Mathematics 131BH
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Instructions: Do nine out of the 12 problems; they are all of equal value. There is plenty of working space, and a blank page at the end. On the first page you will be supplied a list of standard definitions for easy reference.

Unless otherwise specified, we give spaces such as $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{Q}$ the usual metric $d(x, y) := |x - y|$. You are free to use the axiom of choice whenever you wish. (If you do not know what the axiom of choice is, please disregard this notice).

You may enter in a nickname if you want your final score posted. Good luck!

Name: ______________________

Nickame: ______________________

Student ID: ______________________

Signature: ______________________

Problem 1 (10 points). ____________

Problem 2 (10 points). ____________

Problem 3 (10 points). ____________

Problem 4 (10 points). ____________

Problem 5 (10 points). ____________

Problem 6 (10 points). ____________

Problem 7 (10 points). ____________

Problem 8 (10 points). ____________

Problem 9 (10 points). ____________

Problem 10 (10 points). ____________

Problem 11 (10 points). ____________

Problem 12 (10 points). ____________

Best 9 of 12 (90 points): ______________________

Definitions

- **Absolute integrability.** Let Ω be a measurable subset of $\mathbf{R}^n$. A measurable function $f : \Omega \to \mathbf{R}^*$ is said to be *absolutely integrable* if the integral $\int_\Omega |f|$ is finite.

- **Analyticity.** Let $(a-r, a+r)$ be an open interval, and let $f : E \to \mathbf{R}$ be a function defined on a set $E \subseteq \mathbf{R}$ which contains $(a-r, a+r)$. We say that f is *real analytic on* $(a-r, a+r)$ iff there exists a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ centered at a which has a radius of convergence greater than or equal to r, and which converges to f on $(a-r, a+r)$.

- **Boxes.** A (open) *box* B in $\mathbf{R}^n$ is any set of the form

$$B = \prod_{i=1}^{n} (a_i, b_i) := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in (a_i, b_i) \text{ for all } 1 \leq i \leq n\},$$

  where $b_i \geq a_i$ are real numbers. We define the *volume* $vol(B)$ of this box to be the number

$$vol(B) := \prod_{i=1}^{n} (b_i - a_i) = (b_1 - a_1)(b_2 - a_2) \ldots (b_n - a_n).$$

- **Compactness.** A metric space (X, d) is said to be *compact* iff every sequence in (X, d) has at least one convergent subsequence. If Y is a subset of X. We say that Y is *compact* iff the subspace $(Y, d|_{Y \times Y})$ of (X, d) is compact.

- **Connectedness.** Let (X, d) be a metric space. We say that X is *disconnected* iff there exist disjoint non-empty sets V and W in X such that $V \cup W = X$. (Equivalently, X is disconnected if and only if X contains a non-empty set which is simultaneously closed and open). We say that X is *connected* iff is not disconnected.

- **Cover.** Let $\Omega \subseteq \mathbf{R}^n$ be a subset of $\mathbf{R}^n$. We say that a collection $(B_j)_{j \in J}$ of boxes *cover* Ω iff $\Omega \subseteq \bigcup_{j \in J} B_j$.

- **Differentiability.** Let E be a subset of $\mathbf{R}^n$, $f : E \to \mathbf{R}^m$ be a function, $x_0 \in E$ be a point, and let $L : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. We say that f is *differentiable at* x_0 *with derivative* L if we have

$$\lim_{x \to x_0; x \in E - \{x_0\}} \frac{\|f(x) - (f(x_0) + L(x - x_0))\|}{\|x - x_0\|} = 0.$$

  Here $\|x\|$ is the length of x (as measured in the l^2 metric):

$$\|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}.$$

- **Fourier transform.** For any function $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{R})$, and any integer $n \in \mathbf{Z}$, we define the n^{th} *Fourier coefficient of* f, denoted $\hat{f}(n)$, by the formula

$$\hat{f}(n) := \langle f, e_n \rangle = \int_{[0,1]} f(x) e^{-2\pi i n x}\, dx.$$

The function $\hat{f} : \mathbf{Z} \to \mathbf{C}$ is called the *Fourier transform* of f.

- **Integration of simple functions.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a simple function which is non-negative; thus f is measurable and the image $f(\Omega)$ is finite and contained in $[0, \infty)$. We then define the *Lesbegue integral* $\int_\Omega f$ of f on Ω by

$$\int_\Omega f := \sum_{\lambda \in f(\Omega); \lambda > 0} \lambda m(\{x \in \Omega : f(x) = \lambda\}).$$

- **Integration of non-negative measurable functions.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to [0, \infty]$ be measurable and non-negative. Then we define the *Lebesgue integral* $\int_\Omega f$ of f on Ω to be

$$\int_\Omega f := \sup\{\int_\Omega s : s \text{ is simple and non-negative, and minorizes } f\}.$$

- **Integration of absolutely integrable functions.** Let $f : \Omega \to \mathbf{R}^*$ be an absolutely integrable function. We define the *Lebesgue integral* $\int_\Omega f$ of f to be the quantity

$$\int_\Omega f := \int_\Omega f^+ - \int_\Omega f^-.$$

- **Measurable functions.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^m$ be a function. We say that f is *measurable* iff $f^{-1}(V)$ is measurable for every open set $V \subseteq \mathbf{R}^m$. A function $g : \Omega \to \mathbf{R}^*$ is said to be *measurable* iff we $f^{-1}((a, \infty))$ is measurable for every real number a.

- **Measurable sets.** Let E be a subset of $\mathbf{R}$. We say that E is *Lebesgue measurable*, or *measurable* for short, iff we have the identity

$$m^*(A) = m^*(A \cap E) + m^*(A \backslash E)$$

for every subset A of $\mathbf{R}$. If E is measurable, we define the *Lebesgue measure* of E to be $m(E) = m^*(E)$; if E is not measurable, we leave $m(E)$ undefined.

- **Simple functions.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a measurable function. We say that f is a *simple function* if the image $f(\Omega)$ is finite. In other words, there exists a finite number of real numbers $c_1, c_2, \ldots, c_N$ such that for every $x \in \Omega$, we have $f(x) = c_j$ for some $1 \leq j \leq N$.

- **Outer measure.** If Ω is a set, we define the *outer measure* $m^*(\Omega)$ of Ω to be the quantity

$$m^*(\Omega) := \inf\{\sum_{j=1}^{\infty} vol(B_j) : (B_j)_{j \in J} \text{ is a finite or countable cover of } \Omega \text{ by boxes}\}.$$

- **Uniform boundedness.** A sequence $(f_n)_{n=1}^{\infty}$ of functions from a metric space (X, d) to $\mathbf{R}$ is said to be *uniformly bounded* if there exists a constant $M > 0$ (which does not depend on n or x) such that $|f_n(x)| \leq M$ for all $x \in X$ and all positive integers n.

- **Uniform convergence** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^{\infty}$ *converges uniformly to* f *on* X if for every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y(f^{(n)}(x), f(x)) < \varepsilon$ for every $n > N$ and $x \in X$.