Homework 11/21 — 11/25

§ 12.6

3. $\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$

The Ratio Test gives Absolute Convergence:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{10^{n+1}/(n+1)!}{10^n/n!} = \frac{10}{n+1} \to 0 < 1$$

as $n \to \infty$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$

The Alternating series test gives Convergence:

1. $|a_n| = \frac{1}{\sqrt[4]{n}} \to 0$
as $n \to \infty$

2. $\frac{d}{dx}|a_x| = \frac{d}{dx}\left(x^{-1/4}\right) = -\frac{1}{4}x^{-5/4} < 0$

3. $a_n = (-1)^{n+1}|a_n|$

However, $|a_n| = n^{-1/4}$ fails the p-test for convergent series $\to \sum |a_n|$ diverges

So the series converges conditionally

7. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{5+n}$

$$|a_n| = \frac{n}{5+n} \longrightarrow 1 > 0$$

as $n \longrightarrow \infty$

The Divergence test indicates divergence.

9. $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

The Ratio Test gives Absolute Convergence.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n)!}{(2n+1)!} = \frac{1}{2n+1} \longrightarrow 0 < 1$$

11. $\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$

Supposing $n > 1$, so that $e^{1/n} < e$,

$|a_n| \le \frac{e}{n^3}$, which passes the p-test.

So we have absolute convergence.

13. $\sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}} = \sum_{n=1}^{\infty} 4n(3/4)^n(-1)^n$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)(3/4)^{n+1}}{n(3/4)^n} = \frac{n+1}{n} \cdot 3/4 \rightarrow 3/4 < 1$$

as $n \longrightarrow \infty$

② The Ratio Test gives Absolute Convergence

15. $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}} = \sum_{n=1}^{\infty} \frac{1}{4(n+1)} \left(\frac{10}{4^2}\right)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left(\frac{10}{16}\right)^{n+1}(n+1)}{\left(\frac{10}{16}\right)^n (n+2)} = \frac{10}{16}\,\frac{n+1}{n+2} \to \frac{10}{16} < 1$$

as $n \longrightarrow \infty$

The Ratio Test gives Absolute Convergence

§12.8

3. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ — Radius of convergence is 1

Suppose $|x| < 1$ (resp. $|x| > 1$)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}/\sqrt{n+1}}{x^n/\sqrt{n}}\right| = |x|\sqrt{\frac{n}{n+1}} \longrightarrow |x| < 1$$

as $n \longrightarrow \infty$

The RT. gives Abs. Conv. (resp. $|x| > 1$, divergence

Boundaries:

$x = -1$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $\quad |a_n| = \frac{1}{\sqrt{n}} \longrightarrow 0$

as $n \longrightarrow \infty$

$$\frac{d}{dx}|a_x| = \frac{-1}{2}x^{-3/2} < 0$$

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The AST gives convergence.

$\underline{x = 1}$

$$\sum_{n=1}^{\infty} \frac{(1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$ fails the p-test diverges

So $[-1, 1)$ is the interval of conv.

5. $$\sum_{n=1}^{\infty} \cancel{\frac{x^n}{n!}} \frac{(-1)^{n-1} x^n}{n^3}$$ Radius $= 1$

Suppose $|x| < 1$ (resp. $|x| > 1$)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}/(n+1)^3}{x^n/n^3}\right| = |x| \cdot \left(\frac{n}{n+1}\right)^3 \longrightarrow |x| < 1$$

as $n \longrightarrow \infty$

R.T. Gives convergence (resp. divergence if $|x| > 1$)

Boundaries:

$\underline{x = -1}$ $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-1)^n}{n^3} = \sum_{n=1}^{\infty} \frac{-1}{n^3}$$ converges by the p-test

$\underline{x = 1}$ $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (1)^n}{n^3} = \sum \frac{(-1)^{n-1}}{n^3}$$ converges by p-test

So the interval of convergence is

$$[-1, 1]$$

7. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges everywhere

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x}{n+1} \to 0 < 1$$

as $n \to \infty$

for any fixed x.

RT shows that this converges for all values of x.

9. $\sum_{n=1}^{\infty} (-1)^n \, n \, (4x)^n$ has radius $1/4$

Suppose $|x| < 1/4$ (resp. $|x| > 1/4$)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{n} \frac{(4x)^{n+1}}{(4x)}\right| = |4x| \cdot \frac{n+1}{n} \to |4x| < 1$$

as $n \to \infty$

RT shows convergence (resp. div when $|4x| > 1$)

Boundaries:

$\underline{x = -1/4}$ | $\sum_{n=1}^{\infty} (-1)^n (-1)^n n = \sum_{n=1}^{\infty} n$ diverges

$\underline{x = 1/4}$ | $\sum_{n=1}^{\infty} (-1)^n (1)^n n = \sum_{n=1}^{\infty} (-1)^n n$ diverges

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In both case, $|a_n| \not\to 0$ as $n \to \infty$

Therefore the interval of convergence is $(-1/4, 1/4)$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^n}{\sqrt[4]{n}} \qquad R = 1/2$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x)^{n+1}/\sqrt[4]{n+1}}{(2x)^n/\sqrt[4]{n}}\right| = |2x| \sqrt[4]{\frac{n+1}{n}} \to |2x|$$

This series converges when $|x| < 1/2$ and diverges when $|x| > 1/2$

Boundaries:

$\underline{x = -1/2}$ $\displaystyle\sum \frac{(-2(-1/2))^n}{\sqrt[4]{n}} = \sum \frac{1}{\sqrt[4]{n}}$ diverges by p-test

$\underline{x = 1/2}$ $\displaystyle\sum \frac{(-2(1/2))^n}{\sqrt[4]{n}} = \sum \frac{(-1)^n}{\sqrt[4]{n}}$ converges

As in problem 5, § 12.6, the Alternating Series Test gives convergence

So the convergence interval is

$(-1/2, 1/2]$

13. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2} (-1)^n \left(\frac{x}{4}\right)^n \frac{1}{\ln n}$

Has radius 4

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x/4)^{n+1}/\ln n+1}{(x/4)^n/\ln n}\right| = (x/4)\frac{\ln n}{\ln n+1} \to \left|\frac{x}{4}\right|$$

as $n \to \infty$

when $|x| < 4$, converges; $|x| > 4$, diverges.

Boundaries

$\underline{x = -4}$ $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n(-1)^n}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges

$\underline{x = 4}$ $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n(1)^n}{\ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

$$|a_n| = \frac{1}{\ln n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

$$\frac{d}{dx}|a_x| = \frac{-1}{(\ln x)^2} < 0$$

Now, since $a_n = (-1)^n \cdot |a_n|$, the AST gives convergence

So the interval of convergence is

$$(-4, 4]$$

15. $\displaystyle\sum_{n=0}^{\infty} \sqrt{n}\,(x-1)^n \qquad R = 1$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{n+1}\,(x-1)^{n+1}}{\sqrt{n}\,(x-1)^n}\right| = |x-1|\sqrt{\frac{n+1}{n}} \to |x-1|$$

as $n \longrightarrow \infty$

The series converges when $|x-1|<1$ and diverges when $|x-1|>1$

Boundaries:

$\underline{x=0}|$ $\sum_{n=0}^{\infty} \sqrt{n}\,(-1)^n$ does not converge

$|a_n| \not\to 0$

$\underline{x=2}|$ $\sum_{n=0}^{\infty} \sqrt{n}\,(1)^n$ does not converge either

So the interval of convergence is

$(0,2)$

17. $\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n2^n} = \sum_{n=1}^{\infty} (-1)^n \left(\frac{x+2}{2}\right)^n / n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((x+2)/2)^{n+1}/(n+1)}{((x+2)/2)^n/n}\right| = \left|\frac{x+2}{2}\right| \cdot \frac{n}{n+1} \to \left|\frac{x+2}{2}\right|$$

as $n \to \infty$

Converges when $\left|\frac{x+2}{2}\right|<1$, diverges when $\left|\frac{x+2}{2}\right|>1$

Boundaries:

$\underline{x=0}|$ $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{2}\right)^n / n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$|a_n| = 1/n \to 0$ as $n \to \infty$

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$\frac{d}{dx}|a_x| = \frac{-1}{x^2} < 0$

Since $a_n = (-1)^n |a_n|$, the

AST gives convergence

$\underline{x=-4}$ $\displaystyle\sum_{n=1}^{\infty} (-1)^n(-1)^n/n = \sum_{n=1}^{\infty} \frac{1}{n}$

which diverges, by the p-test

So the interval of convergence is

$[-4, 0]$

19. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{x-2}{n}\right)^n$ converges everywhere

$$0 \le \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{x-2}{n+1}\right)^{n+1}}{\left(\frac{x-2}{n}\right)^n}\right| \le \left|\frac{\left(\frac{x-2}{n}\right)^{n+1}}{\left(\frac{x-2}{n}\right)^n}\right|$$

$$= \left|\frac{(x-2)}{n}\right| \to 0 < 1$$

as $n \to \infty$

21. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{x-a}{b}\right)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\left(\frac{x-a}{b}\right)^{n+1}}{n\left(\frac{x-a}{b}\right)^n}\right| = \frac{n+1}{n}\left|\frac{x-a}{b}\right| \to \left|\frac{x-a}{b}\right|$$

as $n \to \infty$

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Converges where $\left|\frac{x-a}{b}\right| < 1$ and

diverges when $\left|\frac{x-a}{b}\right| > 1$

boundaries

$\underline{x = a - b}$ $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{b}{b}\right)^n = \sum_{n=1}^{\infty} n \longrightarrow \infty$ since $|a_n| \not\to 0$

$\underline{x = a + b}$ $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{-b}{b}\right)^n = \sum_{n=1}^{\infty} (-1)^n n$ which diverges, as $|a_n| \not\to 0$

The interval of convergence:

$\therefore$ $(a-b, a+b)$

§12.9

3. $\displaystyle\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$ $R = 1$

$x = -1$ $\displaystyle\sum_{n=0}^{\infty} (1)^n$ diverges

$x = 1$ $\displaystyle\sum_{n=0}^{\infty} (-1)^n$ diverges

Interval: $(-1, 1)$

5. $\displaystyle\frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n}$ $R = 1$

$x = -1$ $\displaystyle\sum_{n=0}^{\infty} (-1)^{3n}$ diverges

$x = 1$ $\displaystyle\sum_{n=0}^{\infty} (1)^{3n}$ diverges

Interval: $(-1, 1)$

7. $\frac{1}{x-5} = \frac{-1/5}{1-x/5} = \frac{-1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n \quad R=5$

$x=-5 \quad \sum_{n=0}^{\infty}(-1)^n$ diverges

$x=5 \quad \sum_{n=0}^{\infty}(1)^n$ diverges

Interval: $(-5,5)$

10. $\frac{x^2}{a^3-x^3} = \frac{x^2/a^3}{1-\left(\frac{x}{a}\right)^3} = \frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\frac{x}{a}\right)^{3n} \quad R=a$

$x=-a \quad \sum(-1)^{3n}$ diverges

$x=a \quad \sum(1)^{3n}$ diverges

Interval: $(-a,a)$

23. $\int\frac{t\,dt}{1-t^8} = \int t\sum_{n=0}^{\infty}t^{8n}\,dt = \int\sum_{n=0}^{\infty}t^{8t+1}\,dt$

$R=1$

$= \sum_{n=0}^{\infty}\frac{t^{8t+2}}{8t+2}$

$\left|\frac{a_{n+1}}{a_n}\right| = |t|^8\cdot\frac{8n+2}{8n+10} \rightarrow |t|^8$

The series converges when $|t|<1$

diverges when $|t|>1$

24. $\int \frac{\ln(1-t)}{t}\,dt$.

$$= \int \frac{\int^t \frac{-dx}{1-x}}{t}\,dt = \int \frac{1}{t}\int^t -\sum_{n=0}^{\infty} x^n\,dx\,dt$$

$$= -\int \frac{1}{t}\sum_{n=0}^{\infty} t^{n+1}/_{n+1}\,dt = -\int \sum_{n=0}^{\infty} t^n/_{n+1}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)^2}$$

$R = 1$

27. $$\int_0^{0.2} \frac{dx}{1+x^5} = \int_0^{0.2} \sum_{n=0}^{\infty} (-x^5)^n\,dx$$

$R = 1$ $$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n+1}}{5n+1}\Bigg|_0^{0.2} = \sum_{n=0}^{\infty} (-1)^n \frac{.2^{5n+1}}{5n+1}$$

Assume $|x| < 1$

$\therefore$ $|a_n| = \frac{|x|^{5n+1}}{5n+1} \longrightarrow 0$ let n

$$\frac{d}{dt}|a_t| = \frac{|x|^{5t+1}\cdot 5\cdot \ln|x|}{5t} - \frac{|x|^{5t+1}}{(5t)^2} < 0$$

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Now, since $a_n = (-1)^n |a_n|$,

$\sum a_n$ is an alternating series, so

the error in a finite subseries

of n terms is less than or equal to the n+1 th term

$a_0 = .2$

$a_1 = -1.07 \times 10^{-5}$

$a_2 = 1.86 \times 10^{-9}$

So it is safe to say that the first two terms are close enough.

$a_0 + a_1 = .199989$

§12.10

3. $\cos x \,|_0 = 1.$

$f' = -\sin x \,|_0 = 0$

$f'' = -\cos x \,|_0 = -1$

$f''' = \sin x \,|_x = 0$

$a_{2n} = (-1)^n$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

The series converges for all x:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+2}/(2n+2)!}{x^{2n}/(2n)!}\right| = \frac{x^2}{(n+1)(n+2)} \to 0<1$$

as $n \to \infty$

4. $\sin 2x \big|_0 = 0$ $\qquad a_{2n+1} = (-1)^n \cancel{x}\, 2^{2n+1}$

$f' = 2\cos 2x \big|_0 = 2$ $\qquad \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$

$f'' = -4\sin 2x \big|_0 = 0$

$f''' = -8\cos 2x \big|_0 = -8$

this series converges for all x

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x)^{2n+3} / (2n+3)!}{(2x)^{2n+1} / (2n+1)!}\right| = \frac{(2x)^2}{(2n+2)(2n+3)}$$

$$\to 0 \text{ as } n \to \infty$$

11. $f(x) = 1 + x + x^2 \big|_2 = 7$

$f'(x) = 1 + 2x \big|_2 = 5$ $\qquad$ Taylor Series

$f''(x) = 2 \big|_2 = 2$

$f'''(x) = 0$

$$f(x) \sim 7 + \frac{5(x-2)}{1!} + \frac{2(x-2)^2}{2!}$$

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$$= 7 + 5(x-2) + (x-2)^2$$

incidentally,

$$= x^2 - 4x + 4 + 5x - 10 + 7 = x^2 + x + 1$$

In general, a polynomial is equal to its Taylor.