Problem 1. A set $X \subset \mathbb{R}^n$ is said to be \textit{star-shaped at the origin} if for every $x \in X$, the line segment $\{tx : 0 \leq t \leq 1\}$ is also contained in $X$.

Show that if $X$ is star-shaped at the origin, then $X$ is simply connected. (Hint: look at loops through the origin).
Problem 2. Let $K, L$ be disjoint compact sets in a normal topological space $X$. Suppose that $f : K \to \mathbb{R}$ and $g : L \to \mathbb{R}$ are bounded and continuous functions on $K, L$ respectively. Show that there exists a bounded continuous function $h : X \to \mathbb{R}$ such that $h(x) = f(x)$ for all $x \in K$ and $h(x) = g(x)$ for all $x \in L$. 
Problem 3. A topological space \( X \) is said to be \textit{locally separable} if for every point \( x \in X \), there is a countable set \( E \) in \( X \) such that \( x \) is in the interior of \( E \).

Show that every compact, locally separable space is separable.
Problem 4.

(a) Show that the continuous image of any connected set is connected.

(b) Let \( \{X_\alpha\}_{\alpha \in A} \) be a collection of topological spaces. Suppose \( \prod_{\alpha \in A} X_\alpha \) is connected. Show that each \( X_\alpha \) is connected.
Problem 5. Let $X$ be a topological space such that, for every $x \in X$, one can find an open neighbourhood $U$ of $x$ which is path-connected.

(a) Show that the path-connected components of $X$ are both open and closed.

(b) Show that every connected component of $X$ is a path-connected component, and vice versa.
Problem 6.

(a) Let $X$ be a locally compact Hausdorff space, and let $X \cup \{\infty\}$ be the one-point compactification of $X$.

Suppose that $X$ is connected and non-compact. Show that $X \cup \{\infty\}$ is connected.

(b) Let $A, B$ be connected subsets of a topological space $X$ such that $A \cap \overline{B}$ is non-empty. Show that $A \cup B$ is connected.
Problem 7. Let $E$ be a covering space of $X$ with covering map $p : E \rightarrow X$, and let $\gamma$ be a path in $X$ which starts at $a$ and ends at $b$.

For each point $e$ in the fiber $p^{-1}(a)$, let $\alpha_e$ be the lift of $\gamma$ starting at $e$, and let $f(e)$ be the final point of $\alpha_e$ (i.e. $f(e) = \alpha_e(1)$).

(a) Show that $f$ is a bijection from $p^{-1}(a)$ to $p^{-1}(b)$.

(b) Show that if $X$ is path-connected, then all the fibers of $E$ have the same cardinality.
Problem 8.

(a) Let $X, Y$ be topological spaces, and define an equivalence relation $\sim$ on $X \times Y$ by defining $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 = x_2$.

Show that $X \times Y / \sim$ is homeomorphic to $X$.

(b) Define an equivalence relation $\sim$ on $[0, 1] \times [0, 1]$ by defining $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 - y_2$ is an integer. Show that $[0, 1] \times [0, 1] / \sim$ is homeomorphic to the cylinder

$$C = \{(x, y, z) : 0 \leq z \leq 1, x^2 + y^2 = 1\}.$$