CLASS NOTES FOR WEEK 8 (MAY 22-26, 2000)

1. Introduction

This week we will cover the topic of *product spaces*. Recall that the Cartesian product of two sets $X \times Y$ is defined as the space of all pairs of elements (x, y) such that $x \in X$, $y \in Y$:

$$X \times Y := \{(x, y) : x \in X, y \in Y\}.$$

More generally, if $X_1, \ldots, X_n$ are a finite collection of sets, the Cartesian product $X_1 \times X_n$ can be defined as

$$X_1 \times \ldots \times X_n := \{(x_1, \ldots, x_n) : x_i \in X_i \text{ for all } 1 \leq i \leq n\}.$$

This product is sometimes abbreviated as

$$\prod_{i=1}^n X_i := X_1 \times \ldots \times X_n.$$

For instance,

$$\prod_{i=1}^3 \{0, 1\} = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$$

is the space of all binary sequences of length 3.

Even more generally, if one has an *arbitrary* collection $\{X_\alpha\}_{\alpha \in A}$ of sets, then one can define the product

$$\prod_{\alpha \in A} X_\alpha := \{(x_\alpha)_{\alpha \in A} : x_\alpha \in X_\alpha \text{ for all } \alpha \in A\}.$$

For instance,

$$\prod_{i=1}^\infty \{0, 1\}$$

is the space of all infinite binary sequences (sequences consisting only of 0s and 1s, such as $(0, 1, 0, 1, \ldots)$.)

We've just defined what a product means for *sets*, but we would also like to define what products mean for *topological spaces*.

For instance, if X and Y are topological spaces, what kind of topology does $X \times Y$ get? What are the open sets, convergent sequences, etc?

We'll first discuss the product of two spaces. The product of n spaces is pretty much the same (just replace 2s by ns throughout). The infinite product case is more tricky conceptually and will be left to the Wednesday class.

2. Product of two spaces

Let X_1 and X_2 be two topological spaces. The space $X_1 \times X_2$ is the set of all pairs (x_1, x_2), where $x_1 \in X_1$ and $x_2 \in X_2$.

Traditionally, we draw X_1 as a horizontal set, X_2 as a vertical set, and $X_1 \times X_2$ as the rectangle with these co-ordinates. This picture can be a bit mis-leading though, since X_1 and X_2 are not necessarily one-dimensional.

To make $X_1 \times X_2$ into a topological space, we need to specify what the open sets of $X_1 \times X_2$ are. Well, if U_1 is open in X_1, and U_2 is open in X_2, then it seems plausible that $U_1 \times U_2$ is open in $X_1 \times X_2$. (The sets $U_1 \times U_2$ look like open rectangles).

Exercise 2.1. *Show that this is actually the case when X_1, X_2 are metric spaces.*

However, these rectangles are not enough to form a topology. For instance, the union of two rectangles is not always a rectangle (e.g. take the union of $(0,1) \times (0,2)$ and $(0,2) \times (0,1)$). However, they do form a base for a topology.

Exercise 2.2. *Show that the set $\{U_1 \times U_2 : U_1 \text{ open in } X_1, U_2 \text{ open in } X_2\}$ satisfies the axioms for a base (see (4.1), (4.2) on p. 70 of the text).*

Definition 2.3. The *product topology* on $X_1 \times X_2$ is defined to be the topology generated by the base $\{U_1 \times U_2 : U_1 \text{ open in } X_1, U_2 \text{ open in } X_2\}$.

In other words, a subset of $X_1 \times X_2$ is considered to be open in the product topology if and only if it is the union of open rectangles of the form $U_1 \times U_2$, where U_1 is open in X_1 and U_2 is open in X_2.

How is the product topology on $X_1 \times X_2$ related to the topology of X_1 and X_2? The most direct relationship comes via the *projection maps* $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ defined by

$$\pi_1(x_1, x_2) := x_1$$
$$\pi_2(x_1, x_2) := x_2.$$

Theorem 2.4. *The projections π_1 and π_2 are continuous and open. (A map is called open if the image of every open set is open; it's like continuity, but in reverse).*

Proof We'll just prove the theorem for π_1, as the proof for π_2 is similar.

To show that π_1 is continuous, we have to show that the inverse image of any open set in X_1 is an open set in $X_1 \times X_2$. So take any open set U_1 in X_1. The inverse image $\pi_1^{-1}(U_1)$ is the set of all points in $X_1 \times X_2$ which project down to X_1. If you think about it, that set is just $U_1 \times X_2$. Since U_1 is open in X_1 and X_2 is open in X_2, $U_1 \times X_2$ is open in $X_1 \times X_2$. So π_1 is continuous.

Now to show that π_1 is open. Let's take any open set V in $X_1 \times X_2$. We have to show that $\pi_1(V)$ is open. Since V is open, it is the union of open rectangles. We can write this as

$$V = \bigcup_{\alpha \in A} U_1^\alpha \times U_2^\alpha$$

where A is an index set (finite or infinite, it doesn't matter) and for each $\alpha \in A$, U_1^α is an open set in X_1 and U_2^α is an open set in X_2. Of course we can assume that the sets U_1^α and U_2^α are non-empty (if any of these sets were empty, they wouldn't contribute anything to the union and we could just throw them out).

It is clear that $\pi_1(U_1^\alpha \times U_2^\alpha) = U_1^\alpha$ for each $\alpha \in A$, so

$$\pi_1(V) = \bigcup_{\alpha \in A} U_1^\alpha$$

(Exercise: prove this!). The right-hand side is the union of open sets, and is therefore open. Thus π_1 is open. ■

So the product topology has the nice property that the projections π_1, π_2 are continuous. In fact, it is the smallest topology with this property; we threw in the barest minumum of open sets in $X_1 \times X_2$ which were required in order to make these maps continuous. If we made the topology any smaller, at least one of π_1 and π_2 would fail to be continuous. (Exercise: prove this!)

The above theorem can be used to prove many theorems of the form

If $X_1 \times X_2$ have [insert property here], then
X_1 and X_2 must individually have [insert property here].

For instance, if $X_1 \times X_2$ is connected, then $\pi_1(X_1 \times X_2)$ is connected, since the image of a connected set under a continuous map remains connected. But $\pi_1(X_1 \times X_2) = X_1$, so X_1 is connected. Similarly X_2 is connected.

Conversely, if X_1 and X_2 both have some topological property, then one can usually prove the same property for the product $X_1 \times X_2$. For instance, the product of two connected sets is connected, two Hausdorff spaces is Hausdorff; the product of two compact sets is compact, and so forth. These are a little trickier to prove, though, and I'll skip over them.

Note that $X_1 \times X_2$ doesn't actually contain X_1 or X_2 directly. ($X_1 \times X_2$ consists of pairs of elements, whereas X_1 and X_2 consist of individual elements). However, for every $x_2 \in X_2$, $X_1 \times X_2$ contains the set $X_1 \times \{x_2\}$, which is homoeomorphic to X_1.

Exercise 2.5. *For each $x_2 \in X_2$, show that the map $f_{x_2} : X_1 \to X_1 \times \{x_2\}$ defined by $f_{x_2}(x_1) = (x_1, x_2)$ is a homeomorphism from X_1 to $X_1 \times \{x_2\}$. (Of course, $X_1 \times \{x_2\}$ is given the relative topology induced by $X_1 \times X_2$).*

Thus $X_1 \times X_2$ consists of many "horizontal slices", each of which is homeomorphic to X_1. One can similarly divide $X_1 \times X_2$ into vertical slices, each of which is homeomorphic to X_2.

Suppose $f : Y \to X_1 \times X_2$ is a continuous map from some topological space Y to a product space $X_1 \times X_2$. This map has two components, $\pi_1 \circ f : Y \to X_1$ and $\pi_2 \circ f : Y \to X_2$. For instance, if $f : [0, 2\pi] \to \mathbf{R} \times \mathbf{R}$ is the curve $f(t) = (\cos(t), \sin(t))$, then we can break it into the components $\pi_1 \circ f(t) = \cos(t)$ and $\pi_2 \circ f(t) = \sin(t)$.

Since f, π_1, and π_2 are all continuous, we see that $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous. In other words, the components of a continuous function are also continuous. The converse is also true:

Theorem 2.6. *Let $f : Y \to X_1 \times X_2$ be such that $\pi_1 \circ f$ and $\pi_2 \circ f$ are both continuous. Then f is also continuous.*

Proof Let V be any open set in $X_1 \times X_2$. Our job is to show that $f^{-1}(V)$ is an open set in Y.

Let y be any point in $f^{-1}(V)$. We have to show that y is an interior point of $f^{-1}(V)$. Well, since $y \in f^{-1}(V)$, we know that $f(y) \in V$. Since V is open, we therefore know that $f(y)$ is an interior point of V. Since the product topology uses open rectangles as a base, there must exist an open rectangle $U_1 \times U_2$ inside V which contains $f(y)$.

Since $U_1 \times U_2$ is inside V and contains $f(y)$, the set $f^{-1}(U_1 \times U_2)$ is inside $f^{-1}(V)$ and contains y. This will allow us to show that y is an interior point of $f^{-1}(V)$ provided that $f^{-1}(U_1 \times U_2)$ is open.

If we knew that f was continuous, then we'd be done, since $U_1 \times U_2$ is continuous. But this is exactly what we're trying to prove! So we can't use that. However, we do know that $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous, so we should try to use that.

Observe that for any $z \in Y$,

$$z \in f^{-1}(U_1 \times U_2) \iff f(z) \in U_1 \times U_2 \iff \pi_1 \circ f(z) \in U_1 \text{ and } \pi_2 \circ f(z) \in U_2$$

$$\iff z \in (\pi_1 \circ f)^{-1}(U_1) \text{ and } z \in (\pi_2 \circ f)^{-1}(U_2).$$

Thus

$$f^{-1}(U_1 \times U_2) = (\pi_1 \circ f)^{-1}(U_1) \cap (\pi_2 \circ f)^{-1}(U_2).$$

Since $\pi_1 \circ f$ and $\pi_2 \circ f$ are open, the sets $(\pi_1 \circ f)^{-1}(U_1)$ and $(\pi_2 \circ f)^{-1}(U_2)$ are open. Since the intersection of two open sets is open, $f^{-1}(U_1 \times U_2)$ is thus open, and we're done. ■

3. The axiom of choice

Now we turn to the subject of infinite Cartesian products, which is a more subtle topic. Conceptually, this is probably the most difficult part of the course. Before we get to the topological aspects of these products, we need to make a digression into one of the foundational axioms of set theory.

Let A be an index set (finite or infinite), and for each $\alpha \in A$ let X_α be a topological space. We define the infinite Cartesian product $\prod_{\alpha \in A} X_\alpha$ to be the space of all objects of the form $(x_\alpha)_{\alpha \in A}$, where for each $\alpha \in A$ x_α is an element of X_α.

For instance, take $A = \mathbf{Z}$, and let $X_\alpha = \mathbf{R}$ for each $\alpha \in Z$. Then the set

$$\prod_{\alpha \in \mathbf{Z}} \mathbf{R}$$

consists of all objects of the form $(x_\alpha)_{\alpha \in \mathbf{Z}}$, where each x_α is a real number. One can think of these objects either as infinite sequences

$$(x_1, x_2, x_3, \dots)$$

of real numbers, or as functions

$$\alpha \mapsto x_\alpha$$

from $\mathbf{Z}$ to $\mathbf{R}$. Mathematically, both perspectives are equally valid.

Another example: take $A \in \mathbf{Z}$, and let $X_\alpha = [\alpha, \alpha + 1]$ for each $\alpha \in \mathbf{Z}$. The set

$$\prod_{\alpha \in \mathbf{Z}} [\alpha, \alpha + 1]$$

consists of all objects of the form $(x_\alpha)_{\alpha \in \mathbf{Z}}$, where each x_α is a real number between α and $\alpha + 1$ inclusive. One can visualize these objects as a function from

$$\alpha \mapsto x_\alpha$$

from $\mathbf{Z}$ to $\mathbf{R}$ such that the value of the function at each α must stay within the interval $[\alpha, \alpha + 1]$.

Now for a more difficult example to visualize. Take $A = \mathbf{R}$, and let $X_\alpha = \mathbf{R}$ for each $\alpha \in A$. Then the set

$$\prod_{\alpha \in \mathbf{R}} \mathbf{R}$$

consists of all objects of the form $(x_\alpha)_{\alpha \in \mathbf{R}}$, where each x_α is a real number. It's a bit difficult to think of this as an infinite sequence as before, because the reals are not countable. However, one can think of such an object instead as a function

$$\alpha \mapsto x_\alpha$$

from $\mathbf{R}$ to $\mathbf{R}$. In other words, one can think of $\prod_{\alpha \in \mathbf{R}} \mathbf{R}$ as the space of all functions from $\mathbf{R}$ to $\mathbf{R}$.

We can now state one of the foundational axioms of mathematics, the *axiom of Choice.*

Axiom of Choice. If $(X_\alpha)_{\alpha \in A}$ is any collection of non-empty sets, then the product space $\prod_{\alpha \in A} X_\alpha$ is non-empty.

Or to put it another way: given any collection $(X_\alpha)_{\alpha \in A}$ of non-empty sets, it is possible to find an object $(x_\alpha)_{\alpha \in A}$ such that for each $\alpha \in A$, x_α is an element of X_α.

Intuitively, this axiom seems very plausible. To show that $\prod_{\alpha \in A} X_\alpha$ is non-empty, one just needs to choose an element x_α of the non-empty set X_α, then put all the x_α together to create the object $(x_\alpha)_{\alpha \in A}$. If A is finite, this is non-controversial. However, if A is infinite, this procedure requires one to make an infinite number of arbitrary choices. Because of this, this axiom comes into conflict with a philosophy of mathematics known as *constructivism*, which insists that one restricts one's attention to objects that can be constructed via precise algorithms, and not rely on arbitrary choices.

Bertrand Russell (1872-1970) described the Axiom of Choice as follows. Suppose you own infinitely many pairs of shoes, and want to select one shoe from each pair. Then one does not need the axiom of choice to do this; one can simply take the left shoe of each pair, and that will achieve your goal. Even a constructivist is happy with that.

However, suppose you also own infinitely pairs of socks, and the two socks in each pair are indistinguishable from each other. Then there is no algorithm for selecting one sock from each pair, and one must make an infinite number of arbitrary choices. The Axiom of Choice asserts that this is actually possible.

Despite being intuitive, the Axiom of Choice does have some counter-intuitive consequences. One of the most famous is the *Banach-Tarski paradox*. Stefan Banach and Alfred Tarski in 1926 showed that one can use the Axiom of Choice to divide the unit ball $B(0, 1)$ (for instance) into a finite number of sets, such that when these sets are translated by some finite amount, their union becomes the ball $B(0, 2)$. This is called a paradox because it contradicts one's intuitive notions about volume; one should not be able to rearrange a ball of length 1 into a ball of length 2 just by cutting and translating! (If you're really interested, I've written up toy version of the Banach-Tarski paradox on the class web page).

The Axiom of Choice is an indispensable tool when working with infinite-dimensional objects, and even in in the ordinary finite-dimensional world. In fact, I've used the Axiom of Choice implicitly a couple times already in this course. (For instance: in a metric space, proving that a point is adherent to a set E if and only if it is the limit of a sequence in E requires a rather mild form of the axiom of choice, known as the *axiom of countable choice*, in which one is only allowed to make a countable number of choices). It is accepted by mainstream mathematicians, but there is still some debate as to its validity by logicians and philosophers. There are formal results in logic which say that the axiom of choice can neither be proven nor disproven from the other axioms of logic and set theory. Furthermore, any "constructive" result which can be proven using Choice can also be proven without Choice, although the proof may be much longer and less intuitive. Because of this, the use of Choice is more a matter of taste than of fundamental truths. However, it is intuitively appealing and very convenient to use, and so we shall continue to use it in this course.

If you are interested in further discussion on this topic, I recommend the URL

`http://math.vanderbilt.edu/ schectex/ccc/choice.html`

4. Zorn's lemma

A particularly useful application of the Axiom of Choice is known as *Zorn's lemma*, discovered by Max Zorn in 1936, and has to do with partially ordered sets.

Definition 4.1. A *partially ordered set* is a set X and a binary relation $<$ between elements of X, which is *anti-symmetric* (if $x < y$, then $y \not< x$) and *transitive* (if $x < y$ and $y < z$, then $x < z$).

Definition 4.2. A *totally ordered set* is a partially ordered set X such that for any two distinct x, y in X, either $x < y$ or $x > y$.

A typical example of a partially ordered set: let X be the collection of all subsets of $\{0, 1, 2\}$, and write $x < y$ if x is a subset of y. This is a partially ordered set with eight elements

$$X = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

It is not totally ordered because not every two elements are comparable, e.g. $\{1\}$ and $\{0, 2\}$ are not comparable.

Often we are interested in the question of whether a partially ordered set X has a *maximal element*. A maximal element is an element x such that $x \neq< y$ for any $y \in X$. For instance, the set X mentioned above has exactly one maximal element, namely $\{0, 1, 2\}$. If one removed this element $\{0, 1, 2\}$ from X, the resulting seven-element set now has three maximal elements, namely $\{0, 1\}$, $\{0, 2\}$, $\{1, 2\}$. As one can see, there can be more than one maximal element if a set is not totally ordered. (It's like how there can be more than one "unbeaten champion" in a contest as long as the unbeaten champions do not play off against each other).

Some sets have no maximal element. For instance the natural numbers $\{1, 2, 3, \dots\}$ with the usual ordering $<$ has no maximal element despite being totally ordered. That's because it contains an infinite chain of increasing elements with no upper bound.

Zorn's lemma is kind of a converse to the above remark:

Lemma 4.3 (Zorn's lemma)**.** *Let X be a partially ordered non-empty set, and suppose that every totally ordered subset Y of X has at least one upper bound (i.e. there exists $x \in X$ such that $x \geq y$ for all $y \in Y$). Then X contains at least one maximal element.*

The proof is extremely technical, and uses the axiom of Choice. (In fact, Zorn's lemma is logically equivalent to the axiom of Choice). I include it here for completeness, but I wouldn't recommend studying it unless you really are interested in this stuff.

Proof (Very optional! Thanks to David Grayson for this proof. I'm skipping some small steps for brevity).

We need another definition. We call a set $Y \subset X$ *well-ordered* if (a) it is totally ordered, and (b) every non-empty subset of Y has a minimal element.

Suppose for contradiction that X did not contain a maximal element. In particular, this means that for every element $x \in X$ there is always some other $x' \in X$ such that $x' > x$. (There's always a bigger fish. - Qui-Gon Jinn).

Every well-ordered subset $Y \subset X$ is totally ordered, and thus has an upper bound. If this upper bound is itself in Y, then there exists a larger upper bound by the preceding paragraph, which cannot then be in Y. Thus every well-ordered subset Y has an upper bound not contained in Y.

By the axiom of Choice, we may thus associate to each well-ordered subset Y an upper bound $g(Y)$ of Y which is not contained in Y. (Note that $g(\emptyset)$ can be any element of X, since any element is an upper bound for the empty set).

Call Y a *g-set* if Y is well-ordered and

$$x = g(\{y \in Y : y < x\})$$

for every $x \in Y$.

Note that if Y is a g-set, then $Y \cup \{g(Y)\}$ is also a g-set.

Also: if Y and Y' are g-sets, then either $Y \subset Y'$ or $Y \subset Y'$. Proof: suppose not. Let x be the least element of $Y - Y'$. One then has $\{y \in Y : y < x\} \subsetneq Y'$. Let x' be the least element of $Y' - \{y \in Y : y < x\}$. Then we have

$$\{y \in Y : y < x\} = \{y \in Y' : y < x'\}.$$

Taking g of both sides we conclude that $x = x'$, contradicting the definition of x and x'.

Let W be the union of all the g-sets. From the above, one may show that W is itself a g-set, and then that $W \cup \{g(W)\}$ is a g-set. But this contradicts the definition of W. ∎

Zorn's lemma is useful for running algorithms which are infinitely long, as it is not always clear that such algorithms do actually terminate. A typical example of how it is used is

Theorem 4.4. *Every vector space V has a basis.*

Proof (Optional) Intuitively, the algorithm to obtain the basis is easy to state. Initialize the basis to be the empty set. Pick any non-zero element of the vector space, and add it to the basis. If it spans, then we're done. Otherwise, we pick another element of the vector space which is linearly independent from the basis, and add it to the basis. If these two span, then we're done. Otherwise, we pick a another element which is linearly independent of the basis, and add it to the basis. Repeat this infinitely often. It may be that even after an infinite number of iterations of this procedure, the basis still has not yet managed to span. In which case, we keep going, adding yet another element to the basis. And so forth. Zorn's lemma ensures us that eventually (possibly after many infinite iterations of this procedure) this algorithm must halt, which can only happen if one actually does find a basis.

More rigorously, let X be the set of all linearly independent subsets of V. At the very least, X contains at least one element, the empty subset $\emptyset$. Thus X is not empty. X is partially ordered if we write $x < y$ for $x \subset y$. Also, we claim every totally ordered set in X has an upper bound. To see this, let Y be a totally ordered subset of X. In other words, Y is a collection of subsets of V, such that each subset consists of linearly independent elements, and for any two subsets in Y, one must contain the other.

Let S denote the union of all the subsets of V which are members of Y. Clearly S is a subset of V, and is an upper bound for Y. Now we claim S is linearly independent. Suppose for contradiction that one could find elements $s_1, \ldots, s_n$ in S and non-zero numbers $a_1, \ldots, a_n$ such that $a_1 s_1 + \ldots + a_n s_n = 0$. Each s_i belongs to some subset in Y. Since Y is totally ordered, one of these subsets is larger than all the others. Thus there is a subset of Y which contains all the s_i. But then this subset would consist of linearly dependent elements, a contradiction. Thus S is linearly independent, and thus in Y.

Since X is non-empty, and every totally ordered set in X has an upper bound. By Zorn's lemma, this means that X contains at least one maximal element, T. T is a linearly independent subset of V. T must span V, otherwise we could pick an element of V not spanned by T and add it to T, contradicting maximality. Since T is both linearly independent and spans, it is a basis. ■

5. Infinite product spaces

We now return to topology, and consider the question of how to define the product topology on an infinite product $\prod_{\alpha \in A} X_\alpha$.

Actually, there are two topologies one could place on this topology. One is called the *weak topology* or *product topology*; the other is called the *strong topology* or *box topology*. The philosophies behind them are different. For the weak topology, one places the bare minimum of open sets that one can get away with; for the strong topology, one puts in as many open sets as one can get away with. It turns out that the weak topology is the more useful, and that is the one we will discuss here. (The box topology is discussed a little bit in the textbook, but we won't ever need it in this course).

To motivate the weak topology, let us introduce the projection operators

$$\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$$

for each $\beta \in A$, in analogy to the projections π_1, π_2 used in the finite case. These projections are defined by

$$\pi_\beta((x_\alpha)_{\alpha \in A}) = x_\beta.$$

In other words, these projections just take out the β component of the object $(x_\alpha)_{\alpha \in A}$ and throw away everything else.

For instance, consider $\prod_{\alpha \in \mathbf{R}} \mathbf{R}$, which is the space of all functions from $\mathbf{R}$ to $\mathbf{R}$. The projection π_β takes any such function f as input, and returns the value of f at β (ignoring all the other aspects of f).

Since the projections π_1 and π_2 were continuous in the finite product case, it seems natural to demand that the π_β are all continuous in the infinite product case. This means that $\pi_\beta^{-1}(U_\beta)$ needs to be open in $\prod_{\alpha \in A} X_\alpha$ whenever U_β is open in X_β.

In the finite product case $X_1 \times X_2$, $\pi_1^{-1}(U_1)$ would be the "vertical slab" $U_1 \times X_2$, while $\pi_2^{-1}(U_2)$ would be a horizontal slab $X_1 \times U_2$. In a triple product $X_1 \times X_2 \times X_3$, things are similar; a set $\pi_1^{-1}(U_1)$ would be the slab $U_1 \times X_2 \times X_3$, and so forth.

More generally, $\pi_\beta^{-1}(U_\beta)$ is the set of all objects $(x_\alpha)_{\alpha \in A}$ in $\prod_{\alpha \in A} X_\alpha$ such that x_β happens to fall inside U_β.

So we'd like the sets $\pi_\beta^{-1}(U_\beta)$ to be open. But this isn't a complete list of open sets. First of all, finite intersections of open sets are supposed to be open. This forces any set of the form

$$\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) \cap \ldots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) \qquad (*)$$

to be open, where $\beta_1, \ldots, \beta_n$ are distinct elements of A and each U_{β_i} is an open set.

The above set is the set of all objects $(x_\alpha)_{\alpha \in A}$ in $\prod_{\alpha \in A} X_\alpha$ such that x_{β_1} falls inside U_{β_1}, x_{β_2} falls inside U_{β_2}, etc.

Following the minimalist philosophy, we can now define the product topology (aka the weak topology) on $\prod_{\alpha \in A} X_\alpha$.

Definition 5.1. The *product topology* on $\prod_{\alpha \in A} X_\alpha$ is the topology generated using the sets of the form (*) as a base.

Exercise 5.2. *Show that the sets of the form (*) do actually obey the axioms for a base.*

Exercise 5.3. *Show that the product topology on $\prod_{\alpha \in \{1,2\}} X_\alpha$ co-incides with the topology on $X_1 \times X_2$ defined in previous notes.*

Because of this definition, we shall refer to sets of the form (*) as *basic open sets.*

To illustrate the product topology we consider the space

$$X := \prod_{k \in \mathbf{Z}} [0, 1];$$

this is the product of a countable number of intervals. Elements of X can be thought of as infinite sequences $(x_1, x_2, x_3, \dots)$ where each component x_k, $k = 1, 2, 3, \dots$ is a number in the interval $[0, 1]$.

There are at least two topologies one can place on X. The first is the product topology which we just defined. Another topology one can define is the uniform (or l^∞) topology, using the l^∞ metric

$$d((x_1, x_2, x_3, \dots), (y_1, y_2, y_3, \dots)) := \sup_{k \geq 1} |x_k - y_k|$$

to define a topology. (There is also the box topology, which is slightly different, but won't be discussed here).

Roughly speaking, the difference between the product topology and the uniform topology is that the product topology is related to pointwise convergence, while the uniform topology is related to uniform convergence.

More precisely, let $x^1, x^2, x^3, \dots$ be a sequence in X, and x be another point in X. Note that the elements of the sequence x^n are themselves sequences:

$$x^1 = (x_1^1, x_2^1, \dots)$$
$$x^2 = (x_1^2, x_2^2, \dots)$$
$$\dots$$

Recall that in order for x^n to converge in x using the l^∞ topology, the sequences x^1, x^2, $\dots$ must converge *uniformly* to x:

$$\sup_{k \geq 1} |x_k^n - x_k| \to 0 \text{ as } n \to \infty.$$

Now let us consider what it means for the sequence x^n to converge in x using the product topology. This means that every neighbourhood of x must contain all but a finite number of elements of the sequence x^n.

In particular, any set of the form $\pi_k^{-1}(U_k)$ that contains x, must also contain all but a finite number of elements of the sequence x^n, since $\pi_k^{-1}(U_k)$ is an open set.

The set $\pi_k^{-1}(U_k)$ contains x if and only if $x_k \in U_k$, and contains x^n if and only if $x_k^n \in U_k$. So, for any $k = 1, 2, 3, \dots$, and any U_k which is open in $[0, 1]$ and contains x_k, we must have that U_k contains all but a finite number of elements of the sequence x_k^n.

In other words, every neighbourhood of x_k in $[0, 1]$ contains all but a finite number of elements of x_k^n. In other words, x_k^n converges to x_k as $n \to \infty$. In other words, the sequence x^n converges *pointwise* to x.

The converse is also true:

Exercise 5.4. *Let $\prod_{\alpha \in A} X_\alpha$ be a product space. If x^n converges pointwise to x, then x^n converges to x in the product topology.*

Proof (Optional) Let x^n converge pointwise to x. We have to show that x^n converges to x in the product topology.

Let V be any neighbourhood of x in the product topology. We have to show that V contains all but a finite number of elements of the sequence x^n.

Since the product topology is given by the basic open sets, and V is a neighbourhood of x, there must exist a basic open set contained in V which contains x. So it suffices to show that all basic open sets which contain x must also contain all but a finite number of elements of the sequence x^n.

So, fix a basic open set which contains x. Such a set has the form (*). In order for this set to contain x, U_{β_i} must contain x_{β_i} for each $i = 1, \ldots, n$. Since x^n converges pointwise to x, $x^n_{\beta_i}$ converges to x_{β_i}, and so U_{β_i} must also contain all but a finite number of elements of the sequence $x^n_{\beta_i}$. Since the number of β_i is also finite, this shows that the basic open set contains all but a finite number of elements of the sequence x^n, and we are done. ∎

Some other properties of the product topology. The projections π_β are all continuous and open (Exercise!). So if $f : Y \to \prod_{\alpha \in A} X_\alpha$ is continuous, then $\pi_\beta \circ f : Y \to X_\beta$ is also continuous for all $\beta \in A$. Conversely, we have the following generalization of Theorem 2.6.

Theorem 5.5. *Let $f : Y \to \prod_{\alpha \in A} X_\alpha$ be such that $\pi_\beta \circ f$ is continuous for all $\beta \in A$. Then f is also continuous.*

We leave the proof as an exercise; it is a modification of the proof of Theorem 2.6.

So, even when the range is an infinite product spaces, a function is continuous if and only if its components are.

6. Tychonoff's theorem

Now we come to one of the most difficult of the fundamental theorems in topology, namely Tychonoff's theorem.

Of all the properties that topological spaces can have (connectedness, compactness, Hausdorff, etc.), the property of being compact is perhaps the most powerful, especially when combined with the Hausdorff property. We've seen some examples of this already. So we would love to have more compact sets out there.

It turns out that the finite product of compact sets is always compact; for instance, since $[0, 1]$ is compact, then any finite cube $[0, 1] \times [0, 1] \times \ldots \times [0, 1]$ is also compact. What about infinite products? One has to be a bit careful. For instance, if one looks at the space

$$X = \prod_{k \in \mathbf{Z}} [0, 1]$$

discussed earlier, this is a product of compact spaces. However, it is not compact in the *uniform* topology. For instance, the sequence

$$x^1 = (1, 0, 0, 0, 0, \dots)$$
$$x^2 = (0, 1, 0, 0, 0, \dots)$$
$$x^3 = (0, 0, 1, 0, 0, \dots)$$
$$x^4 = (0, 0, 0, 1, 0, \dots)$$

are in X, but no subsequence of this sequence converges *uniformly*. On the other hand, this sequence does converge pointwise to the zero sequence $(0, 0, 0, \dots)$. So it still might be possible for X to be compact in the product topology. Fortunately, this is indeed the case.

Theorem 6.1 (Tychonoff's theorem)**.** *If X_α is a compact topological space for every $\alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is compact in the product topology.*

As a typical application, this theorem shows that given any sequence x^1, x^2, ... of bounded sequences, one can always find a subsequence x^{n_1}, x^{n_2}, ... which converge pointwise. Similarly if one replaces "bounded sequences" by "bounded functions".

Suppose the hypotheses of Tychonoff's theorem are true. To show that $\prod_{\alpha \in A} X_\alpha$ is compact, we have to show that every open cover of $\prod_{\alpha \in A} X_\alpha$ has a finite sub-cover. That is actually quite difficult to show, so we'll content ourselves with a weaker version and leave the full version to an appendix.

Definition 6.2. We say that a set V is a *sub-basic* set in $\prod_{\alpha \in A} X_\alpha$ if it is of the form $\pi_\beta^{-1}(U_\beta)$ for some $\beta \in A$ and some set U_β which is open in X_β. We call β the *index* of V, and U_β the *shadow* of V. (This notation is specific to this section, and is not widely used).

Clearly every sub-basic set is open, but of course the converse is not true.

Proposition 6.3 (Baby Tychonoff)**.** *Let X_α is a compact topological space for every $\alpha \in A$, and suppose that $\mathbf{V}$ is an open cover of $\prod_{\alpha \in A} X_\alpha$ such that every member of $\mathbf{V}$ is a sub-basic set. Then $\mathbf{V}$ contains a finite sub-cover.*

This proposition is weaker than the full Tychonoff theorem because it doesn't deal with all possible open covers of $\prod_{\alpha \in A} X_\alpha$; it only handles those open covers which consist entirely of sub-basic sets.

Proof Suppose for contradiction that $\mathbf{V}$ did not contain a finite sub-cover.

Fix some $\beta \in A$, and let $\mathbf{V}_\beta$ denote those sub-basic sets in $\mathbf{V}$ with index β. Each set in $\mathbf{V}_\beta$ has a shadow which is an open set in X_β. Let $\mathbf{W}_\beta$ denote the collection of all such shadows. If $\mathbf{W}_\beta$ covers X_β, then by compactness $\mathbf{W}_\beta$ contains a finite sub-cover of X_β, which implies that $\mathbf{V}_\beta$ contains a finite sub-cover of $\prod_{\alpha \in A} X_\alpha$. This would contradict our assumption that $\mathbf{V}$ has no finite subcover. Hence we may assume that $\mathbf{W}_\beta$ fails to cover X_β for every $\beta \in A$.

By the Axiom of Choice, we can thus pick for every $\beta \in A$ an element x_β in X_β such that x_β is not covered by $\mathbf{W}_\beta$.

Now let x denote the element $x := (x_\beta)_{\beta \in A}$. Certainly x is an element of $\prod_{\alpha \in A} X_\alpha$. Since x_β is not covered by $\mathbf{W}_\beta$, x is not covered by $\mathbf{V}_\beta$ for any $\beta \in A$. But $\mathbf{V}$ is the union of the $\mathbf{V}_\beta$. Thus x is not covered by $\mathbf{V}$. This contradicts the assumption that $\mathbf{V}$ is a cover, and we are done. ■

7. Appendix: Proof of Tychonoff's theorem (Optional)

We now leverage the Baby Tychonoff theorem to the full Tychonoff theorem.

Let X_α be a compact topological space for every $\alpha \in A$. Suppose for contradiction that we could produce an open cover $\mathbf{V}$ of $\prod_{\alpha \in A} X_\alpha$ which had no finite sub-cover.

Let $\mathbf{Z}$ denote the collection of *all* open covers of $\prod_{\alpha \in A} X_\alpha$ which have no finite sub-cover. (This is a set consisting of sets consisting of sets consisting of points of the form $(x_\alpha)_{\alpha \in A}$!) We thus have that $\mathbf{Z}$ is non-empty.

We can make $\mathbf{Z}$ partially ordered by writing $\mathbf{V} < \mathbf{V}'$ for $\mathbf{V} \subset \mathbf{V}'$. In other words, we write $\mathbf{V} < \mathbf{V}'$ if $\mathbf{V}$ is a subcover of $\mathbf{V}'$.

We're going to apply Zorn's lemma to $\mathbf{Z}$. To do this, we need to show that every totally ordered subset $\mathbf{Y}$ of $\mathbf{Z}$ has an upper bound.

Let $\mathbf{Y}$ be a totally ordered subset of $\mathbf{Z}$. In other words, $\mathbf{Y}$ consists entirely of open covers with no finite subcover, such that any given two such covers, one must be a sub-cover of the other. Let $\mathbf{V}_{max}$ be the union of all these covers. In other words, $\mathbf{V}_{max}$ consists of those open sets in $\prod_{\alpha \in A} X_\alpha$ with belong to at least one of the covers in $\mathbf{Y}$.

Clearly $\mathbf{V}_{max}$ is also an open cover, and is an upper bound for $\mathbf{Y}$. But is it in $\mathbf{Z}$? To qualify for membership in $\mathbf{Z}$, $\mathbf{V}_{max}$ needs to contain no finite sub-cover. Well, suppose that $\mathbf{V}_{max}$ did contain a finite subcover $V_1, \ldots, V_n$ of $\prod_{\alpha \in A} X_\alpha$. Each of the V_i must belong to an open cover $\mathbf{V}_i$ in $\mathbf{Y}$, by definition of $\mathbf{V}_{max}$. Since $\mathbf{Y}$ is totally ordered, one of the $\mathbf{V}_i$ is larger than all the others, and thus contains all of $V_1, \ldots, V_n$. But then this $\mathbf{V}_i$ contains a finite sub-cover, contradicting the fact that $\mathbf{V}_i$ is in $\mathbf{Z}$. Thus $\mathbf{V}_{max}$ contains no finite sub-civer and is indeed in $\mathbf{Z}$.

So every totally ordered subset $\mathbf{Y}$ of $\mathbf{Z}$ has an upper bound. By Zorn's lemma, there must therefore exist a maximal element $\mathbf{V}_*$ of $\mathbf{Z}$. This is a really really big open cover with no finite subcover, which is not contained in any other open cover with no finite subcover.

The cover $\mathbf{V}_*$ contains all kinds of open sets. Some are sub-basic open sets; others are not. Let $\mathbf{V}_{**}$ denote the collection of open sets in $\mathbf{V}_*$ which are sub-basic. Then $\mathbf{V}_{**}$ is a subset of $\mathbf{V}_*$. We now claim that $\mathbf{V}_{**}$ is still an open cover of $\prod_{\alpha \in A} X_\alpha$.

To show this, pick a point $x \in \prod_{\alpha \in A} X_\alpha$. Suppose for contradiction that x is not covered by $\mathbf{V}_{**}$. However, it must be covered by $\mathbf{V}_*$, so there exists some open set $V \in \mathbf{V}_*$ which contains x. By definition of the product topology, there must therefore exist a basic open set in V which contains x.

This basic open set is the intersection of a finite number of sub-basic sets $\pi_{\beta_i}^{-1}(U_{\beta_i})$. Consider a single one of these sub-basc sets. This set contains x. It cannot be in $\mathbf{V}_*$, because if it was, then it would be in $\mathbf{V}_{**}$, and we are assuming that x is not covered by $\mathbf{V}_{**}$. So if one adds $\pi_{\beta_i}^{-1}(U_{\beta_i})$ to $\mathbf{V}_*$ one would obtain an open cover which is larger than $\mathbf{V}_*$. By definition of $\mathbf{V}_*$, that means that $\mathbf{V}_* \cup \{\pi_{\beta_i}^{-1}(U_{\beta_i})\}$ must have a finite sub-cover. In particular, this implies that there is a finite subset $\mathbf{V}_i$ of $\mathbf{V}_*$ which covers the complement of $\pi_{\beta_i}^{-1}(U_{\beta_i})$. The set $\mathbf{V}_1 \cup \ldots \mathbf{V}_n$ is thus a finite subset of $\mathbf{V}_*$ which covers the copmlement of the intersection of all the sub-basic sets $\pi_{\beta_i}^{-1}(U_{\beta_i})$, and therefore covers the complement of V. Thus $\mathbf{V}_1 \cup \ldots \cup \mathbf{V}_n \cup \{V\}$ covers $\prod_{\alpha \in A} X_\alpha$, and so $\mathbf{V}_*$ has a finite sub-cover, a contradiction. Thus $\mathbf{V}_{**}$ must cover every point in $\prod_{\alpha \in A} X_\alpha$.

Now we can finally finish the proof. $\mathbf{V}_{**}$ is a cover of $\prod_{\alpha \in A} X_\alpha$ such that every member of $\mathbf{V}_{**}$ is a sub-basic set. Thus by Baby Tychonoff, $\mathbf{V}_{**}$ has a finite sub-cover. Since $\mathbf{V}_*$ contains $\mathbf{V}_{**}$, we thus see that $\mathbf{V}_*$ has a finite sub-cover, and we are (finally!) done.