Assignment 2 (Due April 18). Covers: Week 2 notes

Note: For these assignments you may freely use any material from the textbook (or any other book) or from other courses, especially from 131AH.

- Q1. Prove Lemma 1 from Week 2 notes.
- Q2 (a). Prove Lemma 2 from Week 2 notes.
- Q2 (b). Prove Lemma 3 from Week 2 notes.
- Q3. Prove Proposition 5 from Week 2 notes.
- Q4. Prove Proposition 7. (Hint: prove the completeness and boundedness separately. For both claims, use proof by contradiction).
- Q5 (a). Use the Bolzano-Weierstrass theorem to prove Corollary 6. (Hint: use Proposition 2 from Week 1 notes).
- Q5 (b). Use Corollary 6 to prove the Heine-Borel theorem.
- Q6. Use Theorem 9 to prove Corollary 10. (Hint: work in the compact metric space $(K_1, d|_{K_1 \times K_1})$, and consider the sets $V_n := K_1 \backslash K_n$, which are open on K_1. Assume for contradiction that $\bigcap_{n=1}^{\infty} K_n = \emptyset$, and then apply Theorem 9.)
- Q7. Prove Theorem 11 from Week 2 notes. (For part (c), you may wish to use (b), and first prove that every singleton set is compact.)
- Q8 (a). Prove Theorem 12 from Week 2 notes.
- Q8 (b). Prove Theorem 13 from Week 2 notes. (Note that Theorem 12 already shows that (a) and (b) are equivalent).
- Q8 (c). Let $(\mathbf{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbf{R} \to \mathbf{R}$, and an open set $V \subseteq \mathbf{R}$, such that the image $f(V) := \{f(x) : x \in V\}$ of V is *not* open.
- Q8 (d). Let $(\mathbf{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbf{R} \to \mathbf{R}$, and a closed set $F \subseteq \mathbf{R}$, such that $f(F)$ is *not* closed.

- Q8 (e). Prove Corollary 14 from Week 2 notes.
- Q9 (a). Prove Theorem 15.
- Q9 (b). Use Theorem 15 to prove the Maximum principle. (Hint: what can you say about $\inf f(K)$ and $\sup f(K)$?
- Q9 (c). Prove Theorem 19.
- Q9 (d). Use Theorem 19 to prove the Intermediate Value Theorem.
- Q10. Let (X, d_{disc}) be a metric space with the discrete metric d_{disc}.
- Q10(a). Show that X is always complete.
- Q10(b). When is X compact, and when is X not compact? Prove your claim. (Note: The Heine-Borel theorem will be useless here since that only applies to Euclidean spaces with the Euclidean metric).