Assignment 9 (Due June 6). Covers: Weeks 8-10 notes

- Q1(a) Prove Lemma 15 from Weeks 8-9 notes. (Hint: use Lemma 12 and the σ-algebra property).
- Q1(b) Using Lemma 15, deduce Corollary 16 from Weeks 8-9 notes.
- Q2. Prove Lemma 17 from Weeks 8-9 notes.
- Q3. Prove Lemma 20 from Weeks 8-9 notes. (Hint: use Lemma 15. As a preliminary step, you may need to show that if $f^{-1}((a,\infty))$ is measurable for all a, then $f^{-1}([a,\infty))$ is also measurable for all a.)
- Q4. Prove Lemma 22 from Weeks 8-9 notes.
- Q5. Prove Lemma 23 from Weeks 8-9 notes.
- Q6. Prove Lemma 24 from Weeks 8-9 notes. (Hint: set

  $$f_n(x) := \sup\{\frac{j}{2^n} : j \in \mathbf{Z}, \frac{j}{2^n} \leq \min(f(x), 2^n)\},$$

  i.e. $f_n(x)$ is the greatest integer multiple of 2^{-n} which does not exceed either $f(x)$ or 2^n. (You should draw a picture to see how f_1, f_2, f_3, etc. works). Then prove that f_n obeys all the required properties).
- Q7. Prove Proposition 3 from Week 10 notes. (Hint: Do not attempt to mimic the proof of Proposition 2; rather, try to use Proposition 2 and the sup definition of the Lebesgue integral. For one direction of part (a), start with $\int_\Omega f = 0$ and conclude that $m(\{x \in \Omega : f(x) > 1/n\}) = 0$ for every $n = 1, 2, 3, \ldots$, and then use the countable sub-additivity. To prove (e), first prove it for simple functions.)
- Q8. For each $n = 1, 2, 3, \ldots$, let $f_n : \mathbf{R} \to \mathbf{R}$ be the function $f_n = \chi_{[n,n+1)} - \chi_{[n+1,n+2)}$; i.e. let $f_n(x)$ equal $+1$ when $x \in [n, n+1)$, equal -1 when $x \in [n+1, n+2)$, and 0 everywhere else. Show that

  $$\int_{\mathbf{R}} \sum_{n=1}^{\infty} f_n \neq \sum_{n=1}^{\infty} \int_{\mathbf{R}} f_n.$$

  Explain why this does not contradict Corollary 5 from Week 10 notes.

- Q9. Prove Lemma 6 from Week 10 notes.
- Q10. Prove Proposition 7 from Week 10 notes. (Hint: for (b), break f, g, and $f + g$ up into positive and negative parts, and try to write everything in terms of integrals of non-negative functions only, using Lemma 4).