Assignment 6 (Due May 16). Covers: Week 6 notes

- Q1. Prove Lemma 1 from Week 6 notes. (Hint: For (i), first show that f is bounded on $[0,1]$.)
- Q2. Prove Lemma 2 from Week 6 notes. (Hint: The last part of (ii) is a little tricky. You may need to prove by contradiction, assuming that f is not the zero function, and then show that $\int_{[0,1]} |f(x)|^2$ is strictly positive. You will need to use the fact that f, and hence $|f|$, is continuous, to do this.)
- Q3. Prove Lemma 3 from Week 6 notes. (Hint: Use Lemma 2 frequently. For the Cauchy-Schwarz inequality, begin with the positivity property $\langle f, f\rangle \geq 0$, but with f replaced by the function $f\|g\|_2^2 - \langle f, g\rangle g$, and then simplify using Lemma 2. You may have to treat the case $\|g\|_2 = 0$ separately. Use the Cauchy-Schwarz inequality to prove the triangle inequality).
- Q4. Let $(f_n)_{n=1}^\infty$ be a sequence of functions in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, and let f be another function in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.
- (a) Show that if f_n converges uniformly to f, then f_n also converges to f in the L^2 metric.
- (b) Give an example where f_n converges to f in the L^2 metric, but does not converge to f uniformly. (Hint: take $f = 0$. Try to make the functions f_n large in sup norm.).
- (c) Give an example where f_n converges to f in the L^2 metric, but does not converge to f pointwise. (Hint: take $f = 0$. Try to make the functions f_n large at one point.).
- (d) Give an example where f_n converges to f pointwise, but does not converge to f in the L^2 metric. (Hint: take $f = 0$. Try to make the functions f_n large in L^2 norm).
- Q5(a). Prove Lemma 4 from Week 6 notes.
- Q5(b). Use Lemma 4 to prove Corollary 5. (Hint: for the second identity, either use Pythagoras's theorem and induction, or substitute $f = \sum_{n=-N}^{N} c_n e_n$ and expand everything out).

- Q6. Prove Lemma 6 from Week 6 notes. (Hint: To prove that $f * g$ is continuous, you will have to do something like use the fact that f is bounded, and g is uniformly continuous, or vice versa. To prove that $f * g = g * f$, you will need to use the periodicity to "cut and paste" the interval $[0, 1]$.)

- Q7. Fill in the gaps marked (why?) in Lemma 7 of Week 6 notes. (For the first identity, use the identities $|z|^2 = z\overline{z}$, $\overline{e_n} = e_{-n}$, and $e_n e_m = e_{n+m}$.)

- Q8. Let $f \in C(\mathbf{R}/\mathbf{Z})$, and define the *trigonometric Fourier coefficients* a_n, b_n for $n = 0, 1, 2, 3, \ldots$ by

$$a_n := 2 \int_{[0,1]} f(x) \cos(2\pi n x)\ dx; \quad b_n := 2 \int_{[0,1]} f(x) \sin(2\pi n x)\ dx.$$

- (a) Show that the series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos(2\pi n x) + b_n \sin(2\pi n x))$$

converges in L^2 metric to f. (Hint: use the Fourier theorem, and break up the exponentials into sines and cosines. Combine the positive n terms with the negative n terms).

- (b) Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, then the above series actually converges uniformly to f (not just in L^2 metric). (Hint: use Theorem 8).

- Q9. Let $f(x)$ be the function defined by $f(x) = (1 - 2x)^2$ when $x \in [0, 1)$, and extended to be $\mathbf{Z}$-periodic for the rest of the real line.

  (a) Using Q8, show that the series

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x)$$

converges uniformly to f.

(b) Conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (Hint: Evaluate the above series at $x = 0$.)

(c) Conclude that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. (Hint: Expand the cosines in terms of exponentials, and use Plancherel's theorem.)

- Q10. In this problem we shall develop the theory of Fourier series for functions of any fixed period L.

  Let $L > 0$, and let $f : \mathbf{R} \to \mathbf{C}$ be a complex-valued function which is continuous and L-periodic. Define the numbers c_n for every integer n by

$$c_n := \frac{1}{L} \int_{[0,L]} f(x) e^{-2\pi i n x/L}\, dx.$$

- (a) Show that the series

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/L}$$

  converges in L^2 metric to f. More precisely, show that

$$\lim_{N\to\infty} \int_{[0,L]} |f(x) - \sum_{n=-N}^{N} c_n e^{2\pi i n x/L}|^2\, dx = 0.$$

  (Hint: Apply the Fourier theorem to the function $f(Lx)$).

- (b) If the series $\sum_{n=-\infty}^{\infty} |c_n|$ is absolutely convergent, show that

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/L}$$

  converges uniformly to f.

- (c) Show that

$$\frac{1}{L} \int_{[0,L]} |f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

  (Hint: Apply the Plancherel theorem to the function $f(Lx)$).