PRACTICE FINAL MATH 115A.

WARNING: Your final exam may include topics that are not covered in this practice exam.

Please, e-mail comments to Professor Munoz-Garcia (**munoz@math.ucla.edu**)

EXERCISE 1. Let P_2 be the set of polynomials with real coefficients of degree ≤ 2. We consider the set of polynomials

$$S = \{p(x) \in P_2 \ such \ that \ p(0) + p"(0) = 0\}$$

1.a. Is S a subspace of P_2?

1.b. Find a basis for S and determine the dimension of S.

EXERCISE 2. We consider the matrix $Q = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.
Let $T : M_{2\times 2}(R) \longrightarrow M_{2\times 2}(R)$ be the map defined by $T(M) = QM$.
2.a. Is T a linear transformation?

2.b. What is the Kernel of T? Find the dimension of $Ker\ (T)$.

2.c. Find a basis of the range of T. Find the dimension of $R\ (T)$.

2.d. Is T one-to-one? Is T onto? Is T an isomorphism?

EXERCISE 3. We consider the real vector space $F(R)$ consisting of all functions defined on R. Determine whether or not $f_1(x) = cos\ x$, $f_2(x) = cos\ 2x$ and $f_3(x) = cos\ 3x$ are linearly independent.

EXERCISE 4. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & c & 0 \end{pmatrix}$$

where c is a real number.
4.1. Find the eigenvalues of A.

4.2. For which values of c is the matrix A diagonalizable?

EXERCISE 5. Let M be a real $n \times n$ matrix, let λ be an eigenvalue of M. Let k be an integer with $k \geq 1$.
5.a. Prove that λ^k is an eigenvalue of M^k. What are the corresponding eigenvectors?

5.b. Let A be a real $n \times n$ matrix. Prove that if A is nilpotent, that is, if $A^r = 0$ for some integer $r \geq 1$, then 0 is its only eigenvalue.

EXERCISE 6. Let V be a finite dimensional vector space over a field F. Let W_1 and W_2 be subspaces of V.

Recall that we say that V is the direct sum of W_1 and W_2 if $V = W_1 + W_2$ (that is, every vector $v \in V$ can be written $v = x+y$ where $x \in W_1$ and $y \in W_2$) AND $W_1 \cap W_2 = 0$.

We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.

6.a. Show that $V = W_1 \oplus W_2$ if and only if every vector $v \in V$ can be written $v = x+y$ where $x \in W_1$ and $y \in W_2$ are unique.

6.b. Show that $V = W_1 \oplus W_2$ if and only if there exists a linear transformation

$$T : V \longrightarrow V$$

satisfying the three following conditions:

i. $Ker(T) = W_1$
ii. $R(T) = W_2$
iii. $T^2 = T \circ T = T$.

EXERCISE 7. Regarding the complex numbers as a vector space over the real numbers, define

$$(z_1, z_2) = \frac{1}{2}(z_1\bar{z}_2 + z_2\bar{z}_1)$$

7.a. Show that $(\ ,\ )$ is an inner product.

7.b. If $z_1 = \alpha_1 + \alpha_2 i$ and $z_2 = \beta_1 + \beta_2 i$, show that

$$(z_1, z_2) = \alpha_1\beta_1 + \alpha_2\beta_2$$

Show that (z, z) is the square of the absolute value of the complex number z in the usual sense.

7.c. Let M_a be the linear transformation of C^n into itself defined by $M_a(z) = az$. Show that $M_a(z_1, M_a(z_2) = a\bar{a}(z_1, z_2)$.

7.d. With M_a defined as in (c), show that M_a is an isometry if and only if $|a| = 1$.

7.e. Letting $T(z) = \bar{z}$, show that T is an isometry.

EXERCISE 8. Let U be an orthogonal matrix. Show that the following are equivalent:

8.a. U is symmetric.

8.b. $U^2 = I$.