Math 132 - Week 4
Textbook sections: 3.3, Review
Topics covered:

- Multi-valued functions
- Review for first midterm

Complex powers

- We now look at the question of how to raise one complex number to another, or more precisely how to make sense of an equation such as

$$w = z^\alpha$$

when z and α are complex numbers. We know how to do this in some special cases (e.g. if α is an integer, of if z is e), but we haven't yet shown how to do it in general.

- The answers to this problem are a little unsatisfactory, but the best one we have is to define z^α by the formula

$$z^\alpha = e^{\alpha \log(z)}.$$

- For instance, using this definition we have

$$(1+i)^2 = \exp(2\log(1+i))$$

$$= \exp(2\ln\sqrt{2} + 2i(\pi/4 + 2k\pi))$$

$$= \exp(\ln 2 + \pi i/2 + 4k\pi i)$$

$$= 2i,$$

which agrees with the usual definition. Similarly

$$(1+i)^{1/2} = \exp(\log(1+i)/2)$$

$$= \exp(\ln \sqrt{2}/2 + i(\pi/4 + 2k\pi)/2)$$
$$= \exp(\ln 2^{1/4} + \pi i/8 + k\pi i)$$
$$= 2^{1/4} e^{i\pi/8} \text{ or } 2^{1/4} e^{9i\pi/8}.$$

So $1 + i$ has two square roots, which is consistent with what we've done before.

- However, things get a bit more odd when we take complex powers. For instance, to evaluate i^i:

$$i^i = \exp(i \log(i))$$

$$= \exp(i(\ln 1 + i(\pi/2 + 2k\pi)))$$
$$= \exp(-\pi/2 - 2k\pi)$$
$$= e^{-\pi/2}, e^{-3\pi/2}, e^{-5\pi/2}, e^{\pi/2}, \ldots$$

Thus we get an infinite number of answers for i^i! This is somewhat unsatisfactory.

- In practice, we only use branches of z^α rather than dealing with the multi-valued function directly. Each branch of the logarithm gives a branch of z^α. For instance, the principal branch $\mathrm{Log}(z)$ gives the principal branch of z^α, which is

$$\text{principal branch of } z^\alpha = e^{\alpha \mathrm{Log}(z)};$$

this is sometimes abbreviated as p.v. z^α. For instance, we have

$$\text{p.v. } i^i = e^{i\text{Log}i} = e^{i\pi i/2} = e^{-\pi/2}.$$

- All the branches of z^α are differentiable except at the branch cut. We can compute the derivative of these branches, for instance the principal branch goes like this:

$$\begin{aligned}
\frac{d}{dz}\text{p.v. } z^\alpha &= \frac{d}{dz}e^{\alpha\text{Log}(z)} \\
&= \alpha e^{\alpha\text{Log}(z)}\frac{d}{dz}\text{Log}(z) \\
&= \alpha e^{\alpha\text{Log}(z)} z^{-1} \\
&= \alpha e^{\alpha\text{Log}(z)} e^{-\text{Log}(z)} \\
&= \alpha e^{(\alpha-1)\text{Log}(z)} \\
&= \text{p.v. } \alpha z^{\alpha-1}.
\end{aligned}$$

Inverse trig functions

- Occasionally it is necessary to invert a complex trig function. We won't pursue this in too much detail, but just give one example, in computing the multi-valued function $\cos^{-1}(z)$. This means finding all solutions w to the equation

$$\cos(w) = z.$$

Using the definition of complex cosine, this is

$$\frac{e^{iw} + e^{-iw}}{2} = z,$$

which we rewrite as

$$e^{iw} - 2z + e^{-iw} = 0.$$

Writing $\zeta = e^{iw}$, this is

$$\zeta - 2z + 1/\zeta = 0,$$

or

$$\zeta^2 - 2z\zeta + 1 = 0.$$

From the quadratic formula (which is just as valid for complex numbers as it is for reals) we have

$$\zeta = (2z + (4z^2 - 4)^{1/2})/2 = z + (z^2 - 1)^{1/2},$$

so we have

$$iw = \log(z \pm (z^2 - 1)^{1/2}).$$

Thus we have

$$\cos^{-1}(z) = \frac{1}{i} \log(z + (z^2 - 1)^{1/2}).$$

Since there are two square roots of $z^2 - 1$, and infinitely many logarithms, this is a multi-valued function. To make this a single-valued function we need to pick a branch of log and a branch of the square root function. For instance, we can use the principal branch for each, obtaining

$$\text{p.v. } \cos^{-1}(z) = \frac{1}{i} \mathrm{Log}(z + \text{p.v. } (z^2 - 1)^{1/2}).$$

One can compare this with the more familiar formula

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$$

for real numbers x.

- This branch is differentiable except when $z^2 - 1$ is zero or a negative real, or when $z + \text{p.v. } (z^2 - 1)^{1/2}$ is zero or a negative real.
- We won't deal with inverse trig functions much in this course.

Information on the mid-term

- The mid-term is this Friday (Feb 4), in the usual time and place (i.e. MS 5138 11-11:50). No notes or calculators are allowed.
- There will be 5 questions, each worth 10 points. The midterm as a whole is worth 20
- A sample mid-term is on the class web page

  http : //www.math.ucla.edu/ tao/132.1.00w

  together with (somewhat brief) solutions.
- The mid-term encompasses all topics covered to date (i.e. Chapters 1-3 and Sections 7.3-7.4). The sample midterm also covered this material.

Summary of topics covered in the mid-term

- Arithmetic of complex numbers: addition, multiplication, subtraction, division, magnitude, conjugation. Conversion between Cartesian and polar forms. The difference between arg and Arg, and all the other branches of the argument.

- Complex algebra: Solving simple algebraic equations (e.g. $z^4 = -1$). Powers and roots of complex numbers, and how to make these quantities single-valued. Triangle inequality: if you know the magnitude of z and w, what does this tell you about the magnitude of $z + w$, $z - w$, zw, z/w?

- Complex exponential: how to compute e^z. You can state without proof that e^z is entire.

- Sets in the complex plane: Be able to sketch simple sets such as $\{0 \leq \text{Arg}(z+i) \leq \pi/2\}$, and determine whether they are open, closed, connected, bounded, and/or a domain. Know what interior point, boundary point, exterior point means. (You do not have to give rigorous proofs that a set is open, connected, etc.; verbal explanations will suffice).

- Caveat: You should be aware that certain expressions, e.g. $\{z : 3 < z < 5\}$ do not really make sense, because we do not have a satisfactory notion of "greater than" for complex numbers (is $1 > i$ or $i > 1$?) If you find yourself staring at an object like the one above, something has gone wrong.

- Complex functions: Know how to write a function $w = f(z)$ in Cartesian form $f(x+iy) = u(x+iy) + iv(x+iy)$.

- Limits: Know how to evaluate the horizontal and vertical partial limits of a complex limit $\lim_{z\to z_0} f(z)$; if they don't match, or if at least one of the partial limits doesn't exist, the limit does not exist. (You do not need to know the epsilon-delta definition of limit). Know how to compute a limit if the function inside the limit is continuous, or by using the squeeze test. Knowledge of the limit laws.

- Continuity: Be able to determine from inspection when a function $f(z)$ or $f(x + iy)$ is continuous. (You will not need to formally prove continuity, but you should be aware of the continuity laws, e.g. the composition of two continuous functions is continuous).

- Differentiability: If the Cauchy-Riemann equations are not satisfied at a point, then the function is not differentiable there. If the Cauchy-Riemann equations are satisfied at a point AND the partial derivatives exist and are continuous at that point, then the function is differentiable (you can quote this theorem as a given). This covers 99% of all cases. You can also prove differentiability by first principles, or by using the differentiability laws (e.g. the quotient of two differentiable functions is differentiable as long as the quotient is non-zero). One can compute complex derivatives using the formula $df/dz = \partial f/\partial x = 1/i \partial f/\partial y$.

- Analyticity: If a function is differentiable at a point, and on a small ball surrounding that point, then it is analytic at that point. If it is analytic everywhere, we call it entire.

- Harmonic functions: Know that the real and imaginary parts of an analytic function are harmonic. Given the real part of an analytic function, reconstruct the imaginary part (or vice versa), up to a constant. (You can ignore the stuff in the book about level curves and physical applications).

- Möbius transforms: Know what a Möbius transform is, and how to transform simple objects such as disks, lines, half-planes, etc. How to find Möbius transforms which perform specific tasks (e.g. map a specific disk to another specific disk). Invert Möbius transforms. (Ignore the stuff in the book about cross-ratios and conformality).

- Complex trig and hyperbolic functions: know their definition, and how to differentiate them. Warning: there are some differences between real trig and complex trig functions. For instance, the inequality $\sin(z) \leq 1$ is true for real z, but doesn't even make sense for complex z.

- Complex logarithms: Know the difference between log and Log, and why all the various logarithms have branch cuts. Be able to find the branch cuts of more complicated expressions such as $\mathrm{Log}(i - 3z)$. Know how to differentiate the logarithm.

- Complex powers: Do all the above for complex powers z^α and their branches (e.g. the principal branches).

- Inverse trig functions: Know how to solve equations like $\cos(z) = 2$. (You can ignore stuff about principal branches of $\sin^{-1}$, etc.)