Free Meixner states

`arXiv:math.CO/0702158`

Michael Anshelevich

March 30, 2007

One dimensional free Meixner distributions.

Probability measures on $\mathbb{R}$.

Orthogonal polynomials $\{P_n(x)\}$.

Have a generating function

$$\sum_{n=0}^{\infty} P_n(x) z^n = F(z) \frac{1}{1 - xG(z)}.$$

Proposition. The R-transform R_μ satisfies

$$\frac{R_\mu}{z} = 1 + tR_\mu + cR_\mu^2.$$

for $t \in \mathbb{R}$, $c \geq -1$.

Complete description:

$$\frac{1}{2\pi}\frac{\sqrt{4(1+c)-(x-t)^2}}{1+tx+cx^2}\,dx + \text{ zero, one, or two atoms.}$$

Particular cases:

The semicircular (free Gaussian) distributions

$$\frac{1}{2\pi}\sqrt{4-x^2}\,dx.$$

The Marchenko-Pastur (free Poisson) distributions

$$\frac{1}{2\pi}\frac{\sqrt{4t-(x-t)^2}}{1+tx}\,dx + \text{ possibly one atom.}$$

Spectral distribution of Jacobi (double Wishart) ensemble.

Kesten measures.

Bernoulli distributions $(1-p)\delta_0 + p\delta_1$.

Other appearances:

Szegö (1922), Bernstein (1930), Boas & Buck (1956), Carlin & McGregor (1957), Geronimus (1961), Allaway (1972), Askey & Ismail (1983), Cohen & Trenholme (1984), Kato (1986), Freeman (1998), Saitoh & Yoshida (2001), M.A. (2003), Kubo, Kuo & Namli (2006), Belinschi & Nica (2007), ...

$$\mu(t, c = -1) = \text{ Bernoulli}, \qquad \mathbb{B}_\lambda(\mu(t, -1)) = \mu(t, \lambda - 1).$$

NON-COMMUTATIVE POLYNOMIALS.

$$\mathbb{R}\langle \mathbf{x} \rangle = \mathbb{R}\langle x_1, x_2, \dots, x_d \rangle$$

Involution

$$(x_1\, x_2\, x_1\, x_1\, x_3)^* = x_3\, x_1\, x_1\, x_2\, x_1.$$

φ a state on $\mathbb{R}\langle \mathbf{x} \rangle$: linear functional, $\varphi\,[1] = 1$,

$$\varphi\,[P^*] = \varphi\,[P]\,,$$

$$\varphi\,[P^* P] \geq 0.$$

FREE MEIXNER STATES: FIRST DEFINITION.

Monic orthogonal polynomials

$$P_i(\mathbf{x}) = x_i + \dots,$$
$$P_{ij}(\mathbf{x}) = x_i x_j + \dots$$

$$\{P_{\vec{u}}(\mathbf{x})\} = \Big\{1, P_i(\mathbf{x}), P_{ij}(\mathbf{x}), P_{ijk}(\mathbf{x}), \dots\Big\}$$

Gram-Schmidt: can make orthogonal

$$\langle P_{\vec{u}}, P_{\vec{v}} \rangle = \varphi\left[P_{\vec{u}}^* P_{\vec{v}}\right] = 0$$

unless $\vec{u} = \vec{v}$.

Definition. A state φ is a free Meixner state if its monic orthogonal polynomials have a generating function

$$\sum_{\vec{u}} P_{\vec{u}}(\mathbf{x}) z_{\vec{u}} = 1 + \sum_i P_i(\mathbf{x}) z_i + \sum_{i,j} P_{ij}(\mathbf{x}) z_i z_j + \ldots$$
$$= F(\mathbf{z}) \left(1 - \sum_i x_i G_i(\mathbf{z}) \right)^{-1}$$

for some $F(\mathbf{z}), G_i(\mathbf{z})$.

ANALOG: if instead take commutative polynomials,

$$\sum_{\vec{u}} \frac{1}{\vec{u}!} P_{\vec{u}}(\mathbf{x}) z_{\vec{u}} = F(\mathbf{z}) \exp\left(\sum_i x_i G_i(\mathbf{z}) \right)$$

get Meixner polynomials, quadratic exponential families, Gibbs measures, etc.

Example.

$$\begin{aligned} P_i(\mathbf{x}) &= x_i + \dots &&= U_1(x_i), \\ P_{ij}(\mathbf{x}) &= x_i x_j + \dots &&= U_1(x_i)U_1(x_j), \\ P_{ii}(\mathbf{x}) &= x_i^2 + \dots &&= U_2(x_i), \\ P_{1,2,1,1,1}(\mathbf{x}) &= x_1 x_2 x_1^3 + \dots &&= U_1(x_1)U_1(x_2)U_3(x_1), \end{aligned}$$

where $U_i =$ Chebyshev polynomials. Define φ by $\varphi[1] = 1$,

$$\varphi[P_{\vec{u}}] = 0$$

for all $\vec{u}$. Then

$$\sum_{\vec{u}} P_{\vec{u}}(\mathbf{x}) z_{\vec{u}} = \left(1 - \sum_i x_i z_i + \sum_i z_i^2\right)^{-1}.$$

Operator model.

Will construct a Hilbert space, vector Ω in it, operators $X_1, X_2, \ldots, X_d$ on it such that

$$\varphi\left[P(x_1, \ldots, x_d)\right] = \left\langle \Omega, P(X_1, \ldots, X_d)\Omega \right\rangle .$$

Initial data: instead of numbers $t, c \geq -1$ have

$T_1, \ldots, T_d$ symmetric $d \times d$ matrices.

C diagonal $d^2 \times d^2$ matrix, $I \otimes I + C \geq 0$.

$$(T_i \otimes I)C = C(T_i \otimes I).$$

Let $\mathcal{H} = \mathbb{C}^d$ with an orthonormal basis $e_1, e_2, \ldots, e_d$.

$T_i =$ operator on $\mathcal{H}$, $C =$ operator on $\mathcal{H} \otimes \mathcal{H}$.

$$
\begin{aligned}
\mathcal{F}_{\mathsf{alg}}(\mathcal{H}) &= \bigoplus_{i=0}^{\infty} \mathcal{H}^{\otimes i} = \mathbb{C}\Omega \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus \ldots \\
&= \text{ vector space of non-commutative polynomials in } e_1, e_2, \ldots, e_d.
\end{aligned}
$$

Inner product

$$\langle \eta, \zeta \rangle_C = \langle \eta, K_C \zeta \rangle \, ,$$

$\langle \cdot, \cdot \rangle =$ the usual tensor inner product.

K_C is the non-negative kernel: on $\mathcal{H}^{\otimes 4}$,

$$K_C = \Big(I^{\otimes 2} \otimes (I^{\otimes 2} + C) \Big) \Big(I \otimes (I^{\otimes 2} + C) \otimes I \Big) \Big((I^{\otimes 2} + C) \otimes I^{\otimes 2} \Big).$$

Factor out vectors of length zero, complete, get the Fock space $\mathcal{F}_C(\mathcal{H})$.

Operators

$$\begin{aligned} a_i^+ \quad & \left(e_{u(1)} \otimes e_{u(2)} \otimes \ldots \otimes e_{u(k)}\right) = e_i \otimes e_{u(1)} \otimes e_{u(2)} \otimes \ldots \otimes e_{u(k)}, \\ a_i^- \quad & \left(e_{u(1)} \otimes e_{u(2)} \otimes \ldots \otimes e_{u(k)}\right) = \delta_{i,u(1)} e_{u(2)} \otimes \ldots \otimes e_{u(k)}, \end{aligned}$$

$$\begin{aligned} T_i \quad & = T_i \otimes I^{\otimes(k-1)} \text{ on } \mathcal{H}^{\otimes k}, \\ a_i^- C \quad & = a_i^- (C \otimes I^{\otimes(k-2)}) \text{ on } \mathcal{H}^{\otimes k}. \end{aligned}$$

Define

$$X_i = a_i^+ + a_i^- + T_i + a_i^- C.$$

Lemma. Each X_i factors through to $\mathcal{F}_C(\mathcal{H})$. Each X_i is symmetric and bounded, so self-adjoint.

Define a state $\varphi_{C,\{T_i\}}$ on $\mathbb{R}\langle x_1, \ldots, x_d \rangle$ by

$$\varphi_{C,\{T_i\}}[P(x_1, x_2, \ldots, x_d)] = \langle \Omega, P(X_1, X_2, \ldots, X_d)\Omega \rangle.$$

Theorem. φ is a free Meixner state if and only if

$$\varphi = \varphi_{C,\{T_i\}}$$

for some $C, \{T_i\}$.

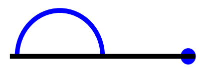
Free probability connection: free cumulants.

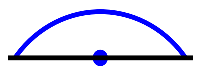
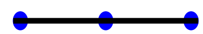
$$\varphi\left[x_{\vec{u}}\right] = \text{ moments of } \varphi.$$

$R_\varphi =$ free cumulant functional, another functional on $\mathbb{R}\langle \mathbf{x} \rangle$.

$$R_\varphi\left[x_{\vec{u}}\right] = \varphi\left[x_{\vec{u}}\right] - \sum_{\substack{\pi \in NC(n) \\ \pi \neq \hat{1}}} \prod_{B \in \pi} R_\varphi\left[\prod_{i \in B} x_{u(i)}\right].$$

$$
\begin{aligned}
R\left[x_i\right] &= \varphi\left[x_i\right], \\
R\left[x_i x_j\right] &= \varphi\left[x_i x_j\right] - R\left[x_i\right] R\left[x_j\right], \\
R\left[x_i x_j x_k\right] &= \varphi\left[x_i x_j x_k\right] - R\left[x_i x_j\right] R\left[x_k\right] - R\left[x_i\right] R\left[x_j x_k\right] \\
&\quad - R\left[x_i x_k\right] R\left[x_j\right] - R\left[x_i\right] R\left[x_j\right] R\left[x_k\right]
\end{aligned}
$$

$= \varphi\left[x_i x_j x_k\right] -$  $-$ 

$-$  $-$  .

For simplicity, assume $\varphi\left[x_i\right] = 0$, $\varphi\left[x_i x_j\right] = \delta_{ij}$: mean zero, identity covariance.

Operator model for the free cumulant functional.

Theorem. $R[x_i] = 0$, and

$$R\left[x_i P(x_1, \ldots, x_d) x_j\right] = \left\langle e_i, P(S_1, \ldots, S_d) e_j \right\rangle,$$

where

$$S_i = a_i^+ + T_i + a_i^- C = X_i - a_i^-.$$

MULTIVARIATE EXAMPLES.

Concentrate on tracial states

$$\varphi[AB] = \varphi[BA].$$

1. Free products.

2. “Simple quadratic exponential families”.

1. Free products. Data $C_{ij} = c_i \delta_{ij}$, $T_i = b_i E_{ii}$.

Here $E_{ii} =$ projection onto e_i.

$$S_i = a_i^+ + b_i E_{ii} + a_i^- C$$

acts only on $\mathrm{Span}\left(e_i^{\otimes k} | k \geq 0\right)$. So

$$R\left[x_{u(1)} \ldots x_{u(k)}\right] = 0$$

unless all $u(j)$ equal. This means

$(X_1, X_2, \ldots, X_d) \sim$ free product of 1-dim free Meixner states.

Rotations of free product states are tracial:

for O an orthogonal matrix, let $O^T(\mathbf{x}) = (\sum O_{i1} x_i, \ldots, O_{id} x_i)$ and

$$\varphi^O [P(\mathbf{x})] = \varphi \left[P(O^T \mathbf{x}) \right].$$

Note: rotations of free Meixner states not always free Meixner.

Proposition. Let $C_{ij} = c_i \delta_{ij}$, T_i arbitrary, φ tracial. Then $\varphi =$ rotation of a free product state.

Example. True for $C = 0$.

Data $C = 0$, $T_i = 0$.

Semicircular: A free product.

Data $C = 0$, T_i arbitrary.

Free Poisson: If tracial, all rotations of free products.

2. Multinomial. $C_{ij} \equiv -1$. (Recall $I \otimes I + C \geq 0$.)

Choose vectors $\{f_1, \ldots, f_d\}$ with

$$\langle f_i, f_i \rangle = p_i(1 - p_i),$$
$$\left\langle f_i, f_j \right\rangle = -p_i p_j,$$

where

$$p_i > 0, \qquad i = 1, 2, \ldots, d, \qquad p_1 + p_2 + \ldots + p_d = 1.$$

Let

$$T_i(f_i) = (1 - 2p_i) f_i,$$
$$T_i(f_j) = -p_i f_j - p_j f_i,$$

and define

$$X_i = a_i^+ + T_i + a_i^- + a_i^- C + p_i.$$

Proposition. $X_i =$ orthogonal projection onto $\mathrm{Span}\,(f_i + p_i\Omega)$.

Orthogonal projections adding up to the identity.

They commute, φ factors through to a state on $\mathbb{R}[x_1, \ldots, x_d]$.

Can be identified with the measure

$$\varphi = p_1\delta_{e_1} + p_2\delta_{e_2} + \ldots p_d\delta_{e_d},$$

the multinomial distribution.

Note that the multinomial distribution is also classical Meixner.

Generalization:

Lemma. If $\varphi_{C,\{T_i\}}$ is tracial, then for all i, j,

$$T_i e_j = T_j e_i$$

and

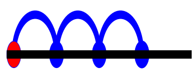
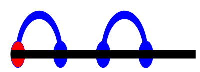
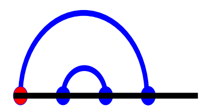
$$[T_i, T_j] = T_i T_j - T_j T_i = C_{ji} E_{ij} - C_{ij} E_{ji}.$$

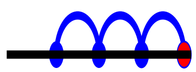
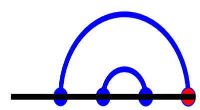
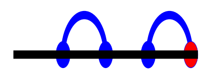
Question. Construct such matrices?

Proposition. If $C_{ij} \equiv c$, then this condition is also sufficient for φ to be a trace.

Proof. φ a trace $\Leftrightarrow$ R_φ a trace.

$$\begin{aligned}\varphi[x_1x_2x_3x_4] &= R[x_1x_2x_3x_4] + R[x_1x_2]\,R[x_3x_4] + R[x_1x_4]\,R[x_2x_3] \\ &= R[x_2x_3x_4x_1] + R[x_2x_1]\,R[x_3x_4] + R[x_2x_3]\,R[x_4x_1]\,.\end{aligned}$$

=  +  + 

=  +  + .

Recall $S_i = a_i^+ + T_i + ca_i^-$.

$$\begin{aligned} R\left[x_1x_2x_3x_4x_5\right] &= \langle e_1, S_2S_3S_4e_5\rangle \\ &= \langle e_1, T_2T_3T_4e_5\rangle + c\,\langle e_1, e_5\rangle\,\langle e_2, T_3e_4\rangle \\ &\quad + c\,\langle e_1, T_4e_5\rangle\,\langle e_2, e_3\rangle + c\,\langle e_1, T_2e_5\rangle\,\langle e_3, e_4\rangle\,. \end{aligned}$$

But

$$\begin{aligned} \langle e_1, T_2T_3T_4e_5\rangle &= \langle e_2, T_1T_3T_4e_5\rangle \\ &= \langle e_2, [T_1, T_3]T_4e_5\rangle + \langle e_2, T_3[T_1, T_4]e_5\rangle + \langle e_2, T_3T_4T_5e_1\rangle \\ &= \langle e_2, T_3T_4T_5e_1\rangle \\ &\quad + c\Big(\langle e_2, e_1\rangle\,\langle e_3, T_4e_5\rangle + \langle e_2, T_3e_1\rangle\,\langle e_4, e_5\rangle\Big) \\ &\quad - c\Big(\langle e_2, e_3\rangle\,\langle e_1, T_4e_5\rangle + \langle e_2, T_3e_4\rangle\,\langle e_1, e_5\rangle\Big) \end{aligned}$$

and

$$c\,\langle e_1, T_2e_5\rangle\,\langle e_3, e_4\rangle = c\,\langle e_2, T_5e_1\rangle\,\langle e_3, e_4\rangle\,,$$

so

$$
\begin{aligned}
&R\left[x_1 x_2 x_3 x_4 x_5\right] \\
&\quad = \langle e_1, T_2 T_3 T_4 e_5 \rangle + c \langle e_1, e_5 \rangle \langle e_2, T_3 e_4 \rangle \\
&\quad\quad + c \langle e_1, T_4 e_5 \rangle \langle e_2, e_3 \rangle + c \langle e_1, T_2 e_5 \rangle \langle e_3, e_4 \rangle \\
&\quad = \langle e_2, T_3 T_4 T_5 e_1 \rangle + c \langle e_2, e_1 \rangle \langle e_3, T_4 e_5 \rangle \\
&\quad\quad + \langle e_2, T_3 e_1 \rangle \langle e_4, e_5 \rangle + c \langle e_2, T_5 e_1 \rangle \langle e_3, e_4 \rangle \\
&\quad = \langle e_2, S_3 S_4 S_5 e_1 \rangle = R\left[x_2 x_3 x_4 x_5 x_1\right].
\end{aligned}
$$

The same calculation works in general.

OPERATOR ALGEBRAS.

1. Free products = free products.

2. $C_{ij} \equiv c$, $T_i = 0$. Ricard:

$$W^*(X_1, \ldots, X_d) = \begin{cases} L(\mathbb{F}_d) \\ L(\mathbb{F}_d) \oplus \mathcal{B}(\ell^2) \end{cases}$$

depending on d, c.