Assignment 7 (Due May 23). Covers: Weeks 7 notes

- Q1. Prove Lemma 2 from Week 7 notes.
- Q2. Prove Lemma 3 from Week 7 notes.
- Q3. Prove Lemma 4 from Week 7 notes. (Hint: Prove by contradiction. If $L_1 \neq L_2$, then there exists a vector v such that $L_1 v \neq L_2 v$; this vector must be non-zero (why?). Now apply the definition of derivative, and try to specialize to the case where $x = x_0 + tv$ for some scalar t, to obtain a contradiction.)
- Q4 (a). Prove Lemma 5 from Week 7 notes. (This will be similar to Q2).
- Q4 (b). Let $f : \mathbf{R}^2 \to \mathbf{R}$ be the function defined by $f(x,y) := \frac{x^3}{x^2+y^2}$ when $(x,y) \neq (0,0)$, and $f(0,0) := 0$. Show that f is not differentiable at $(0,0)$, despite being differentiable in every direction $v \in \mathbf{R}^2$ at $(0,0)$. Explain why this does not contradict Theorem 6.
- Q5 (a). Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Show that there exists a number $M > 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in \mathbf{R}^n$. (Hint: use Lemma 1 to write T in terms of a matrix A, and then set M to be the sum of the absolute values of all the entries in A. Use the triangle inequality often - it's easier than messing around with square roots etc.). Conclude in particular that every linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$ is continuous.
- Q5 (b). Let E be a subset of $\mathbf{R}^n$. Prove that if a function $f : E \to \mathbf{R}^m$ is differentiable at an interior point x_0 of E, then it is also continuous at x_0. (Hint: use Q5(a)).
- Q6. Prove the several variable calculus chain rule. (Hint: you may wish to review the proof of the ordinary chain rule in single variable calculus first. The easiest way to proceed is by using the sequence-based definition of limit (see Proposition 1(b) of Week 3 notes), and use Q5).

- Q7. State and prove some version of the quotient rule for functions of several variables (i.e. functions of the form $f : E \to \mathbf{R}$ for some subset E of $\mathbf{R}^n$). In other words, state a rule which gives a formula for the gradient of f/g. Be sure to make clear what all your assumptions are.

- Q8. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be the function defined by $f(x,y) := \frac{x^3 y}{x^2+y^2}$ when $(x,y) \neq (0,0)$, and $f(0,0) := 0$. Show that f is continuously differentiable, and the double derivatives $\frac{\partial}{\partial y}\frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial x}\frac{\partial f}{\partial y}$ exist, but are not equal to each other at $(0,0)$. Explain why this does not contradict Clairaut's theorem.

- Q9*. Prove the contraction mapping theorem. (Hint: To prove that there is at most one fixed point, argue by contradiction. To prove that there is at least one fixed point, pick any $x_0 \in X$ and define recursively $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$, etc. Prove inductively that $d(x_{n+1}, x_n) \leq c^n d(x_1, x_0)$, and conclude (using the geometric series formula) that the sequence $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence. Then prove that the limit of this sequence is a fixed point of f).

- Q10. Prove Lemma 8 from Week 7 notes. (Hint: invertible functions are one-to-one and onto).