Mathematics 131BH
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Instructions: Try to do all five problems; they are all of equal value. There is plenty of working space, and a blank page at the end. On the first page you will be supplied a list of standard definitions for easy reference.

Unless otherwise specified, you may use all the results from the class notes, textbook, or any other source; you do not need to give precise theorem numbers or page numbers (e.g. saying "by a theorem from the notes" will suffice). You are encouraged to be verbose in your proofs and explanations; a chain of equations with no explanation given may be insufficient for full credit.

You may enter in a nickname if you want your midterm score posted.

Good luck!

Name: ______________________

Nickame: ______________________

Student ID: ______________________

Signature: ______________________

Problem 1. ____________

Problem 2. ____________

Problem 3. ____________

Problem 4. ____________

Problem 5. ____________

Total: ______________________

Reference sheet

This reference page contains some definitions from the Week 4-7 notes which are relevant to the midterm questions.

- **Compactly supported functions.** Let $[a, b]$ be an interval. A function $f : \mathbf{R} \to \mathbf{R}$ is said to be *supported* on $[a, b]$ iff $f(x) = 0$ for all $x \notin [a, b]$. We say that f is *compactly supported* iff it is supported on some interval $[a, b]$. If f is continuous and supported on $[a, b]$, we define the improper integral $\int_{-\infty}^{\infty} f$ to be $\int_{-\infty}^{\infty} f := \int_{[a,b]} f$.

- **Convolution.** Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be continuous, compactly supported functions. We define the *convolution* $f * g : \mathbf{R} \to \mathbf{R}$ of f and g to be the function

$$f * g(x) := \int_{-\infty}^{\infty} f(y) g(x - y)\ dy.$$

- **Fourier series.** For any function $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{R})$, and any integer $n \in \mathbf{Z}$, we define the n^{th} *Fourier coefficient of* f, denoted $\hat{f}(n)$, by the formula

$$\hat{f}(n) := \langle f, e_n \rangle = \int_{[0,1]} f(x) e^{-2\pi i n x}\ dx.$$

  The function $\hat{f} : \mathbf{Z} \to \mathbf{C}$ is called the *Fourier transform* of f.

- **Periodic functions.** A function $f : \mathbf{R} \to \mathbf{C}$ is $\mathbf{Z}$-*periodic*, if we have $f(x + k) = f(x)$ for every integer k. The space of complex-valued continuous $\mathbf{Z}$-periodic functions is denoted $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.

Problem 1. Let $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ be a continuous, 1-periodic function. Suppose also that f is differentiable, and f' is also in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$. Show that $\widehat{f'}(n) = 2\pi i n \hat{f}(n)$ for every integer n. (Hint: use integration by parts).

By definition, we have

$$\widehat{f'}(n) = \int_{[0,1]} f'(x) e^{-2\pi i n x}\ dx.$$

Using the integration by parts formula

$$\int_{[0,1]} u'(x)v(x)\ dx = u(x)v(x)|_{x=0}^{x=1} - \int_{[0,1]} u(x)v'(x)\ dx$$

with $u(x) := f(x)$ and $v(x) := e^{-2\pi i n x}$ (note that both of these functions are continuously differentiable, and so the integration by parts formula is valid), we have

$$\int_{[0,1]} f'(x)e^{-2\pi i n x}\ dx = f(x)e^{-2\pi i n x}|_{x=0}^{x=1} - \int_{[0,1]} f(x)(-2\pi i n e^{-2\pi i n x})\ dx$$

so

$$\widehat{f'(n)} = f(1)e^{-2\pi i n} - f(0)e^0 + 2\pi i n \int_{[0,1]} f(x)e^{-2\pi i n x}\ dx.$$

But since f is 1-periodic, $f(1) = f(0)$. Also, $e^{-2\pi i n} = e^0 = 1$. Thus the first two terms cancel, and the third term is equal to $2\pi i n \hat{f}(n)$, as desired.

Remark. It is also possible to proceed via the Fourier-Plancherel theorem, but to make the argument below rigorous requires much more smoothness on f than is currently assumed (e.g. one may need f to be three times continuously differentiable). The idea is as follows. Starting with the Fourier inversion formula

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e_n(x)$$

one can differentiate both sides with respect to x. If one can interchange the derivative and integral, we will obtain

$$f'(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e_n'(x).$$

But

$$e_n'(x) = \frac{d}{dx}e^{2\pi i n x} = 2\pi i n e^{2\pi i n x} = 2\pi i n e_n$$

and so

$$f'(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) 2\pi i n e_n(x).$$

But from the Fourier inversion formula applied to f', we have

$$f'(x) = \sum_{n=-\infty}^{\infty} \widehat{f'}(n) e_n(x).$$

The claim should then follow by equating Fourier coefficients. (One can modify Corollary 5 from Week 6 notes to show that any given function can have at most one set of Fourier coefficients; if $\sum_{n=-\infty}^{\infty} c_n e_n$ and $\sum_{n=-\infty}^{\infty} d_n e_n$ both converge to the same function f, then $c_n = d_n = \hat{f}(n)$.

Unfortunately, the above proof is not rigorous because one has to justify a number of steps, in particular the interchanging of the derivative and integral. This can be done using Corollary 2 of Weeks 4/5 notes, but only if we know that $\hat{f}(n) 2\pi n$ is an absolutely convergent series. This turns out to be true for three (!) times continuously differentiable functions, but not necessarily true for others.

Problem 2. Obtain a power series for arctan : $\mathbf{R} \to (-\pi/2, \pi/2)$ centered at the origin; indicate the radius of convergence, and justify your reasoning. (You may use without proof the assertion that arctan is differentiable and that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$).

Starting with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

which converges for all $|x| < 1$, we replace x by $-x^2$ to obtain

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

which converges whenever $|-x^2| < 1$, i.e. when $|x| < 1$. The radius of convergence of this series is 1 (as can be seen by e.g. the ratio test). In particular, it converges uniformly on $[-r, r]$ for any $0 < r < 1$. We can then integrate from 0 to r to obtain

$$\int_{[0,r]} \frac{1}{1+x^2}\, dx = r - \frac{r^3}{3} + \frac{r^5}{5} - \dots.$$

But by the fundamental theorem of calculus we have

$$\int_{[0,r]} \frac{1}{1+x^2}\, dx = \arctan(r) - \arctan(0) = \arctan(r).$$

Thus we have the power series expansion

$$\arctan(r) = r - \frac{r^3}{3} + \frac{r^5}{5} - \dots$$

for all $0 < r < 1$. This formula also clearly works for $r = 0$. For $-1 < r < 0$, we replace the integral on $[0, r]$ by the negative integral on $[r, 0]$, and note that

$$-\int_{[r,0]} \frac{1}{1+x^2}\, dx = r - \frac{r^3}{3} + \frac{r^5}{5} - \dots$$

and

$$-\int_{[r,0]} \frac{1}{1+x^2}\, dx = -(\arctan(0) - \arctan(r)) = \arctan(r)$$

so we still have

$$\arctan(r) = r - \frac{r^3}{3} + \frac{r^5}{5} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n r^{2n+1}}{2n+1}$$

when $-1 < r < 0$. Thus this formula is valid for all $-1 < r < 1$.

To compute the radius of convergence of this series, we may use for instance the ratio test. Observe that

$$\lim_{n\to\infty} \frac{|(-1)^{n+1} r^{2(n+1)+1}/(2(n+1)+1)|}{|(-1)^n r^{2n+1}/(2n+1)|} = \lim_{n\to\infty} |r| \frac{2n+1}{2n+3} = |r|,$$

so the series converges when $|r| < 1$ and diverges when $|r| > 1$. Hence the radius of convergence is 1.

Remark: Using this power series expansion, the identity $\arctan(1) = \frac{\pi}{4}$, and Abel's theorem, one can deduce the famous formula

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \dots$$

Problem 3. Let $\vec{x} : \mathbf{R} \to \mathbf{R}^3$ be a differentiable function, and let $r : \mathbf{R} \to \mathbf{R}$ be the function $r(t) := \|\vec{x}(t)\|$, where $\|\vec{x}\|$ denotes the length of $\vec{x}$ as measured in the usual l^2 metric. Let t_0 be a real number. Show that if $r(t_0) \neq 0$, then r is differentiable at t_0, and

$$r'(t_0) = \frac{\vec{x}'(t_0) \cdot \vec{x}(t_0)}{r(t_0)}.$$

(Hint: Use the chain rule).

Let $f : \mathbf{R}^3 \to \mathbf{R}$ be the function $f(\vec{x}) := \|\vec{x}\|$, or in other words

$$f(x_1, x_2, x_3) := \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Then $r(t)$ can be written as $r(t) = \|\vec{x}(t)\| = f(\vec{x}(t))$, i.e. $r = f \circ \vec{x}$. Since $r(t_0)$ is assumed to be non-zero, $\vec{x}(t_0)$ is non-zero, and so f is differentiable at $\vec{x}(t_0)$ (note that $x_1^2 + x_2^2 + x_3^2$ is differentiable everywhere, and $\sqrt{y}$ is differentiable for $y \neq 0$), so the chain rule applies, and we have

$$r'(t_0) = f'(\vec{x}(t_0))\vec{x}'(t_0).$$

Since the partial derivatives

$$\frac{\partial f}{\partial x_1} = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$
$$\frac{\partial f}{\partial x_2} = \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$
$$\frac{\partial f}{\partial x_3} = \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

are continuous away from the origin, we have

$$f'(x_1, x_2, x_3) = \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}} \end{pmatrix}$$

so if we write $\vec{x}(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0))$, we have

$$\begin{aligned} f'(\vec{x}(t_0))\vec{x}'(t_0) &= \frac{x_1(t_0)x_1'(t_0) + x_2(t_0)x_2'(t_0) + x_3(t_0)x_3'(t_0)}{\sqrt{x_1^2(t_0) + x_2^2(t_0) + x_3^2(t_0)}} \\ &= \frac{\vec{x}(t_0) \cdot \vec{x}'(t_0)}{\|x'(t_0)\|} \\ &= \frac{\vec{x}'(t_0) \cdot \vec{x}(t_0)}{r(t_0)} \end{aligned}$$

as desired.

Alternate proof: Observe that

$$r(t)^2 = \|\vec{x}(t)\|^2 = \vec{x}(t) \cdot \vec{x}(t).$$

By the product rule, we thus see that r^2 is differentiable at t_0 and

$$(r^2)'(t_0) = \vec{x}(t_0) \cdot \vec{x}'(t_0) + \vec{x}'(t_0) \cdot \vec{x}(t_0) = 2\vec{x}'(t_0) \cdot \vec{x}(t_0).$$

Since r^2 is differentiable at t_0, and $r^2(t_0)$ is non-zero, we thus see that r is also differentiable at t_0 (this follows from the single-variable calculus chain rule, since r is the square root of r^2, and the square root function is differentiable away from zero). In particular, by the product rule or chain rule we have

$$(r^2)'(t_0) = 2r(t_0)r'(t_0).$$

Equating this with the previous equation we obtain the result.

Problem 4. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be continuous, compactly supported functions. Suppose that f is supported on the interval $[0, 1]$, and g is constant on the interval $[0, 2]$ (i.e. there is a real number c such that $g(x) = c$ for all $x \in [0, 2]$). Show that the convolution $f * g$ is constant on the interval $[1, 2]$.

Let x be any number in $[1, 2]$. We compute

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x - y)\ dy.$$

Since f is supported on $[0, 1]$, so is $f(y)g(x - y)$, and we can rewrite the above integral as

$$f * g(x) = \int_{[0,1]} f(y)g(x - y)\ dy.$$

But if $x \in [1, 2]$ and $y \in [0, 1]$, then $x - y \in [0, 2]$, and hence $g(x - y) = c$ by hypothesis. Thus

$$f * g(x) = \int_{[0,1]} f(y)c\ dy.$$

The right-hand side does not depend on x; thus $f * g$ is constant on the interval $[1, 2]$.

Remark. This result is closely related to both Lemma 6 of Weeks 4/5 notes, and also the remark on page 12 of Week 6 notes regarding convolution with trigonometric polynomials.

Problem 5. Let $f : [0,1] \to \mathbf{R}$ be a continuous function, and suppose that $\int_{[0,1]} f(x)x^n\ dx = 0$ for all non-negative integers $n = 0, 1, 2, \ldots$. Show that f must be the zero function $f \equiv 0$. (Hint: First show that $\int_{[0,1]} f(x)P(x)\ dx = 0$ for all polynomials P. Then, using the Weierstrass approximation theorem, show that $\int_{[0,1]} f(x)f(x)\ dx = 0$.)

First let P be any polynomial, thus $P(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$ for some non-negative integer n and some real numbers $c_0, c_1, \ldots, c_n$. Now compute

$$\int_{[0,1]} f(x)P(x)\ dx = \int_{[0,1]} f(x) \sum_{j=0}^{n} c_j x^j\ dx$$

$$= \sum_{j=0}^{n} c_j \int_{[0,1]} f(x)x^j\ dx = \sum_{j=0}^{n} c_j 0 = 0$$

by hypothesis. Note that we can interchange the sum and integral without difficulty because the sum is finite.

Now we use the Weierstrass approximation theorem. Pick any $\varepsilon > 0$. Since f is continuous on $[0,1]$, we know that there exists a polynomial $P(x)$ on $[0,1]$ such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0,1]$, in other words

$$P(x) - \varepsilon \leq f(x) \leq P(x) + \varepsilon$$

for all $x \in [0,1]$. Multiplying by $f(x)$ (and being careful, because $f(x)$ could be negative), we obtain

$$f(x)P(x) - \varepsilon|f(x)| \leq f(x)f(x) \leq f(x)P(x) + \varepsilon|f(x)|$$

for all $x \in [0,1]$. Integrating over $[0,1]$, we obtain

$$\int_{[0,1]} f(x)P(x)\ dx - \varepsilon \int_{[0,1]} |f(x)|\, dx \leq \int_{[0,1]} f(x)f(x)\ dx \leq \int_{[0,1]} f(x)P(x)\ dx + \varepsilon \int_{[0,1]} |f(x)|\, dx.$$

But we have just proved that $\int_{[0,1]} f(x)P(x)\ dx = 0$, hence we have

$$-\varepsilon \int_{[0,1]} |f(x)|\ dx \leq \int_{[0,1]} f(x)^2\ dx \leq \varepsilon \int_{[0,1]} |f(x)|\ dx.$$

But this is true for any ε, and $\int_{[0,1]} |f(x)|\ dx$ and $\int_{[0,1]} f(x)^2\ dx$ do not depend on ε; hence we must have

$$\int_{[0,1]} f(x)^2\ dx = 0.$$

But this implies that $f \equiv 0$, either by modifying Lemma 2(ii) of Week 6 notes, or observing that if f was not identically 0, then there must be some point x for which $f(x) \neq 0$, say $|f(x)| = c > 0$, then by continuity there would be some ball $B(x,r) \cap [0,1]$ around x for which $|f|$ was larger than $c/2$ (say), which implies that $\int_{[0,1]} f(x)^2$ is strictly greater than 0, contradiction.

Remark. The quantity $\int f(x)x^n\ dx$ is sometimes called the n^{th} *moment* of f. The above problem thus asserts if a (continuous) function has all its moments vanishing, then it must itself vanish. A corollary of this is that if two continuous functions f, g have identical moments, i.e. $\int f(x)x^n\ dx = \int g(x)x^n\ dx$ for all f, g, then they must themselves be identical (to see this, apply the above result to $f - g$). Thus, in principle, one can work out what a function should be just by examining its moments.