Math 131BH - Week 3

Textbook pages covered: 143-152

- Sequences of functions
- Pointwise convergence versus uniform convergence
- Uniform convergence and continuity
- Series of functions; Weierstrass M-test
- Uniform convergence and integrals

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Sequences of functions

- In the last two weeks we have seen what it means for a sequence $(x^{(n)})_{n=1}^\infty$ of points in a metric space (X, d_X) to converge to a limit x; it means that $\lim_{n\to\infty} d_X(x^{(n)}, x) = 0$, or equivalently that for every $\varepsilon > 0$ there exists an $N > 0$ such that $d_X(x^{(n)}, x) < \varepsilon$ for all $n > N$.
- Now, we consider what it means for a sequence of *functions* $(f^{(n)})_{n=1}^\infty$ from one metric space (X, d_X) to another (Y, d_Y) to converge. In other words, we have a sequence of functions $f^{(1)}, f^{(2)}, \ldots$, with each function $f^{(n)} : X \to Y$ being a function from X to Y, and we ask what it means for this sequence of functions to converge to some limiting function $f : X \to Y$.
- It turns out that there are several different concepts of convergence of functions; here we describe the two most important ones, *pointwise convergence* and *uniform convergence*. (There are other types of convergence for functions, such as L^1 convergence, L^2 convergence, convergence in measure, almost everywhere convergence, and so forth, but these are beyond the scope of this course). The two notions are related, but not identical; the relationship between the two is somewhat analogous to the relationship between continuity and uniform continuity.

- Once we work out what convergence means for functions, and thus can make sense of such statements as $\lim_{n\to\infty} f^{(n)} = f$, we will then ask how these limits interact with other concepts. For instance, we already have a notion of limiting values of functions: $\lim_{x\to x_0;x\in X} f(x)$. Can we interchange limits, i.e.

$$\lim_{n\to\infty} \lim_{x\to x_0;x\in X} f^{(n)}(x) = \lim_{x\to x_0;x\in X} \lim_{n\to\infty} f^{(n)}(x)?$$

  As we shall see, the answer depends on what type of convergence we have for $f^{(n)}$. We will also address similar questions involving interchanging limits and integrals, or limits and sums, or sums and integrals.

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Digression: limiting values of functions

- Before we talk about limits of sequences of functions, we should first discuss a similar, but distinct, notion, that of limiting values of functions.

- **Definition** Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, and let $f : X \to Y$ be a function. If $x_0 \in X$ is an adherent point of E, and $L \in Y$, we say that $f(x)$ *converges to* L *in* Y *as* x *converges to* x_0 *in* E, or write $\lim_{x\to x_0;x\in E} f(x) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), L) < \varepsilon$ for all $x \in E$ such that $d_X(x, x_0) < \delta$.

- Comparing this with our definition of continuity from week 2, we see that f is continuous at x_0 if and only if

$$\lim_{x\to x_0;x\in X} f(x) = f(x_0).$$

  Thus f is continuous on X if we have

$$\lim_{x\to x_0;x\in X} f(x) = f(x_0) \text{ for all } x_0 \in X.$$

- Often we shall omit the condition $x \in X$, and abbreviate $\lim_{x\to x_0;x\in X} f(x)$ as simply $\lim_{x\to x_0} f(x)$ when it is clear what space x will range in.

- One can rephrase this definition in terms of sequences:

- **Proposition 1.** Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, and let $f : X \to Y$ be a function. Let $x_0 \in X$ be an adherent point of E and $L \in Y$. Then the following two statements are equivalent.

- (a) $\lim_{x \to x_0; x \in E} f(x) = L$.

- (b) For every sequence $(x^{(n)})_{n=1}^{\infty}$ in E which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to L with respect to the metric d_Y.

- **Proof.** See Week 3 homework. □

- Observe from Proposition 1(b), and from Proposition 4 from Week 1 notes, that a function $f(x)$ can converge to at most one limit L as x converges to x_0. In other words, if the limit $\lim_{x \to x_0; x \in E} f(x)$ exists at all, then it can only take at most one value.

- The requirement that x_0 be an adherent point of E is necessary for the concept of limiting value to be useful, otherwise x_0 will lie in the exterior of E, the notion that $f(x)$ converges to L as x converges to x_0 in E is vacuous (for δ sufficiently small, there are no points $x \in E$ so that $d(x, x_0) < \delta$).

- **Remark.** Strictly speaking, we should write $d_Y - \lim_{x \to x_0; x \in E} f(x)$ instead of $\lim_{x \to x_0; x \in E} f(x)$, since the convergence depends on the metric d_Y. However in practice it will be obvious what the metric d_Y is and so we will omit the d_Y- prefix from the notation.

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Pointwise convergence and uniform convergence

- Note: in the examples in this section, we give $\mathbf{R}$ (and any subsets of $\mathbf{R}$, such as $[0, 1]$) the standard metric $d(x, y) := |x - y|$ unless otherwise specified.

- The most obvious notion of convergence of functions is *pointwise convergence*, or convergence at each point of the domain:

- **Definition** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^{\infty}$ *converges pointwise to* f *on* X if we have

$$d_Y - \lim_{n\to\infty} f^{(n)}(x) = f(x)$$

for all $x \in X$, i.e.

$$\lim_{n\to\infty} d_Y(f^{(n)}(x), f(x)) = 0.$$

Or in other words, for every x and every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y(f^{(n)}(x), f(x)) < \varepsilon$ for every $n > N$. We call the function f the *pointwise limit* of the functions $f^{(n)}$.

- Note that $f^{(n)}(x)$ and $f(x)$ are points in Y, rather than functions, so we are using our prior notion of convergence in metric spaces to determine convergence of functions. Also note that we are not really using the fact that (X, d_X) is a metric space (i.e. we are not using the metric d_X); for this definition it would suffice for X to just be a plain old set with no metric structure. However, later on we shall want to restrict our attention to *continuous* functions from X to Y, and in order to do so we need a metric on X (and on Y).

- **Example** Consider the functions $f^{(n)} : \mathbf{R} \to \mathbf{R}$ defined by $f^{(n)}(x) := x/n$, while $f : \mathbf{R} \to \mathbf{R}$ is the zero function $f(x) := 0$. Then $f^{(n)}$ converges pointwise to f, since for each fixed real number x we have $\lim_{n\to\infty} f^{(n)}(x) = \lim_{n\to\infty} x/n = 0 = f(x)$.

- From Proposition 4 of Week 1 notes we see that a sequence $(f^{(n)})_{n=1}^{\infty}$ of functions from one metric space (X, d_X) to another (Y, d_Y) can have at most one pointwise limit f (this explains why we can refer to f as *the* pointwise limit). However, it is of course possible for a sequence of functions to have no pointwise limit (can you think of an example?), just as a sequence of points in a metric space do not necessarily have a limit.

- Pointwise convergence is a very natural concept, but it has a number of disadvantages:

- **Pointwise convergence does not preserve continuity**. In other words, the pointwise limit of continuous functions is not necessarily continuous. For instance, consider the functions $f^{(n)} : [0,1] \to \mathbf{R}$ defined by $f^{(n)}(x) := x^n$, and let $f : [0,1] \to \mathbf{R}$ be the function defined by setting $f(x) := 1$ when $x = 1$ and $f(x) := 0$ when $0 \leq x < 1$. Then the functions $f^{(n)}$ are continuous, and converge pointwise to f on $[0,1]$ (why? treat the cases $x = 1$ and $0 \leq x < 1$ separately), however the limiting function f is not continuous.

  Note that the same example shows that pointwise convergence does not preserve differentiability either.

- **Pointwise convergence does not preserve limits**. This is a very similar problem to the previous one: if $\lim_{x \to x_0; x \in E} f^{(n)}(x) = L$ for every n, and $f^{(n)}$ converges pointwise to f, this does not mean that $\lim_{x \to x_0; x \in E} f(x) = L$. The counterexample given earlier is also a counterexample here: observe that $\lim_{x \to 1; x \in [0,1)} x^n = 1$ for every n, but x^n converges pointwise to the function f defined in the previous paragraph, and $\lim_{x \to 1; x \in [0,1)} f(x) = 0$. In particular, we see that

  $$\lim_{n \to \infty} \lim_{x \to x_0; x \in X} f^{(n)}(x) \neq \lim_{x \to x_0; x \in X} \lim_{n \to \infty} f^{(n)}(x).$$

- **Pointwise convergence does not preserve integrals**. Suppose we have a sequence of Riemann-integrable functions $f^{(n)} : [a,b] \to \mathbf{R}$ on the interval $[a,b]$. If $\int_{[a,b]} f^{(n)} = L$ for every n, and $f^{(n)}$ converges pointwise to some new function f, this does not mean that $\int_{[a,b]} f = L$. An example comes by setting $[a,b] := [0,1]$, and letting $f^{(n)}$ be the function $f^{(n)}(x) := 2n$ when $x \in [1/2n, 1/n]$, and $f^{(n)}(x) := 0$ for all other values of x. Then $f^{(n)}$ converges pointwise to the zero function $f(x) := 0$ (why?). On the other hand, $\int_{[0,1]} f^{(n)} = 1$ for every n, while $\int_{[0,1]} f = 0$. In particular, we have an example where

  $$\lim_{n \to \infty} \int_{[a,b]} f^{(n)} \neq \int_{[a,b]} \lim_{n \to \infty} f^{(n)}.$$

  One may think that this counterexample has something to do with the $f^{(n)}$ being discontinuous, but one can easily modify this counterexample to make the $f^{(n)}$ continuous (can you see how?).

- Another example in the same spirit is the "moving bump" example. Let $f^{(n)} : \mathbf{R} \to \mathbf{R}$ be the function defined by $f^{(n)}(x) := 1$ if $x \in [n, n+1]$ and $f^{(n)}(x) := 0$ otherwise. Then $\int_{\mathbf{R}} f^{(n)} = 1$ for every n (where $\int_{\mathbf{R}} f$ is defined as the limit of $\int_{[-N,N]} f$ as N goes to infinity). On the other hand, $f^{(n)}$ converges pointwise to the zero function 0 (why?), and $\int_{\mathbf{R}} 0 = 0$. In both of these examples, functions of area 1 have somehow "disappeared" to produce functions of area 0 in the limit.

- These examples show that pointwise convergence is too weak a concept to be of much use. The problem is that while $f^{(n)}(x)$ is converging to $f(x)$ for each x, the *rate* of that convergence varies substantially with x. For instance, consider the first example where $f^{(n)} : [0,1] \to \mathbf{R}$ was the function $f^{(n)}(x) := x^n$, and $f : [0,1] \to \mathbf{R}$ was the function such that $f(x) := 1$ when $x = 1$, and $f(x) := 0$ otherwise. Then for each x, $f^{(n)}(x)$ converges to $f(x)$ as $n \to \infty$; this is the same as saying that $\lim_{n\to\infty} x^n = 0$ when $0 \leq x < 1$, and that $\lim_{n\to\infty} x^n = 1$ when $x = 1$. But the convergence is much slower near 1 than far away from 1. For instance, consider the statement that $\lim_{n\to\infty} x^n = 0$ for all $0 \leq x < 1$. This means, for every $0 \leq x < 1$, that for every ε, there exists an $N \geq 1$ such that $|x^n| < \varepsilon$ for all $n \geq N$ - or in other words, the sequence $1, x, x^2, x^3, \ldots$ will eventually get less than ε, after passing some finite number N of elements in this sequence. But the number of elements N one needs to go out to depends very much on the location of x. For instance, take $\varepsilon := 0.1$. If $x = 0.1$, then we have $|x^n| < \varepsilon$ for all $n \geq 2$ - the sequence gets underneath ε after the second element. But if $x = 0.5$, then we only get $|x^n| < \varepsilon$ for $n \geq 4$ - you have to wait until the fourth element until you get within ε of the limit. And if $x = 0.9$, then one only has $|x^n| < \varepsilon$ when $n \geq 22$. Clearly, the closer x gets to 1, the longer one has to wait until $f^{(n)}(x)$ will get within ε of $f(x)$, although it still will get there eventually. (Curiously, however, while the convergence gets worse and worse as x approaches 1, the convergence suddenly becomes perfect when $x = 1$).

- To put things another way, the convergence of $f^{(n)}$ to f is not *uniform* in x - the N that one needs to get $f^{(n)}(x)$ within ε of f depends on x as well as on ε. This motivates a stronger notion of convergence.

- **Definition** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^\infty$ *converges uniformly to f on X* if for every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y(f^{(n)}(x), f(x)) < \varepsilon$ for every $n > N$ and $x \in X$. We call the function f the *uniform limit* of the functions $f^{(n)}$.

- Note that this definition is subtly different from the definition for pointwise convergence. In the definition of pointwise convergence, N was allowed to depend on x; now it is not.

- It is easy to see that if $f^{(n)}$ converges uniformly to f on X, then it also converges pointwise to the same function f (see homework). However, the converse is not true; for instance the functions $f^{(n)} : [0,1] \to \mathbf{R}$ defined earlier by $f^{(n)}(x) := x^n$ converge pointwise, but do not converge uniformly (see homework).

- **Example.** Let $f^{(n)} : [0,1] \to \mathbf{R}$ be the functions $f^{(n)}(x) := x/n$, and let $f : [0,1] \to \mathbf{R}$ be the zero function $f(x) := 0$. Then it is clear that $f^{(n)}$ converges to f pointwise. Now we show that in fact $f^{(n)}$ converges to f uniformly. We have to show that for every $\varepsilon > 0$, there exists an N such that $|f^{(n)}(x) - f(x)| < \varepsilon$ for every $x \in [0,1]$ and every $n \geq N$. To show this, let us fix an $\varepsilon > 0$. Then for any $x \in [0,1]$ and $n \geq N$, we have

$$|f^{(n)}(x) - f(x)| = |x/n - 0| = x/n \leq 1/n \leq 1/N.$$

  Thus if we choose N such that $N > 1/\varepsilon$ (note that this choice of N does not depend on what x is), then we have $|f^{(n)}(x) - f(x)| < \varepsilon$ for all $n \geq N$ and $x \in [0,1]$, as desired.

- We make one trivial remark here: if a sequence $f^{(n)} : X \to Y$ of functions converges pointwise (or uniformly) to a function $f : X \to Y$, then the restrictions $f^{(n)}|_E : E \to Y$ of $f^{(n)}$ to some subset E of X will also converge pointwise (or uniformly) to $f|_Y$. (Why?)

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Uniform convergence and continuity

- We now give the first demonstration that uniform convergence is significantly better than pointwise convergence. Specifically, we show that the uniform limit of continuous functions is continuous.

- **Theorem 2.** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. Let x_0 be a point in X. If the functions $f^{(n)}$ are continuous at x_0 for each n, then the limiting function f is also continuous at x_0.

- **Proof.** See Week 3 homework. $\square$

- This has an immediate corollary:

- **Corollary 3.** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. If the functions $f^{(n)}$ are continuous on X for each n, then the limiting function f is also continuous on X.

- There is a slight variant of Theorem 2 which is also useful:

- **Proposition 4.** Let (X, d_X) and (Y, d_Y) be metric spaces, with Y complete, and let E be a subset of X. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from E to Y, and suppose that this sequence converges uniformly in E to some function $f : E \to Y$. Let $x_0 \in X$ be an adherent point of E, and suppose that for each n the limit $\lim_{x \to x_0; x \in E} f^{(n)}(x)$ exists. Then the limit $\lim_{x \to x_0; x \in E} f(x)$ also exists, and is equal the limit of the sequence $(\lim_{x \to x_0; x \in E} f^{(n)}(x))_{n=1}^{\infty}$; in other words

$$\lim_{n \to \infty} \lim_{x \to x_0; x \in E} f^{(n)}(x) = \lim_{x \to x_0; x \in E} f(x).$$

- **Proof.** See Week 3 homework. $\square$

- Finally, we have a version of these theorems for sequences: if $x^{(n)}$ converges to x, and $f^{(n)}$ is a sequence of continuous functions which converges *uniformly* to f, then $f^{(n)}(x^{(n)})$ converges to $f(x)$.

- **Proposition 5.** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of continuous functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. Let $x^{(n)}$ be a sequence of points in X which converge to some limit x. Then $f^{(n)}(x^{(n)})$ converges (in Y) to $f(x)$.

- **Proof.** See Week 3 homework. □

- The above proposition sounds very reasonable, but one should caution that it only works if one assumes uniform convergence; pointwise convergence is not enough. (See homework).

- Uniform limits of bounded functions are also bounded. Recall that a function $f : X \to Y$ from one metric space (X, d_X) to another (Y, d_Y) is *bounded* if there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$.

- **Proposition 6.** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. If the functions $f^{(n)}$ are bounded on X for each n, then the limiting function f is also bounded on X.

- **Proof.** See Week 3 homework. □

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The metric of uniform convergence

- We have now developed at least four, apparently separate, notions of limit in this course: (a) limits $\lim_{n\to\infty} x^{(n)}$ of sequences of points in a metric space; (b) limiting values $\lim_{x\to x_0; x\in E} f(x)$ of functions at a point; (c) pointwise limits f of functions $f^{(n)}$; and (d) uniform limits f of functions $f^{(n)}$.

- This proliferation of limits may seem rather complicated. However, we can reduce the complexity slightly by noting that (d) is in fact a special case of (a). The catch is that because we are now dealing with functions instead of points, the convergence is not in X or in Y, but rather in a new space, the space of functions from X to Y.

- **Definition.** Let (X, d_X) and (Y, d_Y) be metric spaces. We let $B(X;Y)$ denote the space of bounded functions from X to Y:

$$B(X;Y) := \{f \big| f : X \to Y \text{ is a bounded function}\}.$$

We define a metric $d_\infty : B(X;Y) \times B(X;Y) \to \mathbf{R}^+$ by defining

$$d_\infty(f,g) := \sup_{x \in X} d_Y(f(x), g(x)) = \sup\{d_Y(f(x), g(x)) : x \in X\}$$

for all $f, g \in B(X;Y)$. This metric is sometimes known as the *sup norm metric* or the L^∞ *metric*. We will also use $d_{B(X;Y)}$ as a synonym for d_∞.

- Notice that the distance $d_\infty(f,g)$ is always finite because f and g are assumed to be bounded on X.

- **Example.** Let $X := [0,1]$ and $Y = \mathbf{R}$. Let $f : [0,1] \to \mathbf{R}$ and $g : [0,1] \to \mathbf{R}$ be the functions $f(x) := 2x$ and $g(x) := 3x$. Then f and g are both bounded functions and thus live in $B([0,1];\mathbf{R})$. The distance between them is

$$d_\infty(f,g) = \sup_{x \in [0,1]} |2x - 3x| = \sup_{x \in [0,1]} |x| = 1.$$

- This space turns out to be a metric space (see Homework). Convergence in this metric turns out to be identical to uniform convergence:

- **Proposition 7.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f^{(n)})_{n=1}^\infty$ be a sequence of functions in $B(X;Y)$, and let f be another function in $B(X;Y)$. Then $(f^{(n)})_{n=1}^\infty$ converges to f in the metric $d_{B(X;Y)}$ if and only if $(f^{(n)})_{n=1}^\infty$ converges uniformly to f.

- **Proof.** See Week 3 homework. □

- Now let $C(X;Y)$ be the space of bounded continuous functions from X to Y:

$$C(X;Y) := \{f \in B(X;Y) : f \text{ is continuous}\}.$$

- This set $C(X;Y)$ is clearly a subset of $B(X;Y)$. Corollary 3 asserts that this space $C(X;Y)$ is closed in $B(X;Y)$ (why?). Actually, we can say a lot more:

- **Theorem 8.** Let (X, d_X) be a metric space, and let (Y, d_Y) be a *complete* metric space. The space $(C(X;Y), d_{B(X;Y)}|_{C(X;Y)\times C(X;Y)})$ is a complete subspace of $(B(X;Y), d_{B(X;Y)})$. In other words, every Cauchy sequence of functions in $C(X;Y)$ converges to a function in $C(X;Y)$.

- **Proof.** See Week 3 homework. □

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Series of functions; the Weierstrass M-test

- Having discussed sequences of functions, we now discuss infinite series $\sum_{n=1}^\infty f_n$ of functions. Now we shall restrict our attention to functions $f : X \to \mathbf{R}$ from a metric space (X, d_X) to the real line $\mathbf{R}$ (which we of course give the standard metric); this is because we know how to add two real numbers, but don't necessarily know how to add two points in a general metric space Y. Functions whose range is $\mathbf{R}$ are sometimes called *real-valued* functions.

- Finite summation is, of course, easy: given any finite collection $f^{(1)}, \ldots, f^{(N)}$ of functions from X to $\mathbf{R}$, we can define the finite sum $\sum_{i=1}^N f^{(i)} : X \to \mathbf{R}$ by

$$(\sum_{i=1}^N f^{(i)})(x) := \sum_{i=1}^N f^{(i)}(x).$$

- **Example.** If $f^{(1)} : \mathbf{R} \to \mathbf{R}$ is the function $f^{(1)}(x) := x$, $f^{(2)} : \mathbf{R} \to \mathbf{R}$ is the function $f^{(2)}(x) := x^2$, and $f^{(3)} : \mathbf{R} \to \mathbf{R}$ is the function $f^{(3)}(x) := x^3$, then $f := \sum_{i=1}^3 f^{(i)}$ is the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x + x^2 + x^3$.

- It is clear that finite sums of bounded functions are bounded. It is also easy to show that finite sums of continuous functions are continuous (see homework).

- Now to add infinite series.

- **Definition** Let (X, d_X) be a metric space. Let $(f^{(n)})_{n=1}^\infty$ be a sequence of functions from X to $\mathbf{R}$, and let f be another function from X to $\mathbf{R}$. If the partial sums $\sum_{n=1}^N f^{(n)}$ converge pointwise to f on X as $N \to \infty$, we say that the infinite series $\sum_{n=1}^\infty f^{(n)}$ *converges pointwise* to f, and write $f = \sum_{n=1}^\infty f^{(n)}$. If the partial sums $\sum_{n=1}^N f^{(n)}$ converge uniformly to f on X as $N \to \infty$, we say that the infinite series $\sum_{n=1}^\infty f^{(n)}$ *converges uniformly* to f, and again write $f = \sum_{n=1}^\infty f^{(n)}$. (Thus when one sees an expression such as $f = \sum_{n=1}^\infty f^{(n)}$, one should look at the context to see in what sense this infinite series converges).

- Note that a series $\sum_{n=1}^\infty f^{(n)}$ converges pointwise to f on X if and only if $\sum_{n=1}^\infty f^{(n)}(x)$ converges to $f(x)$ for *every* $x \in X$. (Thus if $\sum_{n=1}^\infty f^{(n)}$ does not converge pointwise to f, this does not mean that it diverges pointwise; it may just be that it converges for some points x but diverges at other points x.)

- If a series $\sum_{n=1}^\infty f^{(n)}$ converges uniformly to f, then it also converges pointwise to f; but not vice versa, as the following example shows.

- **Example - geometric series formula.** Let $f^{(n)} : (-1, 1) \to \mathbf{R}$ be the sequence of functions $f^{(n)}(x) := x^n$. Then $\sum_{n=1}^\infty f^{(n)}$ converges pointwise, but not uniformly, to the function $x/(1-x)$ (see Homework).

- It is not always clear when a series $\sum_{n=1}^\infty f^{(n)}$ converges or not. However, there is a very useful test that gives at least one test for uniform convergence.

- **Definition** If $f : X \to \mathbf{R}$ is a bounded real-valued function, we define the *sup norm* $\|f\|_\infty$ of f to be the number

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}.$$

- Thus, for instance, if $f : (-2, 1) \to \mathbf{R}$ is the function $f(x) := 2x$, then $\|f\|_\infty = \sup\{|2x| : x \in (-2, 1)\} = 4$ (why?). Notice that when f is bounded then $\|f\|_\infty$ will always be a non-negative real number.

- **Weierstrass M-test** Let (X, d) be a metric space, and let $(f^{(n)})_{n=1}^\infty$ be a sequence of bounded real-valued continuous functions on X such that the series $\sum_{n=1}^\infty \|f^{(n)}\|_\infty$ is absolutely convergent. (Note that this is a series of plain old numbers, not of functions). Then the series $\sum_{n=1}^\infty f^{(n)}$ converges uniformly to some function f on X, and that function f is also continuous.

- **Proof.** See Week 3 homework. □

- To put the Weierstrass M-test succinctly: absolute convergence of sup norms implies uniform convergence of functions.

- **Example - geometric series revisited.** Let $0 < r < 1$ be a real number, and let $f^{(n)} : [-r, r] \to \mathbf{R}$ be the series of functions $f^{(n)}(x) := x^n$. Then each $f^{(n)}$ is continuous and bounded, and $\|f^{(n)}\|_\infty = r^n$ (why?). Since the series $\sum_{n=1}^\infty r^n$ is absolutely convergent (e.g. by the ratio test), we thus see that $f^{(n)}$ converges uniformly in $[-r, r]$ to some continuous function; in the homework we see that this function must in fact be the function $f : [-r, r] \to \mathbf{R}$ defined by $f(x) := x/(1-x)$. In other words, the series $\sum_{n=1}^\infty x^n$ is pointwise convergent, but not uniformly convergent, on $(-1, 1)$, but is uniformly convergent on the smaller interval $[-r, r]$ for any $0 < r < 1$.

- The Weierstrass M-test is especially useful in relation to *power series*, which we will encounter in Week 5.

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Uniform convergence and integration

- We now connect uniform convergence with Riemann integration.

- **Theorem 9.** Let $[a, b]$ be an interval, and for each integer $n \geq 1$, let $f^{(n)} : [a, b] \to \mathbf{R}$ be a Riemann-integrable function. Suppose $f^{(n)}$ converges uniformly on $[a, b]$ to a function $f : [a, b] \to \mathbf{R}$. Then f is also Riemann integrable, and

$$\lim_{n\to\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} f.$$

- **Proof.** We shall use some facts from 131AH about upper and lower Riemann integrals; the reader may consult the Week 9 notes from my 131AH class for these facts.

- We first show that f is Riemann integrable on $[a,b]$. This is the same as showing that the upper and lower Riemann integrals of f match: $\underline{\int}_{[a,b]} f = \overline{\int}_{[a,b]} f$.

- Let $\varepsilon > 0$. Since $f^{(n)}$ converges uniformly to f, we see that there exists an $N > 0$ such that $|f^{(n)}(x) - f(x)| < \varepsilon$ for all $n > N$ and $x \in [a,b]$. In particular we have

$$f^{(n)}(x) - \varepsilon < f(x) < f^{(n)}(x) + \varepsilon$$

for all $x \in [a,b]$. Integrating this on $[a,b]$ we obtain

$$\underline{\int}_{[a,b]} (f^{(n)} - \varepsilon) \le \underline{\int}_{[a,b]} f \le \overline{\int}_{[a,b]} f \le \overline{\int}_{[a,b]} (f^{(n)} + \varepsilon).$$

Since $f^{(n)}$ is assumed to be Riemann integrable, we thus see

$$(\int_{[a,b]} f^{(n)}) - \varepsilon(b-a) \le \underline{\int}_{[a,b]} f \le \overline{\int}_{[a,b]} f \le (\int_{[a,b]} f^{(n)}) + \varepsilon(b-a).$$

In particular, we see that

$$0 \le \overline{\int}_{[a,b]} f - \underline{\int}_{[a,b]} f \le 2\varepsilon(b-a).$$

Since this is true for every $\varepsilon > 0$, we obtain $\underline{\int}_{[a,b]} f = \overline{\int}_{[a,b]} f$ as desired.

- The above argument also shows that for every $\varepsilon > 0$ there exists an $N > 0$ such that

$$|\int_{[a,b]} f^{(n)} - \int_{[a,b]} f| \le 2\varepsilon(b-a)$$

for all $n \ge N$. Since ε is arbitrary, this shows that $\lim_{n\to\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} f$ as desired. □

To rephrase Theorem 9: we can rearrange limits and integrals (on compact intervals $[a,b]$),

$$\lim_{n\to\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} \lim_{n\to\infty} f^{(n)}.$$

provided that the convergence is uniform. (In the beginning of these notes we saw that this statement is not necessarily true of the convergence is merely pointwise).

- There is an analogue of this Theorem for series:

- **Corollary 10** Let $[a, b]$ be an interval, and let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of Riemann integrable functions on $[a, b]$ such that the series $\sum_{n=1}^{\infty} f^{(n)}$ is uniformly convergent. Then we have

$$\sum_{n=1}^{\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} \sum_{n=1}^{\infty} f^{(n)}.$$

- **Proof.** See Week 3 homework. □

- This Corollary works particularly well in conjunction with the Weierstrass M-test.

- **Example.** We already know the identity

$$\sum_{n=1}^{\infty} x^n = x/(1-x)$$

for $x \in (-1, 1)$, and the convergence is uniform (by the Weierstrass M-test) on $[-r, r]$ for any $0 < r < 1$. By adding 1 to both sides we obtain

$$\sum_{n=0}^{\infty} x^n = 1/(1-x)$$

and the converge is again uniform. We can thus integrate on $[0, r]$ and use Corollary 10 to obtain

$$\sum_{n=0}^{\infty} \int_{[0,r]} x^n \, dx = \int_{[0,r]} \frac{1}{1-x} \, dx.$$

The left-hand side is $\sum_{n=0}^{\infty} r^{n+1}/(n+1)$. If we accept for now the use of logarithms (we will justify this use in later weeks), the anti-derivative of $1/(1-x)$ is $-\log(1-x)$, and so the right-hand side is $-\log(1-r)$. We thus obtain the formula

$$-\log(1-r) = \sum_{n=0}^{\infty} r^{n+1}/(n+1)$$

for all $0 < r < 1$.