(Partial) Solutions to Homework 2

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Q4: For this problem we want to find a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ for which $R(T) = N(T)$. As most people saw, this will require $\dim R(T) = \dim N(T) = 1$. Let $\{x\}$ be a basis for $N(T)$. Extending this to a basis $\{x, y\}$ of $\mathbb{R}^2$, we need to have $Ty = \alpha x$ for some $\alpha \in \mathbb{R}$. Giving this a coordinate representation with respect to this (ordered) basis, T is determined by $(a, b) \mapsto (\alpha b, 0)$. Thus, in some sense every such linear map T looks like a truncation composed with a flip $((x, y) \mapsto (0, y) \mapsto (y, 0))$ up to some scalar multiple.

Q7:

Claim. *Let V be an n-dimensional vector space with an ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$. If we define $T : V \to F^n$ by $T(x) = [x]_\beta$, then T is linear.*

Proof: If $x, y \in V$ can be written $x = \sum a_1 v_i$ and $y = \sum b_i v_i$, then

$$[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad [y]_\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Then we have, for $c, d \in F$

$$\begin{aligned} T(cx + dy) &= [cx + dy]_\beta \\ &= \begin{pmatrix} ca_1 + db_1 \\ \vdots \\ ca_n + db_n \end{pmatrix} \\ &= c \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + d \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\ T(cx + dy) &= c[x]_\beta + d[y]_\beta. \end{aligned}$$

It follows that for every $c, d \in F$ and every $x, y \in V$, $T(cx + dy) = cTx + dTy$. Thus T is linear. □

Q8:

Claim. *Let V, W be vector spaces, and let $T : V \to W$ be a linear transformation.*

1. *If* U *is a subspace of* W *then the set*

$$T^{-1}(U) := \{v \in V : T(v) \in U\}$$

is a subspace of V*. Thus* $N(T)$ *is a subspace of* V*.*

2. *If* X *is a subspace of* V *then the set*

$$T(X) := \{Tv : v \in X\}$$

is a subspace of W*. Thus* $R(T)$ *is a subspace of* W*.*

Proof: 1. Note that $0 \in T^{-1}(U)$ since $T0 = 0 \in U$. If $\alpha \in F$ and $v \in T^{-1}(U)$ then $T(\alpha v) = \alpha Tv \in U$ since $Tv \in U$ and U is closed under scalar multiplication. Thus $\alpha v \in T^{-1}(U)$. Similarly, if $v, w \in T^{-1}(U)$ then $T(v + w) = Tv + Tw \in U$ since U is closed under addition. Thus $v + w \in T^{-1}(U)$.

Since $\{0\}$ is a subspace, it follows that $N(T)$ is a subspace.

2. Here $0 \in X$ so $0 \in T(X)$ since $T0 = 0$. If $\alpha \in F$ and $Tv \in T(X)$ then $\alpha Tv = T(\alpha v) \in T(X)$ since $v \in X$ and X is closed under scalar multiplication. Thus $\alpha v \in T(X)$. Similarly, if $Tv, Tw \in T(X)$ then $Tv + Tw = T(v + w) \in T(X)$ since X is closed under addition. Thus $v + w \in T(X)$.

Since V is a subspace, it follows that $T(V) = R(T)$ is a subspace. $\square$