Math 132 - Week 10
Textbook sections: 6.4-6.7
Topics covered:

- Integrals of multi-valued functions
- Rouche's theorem

Integrals of functions with branch cuts

- We now consider a more difficult integral, namely

$$p.v. \int_0^\infty \frac{dx}{\sqrt{x}(x+4)}.$$

  The problem here is going to be that $\sqrt{z}$ is multi-valued, and we must pick a branch of the square root. There is no fixed procedure on how to do this; each integral of this type has to be dealt with using a tailor-made branch and a tailor-made contour. These are the most difficult type of integrals we'll consider in this course.

- Unfortunately, this function is not even, so we cannot replace the $(0, \infty)$ integration with an $(-\infty, \infty)$ integration. Also there is a singularity at 0, so the integral should be interpreted as

$$\lim_{\varepsilon \to 0} \lim_{R \to \infty} \int_\varepsilon^R \frac{dx}{\sqrt{x}(x+4)}.$$

- We would like to replace this with a complex contour integral

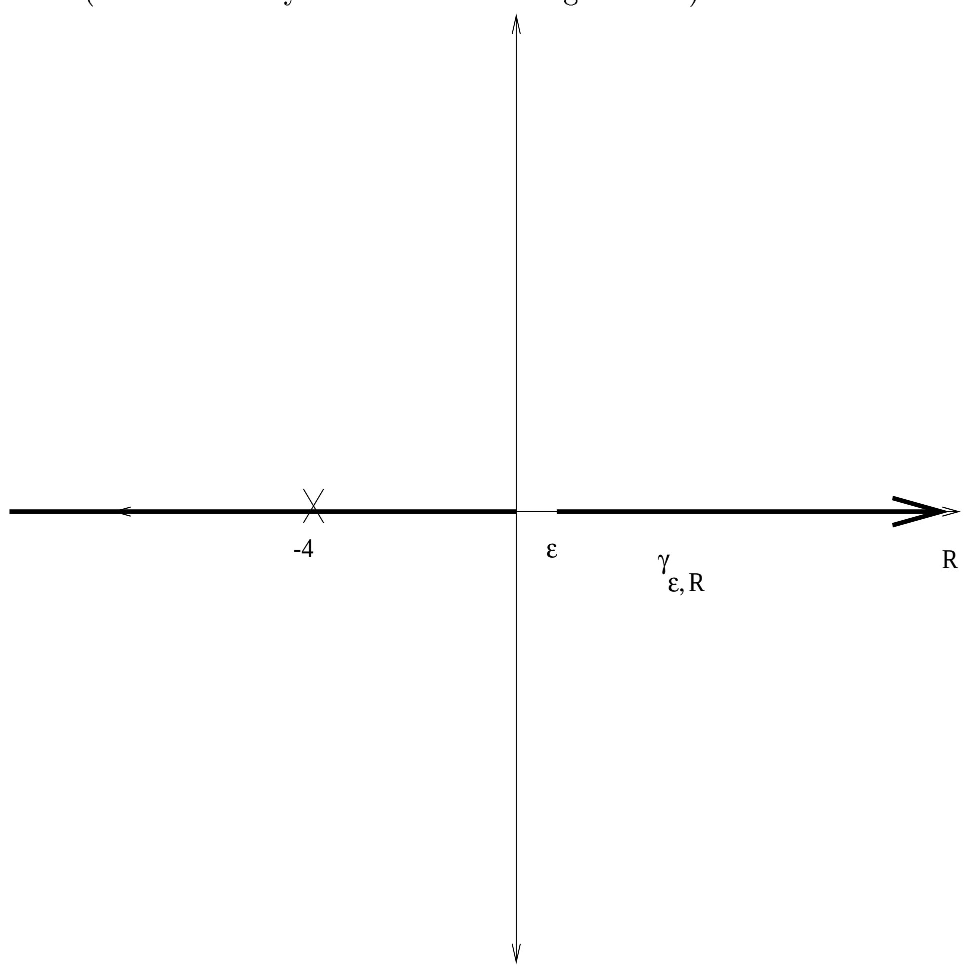
$$\lim_{\varepsilon \to 0} \lim_{R \to \infty} \int_{\gamma_{\varepsilon,R}} \frac{dz}{\sqrt{z}(z+4)}$$

  where $\gamma_{\varepsilon,R}$ is the straight line from $\varepsilon$ to $R$. Unfortunately this is not quite correct, because $\sqrt{z}$ is multi-valued and can equal $+\sqrt{x}$ or $-\sqrt{x}$ when $z = x + i0$. So we have to pick a branch.

- The obvious branch to pick is the principal branch

$$p.v.\sqrt{z} = e^{\frac{1}{2}\text{Log}(z)}.$$

  This function has a branch cut on the negative real axis. However, if one does this then one runs into trouble, because there appears to be no way to close the contour $\gamma_{\varepsilon,R}$ without jumping across the branch cut, which of course would defeat any attempt to use the residue theorem (which can only handle isolated singularities).




- It seems like every choice of branch cut is going to run up against this problem. But there is a sneaky solution - choose the branch cut that goes through the *positive* real axis - straight through the contour of integration! Let's see how this unusual trick can work.

- We let $f(z)$ denote the branch

$$f(z) = e^{\frac{1}{2}\mathrm{Log}_{(0,2\pi]}(z)}$$

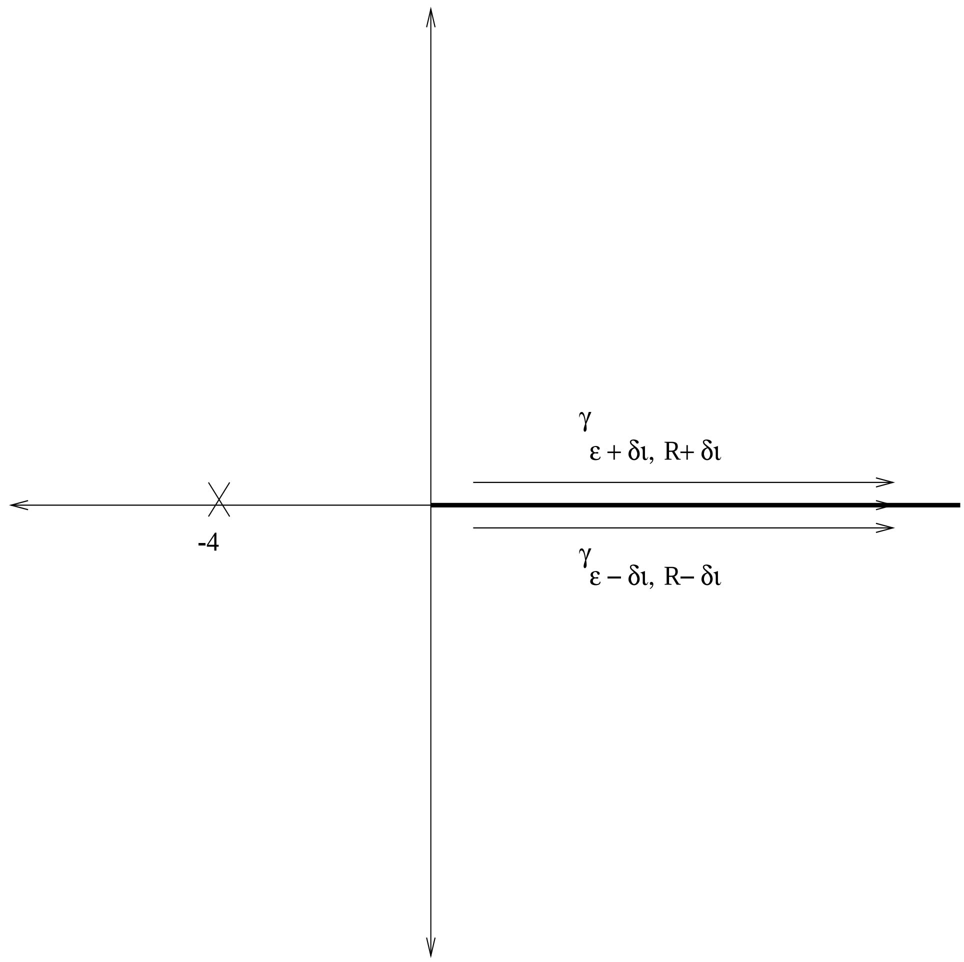
of the square root function $\sqrt{z}$. This has a branch cut on the *positive* real axis. The idea is to use this branch cut to split the contour $\gamma_{\varepsilon,R}$ into two pieces (kind of like splitting a hair with a sharp knife).

- Pick a small number $\delta > 0$, and consider the integrals

$$\int_{\gamma_{\varepsilon+\delta i, R+\delta i}} \frac{dz}{f(z)(z+4)}$$

and

$$\int_{\gamma_{\varepsilon-\delta i, R-\delta i}} \frac{dz}{f(z)(z+4)}$$




- Let's look at the first integral. We can parameterize it by $z = x + \delta i$, $\varepsilon \le x \le R$, so $dz = dx$, and the integral becomes

$$\int_{\varepsilon}^{R} \frac{dx}{f(x+\delta i)(x+4+\delta i)}.$$

Now as $\delta \to 0$, $x+4+\delta i$ approaches $x+4$. What about $f(x+\delta i)$? We have to be a bit careful here because $f$ has a branch cut at the positive real line, and is hence discontinuous at $x$. Well, we have

$$f(x+\delta_i) = \exp(\frac{1}{2}(\ln|x+\delta i| + i\mathrm{Arg}_{(0,2\pi]}(x+\delta i))).$$

As $\delta \to 0$, $\ln|x + \delta i|$ approaches $\ln x$, and $\text{Arg}_{(0,2\pi]}(x + \delta i)$ approaches 0. So

$$f(x + \delta i) \to \exp(\frac{1}{2} \ln x) = x^{1/2} \text{ as } \delta \to 0.$$

This is as one expects, since $f(z)$ is a branch of $z^{1/2}$. Putting these things together we get

$$\lim_{\delta \to 0} \int_{\gamma_{\varepsilon+\delta i, R+\delta i}} \frac{dz}{f(z)(z+4)} = \int_{\varepsilon}^{R} \frac{dx}{\sqrt{x}(x+4)},$$

which is the integral we want.

- Now let's look at the second integral. We can parameterize this by $z = x - \delta i$, $\varepsilon \le x \le R$, so $dz = dx$, and the integral becomes

$$\int_{\varepsilon}^{R} \frac{dx}{f(x - \delta i)(x + 4 - \delta i)}.$$

As before, $x+4-\delta i$ approaches $x+4$ as $\delta \to 0$. What about $f(x-\delta i)$? Well, this is

$$f(x - \delta i) = \exp(\frac{1}{2}(\ln|x - \delta i| + i\text{Arg}_{(0,2\pi]}(x - \delta i))).$$

As $\delta \to 0$, $\ln|x - \delta i|$ approaches $\ln x$. However, $\text{Arg}_{(0,2\pi]}(x - \delta i)$ approaches $2\pi$ instead of 0! (This is due to the branch cut going through $x$). So

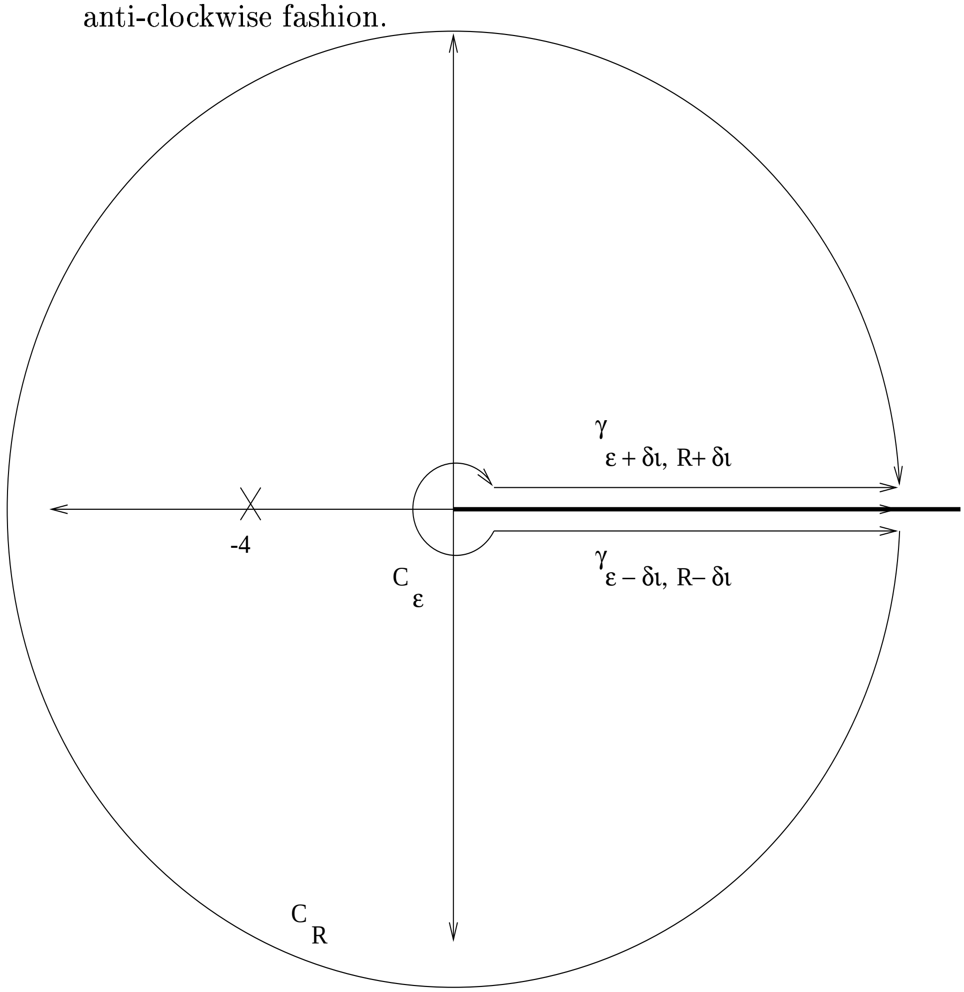
$$f(x - \delta i) \to \exp(\frac{1}{2}(\ln x + 2\pi i)) = -x^{1/2} \text{ as } \delta \to 0.$$

In other words, $f$ approaches $x^{1/2}$ from above $x$, and approaches $x^{-1/2}$ from below $x$. We thus have

$$\lim_{\delta \to 0} \int_{\gamma_{\varepsilon+\delta i, R+\delta i}} \frac{dz}{f(z)(z+4)} = -\int_{\varepsilon}^{R} \frac{dx}{\sqrt{x}(x+4)}.$$

So both of these split integrals are related to the integral that we actually want to compute.

- Now we need to close the contour. Let $C_\varepsilon$ be a nearly complete circle connecting $\varepsilon + \delta i$ to $\varepsilon - \delta i$ in an anti-clockwise fashion, and similarly let $C_R$ be a nearly complete circle connecting $R + \delta i$ to $R - \delta i$ in an anti-clockwise fashion.




- The function $\frac{1}{f(z)(z+4)}$ has a branch cut at 0 and the positive real line, and also has an isolated singularity at -4. The contour

$$\gamma_{\varepsilon+\delta i, R+\delta i} + C_R + -\gamma_{\varepsilon-\delta i, R-\delta i} + -C_\varepsilon$$

is a closed contour which avoids the entire branch cut, and winds once anti-clockwise around $-4$. Thus we can use the residue theorem to conclude

$$\int_{\gamma_{\varepsilon+\delta i,R+\delta i}+C_R+-\gamma_{\varepsilon-\delta i,R-\delta i}+-C_\varepsilon} \frac{dz}{f(z)(z+4)}$$
$$= 2\pi i Res(\frac{1}{f(z)(z+4)}, -4).$$

The function $f(z)$ is analytic and non-zero at $-4$, indeed

$$f(-4) = e^{\frac{1}{2}\mathrm{Log}_{(0,2\pi]}(-4)} = e^{\frac{1}{2}(\ln 4 + i\pi)} = e^{\ln 4/2}e^{i\pi/2} = 2i.$$

Thus the function $\frac{1}{f(z)(z+4)}$ has a simple pole at -4, so

$$Res(\frac{1}{f(z)(z+4)}, -4) = \lim_{z\to 4} \frac{z+4}{f(z)(z+4)}$$
$$= \frac{1}{f(4)} = \frac{1}{2i}.$$

Thus we have

$$\int_{\gamma_{\varepsilon+\delta i,R+\delta i}} \frac{dz}{f(z)(z+4)} + \int_{C_R} \frac{dz}{f(z)(z+4)}$$
$$-\int_{\gamma_{\varepsilon-\delta i,R-\delta i}} \frac{dz}{f(z)(z+4)} - \int_{C_\varepsilon} \frac{dz}{f(z)(z+4)}$$
$$= \pi.$$

- Now take limits as $\delta \to 0$. We've already seen that the first integral converges to $\int_\varepsilon^R \frac{dx}{\sqrt{x}(x+4)}$, and the third integral converges to $-\int_\varepsilon^R \frac{dx}{\sqrt{x}(x+4)}$. The second and fourth integrals barely change as $\delta \to 0$, the only thing that happens is that the almost complete circles become genuinely complete circles. So we have

$$2\int_\varepsilon^R \frac{dx}{\sqrt{x}(x+4)} + \int_{C_R} \frac{dz}{f(z)(z+4)}$$
$$-\int_{C_\varepsilon} \frac{dz}{f(z)(z+4)} = \pi.$$

Now we let $R \to \infty$ and $\varepsilon \to 0$, and hope that the $C_R$ and $C_\varepsilon$ integrals go to zero.

- Let's first look at the $C_R$ integral. When $z$ is on $C_R$, $|z| = R$, so

  $$R - 4 \leq |z + 4| \leq R + 4.$$

  What about $f(z)$? This is one of the two square roots of $z$. Both square roots have magnitude $\sqrt{R}$, so we have $|f(z)| = R^{1/2}$. Thus

  $$|\frac{1}{f(z)(z+4)}| \leq \frac{1}{\sqrt{R}(R-4)}.$$

  Since $C_R$ has length $2\pi R$, we have

  $$|\int_{C_R} \frac{dz}{f(z)(z+4)}| \leq \frac{2\pi R}{\sqrt{R}(R-4)} = \frac{2\pi}{\sqrt{R}(1-\frac{4}{R})}.$$

  As $R \to \infty$, the right-hand side clearly goes to 0, so the $C_R$ integral goes to zero.

- Now we look at the $C_\varepsilon$ integral. When $z$ is in $C_\varepsilon$, $|z| = \varepsilon$, so

  $$4 - \varepsilon \leq |z + 4| \leq 4 + \varepsilon.$$

  Also, we have $|f(z)| = \varepsilon^{1/2}$, so

  $$|\frac{1}{f(z)(z+4)}| \leq \frac{1}{\sqrt{\varepsilon}(4-\varepsilon)}.$$

  Since $C_R$ has length $2\pi\varepsilon$, we have

  $$|\int_{C_\varepsilon} \frac{dz}{f(z)(z+4)}| \leq \frac{2\pi\varepsilon}{\sqrt{\varepsilon}(4-\varepsilon)} = \frac{2\pi\sqrt{\varepsilon}}{4-\varepsilon}.$$

  As $\varepsilon \to 0$, the right-hand side clearly goes to 0, so the $C_\varepsilon$ integral goes to zero.

- Taking limits in the previous expression, we now get

  $$2p.v. \int_0^\infty \frac{dx}{\sqrt{x}(x+4)} = \pi,$$

  so

  $$p.v. \int_0^\infty \frac{dx}{\sqrt{x}(x+4)} = \frac{\pi}{2}.$$

- As you can see, sometimes it takes quite a bit of effort to figure out how to use contour integration to compute integrals! Hopefully you have an idea now of how versatile and powerful the technique of shifting the contour (or trying to make a contour closed) is.

Argument principle and Rouche's theorem - Introduction

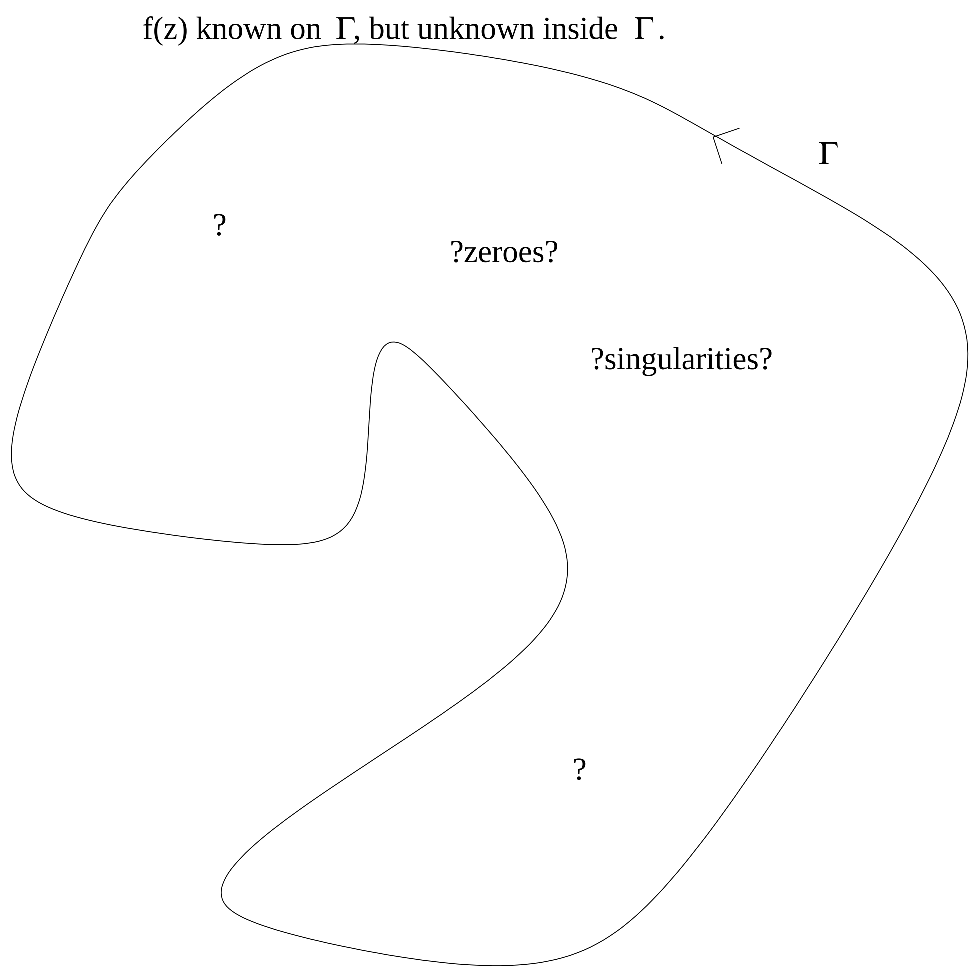
- A basic philosophy in residue calculus is that the behaviour of a function on a closed contour is to a large extent controlled by the singularities inside that contour. We're going to reverse this philosophy, and show how the behaviour of a function on a closed contour can be used to determine the number of singularities and zeroes inside that contour. In fact, there is a very nice geometric relationship called the Argument principle, which states that the number of zeroes of an analytic function $f$ inside a closed contour $\Gamma$ is equal to the number of times $f(\Gamma)$ winds around zero.

- There is a useful application of this principle known as Rouche's theorem, which, roughly speaking, states that if $f$ and $g$ are two analytic functions which are approximately equal on a closed contour $\Gamma$, then $f$ and $g$ have exactly the same number of zeroes inside $\Gamma$. This is useful for counting how many zeroes a function has, even if we can't solve for the zeroes directly.

- As a consequence of this, we'll be able to give a "geometric" proof of the Fundamental Theorem of Algebra which is quite different from the one given earlier.

Counting the number of zeroes or singularities inside a contour

- Let $\Gamma$ be a simple closed anti-clockwise contour, and let $f$ be a function which is analytic on and inside $\Gamma$ except possibly for a finite number of singularities. Suppose we know the value of $f$ on the contour $\Gamma$, but don't know exactly what happens inside the contour $\Gamma$. It is a natural question to ask if we can reconstruct some information about $f$ in the interior of $\Gamma$ just from knowing what happens on the boundary. For instance, we might like to know how many singularities and zeroes $f$ has inside $\Gamma$.




- If we know that $f$ has no singularities inside $\Gamma$, then the Cauchy integral formula allows us to determine $f(z_0)$ for all $z_0$ inside $\Gamma$, by the formula

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, dz.$$

  But this formula only works if there are no singularities inside $\Gamma$, and is not very good for locating zeroes of $f$ (since one would then be faced with trying to solve the equation

$$0 = \int_\Gamma \frac{f(z)}{z - z_0}\, dz$$

for $z_0$.)

- If $f$ has no singularities in $\Gamma$, then the Cauchy-Goursat theorem tells us that $\int_\Gamma f(z)\, dz = 0$. So if we integrate $f$ on $\Gamma$ and find that the integral is non-zero, that tells us that $f$ has at least one singularity inside $\Gamma$. It also tells us what the sum of all the residues of $f$ inside $\Gamma$ are, thanks to the residue theorem. But this doesn't seem to tell us how to get more precise information, such as the number of singularities (or zeroes) that $f$ has.

- Fortunately, there is a trick that allows us to do this. The idea is to look at the integral
$$\int_\Gamma \frac{f'(z)}{f(z)}\, dz.$$
The expression $\frac{f'(z)}{f(z)}$ is known as the *logarithmic derivative*, because it would be the derivative of $\log(f(z))$ were it not for the fact that log is multi-valued.

- Let's try to figure out what this integral is. By the Residue theorem, it is equal to $2\pi i$ times the sum of all the residues of $\frac{f'(z)}{f(z)}$, so we need to find all the singularities of $\frac{f'(z)}{f(z)}$ and their singularities.

- There are two ways that $\frac{f'(z)}{f(z)}$ can develop a singularity at a point $z_0$. The first is when $f(z)$ has a zero at $z_0$. The second is when $f(z)$, and thus $f'(z)$, has a singularity at $z_0$. If $f(z)$ is analytic and non-zero at $z_0$, then $f'/f$ is analytic at $z_0$ (recall that analytic functions are infinitely differentiable), so there is no singularity. Thus the singularities of $f'/f$ occur at the zeroes and singularities of $f$.

- Now to compute the residues of $f'/f$. Let's first suppose that $f(z)$ has a zero of order $m$ at $z_0$:
$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \ldots.$$
This means that $f'$ has a zero of order $m - 1$ at $z_0$:
$$f'(z) = m a_m (z - z_0)^{m-1} + \ldots.$$

Thus $f'/f$ has a simple pole at $z_0$. To find the residue, we use the formula

$$\begin{aligned} Res(f'/f; z_0) &= \lim_{z\to z_0} \frac{f'(z)(z-z_0)}{f(z)} \\ &= \lim_{z\to z_0} \frac{ma_m(z-z_0)^m + \ldots}{a_m(z-z_0)^m + \ldots} \\ &= \lim_{z\to z_0} \frac{m + \ldots}{1 + \ldots} \\ &= m. \end{aligned}$$

Thus $f'/f$ has a residue at $z_0$ equal to the order of the zero of $f$ at $z_0$.

- Now suppose that $f$ has a pole of order $m$ at $z_0$:

$$f(z) = a_{-m}(z-z_0)^{-m} + a_{-m+1}(z-z_0)^{-m+1} + \ldots.$$

Then $f'$ has a pole of order $m+1$ at $z_0$:

$$f'(z) = -ma_{-m}(z-z_0)^{-m-1} + \ldots.$$

Thus $f'/f$ has a simple pole at $z_0$. Again, we compute the residue

$$\begin{aligned} Res(f'/f; z_0) &= \lim_{z\to z_0} \frac{f'(z)(z-z_0)}{f(z)} \\ &= \lim_{z\to z_0} \frac{-ma_{-m}(z-z_0)^{-m} + \ldots}{a_m(z-z_0)^{-m} + \ldots} \\ &= \lim_{z\to z_0} \frac{-m + \ldots}{1 + \ldots} \\ &= -m. \end{aligned}$$

Thus $f'/f$ has a residue at $z_0$ equal to negative the order of the pole of $f$ at $z_0$.

- Removable singularities of $f$ are poles of order 0 and hence have no residue.

- Putting this all together, we get [if $f$ has no essential singularities inside $\Gamma$]

$$\int_\Gamma \frac{f'(z)}{f(z)}\ dz = 2\pi i[$$

$$\text{total order of zeroes inside } \Gamma$$

$$-\text{ total order of poles inside } \Gamma].$$

- In other words, the number of zeroes inside $\Gamma$, minus the number of poles inside $\Gamma$ is equal to

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)\ dz}{f(z)}\ dz,$$

if we count a double zero as two separate zeroes, a triple zero as three separate zeroes etc. (This is referred to as "counting multiplicity". If every zero is counted just once, we say we are "not counting multiplicity").

- We've achieved our aim of counting zeroes and singularities (or more precisely, the difference between the number of zeroes and the number of poles) in terms of the value of $f$ on $\Gamma$. Now let's see if we can simplify the integral. Fortunately, this is very easy thanks to the change of variables

$$w = f(z), \quad dw = f'(z).$$

As $z$ travels along $\Gamma$, $w$ travels along $f(\Gamma)$, the image of $\Gamma$ under $f$. So we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)\ dz}{f(z)} = \frac{1}{2\pi i}\int_{f(\Gamma)} \frac{dw}{w}.$$

Since $\Gamma$ is closed, $f(\Gamma)$ is also closed. Now the function $1/w$ has a simple pole at 0 with residue 1, so by the Residue theorem we have

$$\frac{1}{2\pi i}\int_{f(\Gamma)} \frac{dw}{w} = Wind(f(\Gamma); 0).$$

Summarizing, we have

- **Argument principle.** Let $\Gamma$ be a simple closed anti-clockwise contour, and let $f$ be a function which is analytic on and inside $\Gamma$ except for a finite number of poles and zeroes inside $\Gamma$. Also, suppose that $f$ has no zeroes on $\Gamma$ itself. Then the number of zeroes (counting multiplicity) of $f$ inside $\Gamma$, minus the number of poles inside $\Gamma$, is equal to the number of times $f(\Gamma)$ winds anti-clockwise around the origin. (This is also called the "total Argument of $f(\Gamma)$", hence the name of the principle).

- Let's see some examples. Let $\Gamma$ be the circle $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$ (i.e. the circle $|z| = 1$ traversed once anti-clockwise), and let $f(z) = z^5$. This function has a quintuple zero at 0 and no singularities. So the number of zeroes inside $\Gamma$ is 5, and the number of poles is 0. Now let's look at $f(\Gamma)$. Since $\Gamma$ is given by $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$, and $f(z) = z^5$, the curve $f(\Gamma)$ is given by $e^{5i\theta}, 0 \leq \theta \leq 2\pi$, and this is the unit circle $|z| = 1$ traversed five times anti-clockwise; this curve thus winds around the origin five times. Since $5 - 0 = 5$, we've verified the Argument principle in this case.

- Now let's keep $\Gamma$ the same, but let $f(z) = z^5 + 2$. This function has five zeroes at $2^{1/5}e^{\pi i/5}$, $2^{1/5}e^{3\pi i/5}$, $2^{1/5}e^{5\pi i/5}$, $2^{1/5}e^{7\pi i/5}$ and $2^{1/5}e^{9\pi i/5}$, but these are all outside $\Gamma$. Thus $f$ has no zeroes and no poles inside $\Gamma$. The curve $f(\Gamma)$ is given by $e^{5i\theta} + 2, 0 \leq \theta \leq 2pi$. This curve is the circle $|z - 2| = 1$ traversed five times anti-clockwise; this curve does not wind around 0 at all. Since $0 - 0 = 0$, we've verified the Argument principle in this case.

- As our last example, let's keep $\Gamma$ the same, but let $f(z) = 1/z^5$. This function has five poles at 0 and no zeroes, so $f$ has no zeroes and five poles inside $\Gamma$. The curve $f(\Gamma)$ is given by $e^{-5i\theta}, 0 \leq \theta \leq 2\pi$. This is the circle $|z| = 1$ traversed five times *clockwise*, so the winding number of this curve around 0 is -5. Since $0 - 5 = -5$, we've verified the Argument principle in this case.

- A somewhat oversimplified way to state the argument principle is that every zero of $f$ inside $\Gamma$ twists $f(\Gamma)$ to wind once anti-clockwise around the origin, and every pole twists $f(\Gamma)$ the other way.

- It's natural to ask whether one can compute the number of zeroes and poles directly, rather than just computing the difference between them.

In practical terms this is impossible. To illustrate this, let $\Gamma$ be the unit circle $|z| = 1$ and let $f(z)$ and $g(z)$ be the two functions

$$f(z) = 1, \quad g(z) = \frac{z + 0.0001}{z}.$$

The two functions $f$ and $g$ are very different as far as zeroes and poles are concerned: $f$ has no zeroes or poles whatsoever, while $g$ has a simple zero at $-0.0001$ and a simple pole at 0. However, if we are on $\Gamma$ then $f$ and $g$ are almost identical ($f$ is of course equal to 1, and $g$ only varies from 1 by at most 0.0001). So just by looking at the values on $\Gamma$ it is very hard to distinguish $f$ and $g$, and hence to work out how many poles or zeroes $f$ or $g$ has. (The problem is that the zero and the pole are almost canceling each other).

- Similarly, we always have to count zeroes with multiplicity. The functions

  $$f(z) = z^2, \quad g(z) = z(z + 0.0001)$$

  are almost indistinguishable on the unit circle $|z| = 1$, but $f$ has a single zero (not counting multiplicity) inside this circle and $g$ has two zeroes.

- There is only a very weak analogue of the Argument principle for real analytic functions: knowing what a function $y = f(x)$ does at the endpoints $a, b$ of an interval $[a, b]$ only tells you whether the number of zeroes of $f$ (counting multiplicity) inside the interval $[a, b]$ is even or odd, but doesn't tell you how many zeroes one has. (if $f(a)$ and $f(b)$ are opposite sign, then you know that there are an odd number of zeroes, and if they are the same sign, we have an even number of zeroes.)

Rouche's theorem

- Let's try to apply the Argument principle concretely. Namely, let's try to find how many zeroes (counting multiplicity) of the polynomial $f(z) = z^5 + z + 1$ lie inside the circle $|z| = 2$. (This quintic polynomial turns out to be unsolvable by radicals, which means that you can't write any of its roots in terms of the arithmetic operations $+$, $*$, $-$, $/$, and square roots, cube roots, etc.).

- We can parameterize this circle once anti-clockwise as $z = 2e^{i\theta}$. Since $f$ has no poles, the argument principle tells as that the number of zeroes inside this circle is equal to the number of times the curve $\gamma_1$ given by

$$\gamma_1(\theta) = 32e^{5i\theta} + 2e^{i\theta} + 1, \quad 0 \leq \theta \leq 2\pi$$

  winds anti-clockwise around the origin.

- Let's first look at the related curve $\gamma_2$ given by

$$\gamma_2(\theta) = 32e^{5i\theta}, \quad 0 \leq \theta \leq 2\pi.$$

  This curve traverses the large circle $|z| = 32$ five times anti-clockwise, and so winds around the origin five times.

- The original curve $\gamma_1$ is fairly close to $\gamma_2$; in fact, for any $\theta$, we have

$$|\gamma_2(\theta) - \gamma_1(\theta)| \leq 3.$$

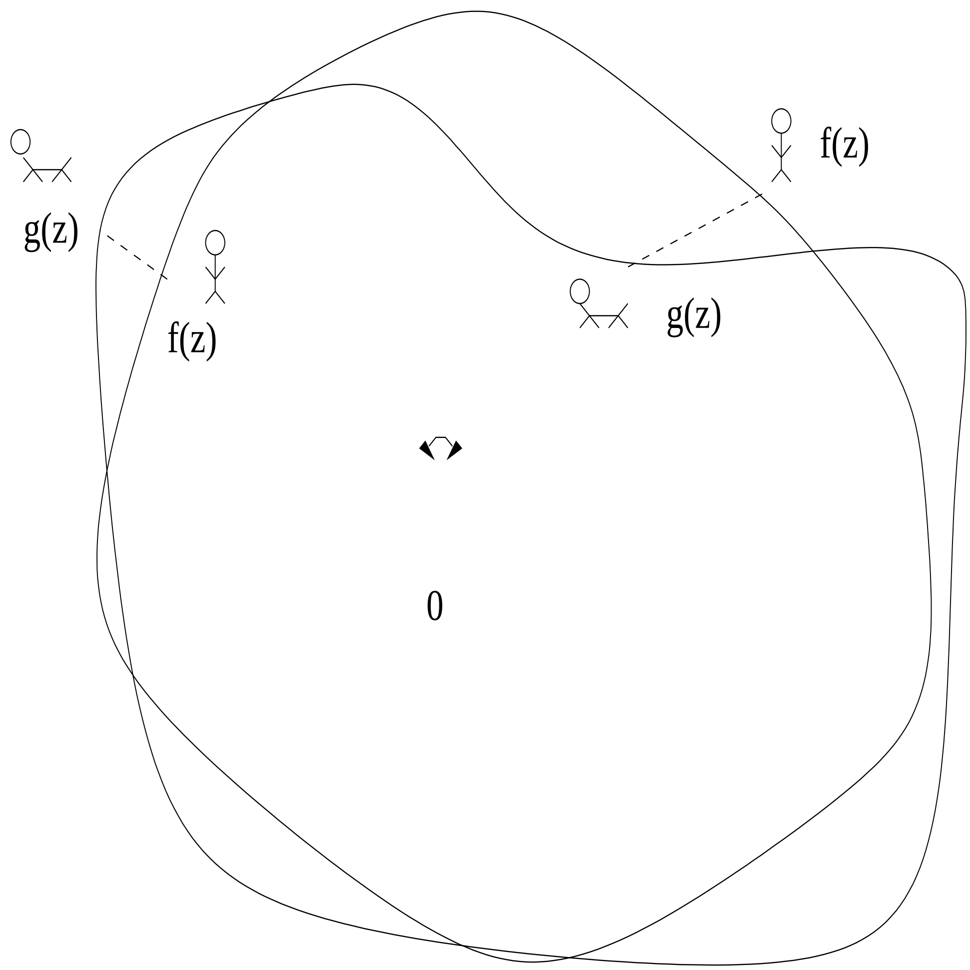
- 3 is much smaller than 32. Intuitively, since $\gamma_2$ goes around the origin five times and $\gamma_1$ stays very close to $\gamma_2$, we expect $\gamma_1$ to also wind around the origin five times, which would imply that $z^5 + z + 1$ has five zeroes inside $|z| = 2$. And this turns out to be correct.

- Another way of saying this is that $z^5$ and $z^5 + z + 1$ have the same number of zeroes inside the circle $|z| = 2$, because the error $z + 1$ between $z^5$ and $z^5 + z + 1$ is always much smaller in magnitude than the function $z^5$ on the circle $|z| = 2$.

- More generally, if $\gamma_1(\theta), a \leq \theta \leq b$ and $\gamma_2, a \leq \theta \leq b$ are two closed contours such that

$$|\gamma_2(\theta) - \gamma_1(\theta)| < |\gamma_1(\theta)|$$

  for all $a \leq \theta \leq b$, we expect $\gamma_2$ to wind around the origin the same number of times that $\gamma_1$ does. This is also true, and is sometimes known as the "walking the dog" theorem, as the picture demonstrates:




- A more rigorous version of the above discussion is
- **Rouche's theorem** Let $\Gamma$ be a simple closed anti-clockwise contour, and let $f$ and $g$ be functions which are analytic on and inside $\Gamma$. If we have
$$|f(z) - g(z)| < |g(z)|$$
for all $z \in \Gamma$, then $f$ and $g$ have the same number of zeroes inside $\Gamma$.
- We shall prove this by letting the leash out a little bit at a time.
- Write $h(z) = f(z) - g(z)$, so that $f(z) = g(z) + h(z)$. $h(z)$ represents the "leash"; note that $|h(z)| < |g(z)|$ for all $z$ on $\Gamma$.

- For every $0 \leq \alpha \leq 1$, consider the expression

$$I(\alpha) = \frac{1}{2\pi i} \int_\Gamma \frac{g'(z) + \alpha h'(z)}{f(z) + \alpha h(z)}\, dz.$$

  Since $|h(z)| < |g(z)|$ and $0 \leq \alpha \leq 1$, we see that $g(z) + \alpha h(z)$ is never zero on $\Gamma$, and so this integral is well-defined. A similar argument shows that $I(\alpha)$ is continuous and has no singularities for $0 \leq \alpha \leq 1$. By the Argument principle, this expression is equal to the number of zeroes of $g(z)+\alpha h(z)$ inside $\Gamma$. In particular, $I(\alpha)$ is always an integer.

- Since $I(\alpha)$ is always an integer, and $I(\alpha)$ is continuous in $\alpha$, this means that $I(0) = I(1)$ (if $I(0) \neq I(1)$, then the intermediate value theorem forces $I(\alpha)$ to take non-integer values for some $\alpha$ between 0 and 1). Thus $g(z)$ and $g(z) + h(z)$ have the same number of zeroes inside $\Gamma$. Since $g(z) + h(z) = f(z)$, we're done.

□

- In practice, Rouche's theorem allows us to work out the number of zeroes that a complicated function $f$ has inside a contour $\Gamma$, providing that we can accurately approximate $f(z)$ by a much simpler function $g(z)$ with known zeroes. Generally, to find $g$ we pick out the terms in $f$ which are "biggest" on $\Gamma$.

- For instance, suppose we want to find out how many zeroes of $f(z) = z^4 - 8z + 10$ are inside the circle $|z| = 1$. On the circle $|z| = 1$, the term $z^4$ has magnitude 1, the term $8z$ has magnitude 8, and 10 has magnitude 10. Thus the 10 term dominates all the others, and we set $g(z) = 10$. We check that

$$|f(z) - g(z)| = |z^4 - 8z| \leq 1 + 8 < 10 = |g(z)|$$

  for all $|z| = 1$, so by Rouche's theorem the number of zeroes of $f$ inside $|z| = 1$ is equal to that of $g$. But since $g(z) = 10$, $g$ clearly has no zeroes anywhere. Thus $f$ also has no zeroes in the disk $|z| \leq 1$.

- Sometimes there is no clear way to apply Rouche's theorem. For instance, it is difficult to work out how many zeroes of $f(z) = z^4 + z + 1$ lie inside the unit circle $|z| = 1$, because there is no clear candidate for $g(z)$. (No single term $z^4$, $z$, 1, or combination of terms such as $z^4 + z$ has the ability to always dominate the remainder). In such cases sometimes the only recourse is to use the Argument principle directly and trace out the shape of $f(\Gamma)$ accurately enough to determine the winding number of $f(\Gamma)$ around the origin. (In this case $f(\Gamma)$ turns out to wind twice around the origin, so we have two zeroes).

- Rouche's theorem can only be directly applied to count zeroes in a bounded domain. We'll see how to count zeroes in an unbounded domain (e.g. figure out how many zeroes lie in the first quadrant) later on.

The fundamental theorem of Algebra revisited.

- We can now give a different proof of

- **Fundamental theorem of Algebra.** Every polynomial of degree $n$ has exactly $n$ roots (counting multiplicity).

- **Proof** Let's write our polynomial as

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

  where $a_n \neq 0$. To count the number of zeroes, we first count the number of zeroes inside a large circle $|z| = R$ and then let $R$ go off to infinity.

- We approximate $f(z)$ by $g(z) = a_n z^n$, and claim that

$$|f(z) - g(z)| < |g(z)|$$

  for all $|z| = R$, if $R$ is big enough.

- To see this, first note that $|g(z)| = |a_n| R^n$. Also,

$$|f(z) - g(z)| = |a_{n-1} z^{n-1} + \ldots + a_0|$$

$$\leq |a_{n-1}| R^{n-1} + \ldots + |a_0|.$$

  Since

$$\frac{|a_{n-1}| R^{n-1} + \ldots + |a_0|}{|a_n| R^n} \to 0 \text{ as } R \to \infty$$

we see that we must eventually have

$$|a_{n-1}|R^{n-1} + \ldots + |a_0| < |a_n|R^n$$

if $R$ is big enough, which proves the claim.

- This means that $f$ and $g$ have the same number of zeroes inside $|z| = R$. But $g$ clearly has $n$ zeroes at $z = 0$ and no other zeroes. So $f(z)$ has $n$ zeroes inside the circle $|z| = R$ when $R$ is large enough. Letting $R \to \infty$ we get the result.

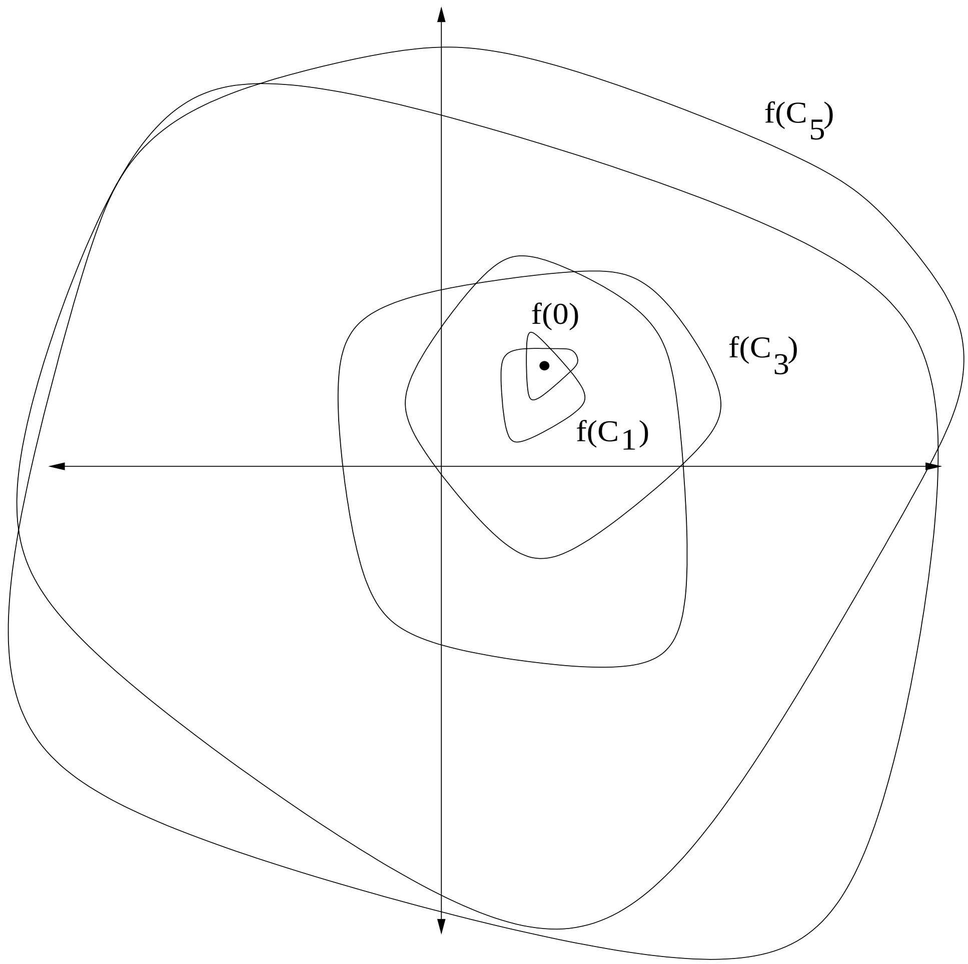
□

- Another way of thinking about this result is as follows. For the sake of argument, let's suppose that $f$ has a constant term, so that $f(0) \neq 0$. (To handle the case $f(z) = 0$, we could just factor out powers of $z$ until $f$ did have a constant term).

- Let $C_R$ be the circle $|z| = R$. When $R$ is really large, $f(z)$ is very close to $g(z)$, so $f(C_R)$ is very close to $g(C_R)$, which is a large circle (of radius $|a_n|R^n$) traversed $n$ times anti-clockwise. So $f(C_R)$ is initially wound around the origin $n$ times.

- Now let's let $R$ shrink all the way to zero. This contracts $C_R$ all the way to a single point (the origin 0), so $f(C_R)$ must also contract to a single point $f(0)$. Since $f(0) \neq 0$, this means that $f(C_R)$ must eventually be totally unwound from the origin as $R \to 0$.

- The only way this can happen is if $f(C_R)$ passes through the origin at least $n$ times before $R$ goes to zero, since you need the curve to pass through the origin once to reduce the winding number by one.




- This shows that $f$ has at least $n$ zeroes. $f$ can't have more than $n$ zeroes, since it is a polynomial of degree $n$ and so can only be factored into at most $n$ factors. Thus $f$ has exactly $n$ roots.