# LECTURE NOTES 7 FOR 254A

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## 1. The metaplectic representation

This material (like the last section) is largely taken from E. Stein's "Harmonic Analysis".

Up until this point we have divided the phase plane $\mathbf{R} \times \mathbf{R}$ into rectangles obeying the uncertainty principle which were parallel to the axes (or more succinctly, *tiles*), or to unions of tiles. The flexibility to choose one's tile decompositions as adapted to the problem at hand is already extremely valuable. However, it is possible to be even more flexible with the phase plane than we already have been doing, and consider regions in the phase plane which are not unions of tiles. In particular we can consider rectangles in phase space whose sides are *not* parallel to the axes, so that their frequency increases or decreases linearly with time over the lifespan of the tile. (Applied mathematicians sometimes refer to such objects as "chirps").

A prototypical example of this are the *imaginary Gaussians* $\exp(\pi i k x^2)$, where $k$ is a real number. At any location $x_0$ in physical space, we have the Taylor expansion

$$\exp(\pi i k x^2) = \exp(\pi i k x_0^2) \exp(2\pi i k x_0 (x - x_0)) + O(|x - x_0|^2),$$

so heuristically one expects this function to have a "frequency" of $kx_0$ at $x_0$. Informally, we thus see that the phase space portrait of this imaginary Gaussian seems to want to be the line $\{(x, \xi) : \xi = kx\}$.

More generally, if $f(x)$ is a function which has frequency $\xi$ at $x_0$ (whatever that means), then $\exp(\pi i k x^2)$ is a function which has frequency $\xi + kx_0$ at $x_0$. Thus the operation

$$f(x) \mapsto \exp(\pi i k x^2) f(x)$$

seems to somehow correspond to the phase space transformation

$$(x, \xi) \mapsto (x, \xi + kx).$$

This is similar to how translations $f(x) \mapsto f(x - x_0)$ correspond to

$$(x, \xi) \mapsto (x + x_0, \xi),$$

modulations $f(x) \mapsto e^{2\pi i x \xi_0} f(x)$ correspond to

$$(x, \xi) \mapsto (x, \xi + \xi_0),$$

and scaling $f(x) \mapsto |\lambda|^{-1/2} f(x/\lambda)$ for any non-zero real $k$ corresponds to

$$(x, \xi) \mapsto (\lambda x, \xi/\lambda).$$

Finally, the Fourier transform $f \mapsto \hat{f}$ corresponds to

$$(x, \xi) \mapsto (\xi, -x).$$

It is now natural to wonder how this could be extended more generally. Given any diffeomorphism of phase space

$$\Phi : (x, \xi) \mapsto \Phi(x, \xi)$$

is it possible to assign a linear operator $T_\Phi$ (possibly an isometry on $L^2$), such that the phase space portrait of $T_\Phi f$ (whatever that means) is the image of the phase space portrait of $f$ under $\Phi$?

To simplify things, let us assume that $\Phi$ is a linear transformation:

$$\Phi(x, \xi)^\dagger = A(x, \xi)^\dagger,$$

where $(x, \xi)^\dagger$ is the column vector with entries $x$, $\xi$, and $A$ is a two-by-two matrix. (Non-linear transformations are also possible, and fall under the broad category of what are known as *Fourier integral operators*. This is a subject in itself, but we will not discuss it here. In many applications we can simulate non-linear transformations by applying linear phase space transformations patched together with phase space cutoffs.)

Modulations by imaginary Gaussians correspond to the matrices

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix};$$

the Fourier transform gives

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and scaling gives

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Then of course one can compose these matrices to obtain further linear transformations. It turns out that one can obtain all (oriented)-area-preserving transformations (i.e. $Sp(2, \mathbf{R})$) in this way. (Note that each generating matrix is area-preserving and orientation preserving). If one also threw in the anti-linear conjugation map $f \mapsto \overline{f}$ then one could in fact get all area-preserving maps.

(There is a question of ambiguity - can one area-preserving matrix correspond to more than one transformation? The answer is yes, but only up to a sign, which is usually not important. The fancy way to say this is that we have a representation not of $Sp(2, \mathbf{R})$, but of the double cover of $Sp(2, \mathbf{R})$.

Why only area preserving maps? It seems very consistent with the uncertainty principle, since that principle singles out rectangles of area 1 as being special. A slightly more rigorous proof proceeds as follows. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Consider the infinitesimal translation and infinitesimal modulation operators $\frac{d}{dx}$ and $2\pi ix$. They should probably get mapped via $T_\Phi$ to the operators $a\frac{d}{dx} + b2\pi ix$ and $c\frac{d}{dx} + d2\pi ix$ respectively. But we have the commutator identity

$$[\frac{d}{dx}, 2\pi ix] = 2\pi i$$

which forces

$$[a\frac{d}{dx} + b2\pi ix, c\frac{d}{dx} + d2\pi ix] = 2\pi i$$

which eventually simplifies to $ad - bc = 1$, i.e. $A$ is area preserving.

More generally, let us move from $\mathbf{R}$ to $\mathbf{R}^n$. Phase space $\{(x,\xi) : x, \xi \in \mathbf{R}^n\}$ is now 2n-dimensional. In order for a linear transformation

$$\Phi(x,\xi)^\dagger = A(x,\xi)^\dagger$$

given by a $2n \times 2n$ matrix $A$ to have a chance of being associated to some reasonable transformation $T_\Phi$, one can repeat the same computation as before and find that $A$ must be - not area preserving, because area doesn't make any sense, but *symplectic*, in that it preserves the symplectic form

$$\omega((x,\xi),(x',\xi')) = x \cdot \xi' - x' \cdot \xi$$

(this is the generalization of the area form in two dimensions). As a consequence it also happens to preserve Lebesgue measure on phase space as this measure turns out to essentially be the top form $\omega^n$.

On the other hand, it turns out that every symplectic matrix (i.e. an element of $Sp(2n, \mathbf{R})$ - not to be confused with $Spin(2n, R)$!) can be written as the product of the higher-dimensional analogues of the generating matrices given earlier (but with $k$ replaced by a symmetric matrix, $\lambda$ by an invertible matrix, and $\lambda^{-1}$ replaced by $(\lambda^t)^{-1}$). This is a nice little exercise in Lie group and Lie algebra theory. Again there is an ambiguity of sign, so actually we get a unitary representation on $L^2$ of the double cover of the symplectic group $Sp(2n, \mathbf{R})$. This representation is known as the *metaplectic representation*. In principle one can write down explicitly the operator corresponding to some matrix (it will be some combination of scalings, modulation by imaginary Gaussian, and the Fourier transform), but this turns out to be unenlightening.

Let us see how the Gaussian $e^{-\pi x^2}$ transforms in one dimension. After scaling and modulation, we are left with a $L^2$-normalized complex Gaussian

$$|\mathrm{Re}(Z)|^{-1/2} \exp(-\pi x Z x)$$

where $Z$ is a complex number with $\mathrm{Re}(Z) > 0$ (we have written the quadratic expressions in this way for easy extension to higher dimensions). One can take the Fourier transform of this using contour integration to obtain

$$|\mathrm{Re}(Z^{-1})|^{-1/2} \exp(-\pi x Z^{-1} x).$$

Thus the Fourier transform of a complex Gaussian is another complex Gaussian. Thus the metaplectic representation preserves the space of complex Gaussians.

## 2. Solvability of vector fields

We now apply the above metaplectic representation (or more precisely, the intuition behind this representation) to the topic of solvability of vector fields in $\mathbf{R}^n$.

Let $X := a(x) \cdot \nabla = \sum_{j=1}^n a_j(x)\partial_{x_j}$ be a real smooth vector field on $\mathbf{R}^n$, so that the $a_j(x)$ are real smooth functions on $\mathbf{R}^n$. Let us also assume the non-degeneracy condition $a(x) \neq 0$ for all $x$.

Suppose we are given a smooth function $f$ and want to solve (locally, at least) the equation

$$Xu = f.$$

This can be easily accomplished by applying a suitable local diffeomorphism to change $X$ to a simple vector field such as $\partial_{x_1}$, and then just using the fundamental theorem of calculus. (One takes the integral curves of $X$, and some reference transverse hyperplane, and constructs the a diffeomorphism which leaves the hyperplane invariant but straightens the integral curves to unit speed parallel lines).

Now suppose that we take a *complex* smooth vector field $X + iY$, where $X$ is as before and $Y := b(x) \cdot \nabla = \sum_{j=1}^n b_j(x)\partial_{x_j}$ for some real smooth $b$. We assume $a(x) + ib(x) \neq 0$. Given the simple resolution to the local solvability of real vector fields it seems natural to conjecture that one also has local solvability of complex vector fields, say at the origin.

In one dimension the problem is really simple: $(a(x) + ib(x))u'(x) = f(x)$ is of course trivial to solve. In two dimensions things are almost as easy, as we shall now see.

First suppose that $X$ and $Y$ are linearly dependent, so that (for instance) $Y = k(x)X$. Then one can easily reduce to a real vector field problem by dividing out by $(1 + ik(x))$. So let's assume that they are linearly independent.

In the two-dimensional case we apply a linear transformation so that $a(0) = e_1$ and $b(0) = e_2$. Let us first consider the constant coefficient case

$$(\partial_{x_1} + i\partial_{x_2})u = f.$$

This can be easily solved by using the Fourier transform:

$$2\pi i(\xi_1 + i\xi_2)\hat{u} = \hat{f}.$$

Since $\xi_1 + i\xi_2$ is always invertible on $\mathbf{R}^2$, one can easily solve for $u$ (globally, even!).

Now consider the variable co-efficient case. At sufficiently close distance to the origin, the problem becomes

$$(\partial_{x_1} + i\partial_{x_2})u + O(\varepsilon|\nabla u|) = f.$$

Being a local solvability problem, we can even insert cutoffs and write this as

$$u = \eta(\partial_{x_1} + i\partial_{x_2})^{-1}(f + O(\eta\varepsilon|\nabla u|)).$$

This type of problem can be solved by iteration in Sobolev spaces ($W^{1,2}$ will do), treating the $O(\eta\varepsilon|\nabla u|)$ term as an error, because of the ellipticity of $(\partial_{x_1} + i\partial_{x_2}) = d/dz$. We leave this as an exercise.

We now return to the general dimensional case. First suppose that $X$ and $Y$ commute, $[X, Y] = 0$. (In general, the commutator $[X, Y]$ of two vector fields does not vanish, but is another vector field

$$[X, Y] = \sum_{i,j} (a_i \partial_{x_i} b_j - b_i \partial_{x_i} a_j) \partial_{x_j}.$$

). Then the flow lines of $X$ and $Y$ commute, which means that we can simultaneously locally straighten both $X$ and $Y$ to, say, $\partial_{x_1}$ and $\partial_{x_2}$. Then the problem becomes

$$(\partial_{x_1} + i\partial_{x_2})u = f.$$

But this is OK from the two-dimensional theory.

Now suppose that $[X, Y]$ is not zero, but is a multiple of $X$: $[X, Y] = k(x)X$. Let us now consider the quantity $[m(x)X, Y]$, where $m$ is some multiplier to be determined later. By the product rule we have

$$[m(x)X, Y] = m(x)[X, Y] - Y(m)X = (km - Y(m))X.$$

If we now choose $m$ to obey the equation

$$Y(m) = km$$

(which can be obtained by exponentiating the local solution to $Y(u) = k$), then $m$ is non-zero and we have that $m(x)X$ and $Y$ commute. Thus we can straighten both of these to $\partial_{x_1}$ and $\partial_{x_2}$, so that the equation becomes

$$(\frac{1}{m}\partial_{x_1} + i\partial_{x_2})u = f.$$

Once again this follows from the two-dimensional theory.

Indeed, a similar thing happens whenever $[X, Y]$, $X$, and $Y$ are linearly dependent. What if these three vectors are linearly independent? A surprising result of Hans Lewy in 1957 states that in this case local solvability can fail. Indeed, we have

**Theorem 2.1.** *[Hörmander] Suppose we are in three or more dimensions. If $[X, Y]$, $X$, and $Y$ are linearly independent at the origin, then there exists smooth $f$ such that $(X + iY)u = f$ is not locally solvable at the origin (even for distributional $u$).*

We shall prove a toy version of this theorem, namely

**Theorem 2.2.** *Suppose we are in three or more dimensions. If $[X, Y]$, $X$, and $Y$ are linearly independent at the origin, then there exists $L^2$ $f$ such that $(X+iY)u = f$ is not locally solvable at the origin for $L^2$ $u$.*

In the next section we give the proof of this toy theorem, guided by the metaplectic representation. Our proof shall be based on some recent notes of Maciej Zworski.

## 3. Proof of Theorem 2.2

To disprove local solvability we first make some remarks about *global* solvability.

Let's start in one dimension. Suppose we have a Schwartz function $f(x)$, and wish to find a Schwartz function $u(x)$ such that $\frac{d}{dx}u = f$. This can only be done if $\int f = 0$, otherwise $u$ will have differing values at $\pm\infty$. This can be viewed as a reflection of the fact that $(\frac{d}{dx})^* 1 = 0$, since

$$\int f = \langle f, 1 \rangle = \langle \frac{d}{dx} u, 1 \rangle = \langle u, (\frac{d}{dx})^* 1 \rangle.$$

Of course, this is only an obstruction to global solvability, and does not prevent local solvability.

Now consider the problem

$$(x - \frac{d}{dx}) u = f$$

with $f$ Schwartz, solving for Schwartz $u$. The analogue of the identity $(\frac{d}{dx})^* 1 = 0$ here is

$$(x - \frac{d}{dx})^* e^{-x^2/2} = (x + \frac{d}{dx}) e^{-x^2/2} = 0$$

and hence

$$\langle f, e^{-x^2/2} \rangle = \langle u, (x + \frac{d}{dx}) e^{-x^2/2} \rangle = 0.$$

This, one can only solve this equation if $f$ is orthogonal to $e^{-x^2/2}$. Again, this is only a global obstruction, not a local one, because $e^{-x^2/2}$ is not compactly supported; however, it is "almost local" in some sense.

The idea will be to generate a large family of almost local obstructions to solvability which become narrower and narrower. If, for instance, we had a class of obstructions of the form

$$\langle f, e^{-M\pi x^2} \rangle = 0$$

for (say) all integers $M$, then we know that we would not have solvability if $f(0) \neq 0$, since

$$\langle f, e^{-M\pi x^2} \rangle = f(0) M^{-1/2} + o(M^{-1/2}).$$

Thus one can convert a bunch of increasingly narrow almost-local obstructions to a genuinely local obstruction.

So, the question is then how to obtain such obstructions. Informally, we would like to somehow transform the vector field $X + iY = \sum_j c_j(x) \partial_{x_j}$ into (inhomogeneous) vector fields such as $Max_1 - a\frac{d}{dx_1}$ (as this would give the localized obstruction $\langle f, e^{-M\pi x_1^2} \psi(x_2, \ldots, x_n) \rangle = 0$ for arbitrary $\psi$).

To do this we shall reason informally. We know that differentiation $\partial_{x_j}$ is like multiplying by $2\pi i \xi_j$ on the Fourier side. Thus the operator

$$X + iY = \sum_j c_j(x) \partial_{x_j}$$

is in some sense multiplying the function in phase space by

$$\sum_j c_j(x)(2\pi i\xi_j)$$

(so that a function with position $x$ and frequency $\xi$ would get multiplied by the above amount). We write the above function as

$$p(x,\xi) + iq(x,\xi)$$

where $p$, $q$ are the real functions

$$p(x,\xi) := -2\pi \sum_j b_j(x)\xi_j; \quad q(x,\xi) := 2\pi \sum_j a_j(x)\xi_j.$$

On the other hand, we would like to transform this somehow to the operator $Max_1 - a\frac{d}{dx_1}$, which corresponds to the multiplier

$$Max_1 - 2\pi i a\xi_1.$$

So the question arises how to transform one to the other, perhaps by using translations, modulations, and the metaplectic operations.

We are given that $X$, $Y$, and $[X,Y]$ are linearly independent at the origin, i.e. the vectors

$$(a_i(0))_{i=1}^n, \quad (b_i(0))_{i=1}^n, \quad (\sum_j a_j(0)\partial_j b_i(0) - b_j(0)\partial_j a_i(0))_{i=1}^n$$

are linearly independent. We can thus find a vector $\xi \neq 0$ which is orthogonal to the first two vectors but not to the third, i.e.

$$\sum_i a_i(0)\xi_i = \sum_i b_i(0)\xi_i = 0; \quad \sum_{i,j}(\sum_j a_j(0)\partial_j b_i(0) - b_j(0)\partial_j a_i(0))\xi_i \neq 0.$$

In other words, we have

$$p(0,\xi) = q(0,\xi) = 0, \quad \{p,q\}(0,\xi) \neq 0$$

where $\{p,q\}$ is the *Poisson bracket* of $p$ and $q$:

$$\{p,q\} := \omega(\nabla_\omega p, \nabla_\omega q) = \sum_i \frac{dp}{dx_i}\frac{dq}{d\xi_i} - \frac{dp}{d\xi_i}\frac{dq}{dx_i}.$$

The quantities $p$, $q$, and $\{p,q\}$ are linear in $\xi$, thus by scaling $\xi$ if necessary we can choose $\{p,q\}(0,\xi) = -2\pi$. Thus the vectors $\nabla_\omega p(0,M\xi)$ and $\nabla_\omega q(0,M\xi)$ have a symplectic inner product of $-2\pi M$.

We now invoke a simple linear algebra lemma, which states that if we have two pairs of vectors $(v,w)$ and $(v',w')$ in $\mathbf{R}^{2n}$ with the same symplectic inner product (i.e. $\omega(v,w) = \omega(v',w')$, then it is possible to find a symplectic linear transformation which maps $v$ and $w$ to $v'$ and $w'$ respectively. (This fact is related to *Darboux's Theorem*, which states that all symplectic structures are locally equivalent). This theorem can be seen by first rotating $v$ to be parallel to $v'$, then dilating $v$ at the expense of some other vector to be equal to $v'$, and then rotating and shearing $w$ to equal $w'$. From this lemma we may therefore find a symplectic $2n \times 2n$ matrix $A = A_M$ such that

$$A\nabla_\omega p(0,M\xi) = (0,e_1) \text{ and } A\nabla_\omega q(0,M\xi) = (2\pi M e_1, 0)$$

which means that

$$d(p \circ \mu \circ A^{-1})(0,0) = (e_1^*, 0) \text{ and } d(q \circ \mu \circ A^{-1}) = (0, -2\pi e_1^*),$$

where $e_i^*$ is the standard dual basis to $e_i$, and $\mu(x,\eta) := (x, M\xi + \eta)$ is the frequency shift by $M\xi$. Since $p$ and $q$ vanish at $(0, M\xi)$, we thus see from Taylor expansion that

$$p \circ \mu \circ A^{-1}(x,\eta) = x + O((|x| + |\eta|)^2); \quad q \circ \mu \circ A^{-1}(x,\eta) = -2\pi M\eta + O((|x| + |\eta|)^2).$$

Thus, the function

$$(p + iq) \circ \mu \circ A^{-1}(x,\eta)$$

is roughly the function $x - 2\pi i M\eta$. Heuristically, this means that the operator

$$T_A e^{-2\pi i M x \cdot \xi}(X + iY) e^{2\pi i M x \cdot \xi} T_A^{-1}$$

is roughly the operator $x - M\frac{d}{dx}$. This means that in order for

$$T_A e^{2\pi i M x \cdot \xi}(X + iY) e^{2\pi i M x \cdot \xi} T_A^{-1} u = f$$

to be solvable, we expect an obstruction to the effect that $f$ must be orthogonal to $e^{-M|x|^2/2}$. Thus, in order for

$$(X + iY)u = f$$

to be solvable, we expect $f$ to be orthogonal to $e^{2\pi i M x \cdot \xi} T_A^* e^{-M|x|^2/2}$ - which is some sort of chirp of thickness about $M^{-1/2}$. As one lets $M \to \infty$ this gives increasingly narrow almost local obstructions, which should hopefully lead to a purely local obstruction to solvability.

It is possible to make this proof more rigorous by explicitly computing $A$, $T_A$, etc. We shall however short-circuit the proof by jumping right to the punchline - namely, that one should use chirps to construct almost-local obstructions.

The idea is going to be as follows. For large $M$, we will find localized chirps $\phi_M$, such that $\|\phi_M\|_2 \sim 1$ and that $|\phi_M(x)| \lesssim M^{-1}$ for $|x| \gtrsim 1$ and

$$\|(X + iY)^* \phi_M\|_2 \to 0 \text{ as } M \to \infty.$$

Assuming this, we can thus find a sequence $M_n$ such that $M_n \geq 100^n$ and

$$\|\phi_{M_n}\|_2 = 1 \text{ and } \|(X + iY)^* \phi_{M_n}\|_2 \leq 100^{-n}.$$

Let $f$ be an $L^2$ function to be chosen later. If we could find an $L^2$ solution $u$ with norm $\|u\|_2 \leq A$ to the equation

$$(X + iY)u = f$$

in a neighbourhood of 0, then we would have

$$\eta(X + iY)u = \eta f$$

for some suitable cutoff $\eta$ around 0. Thus

$$(X + iY)(\eta u) = \eta f + (X(\eta) + iY(\eta))u.$$

If we take the inner product $\phi_{M_n}$ we obtain

$$\langle \eta u, (X + iY)^* \phi_{M_n} \rangle = \langle \eta f, \phi_{M_n} \rangle + \langle (X(\eta) + iY(\eta))u, \phi_{M_n} \rangle.$$

Since $\phi_{M_n}$ is so localized near the origin we see that

$$\langle \eta f, \phi_{M_n} \rangle = \langle f, \phi_{M_n} \rangle + O(M_n^{-1}) = \langle f, \phi_{M_n} \rangle + O(100^{-n})$$

and

$$\langle (X(\eta) + iY(\eta))u, \phi_{M_n}\rangle = O(M_n^{-1}) = O(100^{-n})$$

while from Cauchy-Schwartz we have

$$\langle \eta u, (X+iY)^*\phi_{M_n}\rangle = O(A100^{-n}).$$

Thus we have

$$\langle f, \phi_{M_n}\rangle = O(A100^{-n}).$$

The set of all $f$ with this property is thus nowhere dense in $L^2$. From this and the Baire category theorem, we can thus find an $f$ in $L^2$ which does not obey this property for any $A$. This implies failure of solvability.

It remains to construct these chirps $\phi_M$. As before, we choose $\xi$ such that

$$p(0,\xi) = q(0,\xi) = 0, \quad \{p,q\}(0,\xi) = -2\pi. \tag{1}$$

The $\phi_M$ will not quite be modulated imaginary Gaussians, but will be modified according to the ansatz

$$\phi_M(x) := \eta(x)\tilde{\phi}_M(x) = \eta(x)|\det(M\mathrm{Re}(Z))|^{-1/2}e^{2\pi iMx\cdot\xi}e^{\pi Mx\cdot Zx/2}e^{m(x)+Mn(x)},$$

where $Z$ is a symmetric matrix with $\mathrm{Re}(Z)$ negative definite, $\eta$ is a cutoff to a small neighbourhood of the origin, and $m(x)$, $n(x)$ are some functions, with $n$ vanishing to third order in $x$. None of $Z$, $\eta$, $m$, $n$ depend on $M$.

The $L^2$ normalization is clear, as is the decay away from the origin. Let us now see how we have to choose $Z$, $m$, $n$ in order to obtain the smallness of $(X+iY)^*\phi_M$. We define the complex vector $c(x)$ by $c(x) := a(x) - ib(x)$, so $X + iY = \overline{c(x)}\cdot\nabla$. Thus

$$\begin{aligned}(X+iY)^*\phi_M &= -\nabla\cdot(c(x)\phi_M)\\ &= -(\nabla\cdot c(x)\eta + c(x)\cdot\nabla\eta + c(x)\cdot 2\pi iM\xi + c(x)\cdot\pi MZx\eta + c(x)\cdot\nabla m(x)\eta\\ &\quad + c(x)\cdot M\nabla n(x)\eta)\tilde{\phi}_M.\end{aligned}$$

The $\nabla\eta$ term is extremely small if $\eta$ is sufficiently close to the origin (so that the cubic $Mn(x)$ term does not become huge and positive). So it suffices to show that

$$\|(\nabla\cdot c(x) + c(x)\cdot 2\pi iM\xi + c(x)\cdot\pi MZx + c(x)\cdot\nabla m(x) + c(x)\cdot M\nabla n(x))\phi_M\|_2 = o(1).$$

Given the decay of $\phi_M$ it suffices to show that

$$\nabla\cdot c(x) + c(x)\cdot 2\pi iM\xi + c(x)\cdot\pi MZx + c(x)\cdot\nabla m(x) + c(x)\cdot M\nabla n(x) = o(1) \text{ as } M\to\infty.$$

in the region $|x| \lesssim M^{-1/2+\varepsilon}$ for some small $\varepsilon$.

We Taylor expand

$$c_j(x) = c_j(0) + \partial_k c_j(0)x_k + \partial_k\partial_l c_j(0)x_kx_l + O(M^{-3/2+3\varepsilon}).$$

From (1) we have $c(0)\cdot\xi = 0$. So we reduce to showing that

$$\nabla\cdot c(0) + c(0)\cdot\nabla m(0) + 2\pi iMx\cdot\nabla(\xi\cdot c)(0) + + c(0)\cdot\pi MZx + MQ(x,x) + Mc(0)\cdot\nabla n(x) = o(1)$$

where $Q$ is some quadratic form independent of $M$. One then chooses $m$ so that $\nabla\cdot c(0) + c(0)\cdot\nabla m(0) = 0$, and $n$ so that $c(0)\cdot\nabla n(x) = Q(x,x)$. To finish things we need choose $Z$ so that

$$-2i\nabla(\xi\cdot c) = Zc(0).$$

Since we want $Z$ to have a negative definite real part, this is only possible if

$$\mathrm{Re}\langle -2i\nabla(\xi \cdot c)(0), c(0)\rangle < 0.$$

Conversely, if we have this condition, then we can choose $Z$ correctly (i.e. whenever $\mathrm{Re}\langle v, w\rangle < 0$, one can find a symmetric $Z$ with negative definite real part such that $Zv = w$. To see this, first apply a rotation and scaling to set $v = e_1$ and $w = (-a + ib)e_1 + ke_2$, where $a > 0$, $b$ is real, and $k$ is complex. Then choose the symmetric matrix $Z$ appropriately, with a large negative $e_2, e_2$ co-efficient to ensure negativity.). This condition can be rewritten as

$$\mathrm{Im}\langle \nabla(\xi \cdot (a - ib))(0), a(0) - ib(0)\rangle < 0,$$

but this follows from the third part of (1).

One can modify this argument by adding more correction terms than just the cubic term $n$ in order to make $(X + iY)^* \phi_M$ not just small in $L^2$, but small in negative Sobolev norms as well. This can then be used to improve the failure of solvability result, so that e.g. there exist $W^{k,2}$ data $f$ for which there does not exist $W^{-k,2}$ solutions $u$. If the co-efficients $c$ are real analytic, then one can in fact show that there exists smooth data for which there does not exist distributional solutions $f$, by adding on infinitely many correction terms and keeping everything under control (by techniques very close to the proof of the Cauchy-Kovalevsky theorem).

## 4. Homework for week 9 - due Friday, March 16

- Let $a(x)$, $b(x)$ be real smooth vector-valued functions on $\mathbf{R}^2$ with $a(0) = e_1$ and $b(0) = e_2$. Let $f$ be a smooth function, and $0 < \varepsilon \ll 1$ be a small number. Show that, if $\eta$ is a cutoff on a sufficiently small neighbourhood of the origin, that
$$\|\eta(i(a(x) - e_1)\dot{\nabla} + i(b(x) - e_2)\dot{\nabla})u\|_{L^2} \lesssim \varepsilon \|u\|_{W^{1,2}}$$
and that
$$\|\eta(\partial_{x_1} + i\partial_{x_2})^{-1} F\|_{W^{1,2}} \lesssim \|F\|_{L^2}.$$
(You may find the Poincare inequality useful for the latter inequality). Conclude from this that, if $\varepsilon$ is sufficiently small, the map
$$u \mapsto \eta(\partial_{x_1} + i\partial_{x_2})^{-1}(f - \eta((a(x) - e_1)\dot{\nabla} + i(b(x) - e_2)\dot{\nabla})u)$$
is a contraction on $W^{1,2}$. Conclude from this that there exists a $u \in W^{1,2}$ such that
$$(a(x)\dot{\nabla} + ib(x)\dot{\nabla})u = f$$
at a neighbourhood of the origin. (One can then bootstrap the $W^{1,2}$ regularity to higher regularity, but I won't inflict this on you as an exercise).

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