Mathematics 131BH
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Instructions: Try to do all five problems; they are all of equal value. There is plenty of working space, and a blank page at the end. On the first page you will be supplied a list of standard definitions for easy reference.

Unless otherwise specified, you may use all the results from the class notes, textbook, or any other source; you do not need to give precise theorem numbers or page numbers (e.g. saying "by a theorem from the notes" will suffice). You are encouraged to be verbose in your proofs and explanations; a chain of equations with no explanation given may be insufficient for full credit.

You may enter in a nickname if you want your midterm score posted.

Good luck!

Name: ____________________

Nickame: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Total: ____________________

Reference sheet

This reference page contains some definitions from the Week 1-4 notes which are relevant to the midterm questions.

- **Boundedness.** A function $f : X \to Y$ from one metric space (X, d_X) to another (Y, d_Y) is *bounded* if there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$.

- **Compactness.** A metric space (X, d) is said to be *compact* iff every sequence in (X, d) has at least one convergent subsequence. If Y is a subset of X, we say that Y is *compact* iff the subspace $(Y, d|_{Y \times Y})$ of (X, d) is compact.

- **Disconnectedness.** Let (X, d) be a metric space. We say that X is *disconnected* iff there exist disjoint non-empty sets V and W in X such that $V \cup W = X$. If Y is a subset of X, we say that Y is *disconnected* iff the subspace $(Y, d|_{Y \times Y})$ of (X, d) is disconnected.

- **Discrete metric.** If X is any set, the *discrete metric* $d_{disc} : X \times X \to [0, \infty)$ on X is defined by setting $d_{disc}(x, y) := 0$ when $x = y$, and $d_{disc}(x, y) := 1$ when $x \neq y$.

- **Metric spaces.** A *metric space* (X, d) is a space X of points, together with a metric $d : X \times X \to [0, \infty)$, which obeys the following axioms: (i) For any $x \in X$, we have $d(x, x) = 0$. (ii) (Positivity) For any *distinct* $x, y \in X$, we have $d(x, y) > 0$. (iii) (Symmetry) For any $x, y \in X$, we have $d(x, y) = d(y, x)$. (iv) (Triangle inequality) For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

- **Pointwise convergence.** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^{\infty}$ *converges pointwise to* f *on* X if we have

$$\lim_{n \to \infty} d_Y(f^{(n)}(x), f(x)) = 0$$

  for all $x \in X$.

- **Uniform boundedness.** A sequence of function $(f_n)_{n=1}^{\infty}$ from one metric space (X, d_X) to another (Y, d_Y) is *uniformly bounded* iff there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f_n(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$ and all positive integers n.

- **Uniform convergence.** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^{\infty}$ *converges uniformly to* f *on* X if for every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y(f^{(n)}(x), f(x)) < \varepsilon$ for every $n > N$ and $x \in X$.

Problem 1. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two sequences in a metric space (X, d). Suppose that $(x_n)_{n=1}^{\infty}$ converges to a point $x \in X$, and $(y_n)_{n=1}^{\infty}$ converges to a point $y \in X$. Show that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$. (Hint: use the triangle inequality several times).

Note: to do this problem it is quite helpful to draw a picture. The problem is similar to Q6 from Assignment 1. The key issue here is how to use the triangle inequality. Note that x, y, z are three points in (X, d), then there are *three* useful ways to invoke the triangle inequality: $d(x, z) \le d(x, y) + d(y, z)$, $d(x, y) \le d(x, z) + d(z, y)$, and $d(y, z) \le d(y, x) + d(x, z)$. However, not all of them will be equally useful. Generally speaking, the triangle inequality is useful when trying to estimate one long side by the sum of two shorter sides, but it is not as helpful to try to estimate the short side by the sum of the two long sides.

Let $\varepsilon > 0$ be any real number. We have to show that there exists an $N > 0$ such that

$$|d(x_n, y_n) - d(x, y)| < \varepsilon \text{ for all } n \ge N$$

or in other words that

$$d(x, y) + \varepsilon < d(x_n, y_n) < d(x, y) + \varepsilon.$$

Since x_n converges x, we know that there exists an N_1 such that $d(x_n, x) < \varepsilon/2$ for all $n > N_1$. Similarly there exists an N_2 such that $d(y_n, y) < \varepsilon/2$ for all $n > N_2$. Thus if we define $N = \max(N_1, N_2)$, then we have $d(x_n, x) < \varepsilon/2$ and $d(y_n, y) < \varepsilon/2$ for all $n > N$. In particular we have by the triangle inequality that

$$d(x_n, y_n) \le d(x_n, x) + d(x, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n) < \varepsilon/2 + d(x, y) + \varepsilon/2 = d(x, y) + \varepsilon$$

which is one half of the bounds that we need. The other half comes from doing the above bounds in reverse:

$$d(x, y) \le d(x, x_n) + d(x_n, y) \le d(x, x_n) + d(x_n, y_n) + d(y_n, y) < \varepsilon/2 + d(x_n, y_n) + \varepsilon/2 = d(x_n, y_n) + \varepsilon$$

so that $d(x, y) - \varepsilon \le d(x_n, y_n)$ as desired.

A cautionary note when dealing with inequalities: it is not safe to subtract two inequalities, thus for instance $a < b$ and $c < d$ do not imply $a - c < b - d$.

Problem 2. Let E and F be two compact subsets of $\mathbf{R}$ (with the standard metric $d(x, y) = |x - y|$). Show that the Cartesian product $E \times F := \{(x, y) : x \in E, y \in F\}$ is a compact subset of $\mathbf{R}^2$ (with the Euclidean metric d_{l^2}).

This question is similar to Q5 of Assignment 2.

First Proof. Let $((x_n, y_n))_{n=1}^\infty$ be a sequence in $E \times F$. We need to show that this sequence has a subsequence which converges in $E \times F$.

The sequence $(x_n)_{n=1}^\infty$ lies in E. Since E is compact, we thus have a subsequence $(x_{n_j})_{n=1}^\infty$ which converges in E, say to the point $x \in E$. Then $(x_{n_j}, y_{n_j})_{j=1}^\infty$ is a subsequence of $((x_n, y_n))_{n=1}^\infty$. However this sequence is not yet guaranteed to converge, because only the first component so far is known to converge.

The next step is to look at the sequence $(y_{n_j})_{j=1}^\infty$. This sequence lives in F, and so it must have a convergent subsequence $(y_{n_{j_k}})_{k=1}^\infty$ that converges to some element $y \in F$. Note also that $(x_{n_{j_k}})_{k=1}^\infty$ is a subsequence of $(x_{n_j})_{j=1}^\infty$ and thus must also converge to $x \in F$ (Lemma 1 of Week 2 notes). Thus the sequence $((x_{n_{j_k}}, y_{n_{j_k}}))_{k=1}^\infty$ converges to $(x, y) \in E \times F$ (Proposition 2 of Week 1 notes). This is a subsequence of $((x_n, y_n))_{n=1}^\infty$, and so we are done.

There are many ways to go wrong in the above argument; for instance many of you produced a convergent sequence such as $((x_{n_j}, y_{m_j}))_{j=1}^\infty$ or $((x_{n_j}, y_j))_{j=1}^\infty$, which are not subsequences of $((x_n, y_n))_{n=1}^\infty$.

Second Proof. We use the Heine-Borel theorem. We know that E and F are separately closed and bounded, so it suffices to show that $E \times F$ is also closed and bounded.

First we show that $E \times F$ is closed. Let (x, y) be an adherent point of $E \times F$, i.e. it is the limit of some sequence $((x_n, y_n))_{n=1}^\infty$ in $E \times F$. Then the sequence $(x_n)_{n=1}^\infty$ is a sequence in E which converges to x, and $(y_n)_{n=1}^\infty$ is a sequence in F which converges to y. Thus x is adherent to E and y is adherent to F. Since E, F are closed, this means that $x \in E$ and $y \in F$, and thus $(x, y) \in E \times F$. Thus $E \times F$ contains all of its adherent points and thus is closed.

Now we show that $E \times F$ is bounded. Since E is bounded it is contained in some ball $B(x_0, r)$, i.e. in some interval $(x_0 - r, x_0 + r)$. Similarly F is contained in some interval $(y_0 - s, y_0 + s)$. Thus $E \times F$ lies in the rectangle $(x_0 - r, x_0 + r) \times (y_0 - s, y_0 + s)$. Thus for any $(x, y) \in E \times F$, we have $|x - x_0| < r$ and $|y - y_0| < s$, which implies that

$$d_{l_2}((x, y), (x_0, y_0)) = \sqrt{|x - x_0|^2 + |y - y_0|^2} < \sqrt{r^2 + s^2}.$$

Thus $E \times F$ is contained inside the ball $B_{d_{l^2}}((x_0, y_0), \sqrt{r^2 + s^2})$, and is bounded.

Problem 3. Let (X, d_{disc}) be a metric space with the discrete metric. Let E be a subset of X which contains at least two elements. Show that E is disconnected.

Let x be any element of E, and consider the two sets $V := \{x\}$ and $W := E - \{x\}$. Clearly V is non-empty; since E contains at least two elements, W is also non-empty. Also by definition we have $V \cap W = \emptyset$ and $V \cup W = E$. To conclude the proof that E is disconnected, we have to verify that V and W are both open in E.

Actually, we can show the stronger statement that *every* subset of E is open in E. Indeed, if F is a subset of E, and $x \in F$, then the ball $B_{d_{disc}|_{E \times E}}(x, 1/2) = \{x\}$ is also contained in F, and thus every point in F is an interior point. Thus every set is open, and we are done.

It is also possible to proceed using the alternative definition of disconnectedness as containing a proper non-empty subset which is both open and closed; the point is that in the discrete metric every set is both open and closed. We leave the verification of this to the reader.

Problem 4. Let (X, d_X) a metric space, and for every integer $n \geq 1$, let $f_n : X \to \mathbf{R}$ be a real-valued function. Suppose that f_n converges pointwise to another function $f : X \to \mathbf{R}$ on X (in this question we give $\mathbf{R}$ the standard metric $d(x, y) = |x - y|$). Let $h : \mathbf{R} \to \mathbf{R}$ be a continuous function. Show that the functions $h \circ f_n$ converge pointwise to $h \circ f$ on X, where $h \circ f_n : X \to \mathbf{R}$ is the function $h \circ f_n(x) := h(f_n(x))$, and similarly for $h \circ f$.

We have to show that for every $x \in X$, the sequence $(h \circ f_n(x))_{n=1}^{\infty}$ converges to $h \circ f(x)$. But because we already know that f_n converges pointwise to f, this implies that $(f_n(x))_{n=1}^{\infty}$ converges to $f(x)$. Since h is continuous, this implies that $(h(f_n(x)))_{n=1}^{\infty}$ converges to $h(f(x))$ (Theorem 12 of Week 2), as desired.

Problem 5. Let $f_n : X \to Y$ be a sequence of bounded functions from one metric space (X, d_X) to another metric space (Y, d_Y). Suppose that f_n converges uniformly to another function $f : X \to Y$. Suppose that f is a bounded function. Show that the sequence f_n is uniformly bounded (see Reference Sheet).

This question is similar to Q6(a) of Assignment 3. Note that that homework question shows that the hypothesis that f is bounded is in fact redundant.

Since f_n converges uniformly to f, we know in particular that there exists an $N > 0$ such that $d_Y(f_n(x), f(x)) < 1$ for all $n > N$ and $x \in X$. Also, since f is bounded, there exists a ball $B_{d_Y}(y_0, r)$ such that $f(x) \in B_{d_Y}(y_0, r)$ for all $x \in X$. (Note that we are not assuming Y to be the real line, and so we cannot just write things like $|f(x)| \leq M$; that only makes sense for real-valued functions). In other words, we have $d_Y(f(x), y_0) < r$ for all $x \in X$. By the triangle inequality, we thus have $d_Y(f_n(x), y_0) < r + 1$ for all $x \in X$. Thus $f_n(x) \in B_{d_Y}(y_0, r + 1)$ for all $n \geq N$.

This would give that the f_n are all uniformly bounded, except that we haven't dealt with the cases where $n < N$. However, we also know that each function f_n is individually bounded, thus for each n there is a ball $B_{d_Y}(y_n, r_n)$ such that $f_n(x) \in B_{d_Y}(y_n, r_n)$ for all $x \in X$. Thus we have $d_Y(f_n(x), y_n) < r_n$ for all $x \in X$; by the triangle inequality, this implies that $d_Y(f_n(x), y_0) < r_n + d(y_n, y_0)$. Thus if we define

$$R = \max(r + 1, \max_{n<N} r_n + d(y_n, y_0))$$

then we have $d_Y(f_n(x), y_0) < R$ for all $n < N$, and also $d_Y(f_n(x), y_0) < R$ for all $n > N$. Note that R is a finite real number since it is just the max of a finite number of reals. Thus all of the functions $f_n(x)$ take values in $B_{d_Y}(y_0, R)$, and so the sequence $(f_n)_{n=1}^{\infty}$ is uniformly bounded.