Mathematics 245B
Terence Tao
Midterm, Feb 11, 2003

Instructions: Try to do all three problems; they are all of equal value. There is plenty of working space, and a blank page at the end.

You may enter in a nickname if you want your midterm score posted.

Good luck!

Name: ____________________

Nickame: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1. ____________

Problem 2. ____________

Problem 3. ____________

Total: ____________________

Problem 1. (a) Let X be a compact topological space, and let $\mathcal{F} \subset C(X; \mathbf{R})$ be a collection of continuous functions from X to $\mathbf{R}$ which is pointwise bounded (i.e. the set $\{f(x) : f \in \mathcal{F}\}$ is bounded for all $x \in X$) and equicontinuous. Prove that $\mathcal{F}$ is uniformly bounded (i.e. there exists an $M > 0$ such that $|f(x)| \leq M$ for all $x \in X$ and $f \in \mathcal{F}$).

Proof A: By the Arzela-Ascoli theorem, $\mathcal{F}$ is totally bounded in $C(X; \mathbf{R})$, hence bounded in $C(X; \mathbf{R})$, hence uniformly bounded.

Proof B: Choose any $\varepsilon > 0$ (e.g. $\varepsilon = 1$). For each $x \in X$, there exists an open neighborhood $U_x \subset X$ of x such that $|f(y) - f(x)| \leq \varepsilon$ for all $y \in U_x$ and $f \in \mathcal{F}$, by equicontinuity. We then choose such an U_x for each x (invoking the axiom of choice). The collection of open sets $\{U_x\}_{x \in X}$ covers X (since $x \in U_x$ for all x), thus by compactness we have a finite cover $X = U_{x_1} \cup \ldots \cup U_{x_k}$. For each x_j, we have an $M_j > 0$ such that $|f(x_j)| \leq M_j$ for all $f \in \mathcal{F}$, by pointwise boundedness. This then implies that $|f(y)| \leq M_j + \varepsilon$ for all $y \in U_{x_j}$ and $f \in \mathcal{F}$, by the triangle inequality. This then implies that $|f(y)| \leq \max_{1 \leq j \leq k} M_j + \varepsilon$ for all $y \in X$ and $f \in \mathcal{F}$, which establishes uniform boundedness.

Extra challenge: Suppose we drop the assumption that $\mathcal{F}$ is pointwise bounded for *all* $x \in X$, and just assume instead that $\mathcal{F}$ is pointwise bounded for a *single point* $x \in X$. Also, assume that X is connected. Conclude again that $\mathcal{F}$ is uniformly bounded. (Why do we need connectedness?)

(b) Give an example to show that the results in part (a) fail if the assumption of equicontinuity is dropped.

There are of course many examples. One such is $X = [0, 1]$ and $\mathcal{F} = \{f_n : n \in \mathbf{Z}\}$, where f_n is the function $f_n(x) := \min(n, 1/x)$. Various other examples based on "moving bumps" that get larger but narrower as $n \to \infty$ are possible. Note that we are still assuming X to be compact.

Problem 2. (a) Let X, Y be topological spaces, and suppose that Y is Hausdorff. Let A be a dense subset of X, and let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Show that if f and g agree on A (i.e. $f(x) = g(x)$ for all $x \in A$), then they agree on all of X (i.e. f and g are identical).

Proof A: The function $(f, g) : X \to Y \times Y$ defined by $(f, g)(x) := (f(x), g(x))$ is continuous because each of its component functions are. Since Y is Hausdorff, the diagonal $\Delta := \{(y, y) : y \in Y\}$ is closed in $Y \times Y$ (why?). Thus $(f, g)^{-1}(\Delta)$ is closed in X, i.e. the set $\{x \in X : f(x) = g(x)\}$ is closed. But this set contains A, and hence it contains $\overline{A} = X$, and we are done.

Proof B: Suppose for contradiction that f and g are not identical, then there is an $x \in X$ such that $f(x) \neq g(x)$. By the Hausdorff property there thus exists disjoint open sets U, V in Y containing $f(x)$ and $g(x)$ respectively. By continuity there then exist open neighborhoods W_1, W_2 of x such that $f(W_1) \subseteq U$ and $g(W_2) \subseteq V$. Then f and g are always unequal on $W_1 \cap W_2$, so in particular $W_1 \cap W_2$ cannot intersect A. But this contradicts the fact that A is dense, since $W_1 \cap W_2$ is open and non-empty.

Extra challenge: give an example to show that the statement can fail if the assumption that Y is Hausdorff is dropped. Extra extra challenge: show that the statement can fail if the assumption that Y is Hausdorff is relaxed to Y merely being T_1.

(b) Now suppose that X is normal, and A is a subset of X which is *not* dense. Show that there exist continuous functions $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ which agree on A but do not agree on all of X.

Since A is not dense, there exists an $x \in X$ such that $x \notin \overline{A}$. By Urysohn's lemma, there then exists a continuous function $f : X \to \mathbf{R}$ such that $f \equiv 1$ on the closed set $\overline{A}$ and $f \equiv 0$ on the closed set $\{x\}$. Now take $g \equiv 1$ on all of X, and we are done.

Extra challenge: Give an example to show that this statement can fail if the assumption that X is normal is dropped. Extra extra challenge: Show that the statement can fail if the assumption that X is normal is relaxed to X merely being Hausdorff. (I don't know what happens if X is only assumed to be regular).

Problem 3. Let X be a Banach space, and let $l^1(\mathbf{Z})$ be the space of absolutely summable sequences $(x_n)_{n\in\mathbf{Z}}$, with the l^1 norm $\|(x_n)_{n\in\mathbf{Z}}\|_{l^1(\mathbf{Z})} := \sum_{n\in\mathbf{Z}} |x_n|$. Let $l^1_c(\mathbf{Z})$ be the subspace of $l^1(\mathbf{Z})$ consisting of the compactly supported sequences (those sequences which are only non-zero for finitely many n); we give $l^1_c(\mathbf{Z})$ the norm induced from $l^1(\mathbf{Z})$, of course. For each integer n, let $e_n \in l^1_c(\mathbf{Z})$ be the element of $l^1_c(\mathbf{Z})$ whose n^{th} entry is 1 and whose other entries are zero (thus $(e_n)_m = 1$ if $m = n$ and $(e_n)_m = 0$ otherwise).

Let $T : l^1_c(\mathbf{Z}) \to X$ be a linear transformation. Show that T is continuous if and only if the set $\{T(e_n) : n \in \mathbf{Z}\}$ is a bounded subset of X.

If T is continuous, then it is bounded. Since $\{e_n : n \in \mathbf{Z}\}$ is a bounded subset of $l^1_c(\mathbf{Z})$ (since $\|e_n\|_{l^1(\mathbf{Z})} = 1$, we thus have that $\{T(e_n) : n \in \mathbf{Z}\}$ is a bounded subset of X.

Conversely, suppose that $\{T(e_n) : n \in \mathbf{Z}\}$ is a bounded subset of X, thus there exists an $M > 0$ such that $\|T(e_n)\|_X \leq M$ for all $n \in \mathbf{Z}$. Now take any $x = (x_n)_{n\in\mathbf{Z}}$ in $l^1_c(\mathbf{Z})$. Then observe that $x = \sum_{n\in\mathbf{Z}} x_n e_n$ (this sum has only finitely many non-zero terms and so there are no questions of convergence. Then

$$\|Tx\|_X = \|\sum_{n\in\mathbf{Z}} x_n T e_n\|_X \leq \sum_{n\in\mathbf{Z}} |x_n| \|Te_n\|_X \leq M \sum_{n\in\mathbf{Z}} |x_n| \leq M\|x\|_{l^1_c(X)}$$

which implies that T is bounded, hence continuous.

Extra challenge: if T is continuous, show that $\|T\| = \sup_{n\in\mathbf{Z}} \|T(e_n)\|_X$.

Extra Extra challenge: Given any bounded sequence $\{v_n : n \in \mathbf{Z}\}$ of elements in X, show that there exists precisely one continuous linear operator $T : l^1(\mathbf{Z}) \to X$ such that $Te_n = v_n$ for all $n \in \mathbf{Z}$, and that $\|T\| = \sup_{n\in\mathbf{Z}} \|v_n\|_X$. (This challenge, unlike the previous ones, really uses the fact that X is a Banach space).
