}}^y$ is defined analogously.

Step 2: For each $k, j \in \mathbb{Z}$, we decompose $\psi_{j,k} = \psi_{j,k}^{(1)} + \psi_{j,k}^{(2)}$, where the $\psi_{j,k}^{(i)}$ (for $i = 1, 2$) has stronger cancelation properties than $\psi_{j,k}$. This allows us to decompose (3) into $K = K_1 + K_2$, with

$$K_1(x, y, z) = \sum_{k,j \in \mathbb{Z}} 2^{-2(k+j)} \psi_{k,j}^{(1)} \left(\frac{x}{2^k}, \frac{y}{2^j}, \frac{z}{2^{k+j}} \right),$$

and K_2 defined similarly. Then we have $\widehat{K_1} \in \mathcal{M}_{\mathfrak{z}}^x$ and $\widehat{K_2} \in \mathcal{M}_{\mathfrak{z}}^y$.

Step 3: The key part of the proof is this: the multiplier operator T, corresponding to $m \in M_{\mathfrak{z}}^x$ is bounded on $L^2(\omega)$ for $\omega \in A^2(\mathfrak{z})$. This is done by the use of square functions, adapted to our Zygmund setting, which we now define.

Let $\psi_1 \in C_c^\infty(\mathbb{R})$ be even and $\psi_2 \in C_c^\infty(\mathbb{R}^2)$ be radial, with $\int_{\mathbb{R}} \psi_1 = 0, \int_{\mathbb{R}^2} \psi_2 = 0$. For $s, t > 0$, set

$$\psi_{s,t}(x, y, z) = (st)^{-2} \psi_1 \left(\frac{x}{s} \right) \psi_2 \left(\frac{y}{t}, \frac{z}{st} \right). \tag{4}$$

Then we define the (Zygmund) square function by

$$S_{\mathfrak{z}}^2(f)(x, y, z) = \int \int_{\Gamma_{\mathfrak{z}}(x,y,z)} |f * \psi_{s,t}(u, v, w)|^2 \frac{dudvdwdsdt}{s^3 t^3}, \tag{5}$$

where $\Gamma_{\mathfrak{z}}(x, y, z)$ is the Zygmund rectangle centered at (x, y, z) with respective side lengths s, t, st. Analogous to the classical square functions, we have the following results: Fix $\omega \in A^2(\mathfrak{z})$. Then for all $f \in L^2(\omega)$,

$$\|f\|_{L^2(\omega)} \sim \|S_{\mathfrak{z}}(f)\|_{L^2(\omega)},$$
$$\|S_{\mathfrak{z}}(Tf)\|_{L^2(\omega)} \leq C \|S_{\mathfrak{z}}(f)\|_{L^2(\omega)}.$$

We then have the following chain of inequalities:

$$\|Tf\|_{L^2(\omega)} \leq c_1 \|S_{\mathfrak{z}}(Tf)\|_{L^2(\omega)} \leq c_2 \|S_{\mathfrak{z}}(f)\|_{L^2(\omega)} \leq c_3 \|f\|_{L^2(\omega)}.$$

Having proved this for $p = 2$, the extrapolation theorem extends it to $1 < p < \infty$.

15.2 Sharp weighted inequalities

The second half of the paper deals with sharp bounds of weighted inequalities of various operators. A survey of known sharp inequalities (in 1997) is given, but we will mention one particular case. All weights in the following (subsub)section are Muckenhoupt weights.

15.2.1 A well-known example

Let H be the Hilbert transform. Then we have the following sharp inequality:

$$\|Hf\|_{L^p(dx)} \leq C_p \|f\|_{L^p(dx)} \tag{6}$$

where $C_p = O(p)$ as $p \to \infty$. It turns out that (6), which is a sharp inequality with the Lebesgue measure, is deduced from the following weighted sharp inequality:

$$\|Hf\|_{L^2(\omega)} \leq C\|\omega\|_{A^1}\|f\|_{L^2(\omega)}, \tag{7}$$

using the well-known extrapolation argument of Rubio de Francia of exploiting duality of L^p when $p > 1$, which we now illustrate. By duality, there exists $\varphi \in L^{(p/2)'}(\mathbb{R})$, $\varphi \geq 0$, $\|\varphi\|_{L^{(p/2)'}(\mathbb{R})} = 1$ such that $\|Hf\|_{L^p}^2 \leq \|Hf\|_{L^2(\varphi)}$. Using φ, we define an A^1 weight $\tilde{\varphi}$ by

$$\tilde{\varphi} = \sum_{j=0}^{\infty} \frac{M^j \varphi}{(2\|M\|_{L^{(p/2)'}})^j}, \tag{8}$$

where M is the Hardy-Littlewood maximal operator and $M^j\varphi = M \circ M \circ \cdots \circ M\varphi$, j times. Then $\|\tilde{\varphi}\|_{A^1} \leq 2\|M\|_{L^{(p/2)'}} = O(p)$ as $p \to \infty$. With this, we then have

$$\int (Hf)^2 \varphi dx \leq \int (Hf)^2 \tilde{\varphi} dx \leq C\|\tilde{\varphi}\|_{A^1}^2 \int f^2 \tilde{\varphi} dx \leq C' p^2 \|f\|_{L^p}^2,$$

where (7) is used in the second inequality above. This gives (6).

15.2.2 Two applications of sharp estimates

We have two main results regarding sharp weighted inequalities.

Theorem 3. *(Lebesgue estimate) Let $m \in M_{\mathfrak{z}}^x$ or $M_{\mathfrak{z}}^y$, with T the corresponding multiplier operator. Then $\|T\|_{L^p(\mathbb{R}^3)} = O(p^{5/2})$ as $p \to \infty$.*

The following result is an extension of a result in [1] to the product setting. Appearing first appeared in [6], and a simpler proof is given in this paper. We first define the product space square function, S_{pr}, as follows: Let ψ_1, ψ_2 be nontrivial functions in $C_c^\infty(\mathbb{R}^1)$, with $\int_{\mathbb{R}} \psi_i dx = 0$, and for $s, t > 0$ set $\psi_{s,t}(x, y) = (st)^{-1}\psi_1(x/s)\psi_2(y/t)$, and define

$$S_{pr}^2(f)(x, y) \int \int_{\Gamma(x)\times\Gamma(y)} |f * \psi_{s,t}(u, v)|^2 dudv \frac{dsdt}{s^2t^2}.$$

Theorem 4. *(Product Setting) Let T^2 be the unit cube in $\mathbb{R}^2$. If $S_{pr}(f) \in L^\infty$, then for some small $c > 0$, $e^{c|f|/\|Sf\|_\infty} \in L^1$.*

15.2.3 Outline of theorem 3

Without loss of generality, we assume $m \in M^x_{\mathfrak{z}}$. Theorem 3 follows from the following chain of inequalities:

$$\|T(f)\|_{L^p} \le Cp^{\frac{3}{2}} \|S_{\mathfrak{z}}(Tf)\|_{L^p} \le Cp^{\frac{3}{2}} \|g^*_{\lambda,\mathfrak{z}}(f)\|_{L^p} \le Cp^{\frac{5}{2}} \|f\|_{L^p}, \tag{9}$$

where $S_{\mathfrak{z}}$ is defined in (5). To define $g^*_{\lambda,\mathfrak{z}}$, we first set

$$(\Phi_\lambda)_{s,t}(x,y,z) = (st)^{-2} \left(\frac{1}{1+|x/s|} \right)^\lambda \left(\frac{1}{1+|(\frac{y}{t}, \frac{z}{st})|^2} \right)^\lambda,$$

for $\lambda > 1, s,t > 0$. Using $\psi_{s,t}$ from (4), we define

$$g^*_{\lambda,\mathfrak{z}}(f)^2(x,y,z) = \int_{\mathbb{R}^3} \int_{s,t>0} |f * \psi_{s,t}(u,v,w)|^2 (\Phi_\lambda)_{s,t}(u-x, v-y, w-z) dudvdw \frac{dsdt}{st}.$$

The second inequality in (9) follows from classical arguments, so we focus on the first and third inequalities.

First inequality: $\|f\|_{L^p(\mathbb{R}^3)} \le Cp^{3/2} \|S_{\mathfrak{z}}(f)\|_{L^p(\mathbb{R}^3)}$
This inequality follows in two stages, by first dealing with the $x \in \mathbb{R}$ variable then dealing with $(y,z) \in \mathbb{R}^2$ variables. In each case, we use duality method in (8) to obtain two $\varphi \in L^{(p/2)'}$, $\|\varphi\|_{L^{(p/2)'}} = 1$, one on $\mathbb{R}$ and the other on $\mathbb{R}^2$. This allows us to relate the quantity in question to a known classical weighted norm estimate.

Third inequality: $\|g^*_{\lambda,\mathfrak{z}}(f)\|_{L^p(\mathbb{R}^3)} \le C_\lambda p \|f\|_{L^p(\mathbb{R}^3)}$
First, using a similar argument as theorem 1.1 in [1], we have

$$\int g^*_{\lambda,\mathfrak{z}}(f)^2 \varphi \le C_\lambda \int f^2 M_{\mathfrak{z}} \varphi,$$

for a positive function φ. Again, by the duality argument in (8), for $p > 2$, we have

$$\|g^*_{\lambda,\mathfrak{z}}(f)\|_{L^p(\mathbb{R}^3)} \le C_\lambda \|M_{\mathfrak{z}}\|^{1/2}_{L^{(p/2)'}(\mathbb{R}^3)} \|f\|_{L^p(\mathbb{R}^3)}.$$

Then using a covering lemma in [2], one obtains the estimate $\|M_{\mathfrak{z}}\|_{L^{(p/2)'}(\mathbb{R}^2)} = O(p^2)$.

15.2.4 Outline of Theorem 4

The theorem follows immediately from the following estimate:

$$\|f\|_{L^p(T^2)} \le Cp \|S_{pr}(f)\|_{L^p(T^2)}, \qquad \text{as } p \to \infty, \tag{10}$$

which follows from showing

$$\|f\|_{L^p} \le Cp^{1/2}\|Sf\|_{L^p}, \tag{11}$$

where $f = (f_k)_{k\in\mathbb{N}}$ takes its values in the Hilbert space $L^2(\Gamma; d\gamma)$, S is the classical square function on $\mathbb{R}$, and $|Sf(x)|^2 = \sum_{k\in\mathbb{N}}(Sf_k)^2$. Lastly, (11) follows from the inequality

$$\int_{\mathbb{R}} |f|^2 \omega dx \le C\|\omega\|_{A^1(\mathbb{R})} \int_{\mathbb{R}} Sf^2 \omega dx.$$

(Here, $|f|$ is the Hilbert space norm) The important part of this proof is showing (11) implies (10). This is very similar in dealing with the first inequality in the outline of Theorem 3.

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