LECTURE NOTES 4 FOR 254A

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1. Overview

Let Ω be a bounded open subset in $\mathbf{R}^2$. In this section we apply the paraproduct theory of the previous section to obtain a regularity result for *harmonic maps* on Ω. These maps are the natural generalization of *harmonic functions* on Ω, except that they take values in a Riemannian manifold instead of $\mathbf{R}$.

Recall that a function $u : \Omega \mapsto \mathbf{R}$ is *harmonic* if it satisfies *Laplace's equation*

$$\Delta u = 0$$

where $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian. Alternatively, one can define u as a minimizer to the *energy*

$$E(u) := \int_\Omega |\nabla u|^2.$$

As a minimizer, we mean that

$$E(u) \leq E(u + v)$$

for all $v : \Omega \mapsto \mathbf{R}$ which is supported on a compact subset of Ω. To see why these two definitions of harmonicity are equivalent, we integrate by parts to observe that

$$\begin{aligned} E(u+v) &= E(u) + \int 2\nabla u \cdot \nabla v + \int |\nabla v|^2 \\ &= E(u) - \int 2\Delta u v + \int |\nabla v|^2. \end{aligned}$$

Thus if $\Delta u = 0$, then $E(u)$ is a minimizer. Conversely, if u is a minimizer, then by replacing v by εv for $0 < \varepsilon \ll 1$ and letting $\varepsilon \to 0$ we see that $\int 2\Delta u v = 0$ for all v, which implies that $\Delta u = 0$.

One way to construct harmonic functions is to start with an arbitrary (say smooth) function $\tilde{u}$ on $\overline{\Omega}$, and try to minimize $E(u)$ over all u of the form $u = \tilde{u} + v$, where v has compact support in Ω. (Or, what is essentially equivalently, minimize over all u which have the same boundary value as $\tilde{u}$ on the boundary $\partial\Omega$). Clearly u has finite energy, so that $\nabla u \in L^2$. However, one can say much more:

Theorem 1.1 (Regularity of Laplace's equation)**.** *If u is a harmonic function on Ω with finite energy, then u is C^∞.*

This will be proven in the next section. This argument works for all of $\mathbf{R}^n$, and is not specific to two dimensions.

Harmonic functions take values on $\mathbf{R}$. One can similarly define harmonic functions on Euclidean spaces $\mathbf{R}^m$, but this is a rather trivial generalization as the m components of a vector-valued harmonic function are still harmonic.

A more interesting situation arises when one constrains a vector valued function u to a sub-manifold of $\mathbf{R}^m$; for sake of simplicity we restrict our attention to the sphere S^{m-1}. In this case we are still trying to minimize the energy

$$E(u) := \int_\Omega |\nabla u|^2$$

except that u is now a vector, and we are minimizing subject to the constraint that u stays on the sphere (i.e. $|u| \equiv 1$). Such minimizers are known as *harmonic maps*, and turn up in several equations in physics (such as the Ginzberg-Landau and Einstein equations).

Suppose we perturb u by a small perturbation $u \mapsto u + \varepsilon v + O(\varepsilon^2)$. In order for this perturbation to stay on the sphere we must have $u \cdot v \equiv 0$. As before we have

$$E(u + \varepsilon v + O(\varepsilon^2)) = E(u) + \int 2\varepsilon \Delta u \cdot v + O(\varepsilon^2)$$

which means that if u is to be a minimizer then $\int \Delta u \cdot v = 0$. But this is for arbitrary v such that $u \cdot v \equiv 0$, which means that $\Delta u(x)$ must be parallel to $u(x)$ for all x.

On the other hand, $u \cdot u = 1$, so $\Delta(u \cdot u) = 0$. But the product rule states that $\Delta(u \cdot u) = 2u \cdot \Delta u + 2|\nabla u|^2$. Combining this with the previous, we thus see that harmonic maps must satisfy the equation

$$\Delta u = -u|\nabla u|^2. \tag{1}$$

This equation turns out to be especially interesting in two dimensions $\mathbf{R}^2$. In one dimension, harmonic maps are simply geodesics on the sphere S^{m-1} (i.e. great circles). In dimensions $\mathbf{R}^3$ and higher it is possible for harmonic maps to develop singularities (i.e. the energy is finite but there are still some points where the map is not C^∞). However, in two dimensions things are very closely balanced (the technical term for this is *critical*), because the energy $E(u)$ just barely fails to control the all-important L^∞ norm through Sobolev embedding. Nevertheless, in 1991 Helein proved

Theorem 1.2 (Regularity of two-dimensional harmonic maps to the sphere)**.** *If u is a harmonic map on $\Omega \subset \mathbf{R}^2$ to the sphere S^{m-1} with finite energy, then u is C^∞.*

This result is more difficult than Theorem 1.1 due to the fact that (1) is non-linear. Nevertheless, one can use paraproduct theory, and specifically something called the *div-curl lemma*, to resolve this problem. We will not follow Helein's original approach (which used Hardy spaces, which I don't want to get into), but follow a variant, more in the spirit of some later work of Alice Chang, Lihe Wang, and Paul Yang (while they were all at UCLA!).

2. Regularity of harmonic functions

In this section we prove Theorem 1.1.

Since u has finite energy, we know that u is locally $W^{1,2}$; in other words, for any smooth cutoff function η compactly supported in Ω, that $\eta u \in W^{1,2}$. We now use the equation Δu to bootstrap this. More precisely, we show for any $k \geq 1$:

If u is locally $W^{k,2}$, then u is locally $W^{k+1,2}$.

Iterating this and applying Sobolev embedding we see that u is C^∞.

It remains to prove the claim. Fix k and assume that u is locally $W^{k,2}$. Let η be a smooth cutoff function compactly supported in Ω. We would like to show that

$$\|\eta u\|_{W^{k+1,2}} < \infty.$$

We shall need the *elliptic regularity estimate*

$$\|\eta u\|_{W^{k+1,2}} \lesssim \|\eta u\|_{W^{k-1,2}} + \|\Delta(\eta u)\|_{W^{k-1,2}}. \tag{2}$$

Assuming this estimate for the moment, we apply it together with the calculation

$$\begin{aligned}\Delta(\eta u) &= (\Delta\eta)u + 2\nabla\eta \cdot \nabla u + \eta(\Delta u)\\ &= -(\Delta\eta)u + 2\nabla \cdot ((\nabla\eta)u).\end{aligned}$$

to obtain

$$\|\eta u\|_{W^{k+1,2}} \lesssim \|\eta u\|_{W^{k-1,2}} + \|(\Delta\eta)u\|_{W^{k-1,2}} + \|(\nabla\eta)u\|_{W^{k,2}}.$$

All terms on the right hand side are finite by hypothesis, so the claim is proved.

It remains to prove (2). It is possible to in fact prove the more general estimate

$$\|f\|_{W^{s,p}} \lesssim \|f\|_{W^{s,p}} + \|\Delta f\|_{W^{s-2,p}}$$

using Littlewood-Paley theory and the techniques of the second week's notes; we leave this as an exercise. But because we are in the special case $p = 2$, there are several other ways to prove this estimate. One is to use Plancherel's theorem in the form

$$\|f|_{W^{s,2}} \sim \|\langle\xi\rangle^s \hat{f}(\xi)\|_{L^2(\xi)}$$

and the fact that $\hat{\Delta} f(\xi) = -4\pi^2|\xi|^2\hat{f}(\xi)$. Another way is to integrate by parts to obtain the identity

$$\sum_{i,j=1}^n \int |\frac{\partial^2 f}{\partial x_i \partial x_j}|^2 = \sum_{i,j=1}^n \langle \frac{\partial^2 f}{\partial x_i^2}, \frac{\partial^2 f}{\partial x_j^2}\rangle = \|\Delta f\|_2^2.$$

We leave both of these methods as easy exercises.

3. Harmonic maps: bootstrapping regularity

We now turn to the non-linear analogue of the previous section, in which harmonic functions $\Delta u = 0$ are replaced by harmonic maps, and Ω is a subset of $\mathbf{R}^2$. For any

$s \in \mathbf{R}$, call a function u on Ω *locally in* $W^{s,2}$ if $\eta u \in W^{s,2}$ for all cutoff functions η compactly supported in Ω. The aim of this section is to show for any $\varepsilon > 0$ that

If u is locally $W^{1+\varepsilon,2}$, then u is locally $W^{1+\varepsilon',2}$ for any $0 < \varepsilon' < \min(2\varepsilon, \varepsilon + 1)$.

This is almost enough to finish the proof of Theorem 1.2 by iteration (and Sobolev embedding), except that we only begin with the regularity of $W^{1,2}$; if we could somehow extract an epsilon more regularity than this, then we would have smoothness.

Before we prove this claim, let us see heuristically why it is plausible. Consider a high-frequency component $P_k u$ of u. If u is $W^{1+\varepsilon,2}$, then we expect $\|P_k u\|_2 \lesssim 2^{-k(1+\varepsilon)}$ by cheap Littlewood-Paley. By Bernstein's inequality and the fact that we are in two dimensions we thus have $\|P_k u\|_\infty \lesssim 2^{-\varepsilon k}$. Differentiating this we get $\|\nabla P_k u\|_2 \lesssim 2^{-\varepsilon k}$ and $\|\nabla P_k u\|_\infty \lesssim 2^{(1-\varepsilon)k}$, so that $\||\nabla P_k u|^2\|_2 \lesssim 2^{(1-2\varepsilon)k}$. Since u is bounded, we roughly expect $\|P_k(u|\nabla u|^2)\|_2 \lesssim 2^{(1-2\varepsilon)k}$, so by (1) we have $\|P_k \Delta u\|_2 \lesssim 2^{(1-2\varepsilon)k}$. Since we heuristically have $P_k \Delta u \sim 2^k P_k u$, we thus have $\|P_k u\|_2 \lesssim 2^{-k(1+2\varepsilon)}$, which (thanks to the cheap Littlewood-Paley inequality) shows that $u \in W^{1+\varepsilon',2}$ (at least locally).

(One can reach the endpoint $\varepsilon' = 2\varepsilon$ when $\varepsilon < 1$ by using the non-cheap Littlewood-Paley inequality, but it is not worth our while to do so (since they both iterate equally well)).

Note that if one tried this heuristic calculation starting with just the regularity $W^{1,2}$, then one only ends up back at $W^{1,2}$ again (at best). Because of this, we refer to the regularity $W^{1,2}$ as *critical*, while $W^{1+\varepsilon,2}$ is *subcritical*. As a general rule the sub-critical theory is always a lot easier than the critical theory. For regularities such as $W^{1-\varepsilon,2}$, any attempt to use the equation ends up *losing* regularity instead of gaining it, and so these regularities are called *supercritical*. In many equations, supercritical regularities lead to singularities of some sort of another (e.g. in $n > 2$ the energy regularity $W^{1,2}$ is supercritical for the harmonic map equation). For supercritical equations there are absolutely no techniques (except in the very rare cases that the equation is exactly solvable) to obtain any sort of regularity on solutions. (A typical super-critical problem is the regularity of Navier-Stokes in three dimensions, which happens to be a Clay prize problem.)

OK, let's prove the claim. We need to estimate

$$\|\eta u\|_{W^{1+\varepsilon',2}}.$$

By elliptic regularity and the hypothesis it suffices to estimate

$$\|\Delta(\eta u)\|_{W^{\varepsilon'-1,2}}.$$

We can repeat the argument of the previous section, and all terms are OK except for the term where Δ falls on u:

$$\|\eta \Delta u\|_{W^{\varepsilon'-1,2}}.$$

This term does not vanish any more, but we can use the equation (1) to replace it by

$$\|\eta u |\nabla u|^2\|_{W^{\varepsilon'-1,2}}.$$

We only have local control on u, so we shall replace u by $\tilde{u} := \tilde{\eta}u$ where $\tilde{\eta}$ is a cutoff which equals 1 on the support of η. Clearly this makes no difference.

Suppose we had the estimates

$$\|uv\|_{W^{\varepsilon'-1,2}} \lesssim \|u\|_{W^{1+\varepsilon,2}}\|v\|_{W^{\varepsilon''-1,2}}$$

and

$$\|vw\|_{W^{\varepsilon''-1,2}} \lesssim \|v\|_{W^{\varepsilon,2}}\|w\|_{W^{\varepsilon,2}}$$

for some $\varepsilon'' > \varepsilon'$. Then we would have

$$\|\eta u|\nabla\tilde{u}|^2\|_{W^{\varepsilon'-1,2}} \lesssim \|\eta u\|_{W^{1+\varepsilon,2}}\|\tilde{\eta}u\|^2_{W^{1+\varepsilon,2}}$$

so that we would get $\eta u \in W^{1+\varepsilon',2}$ as desired.

These laws are consequences of

Lemma 3.1 (Cheap Sobolev multiplication law). *If $n \geq 1$ and $s, s_1, s_2 \in \mathbf{R}$, then we have the estimate*

$$\|fg\|_{W^{s,2}} \lesssim \|f\|_{W^{s_1,2}}\|g\|_{W^{s_2,2}}$$

whenever

$$s < s_1, s_2, \quad 0 < s_1 + s_2$$

and

$$s < s_1 + s_2 - n/2.$$

There are endpoint versions of the Sobolev multiplication law: see Exercise 3. There are also analogues of this result for exponents other than $p = 2$ (basically replace $n/2$ by n/p), but we shall not pursue this. For our purposes $n = 2$, and we can get away with the above two estimates if we choose

$$\varepsilon' < \varepsilon'' < \min(2\varepsilon, \varepsilon + 1).$$

Proof We use the cheap Littlewood-Paley inequality to split

$$\|fg\|_{W^{s,2}} \lesssim \|P_{\leq 0}(fg)\|_2 + \sum_{k>0} 2^{sk}\|P_k(fg)\|_2.$$

First consider $P_{\leq 0}(fg)$. We can use the Littlewood-Paley trichotomy to split

$$P_{\leq 0}(fg) = P_{\leq 0}((P_{\leq 5}f)(P_{\leq 5}g) + \sum_{k_1,k_2>5:|k_1-k_2|<3} (P_{k_1}f)(P_{k_2}g)).$$

The "low-low" part is easy to handle by the usual tools of Hölder, Bernstein, and throwing away Littlewood-Paley projections:

$$\begin{aligned}\|P_{\leq 0}((P_{\leq 5}f)(P_{\leq 5}g))\|_2 &\lesssim \|(P_{\leq 5}f)(P_{\leq 5}g)\|_2\\ &\lesssim \|P_{\leq 5}f\|_2\|P_{\leq 5}g\|_\infty\\ &\lesssim \|P_{\leq 5}f\|_2\|P_{\leq 5}g\|_2\\ &\lesssim \|f\|_{W^{s_1,2}}\|g\|_{W^{s_2,2}}.\end{aligned}$$

The "high-high" part is almost as easy, but uses the assumption $s_1 + s_2 > 0$. The only trick is to use Bernstein immediately to get from L^2 to L^1, rather than later, to get from L^∞ to L^2, because it is cheaper to do so immediately than later:

$$\begin{aligned}
\|P_{\leq 0}(\sum_{k_1,k_2>5:|k_1-k_2|<3} (P_{k_1}f)(P_{k_2}g))\|_2 &\lesssim \|P_{\leq 0}(\sum_{k_1,k_2>5:|k_1-k_2|<3} (P_{k_1}f)(P_{k_2}g))\|_1 \\
&\lesssim \sum_{k_1,k_2>5:|k_1-k_2|<3} \|(P_{k_1}f)(P_{k_2}g)\|_1 \\
&\lesssim \sum_{k_1,k_2>5:|k_1-k_2|<3} \|P_{k_1}f\|_2\|P_{k_2}g\|_2 \\
&\lesssim \sum_{k_1,k_2>5:|k_1-k_2|<3} 2^{-s_1k_1}\|f\|_{W^{s_1,2}}2^{-s_2k_2}\|g\|_{W^{s_2,2}} \\
&\lesssim \|f\|_{W^{s_1,2}}\|g\|_{W^{s_2,2}}.
\end{aligned}$$

Indeed, as a general principle it is always best to apply Bernstein's inequality to the lowest frequency term (in this case fg, rather than f or g separately). This is because the factor one loses in Bernstein increases with the frequency.

Now consider $P_k(fg)$. From last week's notes, we can split this into four pieces

$$\begin{aligned}
P_k(fg) =& P_k[(P_{<k-5}f)(P_{k-3\leq\cdot\leq k+3}g) \\
&+ (P_{k-3\leq\cdot\leq k+3}f)(P_{<k-5}g) \\
&+ (P_{k-5\leq\cdot\leq k+5}f)(P_{k-5\leq\cdot\leq k+5}g) \\
&+ \sum_{k',k''>k+5:|k'-k''|\leq 3} (P_{k'}f)(P_{k''}g)].
\end{aligned} \tag{3}$$

Consider the first piece (the "low-high" interactions). Here the lowest frequency term is f, so we will use Bernstein on f to upgrade L^2 to L^∞:

$$\begin{aligned}
\sum_{k>0} 2^{sk}\|P_k((P_{<k-5}f)(P_{k-3\leq\cdot\leq k+3}g))\|_2 &\lesssim \sum_{k>0} 2^{sk}\|P_{<k-5}f\|_\infty\|P_{k-3\leq\cdot\leq k+3}g\|_2 \\
&\lesssim \sum_{k>0}\sum_{k'<k-5} 2^{sk}\|P_{k'}f\|_\infty\|P_{k-3\leq\cdot\leq k+3}g\|_2 \\
&\lesssim \sum_{k>0}\sum_{k'<k-5} 2^{sk}2^{nk'/2}\|P_{k'}f\|_2\|P_{k-3\leq\cdot\leq k+3}g\|_2 \\
&\lesssim \sum_{k>0}\sum_{k'<k-5} 2^{sk}2^{nk'/2}\langle 2^{k'}\rangle^{-s_1}\|f\|_{W^{s_1,2}}2^{-s_2k}\|g\|_{W^{s_2,2}} \\
&\lesssim \sum_{k>0} 2^{sk}\max(1+\log(k), 2^{k(n/2-s_1)}\|f\|_{W^{s_1,2}}2^{-s_2k}\|g\|_{W^{s_2,2}} \\
&\lesssim \|f\|_{W^{s_1,2}}\|g\|_{W^{s_2,2}},
\end{aligned}$$

where in the last line we use the hypotheses $s < s_2$ and $s < s_1 + s_2 - n/2$.

The second and third pieces are treated similarly. The fourth piece, the "high-high" piece, is treated similarly to the "high-high" piece of $P_{\leq 0}(fg)$ and we leave it as an exercise to the reader. ∎

4. Obtaining sub-critical regularity

From the previous section, we see that it suffices to get u in a better regularity space than $W^{1,2}$, locally at least. Roughly speaking, this means that we need $2^k\|P_k u\|_2$ to decay like $2^{-\varepsilon k}$ or better for some $\varepsilon > 0$, as $k \to \infty$. Actually, we can't quite say this, because u is only defined on Ω, not on all of $\mathbf{R}^2$, and so we can't apply the projection operators P_k properly. To get around this we will need to apply a cutoff.

Also, we know that the energy $\int |\nabla u|^2$ is finite. As we shall see, it will be more useful for us if we can make the energy *small* rather than merely finite. This can also be achieved by a (logarithmic!) cutoff. More precisely, we shall need

Lemma 4.1. *Let $x_0 \in \Omega$, $u : \Omega \to S^{m-1}$ be a function with finite energy, and let $\delta > 0$. Then there exists an $0 < r < c_0$ and a C_0^∞ function η supported on a ball $B(x_0, c_0) \subset \Omega$ such that $\eta(x) = 1$ on $B(x_0, r)$, we have the estimates*

$$\int |\nabla(\eta u)|^2 < \delta \tag{4}$$

$$|\eta(x)| \lesssim \frac{\log \frac{2c_0}{r+|x-x_0|}}{\log \frac{2c_0}{r}} \tag{5}$$

$$|\nabla^j \eta(x)| \lesssim \frac{1}{\log \frac{2c_0}{r}} \frac{1}{(r+|x-x_0|)^{j+1}} \tag{6}$$

for $j = 1, 2, \ldots$ and $x \in \Omega$. (The implicit constants may depend on j).

Proof The obvious thing to do is to pick an r so small that

$$\int_{B(x_0,2r)} |\nabla u|^2 \ll \varepsilon$$

and then just let η be a bump function adapted to $B(x_0, 2r)$ which equals 1 on $B(x_0, r)$. However, this will not quite work because $\nabla(\eta u) = \eta \nabla u + (\nabla \eta) u$. The first term is fine, but the second term is bad because $\nabla \eta$ is lousy.

Instead, the trick is as follows. First pick $c_0 > 0$ such that $B(x_0, c_0) \subset \Omega$, then let $0 < r \ll c_0$ be a small number to be chosen later. Let η be a function which equals 1 on $B(x_0, r)$, equals $\frac{\log \frac{2c_0}{r+|x-x_0|}}{\log \frac{2c_0}{r}}$ on $B(x_0, c_0/2) \backslash B(x_0, 2r)$, vanishes outside of $B(x_0, c_0)$, and smoothly interpolates in the two intermediate regions $B(x_0, 2r) \backslash B(x_0, r)$ and $B(x_0, c_0) \backslash B(x_0, c_0/2)$. This η obeys the (5), (6), while

$$\int |\nabla(\eta u)|^2 \lesssim \int \eta^2 |\nabla u|^2 + \int |\nabla \eta|^2 |u|^2.$$

The former integral becomes $\ll \varepsilon$ for sufficiently small r by dominated convergence. The latter integral is OK for sufficiently small r by (6) and the fact that u lies on the sphere. ■

Fix $x_0 \in \Omega$. We need to prove that u is in $W^{1+\varepsilon,2}$ near x_0. Let $0 < \varepsilon \ll 1$ be a number to be chosen later, and let $\tilde{\eta}$ be a function as in the previous lemma with an associated radius $\tilde{r}$. We then use the lemma again and find a function η supported on the ball $B(x_0, \tilde{r})$ with an associated radius r. It thus suffices to show that

$$\|\eta u\|_{W^{1+\delta,2}} < \infty$$

for some $\delta > 0$. To obtain this, we first do a convenient rescaling to make $r = 1$ (and c_0 replaced by c_0/r, etc.). Indeed, note that if we replace $u(x)$ by $u(x_0 + rx)$ and η by $\eta(x_0 + rx)$ this does not disturb the property that u is a harmonic map or the properties (4), (5), (6).

By the cheap Littlewood-Paley inequality, it suffices to show

$$2^k \|P_k(\eta u)\|_2 \lesssim 2^{-\delta k}$$

for $k \gg 1$ and some $\delta > 0$ (not necessarily the same as the previous δ). From (4) and cheap Littlewood-Paley we already have

$$2^k \|P_k(\eta u)\|_2 \lesssim \varepsilon.$$

To improve upon this we use elliptic regularity, or more precisely the estimate

$$2^k \|P_k(\eta u)\|_2 \lesssim 2^{-k} \|P_k \Delta(\eta u)\|_2 \tag{7}$$

(this is essentially proven in the Week 2 notes). By the Leibnitz rule and (1) we have

$$\Delta(\eta u) = (\Delta \eta) u + 2\nabla\eta\nabla u - \eta u |\nabla u|^2.$$

Since u is on the sphere and ∇u has finite energy, the first two terms are bounded by

$$O(\|\Delta\eta\|_2) + O(\|\nabla\eta\|_\infty) = O(1)$$

so we get a net decay of $O(2^{-2k})$ for these terms. It remains to show that

$$2^{-k} \|P_k(\eta u |\nabla u|^2)\|_2 \lesssim 2^{-\delta k}$$

for some $\delta > 0$.

A naive attempt to use the Littlewood-Paley trichotomy now will only yield a bound of 2^{2k} or worse. Helein's observation was to rewrite the expression $u|\nabla u|^2$. Writing u out in co-ordinates u^i, $i = 1, \ldots, m$, we see that the i^{th} component of $u|\nabla u|^2$ is

$$u^i \nabla u^j \cdot \nabla u^j$$

where j is being implicitly summed from 1 to m. On the other hand, since u is on the sphere, ∇u is tangent to u, and so

$$\nabla u^i u^j \cdot \nabla u^j = 0.$$

Thus we can rewrite the i^{th} component of $u|\nabla u|^2$ as

$$R^{ij} \cdot \nabla u^j$$

where R^{ij} is the vector

$$R^{ij} := u^i \nabla u^j - \nabla u^i u^j.$$

The advantage of doing this is that R^{ij} is *divergence free* thanks to (1):

$$\begin{aligned} d_* R^{ij} = \nabla \cdot R^{ij} &= \nabla u^i \cdot \nabla u^j + u_i \Delta u^j - \Delta u^i u^j - \nabla u^i \cdot \nabla u^j \\ &= u^i \Delta u^j - \Delta u^i u^j \\ &= -u^i u^j |\nabla u|^2 + u^i |\nabla u|^2 u^j \\ &= 0 \end{aligned}$$

while ∇u^j is curl-free:

$$dR^{ij} = \nabla \wedge \nabla u^j = 0.$$

There is a general principle, which one generally refers to (by some abuse of notation) as the *div-curl lemma*, which basically asserts that some cancellation always happens when a divergence-free field has an inner product taken against a curl-free field. Specifically, one expects high-high interactions to be smaller than one might first expect. Thus we have a better chance of proving the estimate (suppressing the i, j indices, which are no longer relevant)

$$2^{-k} \|P_k(\eta R \cdot \nabla u)\|_2 \lesssim 2^{-\delta k} \tag{8}$$

than previously.

Heuristically one should think of R as having the same strength as ∇u (since u lies on the sphere and therefore has magnitude 1).

Before we can prove (8), we need some preliminaries on *Hodge theory*, which for our purposes is simply a fancy name for the theory of divergence-free and curl-free vector fields.

5. Hodge theory

Let $u : \mathbf{R}^2 \to \mathbf{R}$ be a vector field with components u_α, $\alpha = 1, 2$. (We are using Greek indices such as α, β to denote indices in the domain (i.e. $1, 2$) and Roman indices i, j to denote indices in the range (i.e. $1, \ldots, m$). The Fourier transform of a vector field would also be a vector field $\hat{u} : \mathbf{R}^2 \to \mathbf{R}$ with components $\hat{u}_\alpha$.

A vector field would be *divergence-free* if

$$2\pi i \xi \cdot \hat{u}(\xi) = 0$$

for all ξ. In other words, at each frequency ξ, the vector $\hat{u}(\xi)$ is orthogonal to ξ.

Similarly, a vector field is *curl-free* if

$$2\pi i \xi \wedge \hat{u}(\xi) = 0$$

for all ξ. In other words, at each frequency ξ, the vector $\hat{u}(\xi)$ is parallel to ξ.

In particular, from Plancherel we have the identity

$$\langle u, v \rangle = 0$$

whenever u and v are divergence-free and curl-free and both in L^2 (so that the inner product makes sense). This already shows that there is some cancellation in the expression uv (it has mean zero).

Let L^2_{df} denote the divergence free component of L^2, and L^2_{cf} the curl-free component. We know these components are orthogonal. Furthermore, they span L^2, basically because for any $\xi \neq 0$, any vector η can be decomposed into a vector parallel to ξ and a vector orthogonal to ξ, each of which has magnitude smaller than or equal to that of η:

$$\eta = \frac{1}{|\xi|^2}\xi(\xi \cdot \eta) + \frac{1}{|\xi|^2}\xi \neg (\xi \wedge \eta).$$

Applying this with $\eta := \hat{u}(\xi)$ we obtain the *Hodge decomposition*

$$u = \Delta^{-1} d(d_* u) + \Delta^{-1} d_*(du) = \Delta^{-1}\nabla(\nabla \cdot u) + \Delta^{-1}\nabla \neg (\nabla \wedge u)$$

(which, given that $\Delta = dd_* + d_* d$, is not so surprising). The first component is curl-free, while the second component is div-free. From Pythagoras's theorem we see that both components have a smaller L^2 norm than the original function u.

The operator $\Delta^{-1}\nabla\neg(\nabla\wedge)$ is denoted $\mathbf{P}$; it is the orthogonal projection from L^2 to L^2_{cf}. It is a (matrix-valued) Fourier multiplier with symbol $\frac{1}{|\xi|^2}\xi\neg(\xi\wedge)$. This is a symbol of order 0, so by the Hörmander-Mihlin multiplier theorem it is bounded on L^p for $1 < p < \infty$. Indeed, the projection $\mathbf{P}$ is quite harmless, and is bounded on most reasonable spaces (e.g. Sobolev spaces $W^{s,p}$). This is consistent with the concept of a projection - it should make things smaller!

The projection $\Delta^{-1}\nabla(\nabla \cdot u)$ onto curl-free functions has no commonly accepted symbol, other than $1 - \mathbf{P}$.

Let us investigate the product $u \cdot v$ where u is div-free and v is curl-free. Of course, we have the Fourier representation

$$\widehat{uv}(\xi) = \int_{\xi=\xi_1+\xi_2} \hat{u}(\xi_1) \cdot \hat{v}(\xi_2).$$

However, we can also write uv as $(\mathbf{P}u)(1-\mathbf{P})v$, so that

$$\widehat{uv}(\xi) = \int_{\xi=\xi_1+\xi_2} \frac{1}{|\xi_1|^2}(\xi_1 \neg(\xi_1 \wedge \hat{u}(\xi_1))) \cdot \frac{1}{|\xi_2|^2}\xi_2(\xi_2 \cdot \hat{v}(\xi_2)).$$

This is a multi-linear operator whose multiplier is the two-form

$$m(\xi_1,\xi_2)(a,b) := \frac{1}{|\xi_1|^2}(\xi_1\neg(\xi_1 \wedge a)) \cdot \frac{1}{|\xi_2|^2}\xi_2(\xi_2 \cdot b).$$

For low-high and high-low interactions, the insertion of this projections doesn't do much. (In these cases there is basically no relation between the orientations of ξ_1 and ξ_2, and so the facts that $\hat{u}(\xi_1)$ is orthogonal to ξ_1 and $\hat{v}(\xi_2)$ is parallel to ξ_2 has no significant impact). However, in the high-high case, when ξ_1 and ξ_2 are nearly equal and opposite, one can obtain a gain by observing that

$$\frac{1}{|\xi_1|^2}(\xi_1\neg(\xi_1 \wedge a)) \cdot \frac{1}{|\xi_2|^2}\xi_1(\xi_2 \cdot b) = 0$$

(the left term is orthogonal to ξ_1, while the right term is parallel ξ). Thus we can write m as

$$m(\xi_1,\xi_2)(a,b) := \frac{1}{|\xi_1|^2}(\xi_1\neg(\xi_1 \wedge a)) \cdot \frac{1}{|\xi_2|^2}\xi(\xi_2 \cdot b),$$

where $\xi := \xi_1 + \xi_2$. In particular we have the bounds

$$|m(\xi_1,\xi_2)| \lesssim \frac{|\xi|}{|\xi_2|}$$

and more generally that (in the high-high interaction case $|\xi_1| \sim |\xi_2|$)

$$|\nabla^j m(\xi_1, \xi_2)| \lesssim \frac{|\xi|}{|\xi_2|^{j+1}}$$

for all $j \geq 0$ (with the constants of course depending on j. This gives a gain of 2^{-a} in the high-high interaction case when $|\xi_1|, |\xi_2| \sim 2^a |\xi|$.

A more specific claim is that

$$\|P_k((P_{k+a}\mathbf{P}f)(P_{k+b}(1-\mathbf{P})g))\|_p \lesssim 2^{-a}\|f\|_q\|g\|_r \tag{9}$$

whenever $1 \leq p, q, r \leq \infty$ are such that $1/p = 1/q + 1/r$, and $a, b > 0$ are such that $a = b + O(1)$. This is a variant of the Coifman-Meyer multiplier theorem; the endpoints 1 and ∞ are now available thanks to the frequency localization.

6. Conclusion of the argument

We now return to the proof of (8). We start by showing the apparently weaker estimate

$$2^{-k}\|P_k(\eta R \cdot \nabla u)\|_2 \lesssim 2^{-\delta k} + \varepsilon \sum_{k' \in \mathbf{Z}} 2^{-2\delta|k-k'|} 2^{k'} \|P_{k'}(\eta u)\|_2. \tag{10}$$

Putting this back into the calculations surrounding (7), we obtain

$$2^k \|P_k(\eta u)\|_2 \lesssim 2^{-\delta k} + \varepsilon \sum_{k' \in \mathbf{Z}} 2^{-2\delta|k-k'|} 2^{k'} \|P_{k'}(\eta u)\|_2$$

for all $k \geq 0$. The contributions of the negative k' to the second term are $O(2^{-\delta k})$ by the finite energy of u, so we have

$$2^k \|P_k(\eta u)\|_2 \lesssim 2^{-\delta k} + \varepsilon \sum_{k' \geq 0} 2^{-2\delta|k-k'|} 2^{k'} \|P_{k'}(\eta u)\|_2.$$

Multiplying by $2^{\delta k}$ and taking suprema we obtain

$$\sup_{k \geq 0} 2^k 2^{\delta k} \|P_k(\eta u)\|_2 \lesssim 1 + \varepsilon \sum_{k' \geq 0} 2^{\delta k} 2^{-2\delta|k-k'|} 2^{k'} \|P_{k'}(\eta u)\|_2.$$

We estimate the right-hand side by

$$1 + \varepsilon (\sup_{k \geq 0} 2^k 2^{\delta k} \|P_k(\eta u)\|_2) \sum_{k' \geq 0} 2^{-\delta|k-k'|}.$$

Thus, if ε is sufficiently small, we have

$$\sup_{k \geq 0} 2^k 2^{\delta k} \|P_k(\eta u)\|_2 \leq C + \frac{1}{2} \sup_{k \geq 0} 2^k 2^{\delta k} \|P_k(\eta u)\|_2,$$

from which we can conclude[1] (8).

It remains to prove (10). The first thing is to deal with the somewhat annoying cutoff η. Heuristically, since η is smooth, it doesn't have much Fourier support

[1] OK, so there is a technical issue in that this supremum could be infinite. This can be dealt with by limiting arguments or by variants of the above argument (e.g. start with the trivial bound $2^k \|P_k(\eta u)\|_2 \lesssim 1$ coming from finite energy, and keep iterating this).

outside the unit ball $|\xi| \lesssim 1$, which means that multiplication with η does not affect the Fourier support much. Because of this, multiplication by η should almost commute with the multiplier P_k for large k. A quantitative version of this is the following *commutator estimate*

Lemma 6.1. *For any u and $k \geq 0$, $1 \leq p \leq \infty$ and u on $\mathbf{R}^2$ we have*

$$\|P_k(\eta u) - \eta P_k u\|_p \lesssim 2^{-k}\|P_{<k+10}u\|_p + 2^{-100k}\|u\|_1.$$

Proof Consider the contribution of $P_{k'}u$ for $k' > k+10$, i.e.

$$\|P_k(\eta P_{k'}u) - \eta P_k P_{k'}u\|_p.$$

The second term vanishes, while the first term can be rewritten as $P_k((P_{>k'-10}\eta)P_{k'}u)$. Hence

$$\|P_k(\eta P_{k'}u) - \eta P_k P_{k'}u\|_p \lesssim \|(P_{>k'-10}\eta)P_{k'}u\|_p \lesssim \|P_{>k'-10}\eta\|_\infty \|P_{k'}u\|_p.$$

The first term is $2^{-Mk'}$ for any k' because η is C^∞, while the second term is bounded by $2^{Ck}\|u\|_1$ for some constant C by Bernstein. The claim follows,

It remains to consider the contribution of $P_{\leq k+10}u$. Since

$$P_k(\eta P_{\leq k+10}u)(x) = \int \hat{\psi}(y)\eta(x - 2^{-k}y)P_{\leq k+10}u(x - 2^{-k}y)\, dy$$

and

$$\eta P_k P_{\leq k+10}u(x) = \int \hat{\psi}(y)\eta(x)P_{\leq k+10}u(x - 2^{-k}y)\, dy.$$

Since η is Lipschitz, we have $\eta(x - 2^{-k}y) - \eta(x) = O(2^{-k}|y|)$, so that

$$\|P_k(\eta P_{\leq k+10}u) - \eta P_k P_{\leq k+10}u\|_p \lesssim 2^{-k}\|\int \hat{\psi}(y)|y|P_{\leq k+10}u(x - 2^{-k}y)\, dy\|_p.$$

The claim then follows by Minkowski since $\hat{\psi}(y)|y|$ is integrable. ■

In practice the two terms on the right-hand side shall always be easy to deal with thanks to the powers of 2^{-k}. Thus we shall be moving η in and out of projections in a quite cavalier manner.

Now we return to (8). We will be using the Littlewood-Paley trichotomy often. This, combined with the errors coming from Lemma 6.1, will result in a lot of cases to consider, but the procedure is straightforward as there are only a limited number of tools available (basically just Bernstein and Hölder).

From the construction of $\tilde{\eta}$ and the fact that u lies on the sphere we have

$$|\tilde{\eta}\nabla u\|_2 \lesssim \varepsilon \tag{11}$$

and

$$|\tilde{\eta}R\|_2 \lesssim \varepsilon. \tag{12}$$

We write the left-hand side of (8) as

$$2^{-k}\|P_k(\eta(\tilde{\eta}R)\cdot(\tilde{\eta}\nabla u))\|_2.$$

We apply Lemma 6.1, obtaining a main term

$$2^{-k}\|\eta P_k((\tilde{\eta}R)\cdot(\tilde{\eta}u))\|_2 \tag{13}$$

and two error terms

$$2^{-2k}\|P_{\leq k+10}((\tilde{\eta}R)\cdot(\tilde{\eta}u))\|_2$$

and

$$2^{-101k}\|(\tilde{\eta}R)\cdot(\tilde{\eta}u))\|_1.$$

The second term is easily dealt with by (11) and (12). For the first term we use Bernstein's inequality to estimate it by

$$2^{-k}\|(\tilde{\eta}R)\cdot(\tilde{\eta}u))\|_1$$

which is also acceptable by (11) and (12).

We split (13) into the low-high term

$$2^{-k}\|\eta P_k(P_{<k-5}(\tilde{\eta}R)\cdot P_{k-3<\cdot<k+3}(\tilde{\eta}\nabla u))\|_2,$$

the high-low term

$$2^{-k}\|\eta P_k(P_{k-3<\cdot<k+3}(\tilde{\eta}R)\cdot P_{<k-5}(\tilde{\eta}\nabla u))\|_2,$$

the low-low term

$$2^{-k}\|\eta P_k(P_{k-5\leq\cdot\leq k+5}(\tilde{\eta}R)\cdot P_{k-5\leq\cdot\leq k+5}(\tilde{\eta}\nabla u))\|_2,$$

and the high-high term

$$2^{-k}\|\sum_{k',k''>k+5;|k'-k''|<3}\eta P_k(P_{k'}(\tilde{\eta}R)\cdot P_{k''}(\tilde{\eta}\nabla u))\|_2.$$

Consider first the low-high term. We move the η back inside the P_k by Lemma 6.1 (the error terms are once again easily dealt with), then discard the P_k, to estimate this by

$$2^{-k}\|\eta P_{<k-5}(\tilde{\eta}R)\cdot P_{k-3<\cdot<k+3}(\tilde{\eta}\nabla u)\|_2.$$

We split this using Hölder and (12) as

$$\lesssim \varepsilon 2^{-k}\|\eta P_{k-3<\cdot<k+3}(\tilde{\eta}\nabla u)\|_\infty.$$

We use the 2^{-k} and Bernstein to swap L^∞ with L^2. We can then move the η inside using Lemma 6.1 (the error terms are easily dealt with) so we end up with a net bound of

$$\lesssim 2^{-\delta k}+\varepsilon\sum_{|k'-k|<3}\|P_{k'}(\eta\nabla u)\|_2.$$

A similar treatment of the high-low term gives a bound of

$$\lesssim 2^{-\delta k}\varepsilon\sum_{|k'-k|<3}\|P_{k'}(\eta R)\|_2, \tag{14}$$

while the low-low term yields

$$\lesssim 2^{-\delta k}\varepsilon\sum_{|k'-k|\leq 5}\|P_{k'}(\eta\nabla u)\|_2.$$

We will set these bounds aside for the moment; now let's go to the high-high term. In this case we shall actually use the div-curl structure. Substituting $k' = k + a$, $k'' = k + b$ and using the triangle inequality, we can bound this by

$$2^{-k} \sum_{a,b>5:|a-b|<3} \|\eta P_k(P_{k+a}(\tilde{\eta} R) \cdot P_{k+b}(\tilde{\eta} u))\|_2.$$

We can easily move the η inside the P_k. Next we move it inside the P_{k+b}. The error in doing this latter maneuver is bounded by

$$2^{-k} \sum_{a,b>5:|a-b|<3} \|P_k(P_{k+a}(\tilde{\eta} R) \cdot (\eta P_{k+b}(\tilde{\eta}\nabla u) - P_{k+b}(\eta\tilde{\eta}\nabla u)))\|_2.$$

We use the 2^{-k} factor and Bernstein to replace the L^2 norm by an L^1 norm, and then Hölder:

$$\sum_{a,b>5:|a-b|<3} \|P_{k+a}(\tilde{\eta} R))\|_2 \|\eta P_{k+b}(\tilde{\eta}\nabla u) - P_{k+b}(\eta\tilde{\eta}\nabla u)\|_2.$$

We estimate the former factor by (12), and the latter factor by Lemma 6.1:

$$\sum_{a,b>5:|a-b|<3} (2^{-k-a}\|\tilde{\eta}\nabla u\|_2 + 2^{-100(k+a)}\|\tilde{\eta}\nabla u\|_1)\varepsilon$$

which is OK by (12) (and Hölder, for the L^1 term).

So now we're left with

$$2^{-k} \sum_{a,b>5:|a-b|<3} \|P_k(P_{k+a}(\tilde{\eta} R) \cdot P_{k+b}(\eta\nabla u))\|_2.$$

We want to use the div-curl lemma, however $\tilde{\eta} R$ is not quite div-free and $\tilde{\eta} u$ is not quite curl-free. However, the curl-free component of ηR is quite small since $\nabla \cdot R = 0$:

$$(1 - \mathbf{P})(\eta R) = \Delta^{-1}\nabla(\nabla \cdot (\tilde{\eta} R)) = \Delta^{-1}\nabla((\nabla\eta)R).$$

This term is small because it is basically an integral of R (so it gains a factor of 2^{-k-a} or so):

$$\|P_{k+a}(\tilde{\eta} R)\|_2 \lesssim 2^{-k-a}.$$

To control the contribution of $(1-\mathbf{P})(\tilde{\eta} R)$ to the above we use the 2^{-k} and Bernstein to swap L^2 for L^1, and then use Hölder, the previous, and (11) to bound this by

$$\sum_{a,b>5:|a-b|<3} 2^{-k-a}$$

which is acceptable. Thus we can replace $\tilde{\eta} R$ by $\mathbf{P}(\tilde{\eta} R)$. Similarly one can replace $\eta\nabla u$ by $(1 - \mathbf{P})(\eta\nabla u)$. We then use the 2^{-k} and Bernstein to replace what's left with

$$\sum_{a,b>5:|a-b|<3} \|P_k((P_{k+a}\mathbf{P}\tilde{\eta} R) \cdot (P_{k+b}(1 - \mathbf{P})\eta\nabla u))\|_1$$

which we rewrite as[2]

$$\sum_{a,b>5:|a-b|<3} \|T_{k,a,b}(\tilde{\eta} R, P_{k+b-3<\cdot<k+b+3}(\eta\nabla u))\|_1$$

[2] We could also attach a $P_{k+a-3<\cdot<k+a+3}$ to the $\tilde{\eta} R$, but we won't need this.

where

$$T_{k,a,b}(f,g) := P_k((P_{k+a}\mathbf{P}f)\cdot(P_{k+b}(1-\mathbf{P})g)).$$

By (9) we may estimate this by

$$\sum_{a,b>5:|a-b|<3} 2^{-a}\|P_{k+a-3<\cdot<k+a+3}(\tilde\eta R)\|_2\|P_{k+b-3<\cdot<k+b+3}\eta\nabla u\|_2$$

which by (12) is bounded by

$$\varepsilon\sum_{k'>k}2^{-|k'-k|}\|P_{k'}(\eta\nabla u)\|_2.$$

Adding together all the terms we see that we have bounded the left-hand side of (10) by

$$2^{-\delta k}+\varepsilon\sum_{k'\in\mathbf{Z}}2^{-|k'-k|}(\|P_{k'}(\eta\nabla u)\|_2+\|P_{k'}(\eta R)\|_2).$$

This is almost right, but we have to replace the stuff in parentheses by $2^{k'}\|P_{k'}(\eta u)\|_2$. Let's first consider the contribution of $\|P_{k'}(\eta\nabla u)\|_2$. We may approximate this expression by $\|\nabla P_{k'}(\eta u)\|_2$, which is acceptable, but there is an error of $\|P_{k'}(\nabla\eta u)\|_2$, which we may estimate by $2^{-k'}\|\nabla(\nabla\eta u)\|_2 = O(2^{-k'})$ by the finite energy hypothesis. So this term is OK.

To deal with the R term it suffices to show (messing around with the δ if necessary)

$$\|P_k(\eta R)\|_2\lesssim 2^{-\delta k}+\sum_{k'\in\mathbf{Z}}2^{-|k'-k|}(\|\nabla P_{k'}(\eta u)\|_2.$$

The matrix R consists of components which look like $u\nabla u$ (suppressing indices). So it will suffice to control

$$\|P_k(\eta(\tilde\eta u)\nabla(\tilde\eta u))\|_2.$$

We move the η outside (the error terms are easily dealt with) and apply the Littlewood-Paley trichotomy, to reduce to the low-high term

$$\|\eta P_k(P_{<k-5}(\tilde\eta u)P_{k-3<\cdot<k+3}\nabla(\tilde\eta u))\|_2,$$

the high-low term

$$\|\eta P_k(P_{k-3<\cdot<k+3}(\tilde\eta u)P_{<k-5}\nabla(\tilde\eta u))\|_2,$$

the low-low term

$$\|\eta P_k(P_{k-5\le\cdot\le k+5}(\tilde\eta u)P_{k-5\le\cdot\le k+5}\nabla(\tilde\eta u))\|_2,$$

and the high-high term

$$\|\eta P_k(\sum_{k',k''>k+5:|k'-k''|\le 3}P_{k'}(\tilde\eta u)P_{k''}\nabla(\tilde\eta u))\|_2.$$

Consider the low-high term. We move the η inside the P_k (commutator terms are easy), discard P_k, and then use the boundedness of u (hence $\tilde\eta u$, hence $P_{<k-5}(\tilde\eta u)$ to estimate this by $\|\eta P_{k-3<\cdot<k+3}\nabla(\tilde\eta u)\|_2$. We then move the η inside the $P_{k-3<\cdot<k+3}$ (again commutator terms are easy), to get $\|P_{k-3<\cdot<k+3}\nabla(\eta u)\|_2$ plus some easy error terms when the derivative falls on η.

Now consider the high-low term. Again we move the η inside the P_k. From the boundedness of u we see that $\|P_{<k-5}\nabla(\tilde\eta u)\|_\infty \lesssim 2^k$, hence we can bound this term by

$$2^k \|\eta P_{k-3<\cdot<k+3}(\tilde\eta u)\|_2.$$

Moving the η inside the projection (commutator terms are still easy) we obtain the desired result.

The low-low term is similar, so we turn to the high-high term. We first move η inside the projection again. We then use Bernstein to replace the previous with

$$2^k \|\eta \sum_{k',k''>k+5:|k'-k''|\leq 3} P_{k'}(\tilde\eta u) P_{k''}\nabla(\tilde\eta u)\|_2.$$

Triangle inequality and Cauchy-Schwarz:

$$2^k \sum_{k',k''>k+5:|k'-k''|\leq 3} \|\eta P_{k'}(\tilde\eta u)\|_2 \|P_{k''}\nabla(\tilde\eta u)\|_2.$$

The second factor is bounded by the finite energy assumption. For the first factor, we move η inside the $P_{k'}$ (error terms easy to handle) to obtain

$$\sum_{k'<k+5} 2^{-|k'-k|} 2^{k'} \|P_{k'}(\eta u)\|_2$$

which is acceptable. This finishes the proof of (10), which implies (8), which implies the $W^{1+\varepsilon,2}$ regularity, and we are done.

7. Homework for week 5 - due Friday, February 16

- Let $s > 0$, and let D^s be the Fourier multiplier
$$\widehat{D^s f}(\xi) = (1+|\xi|^2)^{s/2}\hat f(\xi).$$
  (a) Show that
$$|(1+|\xi+\eta|^2)^{s/2} - (1+|\xi|^2)^{s/2} - (1+|\eta|^2)^{s/2}| \lesssim (1+|\xi|^2)^{s/4}(1+|\eta|^2)^{s/4}$$
  for any $\xi,\eta \in \mathbf{R}^n$. (Hint: split into the cases $|\xi| \sim |\eta|$, $|\xi| \ll |\eta|$, and $|\xi| \gg |\eta|$).
  (b) Prove that
$$\|D^s(fg) - (D^s f)g - f(D^s g)\|_p \lesssim \|D^{s/2} f\|_q \|D^{s/2} g\|_r$$
  whenever $1 < p,q,r < \infty$ is such that $1/p = 1/q + 1/r$. (Hint: use the Coifman-Meyer multiplier theorem). Explain why this is an improvement to the estimate in Question 2 of last week's homework.
- Let Ω be an open subset of $\mathbf{R}^n$, and let L be a second order differential operator of the form
$$L := \sum_{i=1}^n \sum_{j=1}^n a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j},$$
  where the $a_{i,j}$ are C^∞ real functions which are *uniformly elliptic* in the sense that one has the estimates
$$c|\xi|^2 \leq \sum_{i=1}^n \sum_{j=1}^n a_{i,j}(x)\xi_i\xi_j \leq C|\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbf{R}^n$, where $0 < c < C$ are constants. Prove that if u is in locally $W^{1,2}$ on Ω and $Lu = 0$, then u is smooth.

(Hint: Let η be a cutoff function supported on a very small neighbourhood of a point $x_0 \in \Omega$, and approximate L by the constant co-efficient operator

$$L_{x_0} := \sum_{i=1}^n \sum_{j=1}^n a_{i,j}(x_0) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Use the elliptic regularity estimate with Δ replaced by L_{x_0}. The aim will be to get an estimate such as

$$\|\eta u\|_{W^{k+1,2}} \leq C \|\tilde{\eta} u\|_{W^{k,2}} + \frac{1}{2} \|\eta u\|_{W^{k+1,2}}.$$

(You may ignore questions about whether these norms actually exist or not. In practice one can always use limiting arguments to make the a priori assumption that these norms exist and are finite).

- (a) Prove that the Sobolev multiplication law (Lemma 3.1) continues to hold if one assumes $s = s_1$ instead of $s < s_1$.

  (b) Explain why the Sobolev multiplication law fails when $s > s_1$.

  (c) Prove that the Sobolev multiplication law continues to hold if one assumes $s = s_1 + s_2 - n/2$ instead of $s < s_1 + s_2 - n_2$ (but one keeps the assumption $s < s_1$ unchanged).

  (d) Explain why the Sobolev multiplication law fails when $s > s_1 + s_2 - n/2$.

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