(Partial) Solutions to Homework 5

Jon Handy

Q1:

Claim. *Let V and W be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.*

1. *The set $T(V_0)$ is a subspace of W.*
2. $\dim(V_0) = \dim(T(V_0))$.

Proof: 1. This can be found in the solutions for HW3 (Q8).

2. This can also be done using bases, but the quickest way is as follows. Consider the restriction T_{V_0} of T to V_0. Since T_{V_0} is still injective, we have

$$\begin{aligned} \dim N(T_{V_0}) + \dim R(T_{V_0}) &= \dim(V_0) \\ \dim R(T_{V_0}) &= \dim(V_0) \\ \dim(T(V_0)) &= \dim(V_0) \end{aligned}$$

as asserted. □

Q7:

Claim. *Let $T : P_n(\mathbb{R}) \to \mathbb{R}^{n+1}$ be the map*

$$T(f) := (f(0), f(1), f(2), \ldots, f(n)).$$

Then

1. *T is linear, and*
2. *T is an isomorphism.*

Proof: 1. It suffices to show that given $\alpha \in \mathbb{R}$ and $f, g \in P_n(\mathbb{R})$, $T(\alpha f + g) = \alpha T(f) + T(g)$. Here we have

$$\begin{aligned} T(\alpha f + g) &= ((\alpha f + g)(0), (\alpha f + g)(1), \ldots, (\alpha f + g)(n)) \\ &= (\alpha f(0) + g(0), \ldots, \alpha f(n) + g(n)) \\ &= \alpha(f(0), \ldots, f(n)) + (g(0), \ldots, g(n)) \\ T(\alpha f + g) &= \alpha T(f) + T(g), \end{aligned}$$

which is the desired equality.

2. Since $P_n(\mathbb{R})$ and $\mathbb{R}^{n+1}$ are finite-dimensional vector spaces of the same dimension, it suffices to show that T is injective. We recall that a nonzero polynomial of degree n has at most n distinct roots. Thus if $(f(0), \dots, f(n)) = (0, \dots, 0)$ then $f \in P_n(\mathbb{R})$ has $n+1$ zeros, whence f is the zero polynomial. Thus $N(T) = \{0\}$, i.e T is injective. □

Q8:

Claim. *Let A, B be $n \times n$ matrices such that $AB = I_n$, where I_n is the $n \times n$ identity matrix. Than*

1. $L_A L_B = I_{\mathbb{R}^n}$, *where $I_{\mathbb{R}^n}$ is the identity on $\mathbb{R}^n$.*
2. L_B *is a bijection.*
3. $L_B L_A = I_{\mathbb{R}^n}$
4. $BA = I_n$.

Proof: 1. By definition, $L_A x = Ax$ for any $x \in \mathbb{R}^n$. Thus $L_A L_B x = L_A(Bx) = A(Bx) = (AB)x = I_n x = I_{\mathbb{R}^n} x$. It follows that $L_A L_B = I_{\mathbb{R}^n}$.

2. Suppose that $L_B x = L_B y$ for some $x, y \in \mathbb{R}^n$. Then $x = I_{\mathbb{R}^n} x = L_A L_B x = L_A L_B y = I_{\mathbb{R}^n} y = y$, i.e. $x = y$. Thus L_B is injective. Being a map between finite-dimensional vector spaces of the same dimension, L_B must also be surjective. Hence L_B is a bijection.

3. Since L_B is surjective, given $x \in \mathbb{R}^n$ there is $y \in \mathbb{R}^n$ such that $L_B y = x$. Thus $L_B L_A x = L_B L_A L_B y = L_B(L_A L_B) y = L_B(I_{\mathbb{R}^n}) y = L_B y = x$. Thus $L_B L_A x = x$ for every x, whence $L_B L_A = I_{\mathbb{R}^n}$.

4. If $\{e_1, \dots, e_n\}$ is the standard basis in $\mathbb{R}^n$ then $BAe_i = L_B L_A e_i = I_{\mathbb{R}^n} e_i = e_i$, so represented as a matrix in the standard basis, BA is the diagonal matrix with ones on the diagonal, i.e. $BA = I_n$. □

Q10:

Claim. *Let V be a finite dimensional vector space, let $T : V \to V$ be a linear transformation, and let $S : V \to V$ be an invertible linear transformation [i.e. an isomorphism].*

1. $\mathrm{R}(STS^{-1}) = S(\mathrm{R}(T))$ *and* $\mathrm{N}(STS^{-1}) = S\mathrm{N}(T)$
2. $\mathrm{rank}(T) = \mathrm{rank}(STS^{-1})$ *and* $\mathrm{nullity}(T) = \mathrm{nullity}(STS^{-1})$

Proof: 1. Being an isomorphism, S satisfies $S(V) = V$ and $S^{-1}(V) = V$. Thus $\mathrm{R}(STS^{-1}) = STS^{-1}(V) = S(T(S^{-1}(V))) = S(T(V)) = S(\mathrm{R}(T))$.

In a similar way, we may obtain $\mathrm{N}(STS{-}1) = (STS^{-1})^{-1}(0) = ST^{-1}S^{-1}(0) = S(T^{-1}(S^{-1}(0)))$. Since $S^{\pm 1}$ is injective (being an isomorphism), we know that $\mathrm{N}(S^{\pm 1}) = S^{\mp 1}(0) = 0$. Thus we find that $\mathrm{N}(STS^{-1}) = S(T^{-1}(0)) = S\mathrm{N}(T)$. (**Important:** see note below.)

2. Using $Q1$ and (1), we have that $\mathrm{rank}(STS^{-1}) = \dim \mathrm{R}(STS^{-1}) = \dim S(\mathrm{R}(T)) = \dim \mathrm{R}(T) = \mathrm{rank}(T)$. A similar chain of equalities is obtained for the nullity. □

Important note: It is important to note that the equality $(STS^{-1}) = ST^{-1}S^{-1}$ is not true in the sense of a function from V to V–in general T^{-1} is not a function since $T^{-1}(x)$ may not have a single value for x. If $\mathrm{N}(T) \neq 0$, for instance, we would have to have $T^{-1}(0) = x$ and $T^{-1}(0) = y$ for distinct elements $x, y \in \mathrm{N}(T)$. Not being single-valued on an element, $T^{-1} : V \to V$ cannot be a function.

The way we *can* regard T^{-1} is as a function $T^{-1} : \mathcal{P}(V) \to \mathcal{P}(V)$, where $\mathcal{P}(V)$ denotes the power set of V, that is the set of all subsets of V. Thus T^{-1} is viewed as a *set function*, a function from sets to sets. In particular, we regard $T^{-1}(x)$ as $T^{-1}(\{x\}) = \{y \in V | Ty = x\}$. Then $ST^{-1}S^{-1} : \mathcal{P}(X) \to \mathcal{P}(X)$ is regarded as a composition of set functions. This is the fashion in which our proof actually makes sense.