Solutions to the actual midterm

- Q1. (A remark: the linear transformation T is similar to the right-shift operator discussed in class, but note that we are only working on $\mathbb{R}^4$ instead of the space $\mathbb{R}^\infty$ of sequences, so things are a little different.

- (a) The rank of T is the dimension of the range $R(T)$, which is

$$R(T) = \{Tv : v \in \mathbb{R}^4\} = \{(0, x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}.$$

This space is clearly spanned by $\{(0,1,0,0), (0,0,1,0), (0,0,0,1)\}$, which are three independent vectors, and so $R(T)$ is three dimensional, and so $\text{rank}(T) = 3$.

- The nullity of T is the dimension of the null space $N(T)$, which is

$$N(T) = \{v \in \mathbb{R}^4 : Tv = 0\} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : (0, x_1, x_2, x_3) = 0\}$$

$$= \{(0,0,0,x_4) : x_4 \in \mathbb{R}\}$$

(not $\{(x_1, 0, 0, 0) : x_1 \in \mathbb{R}\}$ as some of you claimed!). This space is spanned by $(0,0,0,1)$ and so is one-dimensional, hence $\text{nullity}(T) = 1$.

- Of course, because of the dimension theorem one only needs to compute just one of the rank and nullity of T, the other one can then be obtained by the formula $\text{rank}(T) + \text{nullity}(T) = \dim(\mathbb{R}^4) = 4$.

- (b) Observe that T applied to the first basis vector of β is

$$T(1,0,0,0) = (0,1,0,0)$$

and so the first column of $[T]_\beta^\beta$ will be $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Similarly T applied to the second basis vector of β is

$$T(0,1,0,0) = (0,0,1,0)$$

and so the second column of $[T]_\beta^\beta$ will be $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Similarly T applied to the third basis vector of β is

$$T(0,0,1,0) = (0,0,0,1)$$

and so the third column of $[T]_\beta^\beta$ will be $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Finally, T applied to the last basis vector of β is

$$T(0,0,0,1) = (0,0,0,0)$$

so the fourth column vector of $[T]_\beta^\beta$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Putting this all together we get

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

- To compute $[T^2]_\beta^\beta$, etc., we have two options. One is to multiply the above matrix with itself several times (since $[T^2]_\beta^\beta = [T]_\beta^\beta [T]_\beta^\beta$, etc.). The other is to work out what T^2, T^3, etc. are and then compute matrices by feeding in basis vectors as above. We'll take the latter route (the problem with the former route is that one can make errors in the matrix multiplication which are hard to catch, and what's worse an error in computing T^2 can also mess up T^3 and T^4).

- Since

$$T(x_1, x_2, x_3, x_4) = (0, x_1, x_2, x_3)$$

we see that

$$T^2(x_1, x_2, x_3, x_4) = T(0, x_1, x_2, x_3) = (0, 0, x_1, x_2)$$

(One should check this, in order to make sure that one's substitution skills are in order!) and similarly

$$T^3(x_1, x_2, x_3, x_4) = T(0, 0, x_1, x_2) = (0, 0, 0, x_1)$$

and

$$T^4(x_1, x_2, x_3, x_4) = T(0, 0, 0, x_1) = (0, 0, 0, 0).$$

(One can think of these operations as a double right shift, triple right shift, and quadruple right shift on R^4). Feeding in basis vectors, we obtain

$$[T^2]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$[T^3]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$[T^4]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- Q2. (a) Since $P_3(\mathrm{R})$ is already a vector space, all we need to show here is that V is closed under addition and scalar multiplication.

- Let's show that V is closed under addition. Let f and g be two elements of V; thus $f, g \in P_3(\mathrm{R})$ and $f(0) = f(1) = g(0) = g(1) = 0$. Then the polynomial $f + g$ is also in $P_3(\mathrm{R})$, and

$$(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$$

and

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$$

and so $f + g \in V$. Thus V is closed under addition.

- Now let's show that V is closed under scalar multiplication. Let f be an element of V; thus $f \in P_3(\mathrm{R})$ and $f(0) = f(1) = 0$. Then for any scalar c, the polynomial cf is also in $P_3(\mathrm{R})$, and

$$(cf)(0) = cf(0) = c0 = 0$$

and

$$(cf)(1) = cf(1) = c0 = 0$$

and so $cf \in V$. Thus V is closed under scalar multiplication.

- (b) There are a couple ways to do this problem. One way is to observe that if f is in V, then f must have factors $x(x-1)$, and so (since f has degree at most 3) f must be of the form

$$f(x) = (ax + b)x(x-1) = a(x^3 - x^2) + b(x^2 - x).$$

Thus one can use $(x^3 - x^2, x^2 - x)$ as a basis; these polynomials can easily be checked to be in V, and are also clearly linearly independent, and by the above discussion they span V.

- An alternate approach is to write f in co-ordinates

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0.$$

The condition $f(0) = 0$ then translates to $a_0 = 0$, while the condition $f(1) = 0$ then translates to $a_3 + a_2 + a_1 + a_0 = 0$, which simplifies to $a_3 + a_2 + a_1 = 0$. Thus this space is similar to (indeed, it is isomorphic to) the space $W = \{(a_1, a_2, a_3) \in \mathrm{R}^3 : a_1 + a_2 + a_3 = 0\}$, and any basis of this two-dimensional space W will correspond to a basis for V. For instance, one basis for W is $(1, 0, -1)$ and $(1, -1, 0)$; this corresponds to the polynomials $x - x^3$ and $x - x^2$, which is a perfectly acceptable basis for V. Actually, since V is two-dimensional, any two linearly independent polynomials in V will form a basis.

- Q3. There are several ways to prove this statement. The easiest is to use the dimension theorem - but not quite applied to the original operator $T : V \to W$. If we apply it to T itself, we get

$$\dim(V) = \mathrm{rank}(T) + \mathrm{nullity}(T).$$

Since T is one-to-one, nullity$(T) = 0$, and hence

$$\dim(V) = \text{rank}(T) = \dim(R(T)) = \dim(T(V)).$$

This looks like what we want, except that we have V instead of U. To get around this we have to *restrict* T to U - keep the same map T but reduce the domain from V to U. This restricted map is usually called $T|_U$, and it maps U to W (or if you like, from U onto $T(U)$). Thus $T|_U u := Tu$ when $u \in U$, and $T|_U u$ is undefined when $u \notin U$. Since T is one-to-one, the restriction $T|_U$ is also one-to-one (why?). So we have

$$\dim(U) = \text{rank}(T|_U) + \text{nullity}(T|_U) = \text{rank}(T|_U)$$

$$= \dim(R(T|_U)) = \dim(T|_U(U)) = \dim(T(U))$$

as desired.

- A more direct proof is as follows. We know that U is finite dimensional; let's say it has dimension n. Then U has a basis $(u_1, u_2, \ldots, u_n)$. Since $u_1, \ldots, u_n$ spans U, $Tu_1, Tu_2, \ldots, Tu_n$ span $T(U)$ (see Theorem 3 of Week 3 notes). Also, since $u_1, \ldots, u_n$ are linearly independent and T is one-to-one, $Tu_1, \ldots, Tu_n$ is also linearly independent (see Theorem 4 of Week 3 notes). Thus $Tu_1, \ldots, Tu_n$ is a basis for $T(U)$, which implies that $T(U)$ has dimension n. Thus $\dim T(U) = \dim U$.

- Another argument is as follows: The map T maps U to $T(U)$, and does so in a way which is both one-to-one and onto (the onto-ness comes from the definition of $T(U)$. Thus U and $T(U)$ are isomorphic, and hence have the same dimension (see Lemma 9 of Week 4 notes). One can twist this argument a little bit and argue instead by contradiction: if $T(U)$ had larger dimension than U, then the map from U to $T(U)$ could not be onto (from the dimension theorem), while if $T(U)$ had smaller dimension than U, then the map from U to $T(U)$ could not be one-to-one (again from the dimension theorem). Hence the only remaining possibility is that U and $T(U)$ have the same dimension.

- Q4. A direct way to pursue this question is as follows. Write $w_1 := (1, 1, 0)$, $w_2 := (1, 0, 0)$, $w_3 := (0, 0, 1)$, so that $\gamma = (w_1, w_2, w_3)$.

- From the given matrix $[T]_\beta^\gamma$ we see that

$$Tv_1 = w_3; \quad Tv_2 = w_2; \quad Tv_3 = w_1$$

and hence

$$T(v_1+2v_2+3v_3) = w_3+2w_2+3w_1 = (0,0,1)+(2,0,0)+(3,3,0) = (5,3,1).$$

- Another way to do this is as follows. Write $v := v_1 + 2v_2 + 3v_3$. Since

$$[Tv]^\gamma = [T]_\beta^\gamma [v]^\beta$$

and

$$[v]^\beta = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

we see that

$$[Tv]^\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Thus

$$Tv = 3w_1 + 2w_2 + w_3 = (3,3,0) + (2,0,0) + (0,0,1) = (5,3,1).$$

- Q5. There are several ways to pursue this problem, and there are several possible correct answers for T. (Incidentally, the question is asking for T, and not the matrix $[T]_\beta^\beta$ of T, but for those of you who just supplied the matrix, it was pretty clear what you intended here).

- To specify T it would suffice to specify T on the standard basis vectors, i.e. it would suffice to specify $T(1,0,0)$, $T(0,1,0)$, and $T(0,0,1)$. The vector $T(0,0,1)$ must be zero since $(0,0,1)$ is in the null space. The vectors $T(1,0,0)$ and $T(0,1,0)$ cannot be zero since they are not in the null space, but they must lie in the range $R(T)$. Furthermore, $T(1,0,0)$ and $T(0,1,0)$ must span the range (since $T(0,0,1)$ is zero and thus contributes nothing to the span $R(T)$). Thus we can set $T(0,0,1)$ and $T(0,1,0)$ to be any two linearly independent vectors in

$\{(x, y, z) : x + y + z = 0\}$. For instance, one could specify $T(1, 0, 0) = (1, 0, -1)$ and $T(0, 1, 0) = (0, 1, -1)$, in which case

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix},$$

where β is the standard ordered basis. This corresponds to the transformation

$$T(x, y, z) = (x, y, -x - y)$$

(why is this?), which has the desired properties. Other answers are possible.

- Many of you attempted to use the transformation

$$T(x, y, z) = (x, y, 0).$$

I think the fact that the null space was the z-axis led you to think that the z co-ordinate of $T(x, y, z)$ must always be zero, but this is not the case. This transformation does have the correct null space (the z-axis), but has an incorrect range (the range here is the xy-plane, instead of the plane $\{(x, y, z) : x + y + z = 0\}$).

- Another transformation which almost works, but isn't quite correct, is

$$T(x, y, z) = (x + y, -x - y, 0).$$

This transformation does send the z-axis to zero, but the null space here is NOT the z-axis - it is substantially bigger! In this case $N(T)$ is the plane

$$N(T) = \{(x, y, z) : x + y = 0\}$$

which *contains* the z-axis but is not equal to it. Also, the range is *contained in* $\{(x, y, z) : x + y + z = 0\}$ but is not equal to it; for instance, $(0, 1, -1)$ lies in $\{(x, y, z) : x + y + z = 0\}$ but does not lie in the range of this transformation T.

- This question tested your understanding of the concepts of null space and range, and also of the more fundamental concepts of input and output of a function. The range says something about the output of a function, but the null space is more concerned with the input; perhaps the fact that I used the same letters x, y, z to describe both input and output vectors caused some confusion here. To be clearer about this, let's write

$$T(x, y, z) = (a, b, c)$$

so that the three outputs $a = a(x, y, z)$, $b = b(x, y, z)$, $c = c(x, y, z)$ depend somehow on the three inputs x, y, z. (This describes the most general possible function T from R^3 to R^3). The statement that $N(T)$ is the z-axis means two things; firstly, when $x = y = 0$ (so that (x, y, z) is on the z-axis), then a, b, c all have to vanish; secondly, when x and y are not both zero (so that (x, y, z) does not lie on the z-axis), then a, b, c cannot all vanish (otherwise $N(T)$ would contain something in addition to the z-axis). Meanwhile, the statement that $R(T) = \{(x, y, z) : x + y + z = 0\}$ means two things: first of all, that $a + b + c = 0$ for every choice of input x, y, z, and secondly given any (a_0, b_0, c_0) with $a_0 + b_0 + c_0 = 0$, we can find inputs x, y, z such that $a(x, y, z) = a_0$, $b(x, y, z) = b_0$, and $c(x, y, z) = c_0$. So as you can see the notions of null space and range do require some care to distinguish between the inputs and outputs (here denoted by (x, y, z) and (a, b, c) respectively).

- To compute the nullity, we first compute the null space. A vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is in the null space if

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or in other words that

$$x_1 + x_2 + 2x_3 = 0$$

$$2x_1 + x_2 + 3x_3 = 0$$

$$3x_1 + x_2 + 4x_3 = 0.$$

Now we do some Gaussian elimination. Subtracting two copies of the first row from the second, and subtracting three copies of the first row from the third, we obtain

$$x_1 + x_2 + 2x_3 = 0$$

$$-x_2 - x_3 = 0$$

$$-2x_2 - 2x_3 = 0.$$

Multiplying the second row by -1, and then adding two copies of that row to the third, we get

$$x_1 + x_2 + 2x_3 = 0$$

$$x_2 + x_3 = 0$$

$$0 = 0.$$

We can write this in terms of x_3 as

$$x_2 = -x_3; \quad x_1 = -x_3.$$

Thus the null space is given by

$$N(T) = \{ \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} : x_3 \in \mathrm{R} \}$$

(indeed, one can easily check that every vector of the form $\begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix}$ gets sent to zero by T). This space is one-dimensional (indeed, it is the span of $\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \}$), and so the nullity of T is 1. Hence the rank is 2.