Mathematics 121 Final
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June 10, 1997

Problem 1. A set $X \subset R^n$ is said to be *star-shaped at the origin* if for every $x \in X$, the line segment $\{tx : 0 \leq t \leq 1\}$ is also contained in X.

Show that if X is star-shaped at the origin, then X is simply connected. (Hint: look at loops through the origin).

Problem 2. Let K, L be disjoint compact sets in a normal topological space X. Suppose that $f : K \to \mathbf{R}$ and $g : L \to \mathbf{R}$ are bounded and continuous functions on K, L respectively.

Show that there exists a bounded continuous function $h : X \to \mathbf{R}$ such that $h(x) = f(x)$ for all $x \in K$ and $h(x) = g(x)$ for all $x \in L$.

Problem 3. A topological space X is said to be *locally separable* if for every point $x \in X$ there is a countable set E in X such that x is in the interior of $\overline{E}$.

Show that every compact, locally separable space is separable.

Problem 4.

(a) Show that the continuous image of any connected set is connected.

(b) Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces. Suppose $\prod_{\alpha \in A} X_\alpha$ is connected. Show that eaach X_α is connected.

Problem 5. Let X be a topological space such that, for every $x \in X$, one can find an open neighbourhood U of x which is path-connected.

(a) Show that the path-connected components of X are both open and closed.

(b) Show that every connected component of X is a path-connected component, and vice versa.

Problem 6.

(a) Let X be a locally compact Hausdorff space, and let $X \cup \{\infty\}$ be the one-point compactification of X.

Suppose that X is connected and non-compact. Show that $X \cup \{\infty\}$ is connected.

(b) Let A, B be connected subsets of a topological space X such that $A \cap \overline{B}$ is non-empty. Show that $A \cup B$ is connected.

Problem 7. Let E be a covering space of X with covering map $p : E \to X$, and let γ be a path in X which starts at a and ends at b.

For each point e in the fiber $p^{-1}(a)$, let α_e be the lift of γ starting at e, and let $f(e)$ be the final point of α_e (i.e. $f(e) = \alpha_e(1)$).

(a) Show that f is a bijection from $p^{-1}(a)$ to $p^{-1}(b)$.

(b) Show that if X is path-connected, then all the fibers of E have the same cardinality.

Problem 8.

(a) Let X, Y be topological spaces, and define an equivalence relation $\sim$ on $X \times Y$ by defining $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 = x_2$.

Show that $X \times Y/\sim$ is homeomorphic to X.

(b) Define an equivalence relation $\sim$ on $[0,1] \times [0,1]$ by defining $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 - y_2$ is an integer. Show that $[0,1] \times [0,1]/\sim$ is homeomorphic to the cylinder

$$C = \{(x, y, z) : 0 \leq z \leq 1, x^2 + y^2 = 1\}.$$
