LECTURE NOTES 6 FOR 254A

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1. The Haar basis

Suppose we partition the Walsh phase plane $\mathbf{R}^+ \times \mathbf{R}^+$ into a collection $\mathbf{P}$ of disjoint tiles. Then it is clear that the Walsh wave packets $\{\phi_P : P \in \mathbf{P}\}$ form an orthonormal basis of L^2. (They are clearly orthonormal. To show they span, show that Bessel's inequality is sharp. This can be done, e.g. by using the non-negative phase space density $W(f, f)$ of the previous section. There are many other proofs).

There are, of course, infinitely many ways to partition the phase plane into disjoint tiles, which generates infinitely many orthonormal bases of this type. One such example is the *Walsh-Gabor basis*

$$\{\phi_P : P \text{ is a unit square }\}.$$

To decompose a function f into this basis, one can first split f into (spatial) unit intervals, and do a Walsh-Fourier series decomposition on each interval. (The Fourier analogue of this is the *windowed Fourier transform*, or the closely related *local cosine basis*). This basis is occasionally useful for situations in which the unit spatial scale (or the unit frequency scale) plays a special role. (See Exercise 1).

A more commonly used partition is the *Haar basis*

$$\{\phi_P : \omega_P = [2^k, 2^{k+1}] \text{ for some } k \in \mathbf{Z}\}.$$

Elements of this basis are (now) known as *Haar wavelets* (though Haar predated the coining of the term "wavelet" by about 60 years!). These wavelets are clearly related to the Walsh Littlewood-Paley projections

$$P_k := \pi_{[2^k, 2^{k+1}]};$$

indeed, to split a function f into Haar wavelets, one can first apply the Littlewood-Paley decomposition to split into pieces $P_k f$, then split each piece $P_k f$ into intervals of length 2^{-k}. More precisely, if we split

$$f = \sum_{P \text{ a Haar tile}} a_P \phi_P$$

(so that $a_P := \langle f, \phi_P \rangle$), then we have

$$P_k f = \sum_{P : \omega_P = [2^k, 2^{k+1}]} a_P \phi_P$$

and

$$a_P \phi_P = \chi_{I_P} (P_k f)$$

for all P with $\omega_P = [2^k, 2^{k+1}]$.

The *Haar wavelet transform* maps a function f to the sequence of numbers a_P; clearly this transform is invertible. Thus one can think of a function f as a bunch of numbers living on the Haar tiles. Although I just defined these functions on the half-line $\mathbf{R}^+$, one can easily extend them to all of $\mathbf{R}$ by reflection. (Of course, this slightly compromises the algebraic structure of the Walsh field, but we will no longer be using this structure in the rest of the course).

Note that the Haar tiles P are characterized by their spatial interval I_P.

If P is a Haar tile, then ϕ_P has the explicit formula

$$\phi_P = |I_P|^{-1/2}(\chi_{I'_P} - \chi_{I''_P})$$

where I'_P and I''_P are the two halves of I_P (usually we pick I'_P to be the left half, but the choice of sign is never important). From the Littlewood-Paley inequality

$$\|f\|_p \sim \|(\sum_k |P_k f|^2)^{1/2}\|_p$$

for $1 < p < \infty$ (see last week's exercises), we thus see that we can essentially write the L^p norm of f in terms of the Haar coefficients:

$$\|f\|_p \sim \|(\sum_P |a_P|^2 \frac{\chi_{I_P}}{|I_P|})^{1/2}\|_p.$$

One consequence of this is that if we mess around with the signs of the Haar co-efficients of a function, we really don't change the L^p norm of that function. Another consequence is that the Haar wavelets form an *unconditional basis* for L^p: no matter which order one sums the Haar wavelets of an L^p function f, one ends up converging to f in L^p norm. (This is not true pointwise, and is not true for e.g. Fourier series!).

There are many other spaces than L^p which can be represented using just the magnitudes of the Haar co-efficients, such as the Walsh analogue of the Sobolev spaces. This is sometimes advantageous as it effectively factors out all the oscillation from such spaces.

There are several uses of wavelet bases such as the Haar wavelet basis. They tend to be fairly good at representing functions which appear in real life (e.g. piecewise smooth functions); only a small number of co-efficients will be large, and the rest are so small that they can be safely discarded without losing too accuracy in the function. They are quite localized, in that fluctuations in one region of space only really disturb the wavelet co-efficients near that region of space (in contrast to, say, the Fourier coefficients). Another important advantage of these bases is that they tend to "diagonalize" various standard operators (especially singular integral type operators), making them easier to analyze (or to evaluate, if one is doing numerical work). This is because the Haar wavelet enjoys several moment and support conditions, allowing for good estimates on the matrix co-efficients of an operator with respect to the wavelet basis.

To take a simple example, consider the good old Hilbert transform

$$Hf(x) := p.v. \int \frac{f(y)}{x-y}\, dy.$$

This operator can be viewed as an infinite matrix whose entries in the x^{th} row and y^{th} column is $1/(x-y)$. As such, it is not very diagonal. However, one can view H in the Haar basis instead, transforming H into the infinite matrix

$$(\langle H\phi_P, \phi_Q\rangle)_{P,Q \text{ Haar tiles}}.$$

Since the Haar wavelets form an orthonormal basis, the operator H is unitarily equivalent to the above matrix. Thus, if we wanted to prove (for instance) that H is bounded on L^2, it suffices to show that the above matrix is bounded on l^2(Haar tiles). (Of course, it is easy to prove the L^2 boundedness of H via the Fourier transform, or by complex methods, or even by observing that HH^* is a multiple of the identity. However, the method given here will be much more robust than any of these tricks.) By duality, it suffices to show that

$$|\sum_{P,Q} c_P \langle H\phi_P, \phi_Q\rangle d_Q| \lesssim 1$$

whenever c_P and d_Q are l^2-normalized sequences.

Now let's compute the size of these matrix co-efficients $\langle H\phi_P, \phi_Q\rangle$. We claim the bounds

$$\sum_Q |\langle H\phi_P, \phi_Q\rangle| \lesssim 1$$

for all Haar tiles P, and similarly

$$\sum_P |\langle H\phi_P, \phi_Q\rangle| \lesssim 1$$

for all Haar tiles Q. In other words, the row and columns of the matrix are summable. If this is the case, then the above bound easily follows from the elementary inequality $c_P d_Q \lesssim |c_P|^2 + |d_Q|^2$. (This trick, and generalizations of it, fall under the general heading of what's called *Schur's test*).

To prove these bounds, it suffices by anti-self-adjointness to prove the former. Fix P. To begin with, let us consider the contribution of the Q such that $|I_Q| = 2^k|I_P|$ for some $k \in \mathbf{Z}$. We need to obtain a bound which is decaying in $|k|$.

Fix k, and let x_0, x_1, x_2 denote the left endpoint, midpoint, and right endpoint of I_P. A crude estimation of $H\phi_P$ (exploiting the fact that the kernel $1/(x-y)$ is odd around x) yields the bounds

$$|H\phi_P(x)| \lesssim |I_P|^{-1/2} \log(2 + \frac{|I_P|}{\mathrm{dist}(x, \{x_0, x_1, x_2\})})$$

whenever $\mathrm{dist}(x, I_P) \lesssim |I_P|$, and

$$|H\phi_P(x)| \lesssim |I_P|^{3/2} \mathrm{dist}(x, I_P)^{-2}$$

otherwise. If $k \geq 0$, this gives the bound

$$|\langle H\phi_P, \phi_Q\rangle| \lesssim 2^{-k/2}(1 + \frac{\mathrm{dist}(|I_P|, |I_Q|)}{|I_Q|})^{-2},$$

which sums acceptably.

Now suppose $k \leq 0$. We now have

$$|\langle H\phi_P, \phi_Q\rangle| \lesssim 2^{-|k|/2}(1 + \frac{\operatorname{dist}(|I_P|, |I_Q|)}{|I_P|})^{-2},$$

but this fails to control the sum of all the $|\langle H\phi_P, \phi_Q\rangle|$ with a bound that decays in $|k|$. However, it is enough to deal with the rather annoying Q which border x_0, x_1, x_2, since there are only $O(1)$ of them.

For the other Haar tiles Q, we shall estimate the derivative of $H\phi_P$. Since

$$\phi_P'(x) = \pm|I_P|^{-1/2}(\delta(x - x_0) - 2\delta(x - x_1) + \delta(x - x_2)$$

we have

$$(H\phi_P)'(x) = \pm|I_P|^{-1/2}(\frac{1}{x - x_0} - 2\frac{1}{x - x_1} + \frac{1}{x - x_2}).$$

We now use the bound

$$|\langle f, \phi_Q\rangle| \lesssim |I_Q|^{3/2} \sup_{x \in I_Q} |f'(x)|$$

to obtain the bound

$$|\langle H\phi_P, \phi_Q\rangle| \lesssim 2^{-3|k|/2} \frac{|I_P|}{\operatorname{dist}(I_Q, \{x_0, x_1, x_2\})}$$

when $\operatorname{dist}(I_Q, I_P) \lesssim |I_P|$, and

$$|\langle H\phi_P, \phi_Q\rangle| \lesssim 2^{-3|k|/2}(\frac{|I_P|}{\operatorname{dist}(I_Q, I_P)})^3$$

when $\operatorname{dist}(I_Q, I_P) \gg |I_P|$. This sums easily to get a bound which decays in k.

This shows that the Hilbert transform H is bounded on L^2; by the Calderón-Zygmund theory of earlier weeks this implies that H is bounded on other L^p as well.

2. The Cauchy integral on the real line

In this section we briefly discuss how the Cauchy integral (an object closely related to the Hilbert transform) is used in complex analysis.

Let $f(x)$ be a complex-valued (Schwartz) function on the real line $\mathbf{R}$. In complex analysis it is of interest to extend $f(x)$ analytically to the upper half-plane $\{z : Im(z) \geq 0\}$ in such a way that it decays suitably at infinity. By the Cauchy integral formula, such an extension if it exists, is given above the real line by the formula

$$Cf(z) = \frac{1}{2\pi i} \int_{\mathbf{R}} \frac{f(x)}{x - z}\, dx;$$

the function Cf is then known as the *Cauchy integral* of f. However, such an expression does not need to match f on the real line $\mathbf{R}$:

$$\lim_{\varepsilon \to 0^+} Cf(x + i\varepsilon) \neq f(x) \text{ in general.}$$

Because of this, not every function $f(x)$ has an analytic extension to the upper half-space which decays at infinity.

Similarly, a (Schwartz) function $f(x)$ can be extended to the lower half plane if and only if

$$\lim_{\varepsilon \to 0^+} Cf(x - i\varepsilon) = -f(x).$$

By Louiville's theorem, there is no non-zero function $f(x)$ which can be extended to both half-planes simultaneously. Also, for arbitrary $f(x)$ there is the *Plemelj formula*

$$\lim_{\varepsilon \to 0^+} Cf(x + i\varepsilon) - Cf(x - i\varepsilon) = f(x)$$

(indeed, we see that

$$Cf(x + i\varepsilon) + Cf(x - i\varepsilon) = \frac{1}{2\pi i} \int_{\mathbf{R}} \frac{f(x+t)}{t + i\varepsilon} + \frac{f(x+t)}{t - i\varepsilon}\, dy = \frac{1}{\pi} \int_{\mathbf{R}} \frac{f(x + \varepsilon s)}{s^2 + 1}\, ds$$

$$\to \frac{1}{\pi} \int_{\mathbf{R}} \frac{f(x)}{s^2 + 1}\, ds = f(x).$$

Thus any f can be written as the sum of two functions, one analytically extendable to the upper half plane, and one extendable to the lower half plane.

This turns out to be an orthogonal decomposition. If $f(z)$ lives on the upper half plane and decays at infinity, and $g(z)$ lives on the lower half-plane and decays at infinity, then some integration by parts combined with the Cauchy-Riemann equations shows that

$$\frac{d}{dy} \int_{\mathbf{R}} f(x + iy)\overline{g(x - iy)}\, dx = 0$$

for all $y \geq 0$. Taking limits as $y \to 0$ and $y \to \infty$ we see that f and g are orthogonal on the real line.

All of the above discussion was for Schwartz f. One can extend this to rough f, e.g. $f \in L^2$ by the usual limiting argument, provided that the operators

$$f(x) \mapsto Cf(x + i\varepsilon)$$

are bounded on L^2 uniformly in ε. But this can be verified by a modification of the above arguments. The Hilbert transform is in fact quite closely related to the Cauchy integrals, as the easily verified formula

$$\lim_{\varepsilon \to 0^+} Cf(x + i\varepsilon) + Cf(x - i\varepsilon) = \frac{1}{\pi i} Hf(x)$$

shows.

The L^2 theory can be summarized as follows. Let H^2_+ be the set of all functions $f(x)$ which can be extended to an analytic function $f(z)$ on the upper half-plane such that

$$\sup_{t \geq 0} \|f(x + it)\|_{L^2_x} < \infty.$$

Similarly define H^2_- as the functions extendible to the lower half-plane such that

$$\sup_{t \geq 0} \|f(x - it)\|_{L^2_x} < \infty.$$

Then $L^2(\mathbf{R})$ can be decomposed as the orthogonal sum of H^2_+ and H^2_-.

There is also a similar L^p theory, thanks to the boundedness of the Cauchy integrals on L^p, but we won't discuss it here. (Things get especially interesting when $p = 1$).

3. The Cauchy integral on Lipschitz graphs

We now extend the above complex theory from the real line $\mathbf{R}$ to Lipschitz graphs of the form

$$\gamma = \{x + iA(x) : x \in \mathbf{R}\}$$

where A satisfies the Lipschitz condition

$$|A(x) - A(y)| \lesssim |x - y|$$

(or what is essentially the same thing, $|A'(x)| \lesssim 1$, if one makes the a priori assumption that A is smooth). Note that because of the Lipschitz condition, the arclength measure $|dz|$ on γ is equivalent to the measure dx induced from the real line via the map $x \mapsto x + iA(x)$. Thus we can talk about $L^2(\gamma)$ while being somewhat careless with the choice of measure.

One can set up the Cauchy integral operator C_γ associated to γ by

$$C_\gamma f(w) := \frac{1}{2\pi i} \int_\gamma \frac{f(z)\ dz}{z - w},$$

or in x co-ordinates,

$$C_\gamma f(x + iA(x) + i\varepsilon) = \frac{1}{2\pi i} \int \frac{f(x + iA(y))(1 + iA'(y))\ dy}{x - y + i(A(x) - A(y)) + \varepsilon}.$$

In 1982 Coifman, Meyer, and Semmes showed that C_γ is bounded on $L^2(\gamma)$ to $L^2(\gamma + i\varepsilon)$ uniformly in ε, which allows one to construct a complex theory for the half-spaces above and below γ.

We shall just show this for $\varepsilon = 0$, i.e. show that the operator

$$C_\gamma f(w) = \frac{1}{2\pi i} p.v. \int_\gamma \frac{f(z)\ dz}{z - w}$$

is bounded on $L^2(\gamma)$. The other operators are similar but a bit more messy.

By duality it suffices to show that the form

$$|p.v. \int_\gamma \int_\gamma \frac{f(z) g(w)\ dz dw}{z - w}| \tag{1}$$

is bounded whenever f, g have bounded $L^2(\gamma)$ norm.

Fix f, g. We shall repeat the proof of the L^2 boundedness of the Hilbert transform, except that we need to create Haar wavelets on γ rather than $\mathbf{R}$. The simplest thing to do is to push forward the Haar wavelets on $\mathbf{R}$ via the map $x \mapsto x + iA(x)$, but this turns out to not be the best strategy (it only works when the Lipschitz constant of A is small). Instead, we work on γ directly.

For every dyadic interval I on $\mathbf{R}$, define the γ-dyadic interval

$$\tilde{I} := \{x + iA(x) : x \in I\}.$$

We then define the modified Haar function $\phi_{\tilde{I}}$ by

$$\phi_{\tilde{I}} = (a_I \chi_{\tilde{I}_l} + b_I \chi_{\tilde{I}_r})$$

where I_l, I_r are the left and right halves of I, and a_I and b_I are complex constants such that one has the mean zero condition

$$\int_\gamma \phi_{\tilde{I}}(z)\ dz = 0$$

and the L^2 normalization condition

$$\int \phi_{\tilde{I}}(z)^2\ dz = 1$$

(note the lack of absolute values). One can easily verify that

$$|a_I| \sim |b_I| \sim |I|^{-1/2}$$

(just as with the real line case).

These modified Haar functions are not quite an orthonormal basis of $L^2(\gamma)$, but they are an almost orthonormal basis, in the sense that every $f \in L^2(\gamma)$ can be written as

$$f = \sum_I c_I \phi_{\tilde{I}}$$

for some constants c_I satisfying the almost Bessel inequality

$$(\sum_I |c_I|^2)^{1/2} \lesssim \|f\|_{L^2(\gamma)}. \tag{2}$$

(Actually we can strengthen the $\lesssim$ to a $\sim$, but we only need this direction). Let us assume (2) for the moment. Then, we see that to prove (4) it suffices to show that

$$|\sum_I \sum_J c_I d_J |p.v. \int_\gamma \int_\gamma \frac{\phi_{\tilde{I}}(z)\phi_{\tilde{J}}(w)\ dzdw}{z-w}|$$

whenever c_I, d_J are complex constants (for all dyadic intervals I, J) which are normalized in l^2.

The expression

$$H_{I,J} := p.v. \int_\gamma \int_\gamma \frac{\phi_{\tilde{I}}(z)\phi_{\tilde{J}}(w)\ dzdw}{z-w} \tag{3}$$

is the analogue of $\langle H\phi_P, \phi_Q\rangle$ on the curve γ. Using Schur's test as before, it suffices to show that

$$\sum_J |H_{I,J}| \lesssim 1$$

for all I and
$$\sum_I |H_{I,J}| \lesssim 1$$
for all J.

By anti-symmetry it suffices to prove the first inequality. As before, we fix I, restrict J to those intervals for which $|J| = 2^k|I|$, and obtain a bound which is decaying in $|k|$.

Fix k, and let z_0, z_1, z_2 denote the left endpoint, midpoint, and right endpoint of $\tilde{I}$. A crude estimation of $H\phi_I$ using contour integration to move the integral away from the singularity at $w \in \gamma$ yields the bounds
$$|H\phi_{\tilde{I}}(w)| \lesssim |I|^{-1/2} \log(2 + \frac{|I|}{\operatorname{dist}(w, \{z_0, z_1, z_2\})})$$
whenever $\operatorname{dist}(z, \tilde{I}) \lesssim |I|$, and
$$|H\phi_{\tilde{I}}(x)| \lesssim |I|^{3/2} \operatorname{dist}(z, \tilde{I})^{-2}$$
otherwise (the latter bound uses the mean zero property of $\phi_{\tilde{I}}$). If $k \geq 0$, this gives the bound
$$|H_{I,J}| \lesssim 2^{-k/2}(1 + \frac{\operatorname{dist}(|I|, |J|)}{|I|})^{-2},$$
which sums acceptably.

Now suppose $k < 0$. As before, the above bounds can eliminate those J which border on z_0, z_1, or z_2, but are insufficient to handle the entire sum. However, the mean zero property of ϕ_J lets us use the derivative of $H\phi_I$ as before:
$$|\langle H\phi_{\tilde{I}}, \phi_{\tilde{J}}\rangle| \lesssim |J|^{3/2} \sup_{z \in \tilde{J}} |(H\phi_{\tilde{I}})'(z)|$$
Also, by differentiating under the integral sign (first shifting the contour if necessary) we obtain the bounds
$$(H\phi_P)'(z) = \pm |I_P|^{-1/2}(\frac{1}{|z - z_0|} + \frac{1}{|z - z_1|} + \frac{1}{|z - z_2|})$$
when $\operatorname{dist}(z, \tilde{I}) \lesssim |I|$, and
$$|H\phi_{\tilde{I}}(x)| \lesssim |I|^{3/2} \operatorname{dist}(z, \tilde{I})^{-3}$$
otherwise. This gives bounds similar to that for the Hilbert transform, and we can again sum acceptably.

We now show why the modified Haar wavelet basis is indeed an almost orthogonal basis. We introduce the "inner product"
$$\langle f(z), g(z)\rangle_{dz} := \int f(z)g(z)dz.$$
This is not really an inner product because it is not positive definite (even if one replaces $g(z)$ with $\overline{g(z)}$). However, it is close enough that it is worth using a suggestive notation. From the construction of the Haar wavelets we have the "orthonormal basis property"
$$\langle \phi_{\tilde{I}}, \phi_{\tilde{J}}\rangle_{dz} = \delta_{I,J}$$

where $\delta_{I,J}$ is the Kronecker delta. If the inner product were positive definite we would now be done; as it is we have much work still to do.

We introduce the averaging operators $\tilde{P}_{\leq k}$ on $L^2(\gamma)$ defined by

$$\tilde{P}_{\leq k} f(z_0) := \frac{\int_{\tilde{I}} f(z)\ dz}{\int_{\tilde{I}}\ dz}$$

where I is the dyadic interval of length 2^{-k} which contains z_0. (This is well defined for a.e. z_0). A simple computation shows that

$$\tilde{P}_{\leq k+1} f - \tilde{P}_{\leq k} f = \sum_{I:|I|=2^{-k}} \langle f, \phi_{\tilde{I}} \rangle_{dz} \phi_{\tilde{I}}. \tag{4}$$

Thus if one telescopes this we obtain the reasonable-looking formula

$$f = \sum_I \langle f, \phi_{\tilde{I}} \rangle_{dz} \phi_{\tilde{I}}$$

for test functions f (or L^2 functions f). Thus to show (2) it suffices to show

$$\sum_{k \in \mathbf{Z}} \sum_{|I|=2^{-k}} |\langle f, \phi_{\tilde{I}} \rangle_{dz}|^2 \lesssim \|f\|_2^2.$$

By (4) it suffices to show that

$$\sum_{k \in \mathbf{Z}} \|\tilde{P}_{\leq k+1} f - \tilde{P}_{\leq k} f\|_2^2 \lesssim \|f\|_2^2.$$

We now move everything from γ down to the real line $\mathbf{R}$, in order to try to take advantage of the Haar basis we already have there. On the real line we have the Walsh-Littlewood-Paley operators $P_{\leq k}$ defined by

$$P_{\leq k} f(x) := \frac{\int_I f(x)\ dx}{|I|}$$

where I is the dyadic interval of length 2^{-k} which contains x. If we write $F(x) := f(x + iA(x))$, we observe that

$$\tilde{P}_{\leq k} f(x + iA(x)) = \frac{P_{\leq k}(bF)(x)}{P_{\leq k} b(x)}$$

where $b(x) = dz/dx = 1 + iA'(x)$ is the Jacobian. Thus it suffices to show that

$$\sum_{k \in \mathbf{Z}} \| \frac{P_{\leq k+1}(bF)}{P_{\leq k+1}(b)} - \frac{P_{\leq k}(bF)}{P_{\leq k}(b)} \|_2^2 \lesssim \|F\|_2^2.$$

We write

$$\frac{P_{\leq k+1}(bF)}{P_{\leq k+1}(b)} - \frac{P_{\leq k}(bF)}{P_{\leq k}(b)} = \frac{P_{k+1}(bF) P_{\leq k}(b) - P_{\leq k}(bF) P_{k+1}(b)}{P_{\leq k}(b) P_{\leq k+1}(b)}.$$

Since b is bounded and has real part 1, $P_{\leq k} b$ is bounded and bounded away from zero. Thus it suffices to show that

$$\sum_{k \in \mathbf{Z}} \|P_{k+1}(bF) P_{\leq k}(b) - P_{\leq k}(bF) P_{k+1}(b)\|_2^2 \lesssim \|F\|_2^2.$$

The first term can be bounded by

$$\sum_{k\in\mathbf{Z}} \|P_{k+1}(bF)\|_2^2 = \|bF\|_2^2 \lesssim \|F\|_2^2.$$

Thus it suffices to show the estimate

$$\sum_{k\in\mathbf{Z}} \|P_{<k}(g)P_k(b)\|_2^2 \lesssim \|g\|_2^2.$$

This is a (Walsh version of) an endpoint paraproduct estimate of the type we almost considered when doing the Coifman-Meyer theorem. However, because b is only in L^∞ and not in a smaller L^p, we do not have access to the usual Littlewood-Paley theory, and must instead do some decompositions of a type akin to what was done for Carleson's theorem last week, but somewhat simpler as we only have Haar tiles to deal with. (Actually, the type of argument we're about to give was also pioneered by Carleson, and is generally presented using the notion of a *Carleson measure*).

Using the usual Haar basis ϕ_P, we have

$$P_k(b) = \sum_{P:|I_P|=2^k} \langle b, \phi_P\rangle \phi_P$$

so we can rewrite the above expression as

$$\sum_P \Big(\frac{1}{|I_P|}\int_{I_P} |g|\Big)^2 |\langle b, \phi_P\rangle|^2. \tag{5}$$

If g were in L^∞ and b were in L^2 we'd be easily done by now thanks to Bessel's inequality. Unfortunately the situation is the reverse.

We will now do a "mass decomposition" argument which can be viewed as a simpler version of the arguments of the previous section. For each integer j, consider the contribution of those tiles P such that $2^j \leq \frac{1}{|I_P|}\int_{I_P}|g| \leq 2^{j+1}$. (These are the tiles with "mass" 2^j). The contribution of this portion is bounded by

$$2^{2j} \sum_{P:\int_{I_P}|g|\sim 2^j|I_P|} |\langle b, \phi_P\rangle|^2.$$

Let $\mathbf{P}_j$ denote the collection of tiles P in $\{\int_{I_P}|g| \sim 2^j|I_P|\}$ such that the spatial interval I_P is maximal with respect to inclusion. Then we can estimate the previous by

$$2^{2j} \sum_{Q\in\mathbf{P}_j} \sum_{P:I_P\subseteq I_Q} |\langle b, \phi_P\rangle|^2.$$

Using Bessel's inequality localized to I_Q, we can estimate this as

$$2^{2j} \sum_{Q\in\mathbf{P}_j} \|b\|_{L^2(I_Q)}^2.$$

Since b is bounded, this is bounded by

$$2^{2j} \sum_{Q\in\mathbf{P}_j} |I_Q|.$$

However, on each I_Q, the Hardy-Littlewood maximal function Mg is $\gtrsim 2^j$, by construction. Thus we can bound this by

$$2^{2j}|\{Mg \gtrsim 2^j\}|.$$

Thus, we can estimate (5) by

$$\sum_j 2^{2j}|\{Mg \gtrsim 2^j\}|,$$

which by a simple Fubini argument is comparable to

$$\|Mg\|_2^2.$$

Since the Hardy-Littlewood maximal function is bounded on L^2, the claim follows.

We have now completed the proof of the L^2 boundedness of the Cauchy integral. Once L^2 boundedness is established, one can use the Calderón-Zygmund theory of previous weeks to also establish L^p boundedness, along with a host of other estimates (e.g. weak (1,1) estimates).

This circle of ideas can be extended to more general operators than Cauchy integrals, centering around the $T(1)$ and $T(b)$ theorems of David, Journe, and Semmes. However, we will not discuss this further in this course.

4. Homework for week 8 - due Friday, March 16

- For each non-negative integer k, let f_k be a function on $\mathbf{R}^+$ whose Walsh-Fourier transform $\hat{f}_k$ is supported on $[k, k+1]$. Show the inequality
$$\|(\sum_k |f_k|^2)^{1/2}\|_p \lesssim \|\sum_k f_k\|_p$$
for all $2 \leq p \leq \infty$. (This is known as the *(Walsh) Littlewood-Paley inequality for equally spaced intervals*). Hint: use the Walsh-Gabor basis, and work on each unit spatial interval separately. Use the fact that the L^p norm is larger than the L^2 norm on each unit spatial interval.
- Let f be an element of $H^2_+(\mathbf{R})$. Show that the Fourier transform of f is supported on the positive half-line $\mathbf{R}^+$. Similarly for $H^2_-(\mathbf{R})$ and $\mathbf{R}^-$. (Hint: Take the Fourier transform of the Cauchy-Riemann equation $\frac{\partial}{\partial x} f(x+iy) = -i\frac{\partial}{\partial y} f(x+iy)$ with respect to the x variable). Explain informally why this is consistent with the observation that $e^{2\pi ikz}$ decays at $+i\infty$ if $k > 0$ and decays at $-i\infty$ if $k < 0$.

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