Mathematics 121 Final
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Total ________

Problem 1. A set $X \subset R^n$ is said to be *star-shaped at the origin* if for every $x \in X$, the line segment $\{tx : 0 \leq t \leq 1\}$ is also contained in X.

Show that if X is star-shaped at the origin, then X is simply connected. (Hint: look at loops through the origin).

Suppose X is star-shaped. To show that X is simply connected, it suffices to show that the fundamental group of X is trivial. It is enough to show that all loops through the origin are homotopic to a point.

Let $\gamma_1 : [0, 1] \to X$ be a loop through the origin in X, so that γ_1 is continuous and $\gamma_1(0) = \gamma_1(1) = 0$. Let $\gamma_0 : [0, 1] \to X$ be the origin path, i.e. $\gamma_0(t) = 0$ for all $0 \leq t \leq 1$. We need to show that γ_0 and γ_1 are homotopic.

Let $F : [0, 1] \times [0, 1] \to X$ be the function

$$F(s, t) = t\gamma_1(s).$$

Clearly F is continuous (since t and $\gamma_1(s)$ are individually continuous). Also observe that F does indeed map to X, because $\gamma_1(s)$ is in X and X is star shaped. Finally, we have

$$F(s, 0) = \gamma_0(s)$$

$$F(s, 1) = \gamma_1(s)$$
$$F(0, t) = 0$$
$$F(1, t) = 0$$

for all $0 \leq s \leq 1$ and $0 \leq t \leq 1$. Thus F is a homotopy from γ_0 to γ_1 in X, as desired.

Problem 2. Let K, L be disjoint compact sets in a normal topological space X. Suppose that $f : K \to \mathbf{R}$ and $g : L \to \mathbf{R}$ are bounded and continuous functions on K, L respectively.

Show that there exists a bounded continuous function $h : X \to \mathbf{R}$ such that $h(x) = f(x)$ for all $x \in K$ and $h(x) = g(x)$ for all $x \in L$.

Since X is normal, X is Hausdorff, and so compact sets are closed. In particular, K and L are closed and disjoint.

By Urysohn's lemma, there exists a continuous function $\phi : K \to [0, 1]$ such that $\phi(x) = 1$ for all x in K, and $\phi(x) = 0$ for all x in L.

Define h by $h(x) = f(x)\phi(x) + g(x)(1 - \phi(x))$. Note that $h(x) = f(x)$ for all $x \in K$ and $h(x) = g(x)$ for all $x \in L$. Also, since f, ϕ, g are all bounded and continuous, h is also bounded and continuous.

Problem 3. A topological space X is said to be *locally separable* if for every point $x \in X$ there is a countable set E in X such that x is in the interior of $\overline{E}$.

Show that every compact, locally separable space is separable.

We need to find a countable set $F \subset X$ such that F is dense in X.

Let x be any element of X. By hypothesis, there exists a countable set $E_x \subset X$ (depending on x) such that $x \in int(\overline{E_x})$. The sets $int(\overline{E_x})$ are open (interiors are also open). Also, as x ranges across X, the sets $int(\overline{E_x})$ form a cover X, since every x in X is contained in its own set $int(\overline{E_x})$. By compactness, this means that X can be covered by finitely many sets $int(\overline{E_x})$. In other words, there exists $x_1, \ldots, x_n \in X$ such that

$$X \subseteq int(\overline{E_{x_1}}) \cup \ldots \cup int(\overline{E_{x_n}}). \tag{1}$$

Define the set F by $F = E_{x_1} \cup \ldots \cup E_{x_n}$. Since each E_{x_i} is countable, and the finite union of countable sets is countable (in fact, the countable union of countable sets is also countable!), F is countable.

Now we have to show that F is dense in X. In other words, we want to show that every non-empty open set in X contains at least one element of F. Let V be a non-empty open set in X. Suppose for contradiction that V contained no element of F. Then it contained no element of the E_{x_i}. Since V is open, V therefore cannot contain any element of the closure of E_{x_i}. But this implies that V cannot contain any points in the right-hand side of (1), which contradicts the fact that V is in X and is non-empty. Thus F is both countable and dense, so that X is separable.

Problem 4.

(a) Show that the continuous image of any connected set is connected.

See Chapter 2, Theorem 8.1. (OK, so it was rather silly to put this question in an open-book final).

(b) Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces. Show that the product space $\prod_{\alpha \in A} X_\alpha$ is connected if and only if each X_α is connected.

The easy half:

Suppose $\prod_{\alpha \in A} X_\alpha$ is connected. Since each π_α is continuous for each $\alpha \in A$, we thus have that $\pi_\alpha(\prod_{\alpha \in A} X_\alpha)$ is connected for each $\alpha \in A$. But this space is just X_α.

The difficult half: (Sorry, this one was a bit too tricky for a final!)

Now suppose that each X_α is connected. We have to show that $\prod_{\alpha \in A} X_\alpha$ is connected.

Let $x = (x_\alpha)_{\alpha \in A}$ be a point in $\prod_{\alpha \in A} X_\alpha$. For every finite set $\alpha_1, \ldots, \alpha_n$ of indices in A, the set

$$Y_{\alpha_1,\ldots,\alpha_n} = \{(y_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} X_\alpha : y_\beta = x_\beta \text{ for all } \beta \notin \{\alpha_1, \ldots \alpha_n\}$$

is homeomorphic to $X_{\alpha_1} \times \ldots \times X_{\alpha_n}$, since the map

$$(y_\alpha)_{\alpha \in A} \mapsto (y_{\alpha_1}, \ldots, y_{\alpha_n})$$

is one-to-one, onto, continuous, and open. By Chapter 2, Theorem 10.6, $X_{\alpha_1} \times \ldots \times X_{\alpha_n}$ is connected, and so $Y_{\alpha_1,\ldots,\alpha_n}$ is also connected.

Each of the $Y_{\alpha_1,\ldots,\alpha_n}$ contains x. By Chapter 2, Theorem 8.2, the set

$$\bigcup_{n=1}^{\infty} \bigcup_{\alpha_1,\ldots,\alpha_n \in A} Y_{\alpha_1,\ldots,\alpha_n}$$

is also connected.

Let's call the above set Z. The closure of a connected set is also connected (exercise!) so $\overline{Z}$ is connected.

We now claim that Z is dense in $\prod_{\alpha \in A} X_\alpha$. This would imply that $\overline{Z} = \prod_{\alpha \in A} X_\alpha$, so we'd be done.

To show that Z is dense, it suffices to show that every non-empty basic open set contains an element of Z. So let

$$\pi_{\alpha_1}^{-1}(U_1) \times \ldots \times \pi_{\alpha_n}^{-1}(U_n)$$

be a basic open set with $U_1, \ldots, U_n$ non-empty. Let $(y_\alpha)_{\alpha \in A}$ be defined by choosing y_{α_k} to be an element of U_k for $k = 1, \ldots, n$, and $y_\alpha = x_\alpha$ for all other values of α. Clearly $(y_\alpha)_{\alpha \in A}$ is in the basic open set, but it is also in $Y_{\alpha_1,\ldots,\alpha_n}$, which is in Z, so we are done.

Problem 5. Let X be a topological space such that, for every $x \in X$, one can find an open neighbourhood U of x which is path-connected.

(a) Show that the path-connected components of X are both open and closed.

Let E be a path-connected component of X. We first show that E is open.

Let $x \in E$. By hypothesis, there exists a neighbourhood $U \subset X$ of x with is path connected. By definition of path connected component, U must be in E. Thus x is in the interior of E. Since every element of E is an interior point, E is open.

Now we show that E is closed. Consider all the path connected components other than E. These components are all open, by the previous argument. Thus their union is open. Taking complements, we thus see that E is closed.

(b) Show that every connected component of X is a path-connected component, and vice versa.

By Chapter 2, Corollary 9.3, every connected component of X is a union of path-connected components. Suppose for contradiction that a connected component F of X contained at least two path-connected components. Call one of these path-connected components E. E is both open and closed in X, hence is both open and closed in F. Thus E and $F \backslash E$ are both open in F and are disjoint, so that F is disconnected. This is a contradiction.

Problem 6.

(a) Let X be a locally compact Hausdorff space, and let $X \cup \{\infty\}$ be the one-point compactification of X.

Suppose that X is connected and non-compact. Show that $X \cup \{\infty\}$ is connected.

Suppose for contradiction that $X \cup \{\infty\}$ were not connected. Then there exists disjoint non-empty open sets V, W in $X \cup \{\infty\}$ such that $X \cup \{\infty\} = V \cup W$.

Since V and W are open in $X \cup \{\infty\}$, $V \cap X$ and $W \cap X$ are open in X. (Note that the relative topology on X induced from $X \cup \{\infty\}$ co-incides with the original topology on X, see Chapter 2, Theorem 7.1). Thus $V \cap X$ and $W \cap X$ are disjoint open sets whose union is X. Since X is connected, this is only possible if one of $V \cap X$ or $W \cap X$ is empty. Without loss of generality we may assume that $V \cap X$ is empty. Since V is non-empty, this means that $V = \{\infty\}$. Since V is open, this means that the complement of V must be compact, hence X is compact, a contradiction.

(b) Let A, B be connected subsets of a topological space X such that $A \cap \overline{B}$ is non-empty. Show that $A \cup B$ is connected.

Suppose for contradiction that $A \cup B$ were disconnected. Then there exists disjoint non-empty open sets V, W in $A \cup B$ such that $A \cup B = V \cup W$.

By arguing as in (a) we see that $V \cap A$ and $W \cap A$ are disjoint open sets whose union is A. Thus one of $V \cap A$ and $W \cap A$ must be empty. Similarly one of $V \cap B$ and $W \cap B$ must be empty. Since V and W are non-empty subsets of $A \cup B$, this leaves only two possibilities: $V = A$, $W = B$ or $V = B$, $W = A$. In either case we must have that A and B are both open and disjoint in $A \cup B$.

By hypothesis, there exists a point x which is both in A and $\overline{B}$. Since A is open in $A \cup B$, there exists a neighbourhood of x in $A \cup B$ which is contained in A. Since $x \in \overline{B}$, this neighbourhood must also contain at least one point of B. But this means that A and B are not disjoint, a contradiction.

Problem 7. Let E be a covering space of X with covering map $p : E \to X$, and let γ be a path in X which starts at a and ends at b.

For each point e in the fiber $p^{-1}(a)$, let α_e be the lift of γ starting at e, and let $f(e)$ be the final point of α_e (i.e. $f(e) = \alpha_e(1)$).

(a) Show that f is a bijection from $p^{-1}(a)$ to $p^{-1}(b)$.

It's enough to show that f has an inverse. For every point e' in the fiber $p^{-1}(b)$, let $\beta_{e'}$ be the lift of γ^{-1} starting at e', and let $g(e')$ be the final point of $\beta_{e'}$. Clearly g is a map from $p^{-1}(b)$ to $p^{-1}(a)$. Now we show that $f \circ g$ and $g \circ f$ are the identity. Actually, I'll just show the second one and leave the first to you.

Let $e \in p^{-1}(a)$. We need to show that $g(f(e)) = e$.

The path $\beta_{f(e)}$ starts at $f(e)$, and $p \circ \beta_{f(e)} = \gamma^{-1}$. The path α_e^{-1} also starts at $f(e)$. Since $p \circ \alpha_e = \gamma$, we also have $p \circ \alpha_e^{-1} = \gamma^{-1}$. By Chapter 3, Theorem 5.2, the paths α_e^{-1} and $\beta_{f(e)}$ must be the same. In particular, they must have the same final point. But the final point of α_e^{-1} is e and the final point of $\beta_{f(e)}$ is $g(f(e))$, and we are done.

(b) Show that if X is path-connected, then all the fibers of E have the same cardinality.

Let $p^{-1}(a)$ and $p^{-1}(b)$ be two fibers in X. Since X is path-connected, there is a path γ in X from a to b. By (a), we can find a bijection from $p^{-1}(a)$ to $p^{-1}(b)$, so these two fibers have the same cardinality.

Problem 8.

(a) Let X, Y be topological spaces, and define an equivalence relation $\sim$ on $X \times Y$ by defining $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 = x_2$.

Show that $X \times Y/\sim$ is homeomorphic to Y.

The elements of $X \times Y/\sim$ are equivalence classes of $\sim$, and thus have the form

$$\{(x, y) : y \in Y\}$$

for some x. The map $f : X \to X \times Y/\sim$ defined by

$$f : x \mapsto \{x\} \times Y$$

is thus a bijection. We need to show that it is continuous and open.

Let V be an open set in $X \times Y/\sim$. This means that $\pi^{-1}(V)$ is open in $X \times Y$, where π is the projection from $X \times Y$ to $X \times Y/\sim$ (it maps points (x, y) to equivalence classes $\{x\} \times Y$). Since π_1 is open by Chapter 2 Theorem 10.2, we thus see that $\pi_1(\pi^{-1}(V))$ is open. But this set is the same as $f^{-1}(V)$ (why?). Thus f is continuous.

Now suppose that U is an open set in X. To show that $f(U)$ is open in $X \times Y/\sim$, it suffices to show that $\pi^{-1}(f(U))$ is open in $X \times Y$. But $\pi^{-1}(f(U))$ is the same as $U \times Y$ (why?), which is clearly open.

(b) Define an equivalence relation $\sim$ on $[0, 1] \times [0, 1]$ by defining $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 - y_2$ is an integer. Show that $[0, 1] \times [0, 1]/\sim$ is homeomorphic to the cylinder

$$C = \{(x, y, z) : 0 \leq z \leq 1, x^2 + y^2 = 1\}.$$

Apply Chapter 2, Theorem 13.4 with $f(x, y) = (\cos(2\pi y), \sin(2\pi y), x)$. Observe that this map is continuous and onto from $[0, 1] \times [0, 1]$ to C, and that $f(x_1, y_1) = f(x_2, y_2)$ if and only if $(x_1, y_1) \sim (x_2, y_2)$. Since $[0, 1] \times [0, 1]$ and C are both compact (closed and bounded subsets of $\mathbf{R}^2$ and $\mathbf{R}^3$) and Hausdorff (all metric spaces are Hausdorff), the claim follows.
