PERVERSE EQUIVALENCES

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1. Introduction

In [ChRou], the authors initiated the higher representation theory of Kac-Moody algebras. One of the key constructions was a categorical lift of the adjoint action of simple reflections of the Weyl group. The invertibility of those functors on derived categories was proven by showing that, on weight spaces of simple (or isotypic) representations of $\mathfrak{sl}_2$, suitable shifts of those functors actually induced equivalences of abelian categories. The invertibility in general followed from the fact that the derived categories involved have a filtration whose subquotients correspond to isotypic representations.

This article stems from an attempt to understand this phenomenon, which has been found to occur in many settings. We set up foundations towards a combinatorial theory for triangulated categories. While [ChRou] discussed categorical counterparts of Kac-Moody algebras, our work should be viewed as a step towards a higher representation-theoretic analog of Coxeter group combinatorics. One could hope that tools from geometric group theory can be brought in. Our approach can be viewed as trying to capture combinatorial aspects of Bridgeland's space of stability conditions [Bri1], although we are not able to give precise relations. In a Kac-Moody setting, Bridgeland's approach gives rise to a manifold playing a role similar to a universal covering space for a hyperplane complement, while our approach is related to a combinatorial model for such a subspace, arising from Garside-type structures as originally constructed by Deligne [De].

Consider two abelian categories $\mathcal{A}$ and $\mathcal{A}'$ endowed with filtrations $0 = \mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \cdots \subset \mathcal{A}_r = \mathcal{A}$ and $0 = \mathcal{A}'_{-1} \subset \mathcal{A}'_0 \subset \cdots \subset \mathcal{A}'_r = \mathcal{A}'$ by Serre subcategories. Let $D_{\mathcal{A}_i}(\mathcal{A})$ denote the thick subcategory of $D^b(\mathcal{A})$ of complexes with cohomology in $\mathcal{A}_i$. Consider a map $p : \{0, \dots, r\} \to \mathbf{Z}$.

An equivalence of triangulated categories $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ is perverse if for every i, $F[-p(i)]$ restricts to an equivalence $D^b_{\mathcal{A}_i}(\mathcal{A}) \xrightarrow{\sim} D^b_{\mathcal{A}'_i}(\mathcal{A}')$ and if the induced equivalence between quotient triangulated categories $D^b_{\mathcal{A}_i}(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A}) \xrightarrow{\sim} D^b_{\mathcal{A}'_i}(\mathcal{A}')/D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}')$ restricts to an equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\sim} \mathcal{A}'_i/\mathcal{A}'_{i-1}$. An easy but crucial fact is that given $\mathcal{A}$ and p, the category $\mathcal{A}'$ is unique up to equivalence.

This is best understood in the setting of perverse shifts of t-structures: given a triangulated $\mathcal{T}$ with a filtration $0 = \mathcal{T}_{-1} \subset \mathcal{T}_0 \subset \cdots \subset \mathcal{T}_r = \mathcal{T}$ by thick subcategories, given a t-structure on $\mathcal{T}$ compatible with the filtration, and given $p : \{0, \dots, r\} \to \mathbf{Z}$, there is at most one t-structure t' on $\mathcal{T}$ compatible with the filtration such that the t-structure on $\mathcal{T}_i/\mathcal{T}_{i-1}$ induced by t' is obtained by shifting by $-p(i)$ the one induced by t. Such a t-structure t' need not exist, and part of our work is devoted to finding settings under which such perverse tilts always exist. We achieve this under particular Calabi-Yau and finiteness conditions. Note that the category of perverse sheaves on a stratified space [BBD] is obtained from the category of constructible sheaves by a perverse shift of t-structures, and our work can also be viewed as a generalization of that construction.

When $\mathcal{T} = D^b(\mathcal{A})$ and $\mathcal{A}$ is the category of finite-dimensional representations of a finite-dimensional algebra A over a field, then Serre subcategories of $\mathcal{A}$ are in bijection with finite subsets of the set S of isomorphism classes of simple modules. So, filtrations of $\mathcal{A}$ correspond to filtrations of that set. When A is a symmetric algebra (0-Calabi-Yau condition), we define a set $\mathcal{E}$ parametrizing certain t-structures together with a total order, and we construct commuting actions of $\mathrm{Aut}(D^b(\mathcal{A}))$ and of $\mathrm{Free}(\mathcal{P}'(S)) \rtimes \mathfrak{S}(S)$ on $\mathcal{E}$, where $\mathrm{Free}(\mathcal{P}'(S))$ is the free group on proper subsets of S. We believe this to be an important invariant of the derived category, that could be viewed as a combinatorial counterpart of (part of) the space of stability conditions, but we are not able to say much about it. We show that certain relations occuring in the action are related to homological properties. We show that the same constructions work for algebras of positive Calabi-Yau dimension, under certain conditions. This works for example for $A = k[[V]] \rtimes G$, where V is a finite-dimensional vector space over a field k, G is a finite subgroup of $\mathrm{SL}(V)$ acting freely on $V - \{0\}$ and whose order is invertible in k.

§3 is devoted to the interaction between filtrations of a triangulated category by thick subcategories and t-structures. In §3.3, we study the compatibility of thick subcategories of triangulated categories with t-structures. We discuss the possibility of shifting the t-structure on a quotient in §3.4. The classical torsion theory corresponds to the most basic type of perverse tilt, and every perverse tilt can be obtained as a composition of torsion theories. Sections §3.5 and 3.6 are a preparation for the study of the change of hearts in a shift of t-structures. In §3.7, we provide the key definition of the relative perversity of two t-structures with respect to a perversity function and we discuss in §3.8 the particular case of non-decreasing perversity functions.

Chapter §4 is devoted to perverse equivalences. In §4.1, we introduce the basic definition for derived categories of exact categories, and consider the case of homotopy categories of complexes over additive categories. The important case of derived categories of abelian categories is discussed in §4.2. We provide different characterizations of perverse equivalences and discuss the images of simple and projective objects.

In §5, we consider the case of derived categories of symmetric finite-dimensional algebras. We study in particular the images of simple modules under perverse equivalences corresponding to monotonic perversity functions. We show that perverse equivalences always exist and can be iterated, leading to the construction of a set of "enhanced" t-structures, together with group actions on that set. We prove the existence of certain relations involving the group action.

We provide a similar treatment for Calabi-Yau algebras in §6, under a particular assumption ("isolated" algebra).

We show in §7 that some version of perverse equivalences do take place for stable categories of finite-dimensional symmetric algebras, and for more general triangulated categories, Calabi-Yau of dimension -1.

Finally, §8 is devoted to particular instances of perverse equivalences occurring in the modular representation theory of finite groups.

2. Notations

Let k be a commutative ring and A a k-algebra. We denote by A-Mod the category of A-modules and by A-mod the category of finitely generated A-modules. We denote by A-Proj the category of projective A-modules and by A-proj the category of finitely generated projective A-modules. We write $\otimes$ for $\otimes_k$.

Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a Serre subcategory of $\mathcal{A}$. We denote by $D^b_{\mathcal{B}}(\mathcal{A})$ the full subcategory of $D^b(\mathcal{A})$ of objects with cohomology in $\mathcal{B}$.

We denote by $\operatorname{gldim} \mathcal{A}$ the global dimension of $\mathcal{A}$, *i.e.*, the largest non-negative integer i such that $\operatorname{Ext}^i_{\mathcal{A}}(-,-)$ doesn't vanish. We put $\operatorname{gldim} A = \operatorname{gldim}(A\text{-Mod})$.

Let A be a dg (differential graded) k-algebra. We denote by $D(A)$ the derived category of dg A-modules, by A-perf its full subcategory of perfect complexes (=smallest thick subcategory containing A) and by $D_f(A)$ the full subcategory of $D(A)$ of objects that are perfect as complexes of k-modules.

Given X a variety, we denote by X-coh the category of coherent sheaves over X.

Let $\mathcal{T}$ be a triangulated category and $\mathcal{I}$ a subcategory of $\mathcal{T}$. We say that $\mathcal{I}$ generates $\mathcal{T}$ if $\mathcal{T}$ is the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{I}$.

Given $\mathcal{C}$ a category and $\mathcal{I}$ a subcategory, we denote by $\mathcal{I}^\perp$ (resp. ${}^\perp\mathcal{I}$) the full subcategory of $\mathcal{C}$ of objects M such that $\mathrm{Hom}(\mathcal{I}, M) = 0$ (resp. $\mathrm{Hom}(M, \mathcal{I}) = 0$).

3. t-structures and filtered categories

3.1. **t-structures.** Let $\mathcal{T}$ be a triangulated category. A *left pre-aisle* (resp. a *right pre-aisle*) in $\mathcal{T}$ is a full subcategory $\mathcal{C}$ of $\mathcal{T}$ such that given $C \in \mathcal{C}$, then $C[1] \in \mathcal{C}$ (resp. $C[-1] \in \mathcal{C}$) and such that given a distinguished triangle $X \to Y \to Z \rightsquigarrow$ in $\mathcal{T}$ with $X, Z \in \mathcal{C}$, then $Y \in \mathcal{C}$.

Recall [BBD, §1.3] that a *t-structure* t on $\mathcal{T}$ is the data of full subcategories $\mathcal{T}^{\le i}$ and $\mathcal{T}^{\ge i}$ for $i \in \mathbf{Z}$ with

- $\mathcal{T}^{\le i+1}[1] = \mathcal{T}^{\le i}$ and $\mathcal{T}^{\ge i+1}[1] = \mathcal{T}^{\ge i}$
- $\mathcal{T}^{\le 0} \subset \mathcal{T}^{\le 1}$ and $\mathcal{T}^{\ge 0} \supset \mathcal{T}^{\ge 1}$
- $\mathrm{Hom}(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 1}) = 0$
- given $X \in \mathcal{T}$, there is a distinguished triangle $Y \to X \to Z \rightsquigarrow$ with $Y \in \mathcal{T}^{\le 0}$ and $Z \in \mathcal{T}^{\ge 1}$.

Its *heart* is the intersection $\mathcal{A} = \mathcal{T}^{\le 0} \cap \mathcal{T}^{\ge 0}$. This is an abelian category. The inclusion of $\mathcal{T}^{\le i}$ in $\mathcal{T}$ has a right adjoint $\tau_{\le i}$ and the inclusion of $\mathcal{T}^{\ge i}$ in $\mathcal{T}$ has a left adjoint $\tau_{\ge i}$. We put $H^i = \tau_{\ge i}\tau_{\le i} \simeq \tau_{\le i}\tau_{\ge i} : \mathcal{T} \to \mathcal{A}$. The full subcategory $\mathcal{T}^{\le 0}$ (resp. $\mathcal{T}^{\ge 1}$) is the left (resp. right) *aisle* of the t-structure. Note that $\mathcal{T}^{\ge 1} = (\mathcal{T}^{\le 0})^\perp$, hence the t-structure is determined by $\mathcal{T}^{\le 0}$. Similarly, $\mathcal{T}^{\le -1} = {}^\perp(\mathcal{T}^{\ge 0})$, hence the t-structure is determined by $\mathcal{T}^{\ge 0}$.

Note that a left pre-aisle $\mathcal{T}^{\le 0}$ of $\mathcal{T}$ is the left aisle of a t-structure if and only if the inclusion functor $\mathcal{T}^{\le 0} \to \mathcal{T}$ has a right adjoint [KeVo2, Proposition 1].

Note also that there is a t-structure t^{opp} on $\mathcal{T}^{\mathrm{opp}}$ defined by $(\mathcal{T}^{\mathrm{opp}})^{\le i} = \mathcal{T}^{\ge -i}$ and $(\mathcal{T}^{\mathrm{opp}})^{\ge i} = \mathcal{T}^{\le -i}$.

A t-structure is *bounded* if $\mathcal{A}$ generates $\mathcal{T}$. When t is bounded, the objects of $\mathcal{T}^{\le 0}$ are those $X \in \mathcal{T}$ such that $\mathrm{Hom}(X, M[n]) = 0$ for all $M \in \mathcal{A}$ and $n < 0$, hence $\mathcal{A}$ determines the t-structure. This provides a bijection from the set of bounded t-structures on $\mathcal{T}$ to the set of abelian subcategories $\mathcal{A}$ of $\mathcal{T}$ such that $\mathrm{Hom}(\mathcal{A}, \mathcal{A}[i]) = 0$ for $i < 0$ and $\mathcal{A}$ generates $\mathcal{T}$.

3.2. **Intersections of t-structures.**

Definition 3.1. *Let t, t' and t'' be three t-structures on $\mathcal{T}$. We say that t'' is the* right *(resp.* left*)* intersection *of t and t' if $\mathcal{T}^{\ge'' 0} = \mathcal{T}^{\ge 0} \cap \mathcal{T}^{\ge' 0}$ (resp. $\mathcal{T}^{\le'' 0} = \mathcal{T}^{\le 0} \cap \mathcal{T}^{\le' 0}$).*

We put $t'' = t \cap^r t'$ (resp. $t'' = t \cap^l t'$) when t'' is the right (resp. left) intersection of t and t'. We say that the right (resp. left) intersection of t and t' exists if there is a t'' as above. Note that if the intersection exists, it is unique.

The following lemma is immediate.

Lemma 3.2. *Assume $t'' = t \cap^r t'$. Then $\mathcal{T}^{\le 0} \subset \mathcal{T}^{\le'' 0}$ and $\mathcal{T}^{\le' 0} \subset \mathcal{T}^{\le'' 0}$.*

3.3. **t-structures and thick subcategories.** Let $\mathcal{T}$ be a triangulated category and $\mathcal{I}$ a thick subcategory. Let $Q : \mathcal{T} \to \mathcal{T}/\mathcal{I}$ be the quotient functor.

Consider $t = (\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ a t-structure on $\mathcal{T}$ with heart $\mathcal{A}$ and let $\mathcal{J} = \mathcal{A} \cap \mathcal{I}$. The following lemma expands on [BBD, §1.3.19] (cf also [BelRe, Proposition 2.15] and [BeiGiSch, Remark after Lemma 0.5.1]; in those references, it is claimed incorrectly that (1)⇒(4)).

Lemma 3.3. *The following assertions are equivalent*

(1) $\tau_{\le 0}(\mathcal{I}) \subset \mathcal{I}$
(2) $\tau_{\ge 0}(\mathcal{I}) \subset \mathcal{I}$
(3) $t_{\mathcal{I}} = (\mathcal{I} \cap \mathcal{T}^{\le 0}, \mathcal{I} \cap \mathcal{T}^{\ge 0})$ *is a t-structure on $\mathcal{I}$*

The assertions above hold and $\mathcal{I} \cap \mathcal{A}$ is a Serre subcategory of $\mathcal{A}$ if and only if

(4) $t_{\mathcal{T}/\mathcal{I}} = (Q(\mathcal{T}^{\le 0}), Q(\mathcal{T}^{\ge 0}))$ *is a t-structure on $\mathcal{T}/\mathcal{I}$.*

Proof. Let $X \in \mathcal{I}$. We have a distinguished triangle

$$\tau_{\leq 0}X \to X \to \tau_{\geq 1}X \rightsquigarrow$$

If (1) or (2) holds, then all terms of the triangle are in $\mathcal{I}$, hence (3) holds.

Assume (3) holds: given $X \in \mathcal{I}$, there is a distinguished triangle $X' \to X \to X'' \rightsquigarrow$ with $X' \in \mathcal{I} \cap \mathcal{T}^{\leq 0}$ and $X'' \in \mathcal{I} \cap \mathcal{T}^{\geq 1}$. That implies $X' \simeq \tau_{\leq 0}X$ and $X'' \simeq \tau_{\geq 1}X$. Hence, (1) and (2) hold.

Assume (2) holds and $\mathcal{I} \cap \mathcal{A}$ is a Serre subcategory of $\mathcal{A}$. Let $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$. Consider $f \in \operatorname{Hom}_{\mathcal{T}/\mathcal{I}}(Q(X), Q(Y))$. There is a distinguished triangle $Y'' \to Y \xrightarrow{q} Y' \rightsquigarrow$ and there is $p : X \to Y'$ such that $Y'' \in \mathcal{I}$ and $Q(q)f = Q(p)$. Let $r : Y' \to \tau_{\geq 1}Y'$ be the canonical map. Consider the composition $rq : Y \to \tau_{\geq 1}Y'$. It fits in a distinguished triangle $\bar{Y}'' \to Y \to \tau_{\geq 1}Y' \rightsquigarrow$ and there is an exact sequence

$$0 \to H^0Y' \to H^1Y'' \to H^1\bar{Y}'' \to 0.$$

Since $H^1Y'' \in \mathcal{I} \cap \mathcal{A}$, we deduce that $H^1\bar{Y}'' \in \mathcal{I} \cap \mathcal{A}$. On the other hand, $\tau_{\geq 2}\bar{Y}'' \simeq \tau_{\geq 2}Y'' \in \mathcal{I}$ and $\tau_{\leq 0}\bar{Y}'' = 0$, hence $\bar{Y}'' \in \mathcal{I}$. We have $Q(rq)f = Q(rp)$ and $Q(rq)$ is invertible. Since the composition $X \xrightarrow{p} Y' \xrightarrow{r} \tau_{\geq 1}Y'$ vanishes, it follows that $f = 0$. This shows (4).

$$\begin{tikzcd}
 & & & Y'' \arrow[r] \arrow[dl] & \bar{Y}'' \arrow[d] \\
X \arrow[rr, dotted, "f"] \arrow[dr, "p"'] \arrow[drr, bend right, "0"'] & & Y \arrow[rr, equal] \arrow[dl, "q"] & & Y \arrow[dll] \\
 & Y' \arrow[r, "r"] & \tau_{\geq 1}Y' & &
\end{tikzcd}$$

Assume (4) holds. Given $X \in \mathcal{I}$, we have an isomorphism $Q(\tau_{\geq 1}X) \xrightarrow{\sim} Q(\tau_{\leq 0}X)[1]$: these are objects of $Q(\mathcal{T}^{\geq 1}) \cap Q(\mathcal{T}^{\leq -1}) = 0$. So, $\tau_{\geq 1}X, \tau_{\leq 0}X \in \mathcal{I}$. This shows (1) holds. Consider now an exact sequence

$$0 \to V \to W \to X \to 0$$

in $\mathcal{A}$. If two of V, W and X are in $\mathcal{I}$, then so is the third one. Assume now $W \in \mathcal{I}$. We have an isomorphism $Q(X) \xrightarrow{\sim} Q(V)[1]$. Since $Q(X) \in (\mathcal{T}/\mathcal{I})^{\geq 0}$ and $Q(V)[1] \in (\mathcal{T}/\mathcal{I})^{\leq -1}$, we deduce that $Q(X) = Q(V) = 0$, hence $X, V \in \mathcal{I}$. It follows that $\mathcal{J}$ is a Serre subcategory of $\mathcal{A}$. ☐

Remark 3.4. The assumptions (1)–(3) of Lemma 3.3 show that $\mathcal{J}$ is a full abelian subcategory of $\mathcal{A}$ closed under taking extensions, and that given $f : V \to W$ in $\mathcal{J}$, then $\ker f, \operatorname{coker} f \in \mathcal{J}$. This is not enough to ensure that $\mathcal{J}$ is a Serre subcategory of $\mathcal{A}$. Consider for example k a field and A the quiver algebra of $\bullet \underset{b}{\overset{a}{\rightleftarrows}} \bullet$ modulo the relation $ab = ba = 0$. This is a 4-dimensional indecomposable self-injective algebra with two simple modules (it is unique with this property). Fix P a projective indecomposable A-module. Let $\mathcal{T} = D^b(A\text{-mod})$ and let $\mathcal{I}$ be the full subcategory of $\mathcal{T}$ with objects finite direct sums of shifts of P. This is a thick subcategory. Note that $\mathcal{A} = A$-mod and $\mathcal{I} \cap \mathcal{A}$ has objects finite direct sums of copies of P: this is not a Serre subcategory. Note also that t does not induce a t-structure on $\mathcal{T}/\mathcal{I}$: if S is the simple quotient of P and T the simple submodule of P, then $Q(T) \simeq Q(S)[-1]$, hence there is a non-zero map from an object of $Q(\mathcal{T}^{\leq 0})$ to an object of $Q(\mathcal{T}^{\geq 1})$. This contradicts the claims in [BelRe, Proposition 2.15, (i)⇒(iii)] and [BeiGiSch, Remark after Lemma 0.5.1].

Definition 3.5. *We say that the t-structure t is* compatible with $\mathcal{I}$ *if $t_{\mathcal{T}/\mathcal{I}}$ is a t-structure on $\mathcal{T}/\mathcal{I}$.*

We put $\mathcal{I}^{\le 0} = \mathcal{I} \cap \mathcal{T}^{\le 0}$, $(\mathcal{T}/\mathcal{I})^{\le 0} = Q(\mathcal{T}^{\le 0})$, etc. When t is compatible with $\mathcal{I}$, then the truncation functors commute with the inclusion $\mathcal{I} \to \mathcal{T}$ and the quotient functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{I}$.

The following lemma shows that the gluing of t-structures in a quotient category situation is unique, if it exists. This appears in [BeiGiSch, §0.5], where the equivalent notion of t-exact sequences $0 \to \mathcal{I} \to \mathcal{T} \to \mathcal{T}/\mathcal{I} \to 0$ is studied.

Lemma 3.6. *Fix a t-structure on $\mathcal{T}$ compatible with $\mathcal{I}$. Let $X \in \mathcal{T}$.*
We have $X \in \mathcal{T}^{\le 0}$ if and only $Q(X) \in (\mathcal{T}/\mathcal{I})^{\le 0}$ and $\mathrm{Hom}(X, \mathcal{I}^{>0}) = 0$.
We have $X \in \mathcal{T}^{\ge 0}$ if and only $Q(X) \in (\mathcal{T}/\mathcal{I})^{\ge 0}$ and $\mathrm{Hom}(\mathcal{I}^{<0}, X) = 0$.

Proof. We have a distinguished triangle $\tau_{\le 0}X \to X \to \tau_{>0}X \rightsquigarrow$. If $Q(X) \in (\mathcal{T}/\mathcal{I})^{\le 0}$, then $\tau_{>0}X \in \mathcal{I}$. If $\mathrm{Hom}(X, \mathcal{I}^{>0}) = 0$, then $\mathrm{Hom}(X, \tau_{>0}X) = 0$, hence $X \in \mathcal{T}^{\le 0}$.
The second part of the lemma follows from the first one by replacing $\mathcal{T}$ by $\mathcal{T}^{\mathrm{opp}}$. □

Recall the classical situation of [BBD, Théorème 4.10] (cf for example [Nee, §9.1] for the proof that the other assumptions are automatically satisfied).

Theorem 3.7 (Beilinson-Bernstein-Deligne)**.** *Assume $Q : \mathcal{T} \to \mathcal{T}/\mathcal{I}$ has left and right adjoints. Given t-structures t_1 on $\mathcal{I}$ and t_2 on $\mathcal{T}/\mathcal{I}$, there is a (unique) t-structure t on $\mathcal{T}$ such that $t_1 = t_{\mathcal{I}}$ and $t_2 = t_{\mathcal{T}/\mathcal{I}}$.*

Lemma 3.8. *Let t and t' be two t-structures compatible with $\mathcal{I}$.*
If $\mathcal{I}^{\ge 0} \subset \mathcal{I}^{\ge' 0}$ and $(\mathcal{T}/\mathcal{I})^{\ge 0} \supset (\mathcal{T}/\mathcal{I})^{\ge' 0}$, then $t'' = t \cap^r t'$ exists, it is compatible with $\mathcal{I}$ and we have $t''_{\mathcal{I}} = t_{\mathcal{I}}$, $t''_{\mathcal{T}/\mathcal{I}} = t'_{\mathcal{T}/\mathcal{I}}$ and $\tau_{\ge'' 0} = \tau_{\ge 0} \circ \tau_{\ge' 0}$.
If $\mathcal{I}^{\le 0} \subset \mathcal{I}^{\le' 0}$ and $(\mathcal{T}/\mathcal{I})^{\le 0} \supset (\mathcal{T}/\mathcal{I})^{\le' 0}$, then $t'' = t \cap^l t'$ exists, it is compatible with $\mathcal{I}$ and we have $t''_{\mathcal{I}} = t_{\mathcal{I}}$, $t''_{\mathcal{T}/\mathcal{I}} = t'_{\mathcal{T}/\mathcal{I}}$ and $\tau_{\le'' 0} = \tau_{\le 0} \circ \tau_{\le' 0}$.

Proof. Define $\mathcal{T}^{\le'' r} = \{X \in \mathcal{T} | Q(X) \in (\mathcal{T}/\mathcal{I})^{\le' r}$ and $\mathrm{Hom}(X, \mathcal{I}^{>r}) = 0\}$ and $\mathcal{T}^{\ge'' r} = \{X \in \mathcal{T} | Q(X) \in (\mathcal{T}/\mathcal{I})^{\ge' r}$ and $\mathrm{Hom}(\mathcal{I}^{<r}, X) = 0\}$. We will show that this defines a t-structure t'' and that $\tau_{\ge'' 0} = \tau_{\ge 0} \circ \tau_{\ge' 0}$.
Let $X \in \mathcal{T}$, let $Z = \tau_{\ge 0}\tau_{\ge' 0}(X)$ and let $Y[1]$ be the cone of the composition of canonical maps $X \to \tau_{\ge' 0}X \to Z$.
The octahedron axiom applied to that composition of maps shows there is a distinguished triangle

$$\tau_{<' 0}X \to Y \to \tau_{<0}\tau_{\ge' 0}(X) \rightsquigarrow .$$

We deduce that $\mathrm{Hom}(Y, \mathcal{I}^{>-1}) = 0$ since $\mathcal{I}^{>-1} \subset \mathcal{I}^{>'-1}$. We have $Q(\tau_{<0}\tau_{\ge' 0}(X)) \simeq \tau_{<0}\tau_{\ge' 0}Q(X) = 0$ since $(\mathcal{T}/\mathcal{I})^{\ge' 0} \subset (\mathcal{T}/\mathcal{I})^{\ge 0}$, hence $Q(Y) \in (\mathcal{T}/\mathcal{I})^{<' 0}$. It follows that $Y \in \mathcal{T}^{<'' 0}$.
We have $Q(Z) \simeq \tau_{\ge 0}\tau_{\ge' 0}Q(X) \simeq \tau_{\ge' 0}Q(X) \in (\mathcal{T}/\mathcal{I})^{\ge' 0}$ and $\mathrm{Hom}(\mathcal{I}^{<0}, Z) = 0$, hence $Z \in \mathcal{T}^{\ge'' 0}$. This shows t'' is a t-structure and $\tau_{\ge'' 0} = \tau_{\ge 0} \circ \tau_{\ge' 0}$. We have $\mathcal{I}^{<' 0} \subset \mathcal{I}^{<0}$, so $\mathcal{T}^{\ge'' 0} = \mathcal{T}^{\ge 0} \cap \mathcal{T}^{\ge' 0}$. Finally, $t'' = t \cap^r t'$.
The second statement follows from the first one by replacing $\mathcal{T}$ by $\mathcal{T}^{\mathrm{opp}}$. □

The following result appears in [BelRe, Proposition 2.5].

Lemma 3.9. *Let $\mathcal{A} = \mathcal{T}^{\le 0} \cap \mathcal{T}^{\ge 0}$ be the heart of the t-structure t. If t is compatible with $\mathcal{I}$, then Q induces an equivalence from $\mathcal{A}/(\mathcal{A} \cap \mathcal{I})$ to the heart of $t_{\mathcal{T}/\mathcal{I}}$.*

Proof. The functor Q restricts to an exact functor $\mathcal{A} \to (\mathcal{T}/\mathcal{I})^{\le 0} \cap (\mathcal{T}/\mathcal{I})^{\ge 0}$ with kernel $\mathcal{A} \cap \mathcal{I}$.
Let $V \in (\mathcal{T}/\mathcal{I})^{\ge 0} \cap (\mathcal{T}/\mathcal{I})^{\le 0}$. Let $X \in \mathcal{T}^{\le 0}$ and $Y \in \mathcal{T}^{\ge 0}$ with $Q(X) \simeq Q(Y) \simeq V$. There are $Z \in \mathcal{T}$ and $p : Z \to X$, $q : Z \to Y$ with respective cones X' and Y' in $\mathcal{I}$. Let $\bar{X}'$ be the cone of the composite map $p' : \tau_{\le 0}Z \xrightarrow{\mathrm{can}} Z \xrightarrow{p} X$. Since $Q(p)$ is an isomorphism and $Q(X) \in (\mathcal{T}/\mathcal{I})^{\le 0}$, we deduce that $Q(p')$ is an isomorphism, and so is the image by Q of the composition $q' : \tau_{\le 0}Z \xrightarrow{\mathrm{can}} Z \xrightarrow{q} Y$. The

map q' factors through $H^0(Z) = \tau_{\geq 0}\tau_{\leq 0}Z$ as $\tau_{\leq 0}Z \xrightarrow{\text{can}} H^0(Z) \xrightarrow{q''} Y$. Since $Q(q')$ is an isomorphism and $Q(Y) \in (\mathcal{T}/\mathcal{I})^{\geq 0}$, we deduce that $Q(q'')$ is an isomorphism. So, $Q(H^0(Z)) \simeq V$. We have shown that Q is essentially surjective.

Let $V, W \in \mathcal{A}$ and $f \in \mathrm{Hom}_{\mathcal{A}}(V, W)$. If $Q(f) = 0$, then f factors through an object $X \in \mathcal{I}$, hence $H^0(f) = f$ factors through $H^0(X) \in \mathcal{A} \cap \mathcal{I}$. So, the canonical map $\mathrm{Hom}_{\mathcal{A}/(\mathcal{A}\cap\mathcal{I})}(V, W) \to \mathrm{Hom}_{\mathcal{T}/\mathcal{I}}(V, W)$ is injective.

Let now $g \in \mathrm{Hom}_{\mathcal{T}/\mathcal{I}}(V, W)$. There is $U \in \mathcal{T}$ and maps $a : U \to V$, $b : U \to W$ such that $Q(a)$ is invertible and $Q(b) = gQ(a)$. Let a' be the composition $\tau_{\leq 0}U \xrightarrow{\text{can}} V' \xrightarrow{a} V$. The map $Q(a')$ is an isomorphism. Let b' be the composition $\tau_{\leq 0}U \xrightarrow{\text{can}} V' \xrightarrow{b} W$. The maps a' and b' factor through $H^0(U) = \tau_{\geq 0}\tau_{\leq 0}U$ as $\tau_{\leq 0}U \xrightarrow{\text{can}} H^0(U) \xrightarrow{a''} V$ and $\tau_{\leq 0}U \xrightarrow{\text{can}} H^0(U) \xrightarrow{b''} W$. Furthermore, $Q(a'')$ is an isomorphism, hence $\ker a'' \in \mathcal{A} \cap \mathcal{I}$ and $\operatorname{coker} a'' \in \mathcal{A} \cap \mathcal{I}$. It follows that $g \in \mathrm{Hom}_{\mathcal{A}/(\mathcal{A}\cap\mathcal{I})}(V, W)$. □

In the following, we will identify $\mathcal{A}/(\mathcal{A} \cap \mathcal{I})$ with its essential image in $\mathcal{T}/\mathcal{I}$.

Lemma 3.10. *Let $\mathcal{T}$ be a triangulated category with a bounded t-structure. Let $\mathcal{A}$ be its heart.*

There is a bijection between the set of thick subcategories $\mathcal{I}$ of $\mathcal{T}$ compatible with the t-structure and the set of Serre subcategories of $\mathcal{A}$ given by $\mathcal{I} \mapsto \mathcal{I}\cap\mathcal{A}$, with inverse $\mathcal{J} \mapsto \{C \in \mathcal{T} \mid H^i(C) \in \mathcal{J}\ \forall i \in \mathbf{Z}\}$.

Proof. Let $\mathcal{J}$ be a Serre subcategory of $\mathcal{A}$. Let $\mathcal{I}$ be the full subcategory of $\mathcal{T}$ of objects X such that $H^i(X) \in \mathcal{J}$ for all i. Consider a morphism $X \to Y$ in $\mathcal{I}$ and let Z be its cone. We have an exact sequence $H^i(Y) \to H^i(Z) \to H^{i+1}(X)$, hence $H^i(Z) \in \mathcal{J}$ for all i. It follows that $\mathcal{I}$ is a thick subcategory of $\mathcal{T}$. Lemma 3.3 shows that t is compatible with $\mathcal{I}$.

Conversely, let $\mathcal{I}$ be a thick subcategory of $\mathcal{T}$ compatible with the t-structure. By Lemma 3.3, $\mathcal{A}\cap\mathcal{I}$ is a Serre subcategory of $\mathcal{A}$, and $H^i(C) \in \mathcal{A} \cap \mathcal{I}$ for all $C \in \mathcal{I}$ and $i \in \mathbf{Z}$. Conversely, let $C \in \mathcal{T}$ such that $H^i(C) \in \mathcal{A} \cap \mathcal{I}$ for all $i \in \mathbf{Z}$. We have $H^i(Q(C)) = 0$ for all $i \in \mathbf{Z}$, hence $Q(C) = 0$ and $C \in \mathcal{I}$. It follows that $\mathcal{I}$ is the full subcategory of $\mathcal{T}$ of objects C such that $H^i(C) \in \mathcal{A} \cap \mathcal{I}$ for all $i \in \mathbf{Z}$. □

Lemma 3.11. *Let $\mathcal{I}'$ be a thick subcategory of $\mathcal{T}$ containing $\mathcal{I}$. The following assertions are equivalent:*

- *t is compatible with $\mathcal{I}$ and $\mathcal{I}'$*
- *t is compatible with $\mathcal{I}'$ and $t_{\mathcal{I}'}$ is compatible with $\mathcal{I}$*
- *t is compatible with $\mathcal{I}$ and $t_{\mathcal{T}/\mathcal{I}}$ is compatible with $\mathcal{I}'/\mathcal{I}$.*

Proof. Let $\mathcal{J}' = \mathcal{A} \cap \mathcal{I}'$.

Note that $t_{\mathcal{I}} = (t_{\mathcal{I}'})_{\mathcal{I}}$. Assume t is compatible with $\mathcal{I}'$. Assume this is a t-structure on $\mathcal{I}$. We have inclusions $\mathcal{J} \subset \mathcal{J}' \subset \mathcal{A}$ where $\mathcal{J}$ is a full abelian subcategory of $\mathcal{J}'$ closed under extensions and $\mathcal{J}'$ is a Serre subcategory of $\mathcal{A}$. Given $V \in \mathcal{J}$, the and subobjects in $\mathcal{A}$ of V are in $\mathcal{J}'$. It follows that $\mathcal{J}$ is a Serre subcategory of $\mathcal{A}$ if and only if it is a Serre subcategory of $\mathcal{J}'$.

Assume t is compatible with $\mathcal{I}$ and $t_{\mathcal{T}/\mathcal{I}}$ is compatible with $\mathcal{I}'/\mathcal{I}$. Let $X \in \mathcal{I}'$. We have $Q(\tau_{\leq 0}(X)) \simeq \tau_{\leq 0}Q(X) \in \mathcal{I}'/\mathcal{I}$. It follows that $\tau_{\leq 0}(X) \in \mathcal{I}'$, hence $\tau_{\leq 0}(\mathcal{I}') \subset \mathcal{I}'$.

Let $V \in \mathcal{J}'$ and let V' be a subobject of V in $\mathcal{A}$. Then $Q(V') \in Q(\mathcal{A}) \cap (\mathcal{I}'/\mathcal{I})$, so $V' \in \mathcal{I}'$, hence $V' \in \mathcal{J}'$. So, $\mathcal{J}'$ is a Serre subcategory of $\mathcal{A}$. It follows that t is compatible with $\mathcal{I}'$

Assume now t is compatible with $\mathcal{I}$ and with $\mathcal{I}'$. Let $Y \in \mathcal{I}'/\mathcal{I}$ and $X \in \mathcal{I}'$ with $Q(X) = Y$. We have $\tau_{\leq 0}(X) \in \mathcal{I}'$, hence $\tau_{\leq 0}Y \simeq Q(\tau_{\leq 0}(X)) \in \mathcal{I}'/\mathcal{I}$. So, $\tau_{\leq 0}(\mathcal{I}'/\mathcal{I}) \subset \mathcal{I}'/\mathcal{I}$.

Let $W \in Q(\mathcal{A}) \cap (\mathcal{I}'/\mathcal{I})$ and W' a subobject of W in $Q(\mathcal{A})$. Let $V \in \mathcal{A}$ and V' a subobject of V in $\mathcal{A}$ with $Q(V) = W$ and $Q(V') = W'$. We have $V \in \mathcal{I}'$, hence $V \in \mathcal{J}'$. It follows that $V' \in \mathcal{J}'$, hence $Q(V') \in \mathcal{I}'/\mathcal{I}$. So, $Q(\mathcal{A}) \cap (\mathcal{I}'/\mathcal{I})$ is a Serre subcategory of $Q(\mathcal{A})$. It follows that $t_{\mathcal{T}/\mathcal{I}}$ is compatible with $\mathcal{I}'/\mathcal{I}$. □

3.4. **Shifts of t-structures.** Let $\mathcal{T}$ be a triangulated category, $\mathcal{I}$ a thick subcategory and t a t-structure on $\mathcal{T}$ compatible with $\mathcal{I}$. Let $\mathcal{J} = \mathcal{A} \cap \mathcal{I}$. Let $n \in \mathbf{Z}$. Define a candidate t-structure t' by

- $\mathcal{T}^{\leq' r} = \{X \in \mathcal{T} | Q(X) \in (\mathcal{T}/\mathcal{I})^{\leq n+r} \text{ and } \mathrm{Hom}(X, \mathcal{I}^{>r}) = 0\}$
- $\mathcal{T}^{\geq' r} = \{X \in \mathcal{T} | Q(X) \in (\mathcal{T}/\mathcal{I})^{\geq n+r} \text{ and } \mathrm{Hom}(\mathcal{I}^{<r}, X) = 0\}$.

Lemma 3.12. *We have $\mathcal{I}^{\leq' r} = \mathcal{I}^{\leq r}$ and $\mathcal{I}^{\geq' r} = \mathcal{I}^{\geq r}$. Assume t' defines a t-structure on $\mathcal{T}$. Let $\mathcal{A}$ be the heart of t and $\mathcal{A}'$ be the heart of t'. Then*

- *t' is compatible with $\mathcal{I}$*
- *$t'_{\mathcal{I}} = t_{\mathcal{I}}$ and $t_{\mathcal{T}/\mathcal{I}} = t'_{\mathcal{T}/\mathcal{I}}[n]$*
- *$\mathcal{A} \cap \mathcal{I} = \mathcal{A}' \cap \mathcal{I}$ and $\mathcal{A}/(\mathcal{A} \cap \mathcal{I}) = \mathcal{A}'/(\mathcal{A}' \cap \mathcal{I})[n]$*

Proof. The statement about $\mathcal{I}$ is immediate. Assume t' defines a t-structure on $\mathcal{T}$. Let $X \in \mathcal{T}$ such that $Q(X) \in (\mathcal{T}/\mathcal{I})^{\leq n}$. There is a distinguished triangle $\tau'_{\leq 0} X \to X \to \tau'_{>0} X \rightsquigarrow$. It induces a distinguished triangle $Q(\tau'_{\leq 0} X) \to Q(X) \xrightarrow{f} Q(\tau'_{>0} X) \rightsquigarrow$. We have $Q(\tau'_{>0} X) \in (\mathcal{T}/\mathcal{I})^{>n}$, hence $f = 0$. Consequently, $Q(\tau'_{>0} X)$ is a direct summand of $Q(\tau'_{\leq 0} X)[1]$. The latter is in $(\mathcal{T}/\mathcal{I})^{\leq n-1}$, hence $Q(\tau'_{>0} X) = 0$, so $Q(X) \in Q(\mathcal{T}^{\leq' 0})$. It follows that $Q(\mathcal{T}^{\leq' 0}) = (\mathcal{T}/\mathcal{I})^{\leq n}$. Similarly, $Q(\mathcal{T}^{\geq' 0}) = (\mathcal{T}/\mathcal{I})^{\geq n}$. The last statement follows from Lemma 3.9. □

Definition 3.13. *If t' defines a t-structure on $\mathcal{T}$, we call t' the* n-shift *of t.*

Lemma 3.14. *We have $\mathrm{Hom}(\mathcal{T}^{\leq' 0}, \mathcal{T}^{\geq' 1}) = 0$. Assume that given $X \in \mathcal{T}$, there is a distinguished triangle $Y \to X \to Z \rightsquigarrow$ with $Y \in \mathcal{T}^{\leq' 0}$ and $Z \in \mathcal{T}^{\geq' 1}$. Then t' defines a t-structure on $\mathcal{T}$.*

Proof. Let $X \in \mathcal{T}^{\leq' 0}$, $Y \in \mathcal{T}^{\geq' 1}$ and $f : X \to Y$. We have $Q(f) = 0$, hence f factors through an object $Z \in \mathcal{I}$ as $X \xrightarrow{f_1} Z \xrightarrow{f_2} Y$. We have $\mathrm{Hom}(X, \tau_{>0} Z) = 0$, hence f_1 factors through $\tau_{\leq 0} Z$. On the other hand, $\mathrm{Hom}(\tau_{\leq 0} Z, Y) = 0$, hence $f = 0$.

The second part of the lemma is clear. □

Lemma 3.15. *If $n \geq 0$, then*

- *$\mathcal{T}^{\geq' 0} \subset \mathcal{T}^{\geq 0} \subset \mathcal{T}^{\geq' -n}$ and $\mathcal{T}^{\leq' -n} \subset \mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq' 0}$*
- *$\mathcal{T}^{\leq' 0} = \{X \in \mathcal{T}^{\leq n} | \mathrm{Hom}((\tau_{>0} X)[1], \mathcal{I}^{\geq 0}) = 0\}$*
- *$\mathcal{T}^{\geq' 0} = \{X \in \mathcal{T}^{\geq 0} | H^i(X) \in \mathcal{J} \text{ for } 0 \leq i \leq n-1\}$.*

If $n \leq 0$, then

- *$\mathcal{T}^{\leq' 0} \subset \mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq' -n}$ and $\mathcal{T}^{\geq' -n} \subset \mathcal{T}^{\geq 0} \subset \mathcal{T}^{\geq' 0}$*
- *$\mathcal{T}^{\leq' 0} = \{X \in \mathcal{T}^{\leq 0} | H^i(X) \in \mathcal{J} \text{ for } 1+n \leq i \leq 0\}$*
- *$\mathcal{T}^{\geq' 0} = \{X \in \mathcal{T}^{\geq n} | \mathrm{Hom}(\mathcal{I}^{\leq 0}, (\tau_{<0} X)[-1]) = 0\}$.*

Proof. Assume $n \geq 0$. The inclusions are clear. Let $X \in \mathcal{T}^{\leq n}$. The canonical map $\mathrm{Hom}(\tau_{>0} X, Y) \to \mathrm{Hom}(X, Y)$ is an isomorphism for $Y \in \mathcal{I}^{>0}$. We deduce that from Lemma 3.6 that $X \in \mathcal{T}^{\leq' 0}$ if and only if $\mathrm{Hom}(\tau_{>0} X, Y)$ for all $Y \in \mathcal{I}^{>0}$.

Let $X \in \mathcal{T}^{\geq 0}$. We have $X \in \mathcal{T}^{\geq' 0}$ if and only if $\tau_{<n} X \in \mathcal{I}$. Since $t_{|\mathcal{I}}$ is a t-structure with heart $\mathcal{A} \cap \mathcal{I}$, we have $\tau_{<n} X \in \mathcal{I}$ if and only if $H^i(X) \in \mathcal{A} \cap \mathcal{I}$ for $i < n$.

The case $n < 0$ follows from the previous case applied to $\mathcal{T}^{\mathrm{opp}}$. □

Proposition 3.16. *Let $m \in \mathbf{Z}$ with $0 \leq m \leq n$. Assume t' is the n-shift of t. Then, there is an m-shift t'' of t and we have*

- *$\tau_{\geq'' 0} \simeq \tau_{\geq 0} \circ \tau_{\geq' m-n}$ and $\tau_{\leq'' 0} \simeq \tau_{\leq' 0} \circ \tau_{\leq m}$*
- *$\mathcal{T}^{\geq'' 0} = \mathcal{T}^{\geq 0} \cap \mathcal{T}^{\geq' m-n}$ and $\mathcal{T}^{\leq'' 0} = \mathcal{T}^{\leq' 0} \cap \mathcal{T}^{\leq m}$*
- *$t'' = t \cap^r (t'[n-m]) = (t[-m]) \cap^l t'$.*

Proof. This follows immediately from Lemma 3.8. ☐

Given $\mathcal{C}$ an abelian category, a pair $(\mathcal{C}_{\text{torsion}}, \mathcal{C}_{\text{free}})$ of full subcategories is a torsion pair if

- $\operatorname{Hom}(\mathcal{C}_{\text{torsion}}, \mathcal{C}_{\text{free}}) = 0$
- given any $M \in \mathcal{C}$, there is an exact sequence

$$0 \to T \to M \to F \to 0 \quad \text{with } T \in \mathcal{C}_{\text{torsion}} \text{ and } F \in \mathcal{C}_{\text{free}}.$$

Given a torsion pair, we have $\mathcal{C}_{\text{free}} = \mathcal{C}_{\text{torsion}}^{\perp}$, hence the torsion pair is determined by its torsion part and we say that $\mathcal{C}_{\text{torsion}}$ defines a torsion pair.

The following proposition is due (for bounded t) to Happel-Reiten-Smalö [HaReSm, Proposition 2.1] (cf also [Bri2, Proposition 2.5]) and to Beligiannis-Reiten [BelRe, Theorem 3.1] (second part of the proposition).

Proposition 3.17. *Let $n = -1$. The data t' is a t-structure if and only if $\mathcal{J}$ defines a torsion theory.*

Proof. Assume t' is a t-structure. Let $M \in \mathcal{A}$. There is a distinguished triangle $Y \to M \to Z \rightsquigarrow$ with $Y \in \mathcal{T}^{\leq' 0}$ and $Z \in \mathcal{T}^{>' 0}$. Since $\mathcal{T}^{\leq' 0} \subset \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{>' 0} \subset \mathcal{T}^{\geq 0}$ (Lemma 3.15), we deduce that $Y \simeq H^0(Y)$ and $Z \simeq H^0(Z)$. Lemma 3.15 shows that $H^0(Y) \in \mathcal{J}$ and $\operatorname{Hom}(\mathcal{J}, H^0(Z)) = 0$. The first part of the lemma follows.

Assume $(\mathcal{J}, \{M \in \mathcal{A} | \operatorname{Hom}(\mathcal{J}, M) = 0\})$ is a torsion pair. Let $X \in \mathcal{T}$. Consider an exact sequence $0 \to T \to H^0(X) \to F \to 0$ with $T \in \mathcal{J}$ and $\operatorname{Hom}(\mathcal{J}, F) = 0$. Let Y be the cocone of the composition $\tau_{\leq 0} X \to H^0(X) \to F$. There is a distinguished triangle $\tau_{<0} X \to Y \to T \rightsquigarrow$. We deduce that $Y \in \mathcal{T}^{\leq' 0}$ (Lemma 3.15). Let Z be the cone of the composition $Y \to \tau_{\leq 0} X \to X$. There is a distinguished triangle $F \to Z \to \tau_{>0} X \rightsquigarrow$. We have $\operatorname{Hom}(\mathcal{J}, F) = 0$, hence $\operatorname{Hom}(\mathcal{I}^{\leq 0}, Z) = 0$ and finally $Z \in \mathcal{T}^{>' 0}$ by Lemma 3.15. It follows that t' is a t-structure. ☐

Example 3.18. Let $\mathcal{T} = D^b(\mathbf{Z}\text{-mod})$ be the bounded derived category of finitely generated abelian groups. Let $\mathcal{J}$ be the category of finitely generated torsion abelian groups. This defines a torsion theory of $\mathbf{Z}$-mod, with $\mathcal{J}^{\perp}$ the free abelian groups of finite rank. Let $\mathcal{I}$ be the thick subcategory of $\mathcal{T}$ of complexes with cohomology in $\mathcal{I}$. The t-structure t' is the image by the duality $R\operatorname{Hom}_{\mathbf{Z}}(-, \mathbf{Z})$ of the standard t-structure.

3.5. **Serre quotients and minimal continuations.** Let $\mathcal{A}$ be an abelian category and $\mathcal{J}$ a Serre subcategory. Let $Q : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ be the quotient functor. Let $\mathcal{J} - \mathrm{loc}$ be the full subcategory of $\mathcal{A}$ of objects M such that $\operatorname{Hom}(M, V) = \operatorname{Hom}(V, M) = 0$ for all $V \in \mathcal{J}$.

Lemma 3.19. *The quotient functor restricts to a fully faithful functor $\mathcal{J} - \mathrm{loc} \to \mathcal{A}/\mathcal{J}$.*

Proof. This is clear, since given $M, N \in \mathcal{A}$, we have

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{J}}(M, N) = \operatorname{colim}_{M' \to M, N' \to N} \operatorname{Hom}_{\mathcal{A}}(M', N/N')$$

where $M' \to M$ (resp. $N' \to N$) runs over injective (resp. surjective) maps in $\mathcal{A}$ whose cokernel (resp. kernel) is in $\mathcal{J}$. ☐

Definition 3.20. *A* minimal continuation *of an object $M \in \mathcal{A}/\mathcal{J}$ is an object $\tilde{M} \in \mathcal{J} - \mathrm{loc}$ endowed with an isomorphism $Q(\tilde{M}) \xrightarrow{\sim} M$.*

Lemma 3.19 shows the uniqueness of minimal continuations.

Lemma 3.21. *A minimal continuation is unique up to unique isomorphism, if it exists.*

The following lemma is obvious.

Lemma 3.22. *Let V be a simple object of $\mathcal{A}$. If $V \notin \mathcal{J}$, then $V \in \mathcal{J} - \mathrm{loc}$, i.e., V is a minimal continuation of $Q(V)$.*

Assume now Q has a left adjoint L and a right adjoint R. The unit is an isomorphism $1_{\mathcal{A}/\mathcal{J}} \xrightarrow{\sim} QL$, as well as the counit $QR \xrightarrow{\sim} 1_{\mathcal{A}/\mathcal{J}}$. The inverse map $1 \xrightarrow{\sim} QR$ induces by adjunction a map $L \to R$. Let F be the image of that map. Note that the canonical maps $L \twoheadrightarrow F \hookrightarrow R$ induce isomorphisms $QL \xrightarrow{\sim} QF \xrightarrow{\sim} QR$. Composing with the counit $QR \xrightarrow{\sim} 1$, we obtain an isomorphism $QF \xrightarrow{\sim} 1$.

Lemma 3.23. *The canonical functor $\mathcal{J}-\mathrm{loc} \to \mathcal{A}/\mathcal{J}$ is an equivalence with inverse F. In particular, the minimal continuation of $M \in \mathcal{A}/\mathcal{J}$ is $F(M)$.*

Proof. The only thing left to prove is that $F(M) \in \mathcal{J}-\mathrm{loc}$ for $M \in \mathcal{A}/\mathcal{J}$. Let $V \in \mathcal{J}$. We have $\mathrm{Hom}(V, F(M)) \hookrightarrow \mathrm{Hom}(V, R(M)) \xrightarrow{\sim} \mathrm{Hom}(Q(V), M) = 0$. Similarly, $\mathrm{Hom}(F(M), V) \hookrightarrow \mathrm{Hom}(L(M), V) \xrightarrow{\sim} \mathrm{Hom}(M, Q(V)) = 0$. This shows the required property. □

Example 3.24. Let $(X, \mathcal{O})$ be a ringed space, Z a closed subspace, $\mathcal{A}$ the category of $\mathcal{O}$-modules, $\mathcal{J}$ the Serre subcategory of $\mathcal{O}$-modules with support contained in Z. Let $j : U = X - Z \to X$ be the open embedding. The functor $j^* : \mathcal{O}_X\text{-Mod} \to \mathcal{O}_U\text{-Mod}$ is the quotient functor Q by $\mathcal{J}$. It has a left adjoint $L = j_!$ and a right adjoint $R = j_*$. The canonical map $L \to R$ is injective, so $F = j_!$. The category $\mathcal{J}-\mathrm{loc}$ is the full subcategory of $\mathcal{A}$ of sheaves with support contained in U.

Example 3.25. Let $\mathcal{A}$ be an abelian category all of whose objects have finite composition series. Serre subcategories of $\mathcal{A}$ are determined by the simple objects they contain and this defines a bijection from the set of Serre subcategories to the set of subsets of the set S of isomorphism classes of simple objects of $\mathcal{A}$. Let $J \subset S$ and $\mathcal{J}$ the Serre subcategory of $\mathcal{A}$ it generates. The category $\mathcal{J}-\mathrm{loc}$ consists of objects with no submodule nor quotient in J. Let $M \in \mathcal{A}$. Let N be the smallest subobject of M such that all composition factors of M/N are in J. Let V be the largest subobject of N all of whose composition factors are in J. Then, N/V is the minimal continuation of $Q(M)$ and $Q(M) \mapsto N/V$ defines an inverse to the equivalence $\mathcal{J}-\mathrm{loc} \xrightarrow{\sim} \mathcal{A}/\mathcal{J}$.

Let $\mathcal{T}$ be a triangulated category with a thick subcategory $\mathcal{I}$. Consider t, t' two t-structures compatible with $\mathcal{I}$. We assume t' is the n-shift of t. We denote by $\mathcal{A}$ (resp. $\mathcal{A}'$) the heart of t (resp. t'). We put $\mathcal{J} = \mathcal{A} \cap \mathcal{I} = \mathcal{A}' \cap \mathcal{I}$. We have $\mathcal{A}/\mathcal{J} = (\mathcal{A}'/\mathcal{J})[n] \subset \mathcal{T}/\mathcal{I}$.

The following lemma is a variation on [BBD, Proposition 1.4.23].

Lemma 3.26. *Let $X \in \mathcal{A}'/\mathcal{J}$.*

If $Q : \mathcal{T} \to \mathcal{T}/\mathcal{I}$ has a left adjoint L and $n > 0$, then $\tau_{\geq 1}(L(X)) \in \mathcal{A}'$ is a minimal continuation of X.

If $Q : \mathcal{T} \to \mathcal{T}/\mathcal{I}$ has a right adjoint R and $n < 0$, then $\tau_{\leq -1}(R(X)) \in \mathcal{A}'$ is a minimal continuation of X.

Proof. Assume Q has a left adjoint L. The unit $\mathrm{Id}_{\mathcal{T}/\mathcal{I}} \to QL$ is an isomorphism. We have $X \simeq \tau_{\geq 1}(X)$, hence $X \simeq Q(\tau_{\geq 1}(L(X)))$. We have a distinguished triangle $L(X) \to \tau_{\geq 1}(L(X)) \to (\tau_{<1}(L(X)))[1] \rightsquigarrow$. We have $\mathrm{Hom}(L(X), \mathcal{I}) = 0$ and $\mathrm{Hom}((\tau_{<1}(L(X)))[1], \mathcal{I}^{\geq 0}) = 0$, hence $\mathrm{Hom}(\tau_{\geq 1}(L(X)), \mathcal{I}^{\geq 0}) = 0$. On the other hand, we have $\mathrm{Hom}(\mathcal{I}^{\leq 0}, \tau_{\geq 1}(L(X))) = 0$ and it follows from Lemma 3.6 that $\tau_{\geq 1}(L(X)) \in \mathcal{A}'$ and it is a minimal continuation of X.

The second part of the lemma follows from the first part by replacing $\mathcal{T}$ by $\mathcal{T}^{\mathrm{opp}}$. □

Intermediate extensions are minimal continuations[BBD, Corollaire 1.4.25]:

Proposition 3.27. *Assume Q has a left adjoint L and a right adjoint R. Given $X \in \mathcal{A}/\mathcal{J}$, then the image of $H^0(L(X))$ in $H^0(R(X))$ is a minimal continuation of X.*

3.6. **Maximal extensions.**

Definition 3.28. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{L}$ a set of objects of $\mathcal{T}$. Let $f : M \to N$ be a morphism in $\mathcal{T}$.*

We say that f (or N) is a maximal $\mathcal{L}$-extension by M *if* $\mathrm{cone}(f) \in \mathcal{L}$ *and if given* $L \in \mathcal{L}$*, then the canonical map* $\mathrm{Hom}(L, \mathrm{cone}(f)) \xrightarrow{\sim} \mathrm{Hom}(L, M[1])$ *is an isomorphism.*

We say that f (or M) is a maximal extension of N by $\mathcal{L}$ *if* $\mathrm{cone}(f)[-1] \in \mathcal{L}$ *and if given* $L \in \mathcal{L}$*, then the canonical map* $\mathrm{Hom}(\mathrm{cone}(f), L[1]) \xrightarrow{\sim} \mathrm{Hom}(N, L[1])$ *is an isomorphism.*

Note that the two notions in the definition are swapped by passing to $\mathcal{T}^{\mathrm{opp}}$.

Lemma 3.29. *Let $M \in \mathcal{T}$. If a maximal $\mathcal{L}$-extension by M exists, it is unique. If* $\mathrm{Hom}(L, M) = 0$ *for all $L \in \mathcal{L}$, then it is unique up to unique isomorphism.*

If a maximal extension of M by $\mathcal{L}$ exists, it is unique. If $\mathrm{Hom}(M, L) = 0$ *for all $L \in \mathcal{L}$, then it is unique up to unique isomorphism.*

Proof. Let $f : M \to N$ and $f' : M \to N'$ be two maximal extensions, with cones L and L'. Since the canonical map $\mathrm{Hom}(L, L') \to \mathrm{Hom}(L, M[1])$ is an isomorphism, the canonical map $L \to M[1]$ factors uniquely as a composite $L \xrightarrow{\alpha} L' \xrightarrow{\mathrm{can}} M[1]$. There is a map $u : N \to N'$ making the following diagram commutative

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "1"'] & N \arrow[r] \arrow[d, dashed, "u"'] & L \arrow[r] \arrow[d, "\alpha"'] & M[1] \arrow[d, "1"'] \\
M \arrow[r, "f'"] & N' \arrow[r] & L' \arrow[r] & M[1]
\end{tikzcd}$$

Similarly, we construct a map $\beta : L' \to L$ and a map $v : N' \to N$. The composite $L \xrightarrow{\beta\alpha - 1} L \xrightarrow{\mathrm{can}} M[1]$ vanishes, hence $\beta\alpha = 1$. Similarly, $\alpha\beta = 1$. We deduce that u and v are isomorphisms. If $\mathrm{Hom}(L, M) = 0$, then the map u is unique.

The second statement follows from the first one by passing to $\mathcal{T}^{\mathrm{opp}}$. ☐

Lemma 3.30. *Assume $\mathcal{L}$ is closed under extensions,* i.e.*, given a distinguished triangle $M_1 \to M_2 \to M_3 \rightsquigarrow$ in $\mathcal{T}$ with $M_1, M_3 \in \mathcal{L}$, we have $M_2 \in \mathcal{L}$.*

- *Let $N \in \mathcal{T}$. Assume* $\mathrm{Hom}(N, L) = 0$ *for all $L \in \mathcal{L}$.*
  A maximal extension of N by $\mathcal{L}$ is an object M of $\mathcal{T}$ endowed with a map $f : M \to N$ such that $\mathrm{cone}\, f[-1] \in \mathcal{L}$ *and* $\mathrm{Hom}(M, L) = \mathrm{Hom}(M, L[1]) = 0$ *for all $L \in \mathcal{L}$.*
- *Let $M \in \mathcal{T}$. Assume* $\mathrm{Hom}(L, M) = 0$ *for all $L \in \mathcal{L}$.*
  A maximal $\mathcal{L}$-extension by M is an object N of $\mathcal{T}$ endowed with a map $f : M \to N$ such that $\mathrm{cone}\, f \in \mathcal{L}$ *and* $\mathrm{Hom}(L, N) = \mathrm{Hom}(L, N[1]) = 0$ *for all $L \in \mathcal{L}$.*

Proof. Let $f : M \to N$ be a maximal extension of N by $\mathcal{L}$. Let $V = \mathrm{cone}(f)[-1]$. We have $V \in \mathcal{L}$. Let $L \in \mathcal{L}$. We have an exact sequence

(1) $\mathrm{Hom}(N, L) \to \mathrm{Hom}(M, L) \to \mathrm{Hom}(V, L) \to \mathrm{Hom}(N, L[1]) \to \mathrm{Hom}(M, L[1]) \to \mathrm{Hom}(V, L[1])$

We deduce that $\mathrm{Hom}(M, L) = 0$. Let $\zeta \in \mathrm{Hom}(M, L[1])$ and ϕ be the composition $V \xrightarrow{\mathrm{can}} M \xrightarrow{\zeta} L[1]$. Let $L'[1]$ be the cone of ϕ. There is $\zeta' : N \to L'[1]$ giving rise to a morphism of distinguished triangles as in the diagram below. Since f is a maximal extension of N by $\mathcal{L}$ and $L' \in \mathcal{L}$, we deduce that ζ' factors through the canonical map $N \to V[1]$. It follows that $\zeta' f = 0$, hence ζ factors through a map $M \to V$. By assumption, that map vanishes, hence $\zeta = 0$ and $\mathrm{Hom}(M, L[1]) = 0$.

$$\begin{tikzcd}
V \arrow[r] \arrow[d, equal] & M \arrow[r, "f"] \arrow[d, "\zeta"'] & N \arrow[r] \arrow[d, dashed, "\zeta'"'] & V[1] \arrow[d, equal] \arrow[dl, dotted] \\
V \arrow[r, "\phi"'] & L[1] \arrow[r] & L'[1] \arrow[r] & V[1]
\end{tikzcd}$$

Conversely, consider a distinguished triangle $V \to M \to N \rightsquigarrow$, where $V \in \mathcal{L}$ and assume $\mathrm{Hom}(M,L) = \mathrm{Hom}(M,L[1]) = 0$ for all $L \in \mathcal{L}$. The exact sequence (1) shows that M is a maximal extension of N by $\mathcal{L}$.

The second part of the lemma follows by passing to $\mathcal{T}^{\mathrm{opp}}$. □

The previous lemma takes a more classical form for abelian categories.

Lemma 3.31. *Let $\mathcal{A}$ be an abelian category, $\mathcal{T} = D(\mathcal{A})$ and $\mathcal{L}$ a full subcategory of $\mathcal{A}$ closed under extensions.*

- *Let $N \in \mathcal{A}$. Assume $\mathrm{Hom}(N,L) = 0$ for all $L \in \mathcal{L}$.*
  A maximal extension of N by $\mathcal{L}$ is an object M of $\mathcal{A}$ endowed with a surjective map $f : M \to N$ such that $\ker f \in \mathcal{L}$ and $\mathrm{Hom}(M,L) = \mathrm{Ext}^1(M,L) = 0$ for all $L \in \mathcal{L}$.
- *Let $M \in \mathcal{A}$. Assume $\mathrm{Hom}(L,M) = 0$ for all $L \in \mathcal{L}$.*
  A maximal $\mathcal{L}$-extension by M is an object N of $\mathcal{A}$ endowed with a injective map $f : M \to N$ such that $\mathrm{coker}\, f \in \mathcal{L}$ and $\mathrm{Hom}(L,M) = \mathrm{Ext}^1(L,M) = 0$ for all $L \in \mathcal{L}$.

3.7. **Filtrations, perversities and t-structures.** Let $\mathcal{T}$ be a triangulated category and t, t' be two t-structures on $\mathcal{T}$. Consider a filtration of $\mathcal{T}$ by thick subcategories $0 = \mathcal{T}_{-1} \subset \mathcal{T}_0 \subset \cdots \subset \mathcal{T}_r = \mathcal{T}$. We say that t is compatible with the filtration if it is compatible with $\mathcal{T}_i$ for all i. Lemma 3.11 shows that $t_{\mathcal{T}_{i+1}}$ is compatible with $\mathcal{T}_i$ for all i.

Consider a function $p : \{0,\ldots,r\} \to \mathbf{Z}$.

Definition 3.32. *We say that $(t,t',\mathcal{T}_\bullet,p)$ is* perverse *(or that t' is a p-tilt of t) if t and t' are compatible with $\mathcal{T}_\bullet$ and $t_{\mathcal{T}_i/\mathcal{T}_{i-1}} = t'_{\mathcal{T}_i/\mathcal{T}_{i-1}}[p(i)]$ for all i.*

The most important property of perverse data is that t' is determined by t, $\mathcal{T}_\bullet$ and p.

Lemma 3.33. *Let $(t,t',\mathcal{T}_\bullet,p)$ and $(t,t'',\mathcal{T}_\bullet,p)$ be two perverse data. Then, $t'' = t'$.*

If $(t,t',\mathcal{T}_\bullet,p)$ is a perverse data and p is constant of value n, then $t' = t[-n]$.

Proof. We proceed by induction on i to show that $t'_{\mathcal{T}_i} = t''_{\mathcal{T}_i}$. Assume this holds for i. We have

$$t'_{\mathcal{T}_{i+1}/\mathcal{T}_i} = t_{\mathcal{T}_{i+1}/\mathcal{T}_i}[-p(i)] = t''_{\mathcal{T}_{i+1}/\mathcal{T}_i}.$$

It follows from Lemma 3.6 that $t'_{\mathcal{T}_{i+1}} = t''_{\mathcal{T}_{i+1}}$.

The second part of the lemma follows immediately. □

The following lemmas are clear.

Lemma 3.34. *Let $(t,t',\mathcal{T}_\bullet,p)$ and $(t',t'',\mathcal{T}_\bullet,p')$ be two perverse data. Then,*

- *$(t,t'',\mathcal{T}_\bullet,p+p')$ is a perverse data*
- *$(t',t,\mathcal{T}_\bullet,-p)$ is a perverse data*
- *$(t^{\mathrm{opp}},t'^{\mathrm{opp}},\mathcal{T}_\bullet^{\mathrm{opp}},-p)$ is a perverse data.*

Lemma 3.35. *Let $\mathcal{T}_\bullet$ be a filtration of $\mathcal{T}$ by thick subcategories and let t,t' be t-structures. Fix i such that t and t' are compatible with $\mathcal{T}_i$. Consider $\bar{\mathcal{T}} = \mathcal{T}/\mathcal{T}_i$ with the filtration $0 = \mathcal{T}_i/\mathcal{T}_i \subset \mathcal{T}_{i+1}/\mathcal{T}_i \subset \cdots \subset \mathcal{T}_r/\mathcal{T}_i$ and induced t-structures $\bar{t}$ and $\bar{t}'$. Consider $\bar{p} : \{0,\ldots,r-i\} \to \mathbf{Z}$ given by $\bar{p}(j) = p(j+i)$.*

The data $(t,t',\mathcal{T}_\bullet,p)$ is perverse if and only if $(t_{\mathcal{T}_i},t'_{\mathcal{T}_i},\mathcal{T}_{\le i},p_{\le i})$ and $(\bar{t},\bar{t}',\bar{\mathcal{T}}_\bullet,\bar{p})$ are perverse data.

Lemma 3.36. *Let $(t,t',\mathcal{T}_\bullet,p)$ be a perverse data and let $i \in \{0,\ldots,r\}$. We have*

$$\mathcal{T}_i^{\ge \max\{p(0),\ldots,p(i)\}} \subset \mathcal{T}_i^{\ge' 0} \subset \mathcal{T}_i^{\ge \inf\{p(0),\ldots,p(i)\}}$$

and

$$(\mathcal{T}/\mathcal{T}_i)^{\ge \max\{p(i+1),\ldots,p(r)\}} \subset (\mathcal{T}/\mathcal{T}_i)^{\ge' 0} \subset (\mathcal{T}/\mathcal{T}_i)^{\ge \inf\{p(i+1),\ldots,p(r)\}}.$$

The next lemma shows that a perverse tilt corresponds to shifts of the successive quotients of the filtration of the heart.

Lemma 3.37. *Let $(t,t',\mathcal{T}_\bullet,p)$ be a perverse data and let $i\in\{0,\ldots,r\}$. We have $(\mathcal{A}\cap\mathcal{T}_i)/(\mathcal{A}\cap\mathcal{T}_{i-1})=(\mathcal{A}'\cap\mathcal{T}_i)/(\mathcal{A}'\cap\mathcal{T}_{i-1})[p(i)]$.*

Proof. This follows from Lemma 3.9. □

Proposition 3.38. *Let $\tilde{\mathcal{T}}_\bullet=(0=\tilde{\mathcal{T}}_{-1}\subset\cdots\subset\tilde{\mathcal{T}}_{\tilde{r}}=\mathcal{T})$ be a filtration refining $\mathcal{T}_\bullet$: there is an increasing map $f:\{0,\ldots,r\}\to\{0,\ldots,\tilde{r}\}$ such that $\mathcal{T}_i=\tilde{\mathcal{T}}_{f(i)}$. Let $p:\{0,\ldots,r\}\to\mathbf{Z}$ and $\tilde{p}:\{0,\ldots,\tilde{r}\}\to\mathbf{Z}$ be two maps such that $\tilde{p}(j)=p(i)$ for any $j\in\{f(i-1)+1,\ldots,f(i)\}$ and any i (where $f(-1)=-1$).*

Let t' be a t-structure on $\mathcal{T}$. Then, $(t,t',\tilde{\mathcal{T}}_\bullet,\tilde{p})$ is a perverse data if and only if $(t,t',\mathcal{T}_\bullet,p)$ is a perverse data and t is compatible with $\tilde{\mathcal{T}}_\bullet$.

Proof. It is clear that if $(t,t',\tilde{\mathcal{T}}_\bullet,\tilde{p})$ is a perverse data, then so is $(t,t',\mathcal{T}_\bullet,p)$.

Assume first $r=0$. We have a filtration $0=\tilde{\mathcal{T}}_{-1}\subset\cdots\subset\tilde{\mathcal{T}}_{\tilde{r}}=\mathcal{T}$ and the function $\tilde{p}$ is constant, with value $p(0)$. The data $(t,t',\mathcal{T}_\bullet,p)$ is perverse if and only if $t=t'[-p(0)]$. If $t=t'[-p(0)]$ and t,t' are compatible with $\mathcal{T}_\bullet$, then $(t,t',\tilde{\mathcal{T}}_\bullet,\tilde{p})$ is perverse. Conversely, if $(t,t',\tilde{\mathcal{T}}_\bullet,\tilde{p})$ is perverse, then $t'=t[-p(0)]$ (Lemma 3.33), hence $(t,t',\mathcal{T}_\bullet,p)$ is perverse.

Assume now $(t,t',\mathcal{T}_\bullet,p)$ is a perverse data and t is compatible with $\tilde{\mathcal{T}}_\bullet$. The case $i=0$ above shows that

$$(t_{\mathcal{T}_{i+1}/\mathcal{T}_i},t'_{\mathcal{T}_{i+1}/\mathcal{T}_i},\tilde{\mathcal{T}}_{\{f(i),\ldots,f(i+1)\}}/\mathcal{T}_i,\tilde{p}_{|\{f(i)+1,\ldots,f(i+1)\}})$$

is a perverse data, and we deduce that $(t,t',\tilde{\mathcal{T}}_\bullet,\tilde{p})$ is a perverse data. □

Proposition 3.38 shows that the filtration can always be replaced by a coarser one for which $p(i)\neq p(i+1)$ for all i.

Example 3.39. The motivating example is that of perverse sheaves [BBD]. Let $(X,\mathcal{O})$ be a ringed space, $\emptyset=X_{-1}\subset\cdots\subset X_r=X$ a filtration by closed subspaces and $p:\{0,\ldots,r\}\to\mathbf{Z}$. We have a stratification $X=\coprod_{i\geq 1}(X_i-X_{i-1})$. Let $\mathcal{T}=D(X,\mathcal{O})$ and $\mathcal{T}_r=D_{X_r}(X,\mathcal{O})$. Let t be the natural t-structure on $\mathcal{T}$. Consider the t-structure t' of perverse sheaves relative to p. Then, $(t,t',\mathcal{T}_\bullet,-p)$ defines a perverse data.

Remark 3.40. The definition of perversity can be made for filtrations indexed by more general posets. Let $\mathcal{P}$ be a poset. A *$\mathcal{P}$-filtration* $\mathcal{T}_\bullet$ of $\mathcal{T}$ is the data of thick subcategories $\mathcal{T}_\lambda$ for $\lambda\in\mathcal{P}$ such that $\mathcal{T}_\mu\subset\mathcal{T}_\lambda$ if $\mu<\lambda$. Given $\lambda\in\mathcal{P}$, we denote by $\mathcal{T}_{<\lambda}$ the thick subcategory of $\mathcal{T}$ generated by the $\mathcal{T}_\mu$ for $\mu<\lambda$.

We say that a t-structure t is *compatible* with $\mathcal{T}_\bullet$ if it is compatible with $\mathcal{T}_\lambda$ for all $\lambda\in\mathcal{P}$.

Let $p:\mathcal{P}\to\mathbf{Z}$ be a map. We say that $(t,t',\mathcal{T}_\bullet,p)$ is a *perverse data* if t and t' are compatible with $\mathcal{T}_\bullet$ for all $\lambda\in\mathcal{P}$ and given $\lambda\in\mathcal{T}$, then $t_{\mathcal{T}_\lambda/\mathcal{T}_{<\lambda}}=t'_{\mathcal{T}_\lambda/\mathcal{T}_{<\lambda}}[p(\lambda)]$ for all $\lambda\in\mathcal{P}$.

In §8.2, we describe an example where $P=\mathbf{Z}_{\geq 0}$.

Remark 3.41. One can consider a more general theory where the perversity function takes values in $\mathrm{Aut}(\mathcal{T})$ (instead of just the subgroup generated by translations) and where the filtration is stable under the self-equivalences involved.

3.8. **Non-decreasing perversities.** Assume p is non-decreasing.

Lemma 3.42. *Assume t' is a p-tilt of t. Let $q:\{0,\ldots,r\}\to\mathbf{Z}$ be a non-decreasing map.*

If $q(i)-q(i-1)\leq p(i)-p(i-1)$ for $1\leq i\leq r$, then there exists a q-tilt of t.

Proof. Replacing p by $p-p(0)$ and q by $q-q(0)$, we can assume that $p(0)=q(0)=0$.

We proceed by induction on r, then on $\sum_i p(i)$ to prove the lemma.

Assume $p(1)=q(1)=0$. Replacing the filtration of $\mathcal{T}$ by $0=\mathcal{T}_{-1}\subset\mathcal{T}_1\subset\mathcal{T}_2\subset\cdots\subset\mathcal{T}_r$ (cf Proposition 3.38), we can use our induction hypothesis and we are done.

Assume $p(1)>q(1)=0$. Let $t''=t\cap^r t'[p(1)]$ and $p':\{0,\ldots,r\}\to\mathbf{Z}$ given by $p'(0)=0$ and $p'(i)=p(i)-p(1)$ for $i>0$. This defines a t-structure by Lemmas 3.8 and 3.36 and this is a p'-tilt of t. Now, we can apply the induction hypothesis to (t,t'',p') and q.

Assume $q(1)>0$. Let $t''=t'\cap^l t[-q(1)]$ and $p':\{0,\ldots,r\}\to\mathbf{Z}$ given by $p'(0)=0$ and $p'(i)=q(1)$ for $i>0$. This defines a t-structure by Lemmas 3.8 and 3.36 and this is a p'-tilt of t. The induction hypothesis applies to $(t'',t',p-p')$ and $q-p'$. It provides a t-structure t''' that is a $(q-p')$-tilt of t'', hence a q-tilt of t. □

We can now decompose any non-decreasing tilt into a sequence of elementary ones.

Proposition 3.43. *Assume $p(0)=0$. Then, there is a sequence of t-structures $t_0=t,t_1,\ldots,t_{p(r)}=t'$ such that t_i is the tilt of t_{i-1} relative to the function p_i given by $p_i(j)=0$ if $p(j)<i$ and $p_i(j)=1$ if $p(j)\geq i$.*

There is also a sequence of t-structures $t_0=t,t_1,\ldots,t_{p(r)}=t'$ such that t_i is the tilt of t_{i-1} relative to the function p_i given by $p_i(j)=0$ if $p(j)\leq p(r)-i$ and $p_i(j)=1$ if $p(j)>p(r)-i$.

Proof. We proceed by induction on $p(r)$ to prove the first part of the proposition. By Lemma 3.42, there exists a t-structure t'' that is a p_1-tilt of t. The induction hypothesis applied to $(t'',t',p-p_1)$ gives a sequence $t''_0,\ldots,t''_{r-1}$ and the sequence $t,t''_0,\ldots,t''_{r-1}$ gives the solution.

The second statement follows by applying the first statement to $(t'^{\mathrm{opp}},t^{\mathrm{opp}},p)$. □

The following result shows how to relate minimal continuations in two different t-structures.

Proposition 3.44. *Let $(t,t',\mathcal{T}_\bullet,p)$ be a perverse data where p is non-decreasing and $p(0)=0$. Let $-1\leq j<i\leq r$ and let $X\in(\mathcal{A}'\cap\mathcal{T}_i)/(\mathcal{A}'\cap\mathcal{T}_j)$. Assume X has a minimal continuation $W\in\mathcal{A}'\cap\mathcal{T}_i$ and assume $X[p(i)]$ has a minimal continuation $V\in\mathcal{A}\cap\mathcal{T}_i$.*

Let $U_1=V$ and $U_{l+1}=\tau^{\geq p(i)-l+1}(W)$ for $1\leq l\leq p(i)$. We have $U_{p(i)+1}=W$ and $U_{l+1}[1]$ is the maximal extension of U_l by $(\mathcal{A}\cap\mathcal{T}_{\phi(l)})$ for $1\leq l\leq p(i)$, where $\phi(l)=\max\{m\leq j|p(m)\leq p(i)-l\}$.

Let $U'_1=W$ and $U'_{l+1}=(\tau')^{\leq r-1-p(i)}(V)$ for $1\leq l\leq p(i)$. We have $U'_{p(i)+1}=V$ and that $U'_{l+1}[-1]$ is the maximal $(\mathcal{A}'\cap\mathcal{T}_{\phi(l)})$-extension by U'_l for $1\leq l\leq p(i)$, where $\phi(l)=\max\{m\leq j|p(m)\leq p(i)-l\}$.

Proof. If $p(j)=p(i)$, let $j'<j$ be maximal such that $p(j')<p(i)$. Let X' be the image of W in $(\mathcal{A}'\cap\mathcal{T}_i)/(\mathcal{A}'\cap\mathcal{T}_{j'})$. We have $(\mathcal{A}'\cap\mathcal{T}_i)/(\mathcal{A}'\cap\mathcal{T}_{j'})[p(i)]=(\mathcal{A}\cap\mathcal{T}_i)/(\mathcal{A}\cap\mathcal{T}_{j'})$ by Lemma 3.37. Both $X'[p(i)]$ and the image of V in $(\mathcal{A}\cap\mathcal{T}_i)/(\mathcal{A}\cap\mathcal{T}_{j'})$ are continuations extensions of $X[p(i)]$, hence they are isomorphic. It follows that V is a minimal continuation of $X'[p(i)]$. So, if the proposition holds for (X',j'), then it holds for (X,j). So, we can assume $p(i)>p(j)$. Replacing the filtration by $0=\mathcal{T}_{-1}\subset\mathcal{T}_0\subset\cdots\subset\mathcal{T}_{j-1}\subset\mathcal{T}_j\subset\mathcal{T}_i$, we can assume $i=r$, $j=r-1$ and $p(r)>p(r-1)$.

We prove now the proposition by induction on $n=p(r)$. Let $\mathcal{I}=\mathcal{T}_{r-1}$ and $\mathcal{J}=\mathcal{A}\cap\mathcal{I}$. We denote by $Q:\mathcal{T}\to\mathcal{T}/\mathcal{I}$ the quotient functor.

Let t'' be the tilt of t with respect to the perversity function p' given by $p'(i)=0$ for $i\leq r$ and $p'(r)=1$ (the existence is provided by Proposition 3.43). We have $t''_{\mathcal{I}}=t_{\mathcal{I}}$ and $t''_{\mathcal{T}/\mathcal{I}}=t_{\mathcal{T}/\mathcal{I}}[-1]$. Let $U'=(\tau_{\geq n}W)[n]$ and $U=U'[-1]$. We have $\mathcal{T}^{\leq' 0}\subset\mathcal{T}^{\leq n}$, hence $U'\in\mathcal{A}$. We have $Q(W)\simeq X\in\mathcal{A}/\mathcal{J}[-n]$, hence the canonical map $W[n-1]\to U$ induces an isomorphism $X[n-1]\xrightarrow{\sim}Q(U)$.

We have $\mathrm{Hom}(W,\mathcal{I}^{\geq' 0})=0$, hence $\mathrm{Hom}(W[n-1],\mathcal{I}^{\geq' -n+1})=0$ and finally $\mathrm{Hom}(W[n-1],\mathcal{I}^{\geq 0})=0$ because $\mathcal{I}^{\geq 0}\subset\mathcal{I}^{\geq' -p(r-1)}\subset\mathcal{I}^{\geq' -p(r)+1}$. We have a distinguished triangle

$$W[n-1]\to U\to(\tau_{<n}W)[n]\rightsquigarrow .$$

Since $\mathrm{Hom}(\tau_{<n}W)[n], \mathcal{I}^{\geq 0}) = 0$, we deduce that $\mathrm{Hom}(U, \mathcal{I}^{\geq 0}) = 0$. We have $\mathrm{Hom}(\mathcal{I}^{\leq 0}, U) = 0$, since $U \in \mathcal{A}[-1]$. Finally, we have $Q(U) \in \mathcal{A}/\mathcal{J}[-1]$ and it follows that $U \in \mathcal{A}''$ and U is the minimal continuation of $X[n-1]$.

We have $\mathrm{Hom}(U', \mathcal{J}) = 0$. So, the canonical isomorphism $Q(U') \xrightarrow{\sim} X[n]$ lifts uniquely to a surjective morphism $U' \to V$ in $\mathcal{A}$, with a kernel in $\mathcal{J}$. We have $\mathrm{Hom}(U', \mathcal{I}^{\geq -1}) = 0$, hence $\mathrm{Ext}^1(U', \mathcal{J}) = 0$. It follows that U' is the maximal extension of V by $\mathcal{J}$. The first part of the proposition follows by induction.

The second statement follows from the first one applied to $(t'^{\mathrm{opp}}, t^{\mathrm{opp}}, \mathcal{T}_\bullet^{\mathrm{opp}}, p)$ (cf Lemma 3.34). ☐

4. Perverse equivalences

4.1. Definition.

4.1.1. *Exact categories.* Recall that an exact category is a category endowed with a class of exact sequences and satisfying certain properties [GaRoi, §9.1]. Let $\mathcal{E}$ be an exact category and $\mathcal{J}$ a full subcategory. We say that $\mathcal{J}$ is a *Serre* subcategory if given any exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{E}$, then $M \in \mathcal{J}$ if and only if $L, N \in \mathcal{J}$. We denote by $\langle \mathcal{J} \rangle$ the thick subcategory of $D^b(\mathcal{E})$ generated by $\mathcal{J}$. We denote by $\mathcal{E}/\mathcal{J}$ the full subcategory of $D^b(\mathcal{E})/\langle \mathcal{J} \rangle$ with object set $\mathcal{E}$.

Let $\mathcal{E}$ and $\mathcal{E}'$ be two exact categories. Consider filtrations $0 = \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$ and $0 = \mathcal{E}'_{-1} \subset \mathcal{E}'_0 \subset \cdots \subset \mathcal{E}'_r = \mathcal{E}'$ by Serre subcategories and consider $p : \{0, \ldots, r\} \to \mathbf{Z}$.

Definition 4.1. *An equivalence $F : D^b(\mathcal{E}) \xrightarrow{\sim} D^b(\mathcal{E}')$ is* perverse *relative to $(\mathcal{E}_\bullet, \mathcal{E}'_\bullet, p)$ if*

- *F restricts to equivalences $\langle \mathcal{E}_i \rangle \xrightarrow{\sim} \langle \mathcal{E}'_i \rangle$*
- *$F[-p(i)]$ induces equivalences $\mathcal{E}_i/\mathcal{E}_{i-1} \xrightarrow{\sim} \mathcal{E}'_i/\mathcal{E}'_{i-1}$.*

The following lemmas are clear.

Lemma 4.2. *If F is perverse relative to $(\mathcal{E}_\bullet, \mathcal{E}'_\bullet, p)$, then F^{-1} is perverse relative to $(\mathcal{E}'_\bullet, \mathcal{E}_\bullet, -p)$.*

Lemma 4.3. *If F is perverse relative to $(\mathcal{E}_\bullet, \mathcal{E}'_\bullet, p)$, then the induced equivalence $D^b(\mathcal{E}^{\mathrm{opp}}) \xrightarrow{\sim} D^b((\mathcal{E}')^{\mathrm{opp}})$ is perverse relative to $(\mathcal{E}_\bullet^{\mathrm{opp}}, \mathcal{E}_\bullet'^{\mathrm{opp}}, -p)$.*

Lemma 4.4. *Let $\mathcal{E}''$ be an exact category with a filtration $0 = \mathcal{E}''_{-1} \subset \cdots \subset \mathcal{E}''_r = \mathcal{E}$ by Serre subcategories. Let $p' : \{0, \ldots, r\} \to \mathbf{Z}$ be a map. Let $F' : D^b(\mathcal{E}') \xrightarrow{\sim} D^b(\mathcal{E}'')$ be an equivalence.*

If F is perverse relative to $(\mathcal{E}_\bullet, \mathcal{E}'_\bullet, p)$ and F' is perverse relative to $(\mathcal{E}'_\bullet, \mathcal{E}''_\bullet, p')$, then $F' \circ F$ is perverse relative to $(\mathcal{E}_\bullet, \mathcal{E}''_\bullet, p + p')$.

Note that a functor inducing equivalences on subquotients of a filtration of a triangulated category will be an equivalence, under certain conditions, as the following lemma shows.

Lemma 4.5. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulated categories with thick subcategories $\mathcal{I}$ and $\mathcal{I}'$. Let $F : \mathcal{T} \to \mathcal{T}'$ be a functor admitting a left and a right adjoint.*

If F restricts to an equivalence $\mathcal{I} \xrightarrow{\sim} \mathcal{I}'$ and induces an equivalence $\mathcal{T}/\mathcal{I} \xrightarrow{\sim} \mathcal{T}'/\mathcal{I}'$, then F is an equivalence.

Proof. Let E be a left adjoint and G a right adjoint of F. Let $M \in \mathcal{T}$, $N \in \mathcal{I}$ and $n \in \mathbf{Z}$. The composition of canonical maps

$$\mathrm{Hom}(N[n], M) \longrightarrow \mathrm{Hom}(N[n], GF(M)) \xrightarrow{\sim} \mathrm{Hom}(F(N[n]), F(M)) \xrightarrow{\sim} \mathrm{Hom}(EF(N[n]), M)$$

is the canonical map, hence it is an isomorphism. It follows that $\mathrm{Hom}(N, C) = 0$, where C is the cone of the canonical map $M \to GF(M)$. On the other hand, $C \in \mathcal{I}$, hence $C = 0$. One shows similarly that the canonical map $FG(M) \to M$ is an isomorphism. ☐

It is possible to define perverse equivalences given a filtration of only one of the two triangulated categories.

Let $\mathcal{E}$ and $\mathcal{E}'$ be two exact categories. Consider a filtration $0 = \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$ by Serre subcategories and consider $p : \{0, \ldots, r\} \to \mathbf{Z}$. Let $F : D^b(\mathcal{E}) \xrightarrow{\sim} D^b(\mathcal{E}')$ be an equivalence. Let $\mathcal{E}'_i = \mathcal{E}' \cap F(\langle \mathcal{E}_i \rangle)$: this is an extension-closed full subcategory of $\mathcal{E}'$.

Definition 4.6. *We say that F is a* perverse *equivalence relative to $(\mathcal{E}_\bullet, p)$ if the subcategories $\mathcal{E}'_i$ of $\mathcal{E}'$ are Serre subcategories and F is perverse relative to $(\mathcal{E}_\bullet, \mathcal{E}'_\bullet, p)$.*

4.1.2. *Additive categories.* Let $\mathcal{C}$ be an additive category. We endow it with a structure of exact category via the split exact sequences. We have $D^b(\mathcal{C}) = \mathrm{Ho}^b(\mathcal{C})$. A Serre subcategory $\mathcal{J}$ of $\mathcal{C}$ is a full additive subcategory closed under taking direct summands. Given $\mathcal{J}'$ a full subcategory of $\mathcal{J}$ closed under taking direct summands, the full subcategory $\mathcal{J}/\mathcal{J}'$ of $\mathrm{Ho}^b(\mathcal{C})/\langle \mathcal{J}' \rangle$ is isomorphic to the additive category quotient of $\mathcal{J}$ by $\mathcal{J}'$.

Let $\mathcal{C}$ and $\mathcal{C}'$ be additive categories. Assume $\mathcal{C}$ is endowed with a filtration $0 = \mathcal{C}_{-1} \subset \mathcal{C}_0 \subset \cdots \subset \mathcal{C}_r = \mathcal{C}$ by full additive subcategories closed under taking direct summands and consider $p : \{0, \ldots, r\} \to \mathbf{Z}$.

Lemma 4.7. *Let $F : \mathrm{Ho}^b(\mathcal{C}) \xrightarrow{\sim} \mathrm{Ho}^b(\mathcal{C}')$ be an equivalence. Let $\mathcal{C}'_i = \mathcal{C}' \cap F(\langle \mathcal{C}_i \rangle)$. This is a Serre subcategory of $\mathcal{C}'$ and the equivalence F is perverse relative to $(\mathcal{C}_\bullet, p)$ if and only if it is perverse relative to $(\mathcal{C}_\bullet, \mathcal{C}'_\bullet, p)$.*

Proof. Let $M'_1, M'_2 \in \mathcal{C}'$ such that $M'_1 \oplus M'_2 \in \mathcal{C}'_i$. We have $F^{-1}(M'_1 \oplus M'_2) \in \langle \mathcal{C}_i \rangle$, hence $F^{-1}(M'_1), F^{-1}(M'_2) \in \langle \mathcal{C}_i \rangle$, so $M_1, M'_2 \in \mathcal{C}'_i$. We deduce that $\mathcal{C}'_i$ is a Serre subcategory of $\mathcal{C}'$. $\square$

We say that $\mathcal{C}$ satisfies the Krull-Schmidt property if given any $M \in \mathcal{C}$, then the following holds:

- any idempotent of $\mathrm{End}(M)$ has an image
- if M is indecomposable, then $\mathrm{End}(M)$ is local
- there is a decomposition of M into a finite direct sum of indecomposable objects.

Assume $\mathcal{C}$ is Krull-Schmidt. It follows that $\mathrm{Comp}^b(\mathcal{C})$ and $\mathrm{Ho}^b(\mathcal{C})$ are Krull-Schmidt. Given $C \in \mathrm{Comp}^b(\mathcal{C})$, there is $C_{\min} \in \mathrm{Comp}^b(\mathcal{C})$ unique up to isomorphism such that $C \simeq C_{\min}$ in $\mathrm{Ho}^b(\mathcal{C})$ and $C_{\min}$ has no non-zero direct summand that is homotopy equivalent to 0.

Let I be the set of indecomposable objects of $\mathcal{C}$, taken up to isomorphism. A Serre subcategory of $\mathcal{C}$ is determined by the subset of I of indecomposable objects in contains. This correspondence defines a bijection $\mathcal{I} \mapsto [\mathcal{I}]$ from Serre subcategories of $\mathcal{C}$ to subsets of I.

We denote by I' the set of indecomposable objects of $\mathcal{C}'$. Consider a filtration $0 = \mathcal{C}'_{-1} \subset \mathcal{C}'_0 \subset \cdots \subset \mathcal{C}'_r = \mathcal{C}'$ by full additive subcategories closed under taking direct summands

Lemma 4.8. *An equivalence $F : \mathrm{Ho}^b(\mathcal{C}) \xrightarrow{\sim} \mathrm{Ho}^b(\mathcal{C}')$ is perverse relative to $(\mathcal{C}_\bullet, \mathcal{C}'_\bullet, p)$ if and only if*

- *given $M \in [\mathcal{I}_i] - [\mathcal{I}_{i-1}]$, we have $\big(F(M)_{\min}\big)^r \in \mathcal{I}'_{i-1}$ for $r \neq -p(i)$ and $\big(F(M)_{\min}\big)^{-p(i)} = M' \oplus L$ for some $M' \in [\mathcal{I}'_i] - [\mathcal{I}'_{i-1}]$ and $L \in \mathcal{I}'_{i-1}$.*
- *The map $M \mapsto M'$ induces a bijection $[\mathcal{I}_i] - [\mathcal{I}_{i-1}] \xrightarrow{\sim} [\mathcal{I}'_i] - [\mathcal{I}'_{i-1}]$.*

Proof. Note that $\mathrm{Ho}^b(\mathcal{C}')$ is Krull-Schmidt, hence $\mathcal{C}'$ is Krull-Schmidt as well.

Assume F is perverse. Let $M \in [\mathcal{I}_i] - [\mathcal{I}_{i-1}]$. The image of $F(M)$ in $\mathrm{Ho}^b(\mathcal{C}')/\langle \mathcal{I}'_{i-1} \rangle$ is isomorphic to $M'[p(i)]$ for some $M' \in [\mathcal{I}'_i] - [\mathcal{I}'_{i-1}]$. So, there are morphisms of complexes $p : X \to F(M)$ and $q : X \to M'[p(i)]$ whose cones C and D are in $\langle \mathcal{I}'_{i-1} \rangle$. We can assume $D = D_{\min}$. Then, $D \in \mathrm{Comp}^b(\mathcal{I}'_{i-1})$. Let $Y[1]$ be the cone of the composition of canonical maps $C_{\min} \to C \to X[1]$. We have $F(M) \simeq Y$ in $\mathrm{Ho}^b(\mathcal{C}')$. On the other hand, $Y^r \in \mathcal{I}'_{i-1}$ for $r \neq -p(i)$ and $Y^r \simeq M' \oplus L$ for some $L \in \mathcal{I}'_{i-1}$. Since $F(M)_{\min}$ is a direct summand of Y, it has the description predicted by the lemma. We have $[\mathcal{I}_i/\mathcal{I}_{i-1}] = [\mathcal{I}_i] - [\mathcal{I}_{i-1}]$, and the second statement follows.

Let us consider now the converse statement of the lemma. The functor F restricts to a fully faithful functor $F_i : \langle \mathcal{I}_i \rangle \xrightarrow{\sim} \langle \mathcal{I}'_i \rangle$.

Assume that $F[-p(i)]$ restricts to an equivalence $\langle \mathcal{I}_{i-1} \rangle \xrightarrow{\sim} \langle \mathcal{I}'_{i-1} \rangle$. The functor $F[-p(i)]$ induces a fully faithful functor $\bar{F}_i[-p(i)] : \mathcal{I}_i/\mathcal{I}_{i-1} \to \mathcal{I}'_i/\mathcal{I}'_{i-1}$. Since the image contains $[\mathcal{I}'_i] - [\mathcal{I}'_{i-1}]$, it follows that $\bar{F}_i[-p(i)]$ is an equivalence and that F_i is an equivalence. We deduce by induction on i that F is perverse. $\square$

4.2. **Abelian categories.**

4.2.1. *Characterizations of perverse equivalences.* Let $\mathcal{A}$ and $\mathcal{A}'$ be two abelian categories. Consider filtrations $0 = \mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \cdots \subset \mathcal{A}_r = \mathcal{A}$ and $0 = \mathcal{A}'_{-1} \subset \mathcal{A}'_0 \subset \cdots \subset \mathcal{A}'_r = \mathcal{A}'$ by Serre subcategories and consider $p : \{0, \ldots, r\} \to \mathbf{Z}$.

The canonical t-structure on $D^b(\mathcal{A})$ induces a t-structure on $D^b_{\mathcal{A}_i}(\mathcal{A})$, with heart $\mathcal{A}_i$: this in turn induces a t-structure on $D^b_{\mathcal{A}_i}(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A})$ with heart $\mathcal{A}_i/\mathcal{A}_{i-1}$ (Lemma 3.9). Note that $\mathcal{A}_i/\mathcal{A}_{i-1}$ generates $D^b_{\mathcal{A}_i}(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A})$ as a triangulated category.

Remark 4.9. Note that given an equivalence F perverse relative to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$, then the filtration $\mathcal{A}'_\bullet$ is determined by $\mathcal{A}_\bullet$ and F. We have $\mathcal{A}'_i = \mathcal{A}' \cap F(D^b_{\mathcal{A}_i}(\mathcal{A}))$. The function p is determined by F and $\mathcal{A}_\bullet$ as long as $\mathcal{A}_{i-1}$ is a proper subcategory of $\mathcal{A}_i$ for all i.

Lemma 4.10. *An equivalence $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ is* perverse *relative to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$ if and only if given $i \in \{0, \ldots, r\}$, then*

- *for any $M \in \mathcal{A}_i$, we have $H^r(F(M)) \in \mathcal{A}'_{i-1}$ for $r \neq -p(i)$ and $H^{-p(i)}(F(M)) \in \mathcal{A}'_i$*
- *for any $M' \in \mathcal{A}'_i$, we have $H^r(F^{-1}(M')) \in \mathcal{A}_{i-1}$ for $r \neq p(i)$ and $H^{p(i)}(F^{-1}(M')) \in \mathcal{A}_i$.*

Proof. Assume F is perverse. Let $Q : D^b_{\mathcal{A}'_i}(\mathcal{A}') \to D^b_{\mathcal{A}'_i}(\mathcal{A}')/D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}')$ be the quotient functor. We have $\tau_{<-p(i)}QF(M) = 0$, hence $Q\tau_{<-p(i)}F(M) = 0$, so $\tau_{<-p(i)}F(M) \in D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}')$. Similarly, $\tau_{>-p(i)}F(M) \in D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}')$. This shows the first statement. The second statement follows from the fact that F^{-1} is perverse relative to $(\mathcal{A}'_\bullet, \mathcal{A}_\bullet, -p)$.

Consider now the converse. We have $F(D^b_{\mathcal{A}_i}(\mathcal{A})) \subset D^b_{\mathcal{A}'_i}(\mathcal{A}')$ and $F^{-1}(D^b_{\mathcal{A}'_i}(\mathcal{A}')) \subset D^b_{\mathcal{A}_i}(\mathcal{A})$, hence F restricts to an equivalence $D^b_{\mathcal{A}_i}(\mathcal{A}) \xrightarrow{\sim} D^b_{\mathcal{A}'_i}(\mathcal{A}')$. Similarly, $F[-p(i)](\mathcal{A}_i/\mathcal{A}_{i-1}) \subset \mathcal{A}'_i/\mathcal{A}'_{i-1}$ and $(F[-p(i)])^{-1}(\mathcal{A}'_i/\mathcal{A}'_{i-1}) \subset \mathcal{A}_i/\mathcal{A}_{i-1}$, hence the equivalence $F[-p(i)] : D^b_{\mathcal{A}_i}(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A}) \xrightarrow{\sim} D^b_{\mathcal{A}'_i}(\mathcal{A}')/D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}')$ restricts to an equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\sim} \mathcal{A}'_i/\mathcal{A}'_{i-1}$. So, F is perverse. $\square$

Lemma 4.11. *An equivalence $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ is* perverse *relative to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$ if and only if given $i \in \{0, \ldots, r\}$, then*

- *for any $M \in \mathcal{A}_i$, we have $H^r(F(M)) \in \mathcal{A}'_{i-1}$ for $r \neq -p(i)$ and $H^{-p(i)}(F(M)) \in \mathcal{A}'_i$*
- *the functor $H^{-p(i)} \circ F : \mathcal{A}_i \to \mathcal{A}'_i/\mathcal{A}'_{i-1}$ is essentially surjective.*

Proof. Assume F is perverse. Lemma 4.10 shows the first statement. The second statement follows from the fact that the functor $H^{-p(i)} \circ F : \mathcal{A}_i \to \mathcal{A}'_i/\mathcal{A}'_{i-1}$ factors as the composition of the quotient functor $\mathcal{A}_i \to \mathcal{A}_i/\mathcal{A}_{i-1}$ with $F[-p(i)] : \mathcal{A}_i/\mathcal{A}_{i-1} \to \mathcal{A}'_i/\mathcal{A}'_{i-1}$.

Let us now prove the converse assertion. We proceed by induction on i.

The thick subcategory $D^b_{\mathcal{A}_i}(\mathcal{A})$ is generated by $\mathcal{A}_i$. By assumption, $F(\mathcal{A}_i) \subset D^b_{\mathcal{A}'_i}(\mathcal{A}')$, hence F restricts to a functor $D^b_{\mathcal{A}_i}(\mathcal{A}) \to D^b_{\mathcal{A}'_i}(\mathcal{A}')$. This functor is still fully faithful. By induction, it restricts to an equivalence $D^b_{\mathcal{A}_{i-1}}(\mathcal{A}) \xrightarrow{\sim} D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}')$. So, F induces a fully faithful functor

$$F_i : D^b_{\mathcal{A}_i}(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A}) \to D^b_{\mathcal{A}'_i}(\mathcal{A}')/D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}')$$

By assumption, $F_i[-p(i)]$ restricts to an essentially surjective functor $\mathcal{A}_i/\mathcal{A}_{i-1} \to \mathcal{A}'_i/\mathcal{A}'_{i-1}$, hence it restricts to an equivalence. We deduce now that $F_i : D^b_{\mathcal{A}_i}(\mathcal{A}) \to D^b_{\mathcal{A}'_i}(\mathcal{A}')$ is an equivalence, since it is fully faithful and its image contains $\mathcal{A}'_i$. $\square$

When the filtrations arise by taking orthogonals, there is a criterion ensuring that an equivalence is perverse.

Lemma 4.12. *Let $\mathcal{F}_0, \ldots, \mathcal{F}_r$ (resp. $\mathcal{F}'_0, \ldots, \mathcal{F}'_r$) be subcategories of $\mathcal{A}$ (resp. $\mathcal{A}'$) such that*

(i) *$F(V)[-p(i)] \in \mathcal{F}'_i$ for all $V \in \mathcal{F}_i$.*
(ii) *$F^{-1}(V')[p(i)] \in \mathcal{F}_i$ for all $V' \in \mathcal{F}'_i$.*
(iii) *The following assertions are equivalent for $M \in \mathcal{A}$:*
  - *$M \in \mathcal{A}_i$*
  - *$\mathrm{Hom}(M, V[r]) = \mathrm{Hom}(V, M[r]) = 0$ for all $V \in \mathcal{F}_j$, $j > i$ and $r \in \mathbf{Z}$*
  - *$\mathrm{Hom}(M, V) = 0$ for all $V \in \mathcal{F}_j$, $j > i$*
  - *$\mathrm{Hom}(V, M) = 0$ for all $V \in \mathcal{F}_j$, $j > i$*
(iv) *Same as (iii) with $\mathcal{A}$ and $\mathcal{A}'$ swapped.*

Then, F is a perverse equivalence relative to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$.

Proof. Let $M \in \mathcal{A}_i$. Let $j \geq i$, $V' \in \mathcal{F}'_j$ and $r \in \mathbf{Z}$. We have

$$\mathrm{Hom}(V', F(M)[r]) \simeq \mathrm{Hom}(F^{-1}(V')[p(j)], M[r + p(j)])$$

and the last space vanishes when $j > i$, since $F^{-1}(V')[p(j)] \in \mathcal{F}_j$. When $j = i$, the last space vanishes if $r < -p(i)$. Let d be minimal such that $H^d F(M) \notin \mathcal{A}'_{i-1}$. We have $\mathrm{Hom}(V', (\tau_{<d} F(M))[r]) = 0$, since $\mathrm{Hom}(V', H^n F(M)[r']) = 0$ for all r' and $n < d$. It follows that $\mathrm{Hom}(V', H^d F(M)) = 0$ if $j > i$, so $H^d F(M) \in \mathcal{A}'_i$. On the other hand, $\mathrm{Hom}(V', H^d F(M)) = 0$ if $j = i$ and $d < -p(i)$, so $d \geq -p(i)$.

Similarly, one shows that $H^n F(M) \in \mathcal{A}'_{i-1}$ for $n > p(i)$. We deduce that $H^n F(M) \in \mathcal{A}'_{i-1}$ for $n \neq -p(i)$ and $H^{-p(i)} F(M) \in \mathcal{A}'_i$.

Replacing F by F^{-1}, we obtain that given $M' \in \mathcal{A}'_i$, then $H^n F^{-1}(M) \in \mathcal{A}_{i-1}$ for $n \neq p(i)$ and $H^{p(i)} F^{-1}(M) \in \mathcal{A}_i$. Lemma 4.10 shows that F is perverse. $\square$

Remark 4.13. More generally, we say that a (triangulated) functor $F : D^b(\mathcal{A}) \to D^b(\mathcal{A}')$ is perverse relative to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$ if $\{M \in D^b(\mathcal{A}) | F(M) \in D^b_{\mathcal{A}'_i}(\mathcal{A}')\} = D^b_{\mathcal{A}_i}(\mathcal{A})$ and $\{M \in \mathcal{A}/\mathcal{A}_{i-1} | F(M)[-p(i)] \in \mathcal{A}'_i/\mathcal{A}'_{i-1}\} = \mathcal{A}_i/\mathcal{A}_{i-1}$.

If in addition the functor is an equivalence, then it is a perverse equivalence.

4.2.2. *Perverse equivalences and perverse data.* We consider $\mathcal{A}$ and $\mathcal{A}'$ two abelian categories and we assume $\mathcal{A}$ is equiped with a filtration by Serre subcategories $0 = \mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \cdots \subset \mathcal{A}_r = \mathcal{A}$. Consider $p : \{0, \ldots, r\} \to \mathbf{Z}$.

Let $\mathcal{T} = D^b(\mathcal{A})$ and $\mathcal{T}_i = D^b_{\mathcal{A}_i}(\mathcal{A})$. Let t (resp. t') be the canonical t-structure on $D^b(\mathcal{A})$ (resp. $D^b(\mathcal{A}')$).

Let $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ be an equivalence and let $\mathcal{A}'_i = \mathcal{A}' \cap F(\mathcal{T}_i)$.

Lemma 4.14. *The following conditions are equivalent:*

(1) *F is a perverse equivalence relative to $(\mathcal{A}_\bullet, p)$*
(2) *$(t, F^{-1}(t'), \mathcal{T}_\bullet, p)$ is a perverse data*
(3) *given $i \in \{0, \ldots, r-1\}$ and $M \in \mathcal{A}_i$, then $\tau_{<-p(i)} F(M)$ and $\tau_{>-p(i)} F(M)$ are in $F(\mathcal{T}_{i-1})$.*

Proof. Let $\mathcal{T}'_i$ be the thick subcategory of $\mathcal{T}'$ generated by $\mathcal{A}'_i$. We have $\mathcal{T}'_i \subset F(\mathcal{T}_i)$. Note that if $\mathcal{A}'_i$ is a Serre subcategory of $\mathcal{A}'$, then $\mathcal{T}'_i = D^b_{\mathcal{A}'_i}(\mathcal{A}')$.

The t-structure $F^{-1}(t')$ is compatible with $\mathcal{T}_i$ if and only if t' is compatible with $F(\mathcal{T}_i)$. Assume it is compatible. Lemma 3.9 shows that $\mathcal{A}'_i$ is a Serre subcategory of $\mathcal{A}'$. Given $M \in F(\mathcal{T}_i)$, then $H^n(M) = \tau_{\geq 0}\tau_{\leq 0}M[-n] \in \mathcal{A}' \cap F(\mathcal{T}_i) = \mathcal{A}'_i$, hence $M \in \mathcal{T}'_i$.

Conversely, assume $\mathcal{A}'_i$ is a Serre subcategory of $\mathcal{A}'$ and $F(\mathcal{T}_i) \subset D^b_{\mathcal{A}'_i}(\mathcal{A}')$. Then $\tau_{\leq 0}F(\mathcal{T}_i) \subset F(\mathcal{T}_i)$, hence t' is compatible with $F(\mathcal{T}_i)$. So, we have shown that $F^{-1}(t')$ is compatible with $\mathcal{T}_i$ if and only if $\mathcal{A}'_i$ is a Serre subcategory of $\mathcal{A}'$ and $D^b_{\mathcal{A}'_i}(\mathcal{A}') = F(\mathcal{T}_i)$.

Assume that $D^b_{\mathcal{A}'_i}(\mathcal{A}') = F(\mathcal{T}_i)$ for all i. Then $(F[-p(i)])^{-1}(t')$ induces the same t-structure as t on $\mathcal{T}_i/\mathcal{T}_{i-1}$ if and only if $F[-p(i)]$ induces an equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\sim} \mathcal{A}'_i/\mathcal{A}'_{i-1}$.

We deduce that (1) is equivalent to (2).

Assume (3) holds. Let us show by induction on i the conditions of Definition 4.1.

Let us start with $i = 0$. By assumption, $F[-p(0)]$ restricts to a fully faithful functor $\mathcal{A}_0 \to \mathcal{A}'_0$, and given $M \in \mathcal{T}_0$, if $F(M)[-p(0)] \in \mathcal{A}'$, then $M \in \mathcal{A}$. So, $F[-p(0)]$ restricts to an equivalence $\mathcal{A}_0 \xrightarrow{\sim} \mathcal{A}'_0$ and F restricts to an equivalence $D^b_{\mathcal{A}_0}(\mathcal{A}) \xrightarrow{\sim} D^b_{\mathcal{A}'_0}(\mathcal{A})$.

Assume $i \geq 1$. By induction, t' is compatible with $F(\mathcal{T}_{i-1})$, hence it induces a t-structure t'' on $\mathcal{T}'/F(\mathcal{T}_{i-1})$. In order to show that t' is compatible with $F(\mathcal{T}_i)$, it is enough to show that t'' is compatible with $F(\mathcal{T}_i/\mathcal{T}_{i-1})$: that is known by the case $i = 0$ treated above. We deduce also that $F[-p(i)]$ induces an equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\sim} \mathcal{A}'_i/\mathcal{A}'_{i-1}$. $\square$

The next lemma shows that filtrations in perverse equivalences can be refined, and that the filtrations can be chosen to be minimal (*i.e.*, $p(i) \neq p(i+1)$ for all i).

Lemma 4.15. *Let $\tilde{\mathcal{A}}_\bullet = (0 = \tilde{\mathcal{A}}_{-1} \subset \cdots \subset \tilde{\mathcal{A}}_{\tilde{r}} = \mathcal{A})$ be a filtration refining $\mathcal{A}_\bullet$: there is an increasing map $f : \{0, \ldots, r\} \to \{0, \ldots, \tilde{r}\}$ such that $\mathcal{A}_i = \tilde{\mathcal{A}}_{f(i)}$. Let $\tilde{p} : \{0, \ldots, \tilde{r}\} \to \mathbf{Z}$ be a map such that $\tilde{p}(j) = p(i)$ for any $j \in \{f(i-1)+1, \ldots, f(i)\}$ and any i (where $f(-1) = -1$).*

An equivalence $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ is perverse relative to $(\mathcal{A}_\bullet, p)$ if and only if it is perverse relative to $(\tilde{\mathcal{A}}_\bullet, \tilde{p})$.

Note a special case of Lemma 4.15:

Lemma 4.16. *Let F be a perverse equivalence with $p = 0$. Then, F restricts to an equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}'$.*

We deduce from Lemmas 4.4 and 4.16 the following Proposition which says that the filtration $\mathcal{A}_\bullet$ and the function p determine $\mathcal{A}'$, up to equivalence.

Proposition 4.17. *Let $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ and $\tilde{F} : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\tilde{\mathcal{A}}')$ be perverse equivalences relative to $(\mathcal{A}_\bullet, p)$. Then the composition $\tilde{F}F^{-1}$ restricts to an equivalence $\mathcal{A}' \xrightarrow{\sim} \tilde{\mathcal{A}}'$.*

4.2.3. *Perverse equivalences and simple objects.* Let us assume that every object of $\mathcal{A}$ (resp. of $\mathcal{A}'$) has a finite composition series. Let S (resp. S') the set of isomorphism classes of simple objects of $\mathcal{A}$ (resp. $\mathcal{A}'$).

Consider

- a filtration $\emptyset = S_{-1} \subset S_0 \subset \cdots \subset S_r = S$
- a filtration $\emptyset = S'_{-1} \subset S'_0 \subset \cdots \subset S'_r = S'$
- and a function $p : \{0, \ldots, r\} \to \mathbf{Z}$.

Let $\mathcal{A}_i$ (resp. $\mathcal{A}'_i$) be the Serre subcategory of $\mathcal{A}$ (resp. $\mathcal{A}'$) generated by S_i (resp. S'_i).

Definition 4.18. *An equivalence $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ is* perverse *relative to $(S_\bullet, S'_\bullet, p)$ if it is perverse relative to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$.*

Lemma 4.11 gives the following criterion for perversity.

Lemma 4.19. *An equivalence $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ is* perverse *relative to $(S_\bullet, S'_\bullet, p)$ if and only if for every i, the following holds:*

- *given $V \in S_i - S_{i-1}$, then the composition factors of $H^r(F(V))$ are in S'_{i-1} for $r \neq -p(i)$ and there is a filtration $L_1 \subset L_2 \subset H^{-p(i)}(F(V))$ such that the composition factors of L_1 and of $H^{-p(i)}(F(V))/L_2$ are in S'_{i-1} and $L_2/L_1 \in S'_i - S'_{i-1}$.*
- *The map $V \to L_2/L_1$ induces a bijection $S_i - S_{i-1} \xrightarrow{\sim} S'_i - S'_{i-1}$.*

Proof. Assume the two conditions hold. The simple modules in $S'_i - S'_{i-1}$ are in the image of $H^{-p(i)} \circ F$. On the other hand, that functor is the restriction of the fully faithful functor $F[-p(i)]$: $D^b_{\mathcal{A}_i}(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A}) \to: D^b_{\mathcal{A}'_i}(\mathcal{A}')/D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}')$ and given $V, W \in \mathcal{A}'_i$, we have a canonical isomorphism

$$\mathrm{Ext}^1_{\mathcal{A}'_i/\mathcal{A}'_{i-1}}(V, W) \xrightarrow{\sim} \mathrm{Hom}_{D^b_{\mathcal{A}'_i}(\mathcal{A}')/D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}')}(V, W[1]).$$

It follows that $F[-p(i)] : \mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\sim} \mathcal{A}'_i/\mathcal{A}'_{i-1}$ is an equivalence, hence F is perverse by Lemma 4.11.

Conversely, assume F is perverse. The functor $H^{p(i)} \circ F : \mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\sim} \mathcal{A}'_i/\mathcal{A}'_{i-1}$ is an equivalence. It follows that given $V \in S_i - S_{i-1}$, then the image of $H^{p(i)} \circ F$ in $\mathcal{A}'_i/\mathcal{A}'_{i-1}$ is simple, hence there is a filtration as stated in the lemma, and the equivalence induces a bijection $S_i - S_{i-1} \xrightarrow{\sim} S'_i - S'_{i-1}$. $\square$

The construction of Lemma 4.19 shows that a perverse equivalence gives rise to a bijection $S \xrightarrow{\sim} S'$ compatible with the filtrations.

The following lemma follows immediately from Lemma 4.3.

Lemma 4.20. *Let A and A' be two finite-dimensional algebras over a field k, let $\mathcal{A} = A$-mod and $\mathcal{A}' = A'$-mod. Let $F : D^b(A) \xrightarrow{\sim} D^b(A')$ be an equivalence perverse relative to $(S_\bullet, S'_\bullet, p)$. Then, the composition*

$$D^b(A^{\mathrm{opp}}) \xrightarrow[\sim]{(-)^*} D^b(A)^{\mathrm{opp}} \xrightarrow[\sim]{F} D^b(A')^{\mathrm{opp}} \xrightarrow[\sim]{(-)^*} D^b(A'^{\mathrm{opp}})$$

is an equivalence perverse relative to $(S_\bullet, S'_\bullet, -p)$.

4.2.4. *Projective objects.* We assume here that every object of $\mathcal{A}$ (and of $\mathcal{A}'$) has a finite composition series and a projective cover.

We put $\mathcal{E} = \mathcal{A}$-proj and $\mathcal{E}' = \mathcal{A}'$-proj. We denote by $\mathcal{E}_i$ the additive full subcategory of $\mathcal{E}$ generated by the projective objects P_V, where $V \in S - S_{r-i-1}$. We have $\mathcal{E}_i = \mathcal{E} \cap {}^\perp\mathcal{A}_{r-i}$. We define similarly $\mathcal{E}'_i$. We define $\bar{p}$ by $\bar{p}(i) = p(r-i)$.

We consider an equivalence $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ that restricts to an equivalence $\bar{F} : \mathrm{Ho}^b(\mathcal{E}) \xrightarrow{\sim} \mathrm{Ho}^b(\mathcal{E}')$.

Lemma 4.21. *The equivalence F is perverse relative to $(S_\bullet, S'_\bullet, p)$ if and only if $\bar{F}$ is perverse relative to $(\mathcal{E}_\bullet, \mathcal{E}'_\bullet, \bar{p})$.*

Proof. Let $V \in S_i$, $W' \in S'_j$ and $n \in \mathbf{Z}$. We have $\mathrm{Hom}(P_{W'}, F(V)[n]) \xrightarrow{\sim} \mathrm{Hom}(\bar{F}^{-1}(P_{W'}), V[n])$.

Assume F is perverse. Let $W' \in S'_j - S'_{j-1}$. Let $C = \bar{F}^{-1}(P_{W'})_{\min}$.

If $V \in S_{j-1}$, then $\mathrm{Hom}(P_{W'}, F(V)[n]) = 0$, hence C^{-n} is a direct sum of P_W's with $W \in S - S_{j-1}$. If $V \in S_j - S_{j-1}$, then $\mathrm{Hom}(P_{W'}, F(V)[n]) = 0$ for $n \neq -p(j)$, hence C^{-n} is a direct summand of P_W's with $W \in S - S_j$. We have $\mathrm{Hom}(P_{W'}, F(V)[p(j)]) \simeq \delta_{VW}\,\mathrm{End}(W') \simeq \delta_{VW}\,\mathrm{End}(V)$ (Lemma 4.19), hence $C^{p(j)} \simeq P_V \oplus \bigoplus_{W \in S-S_j} P_W^{a_W}$ for some integers a_W. Lemma 4.8 shows that $\bar{F}^{-1}$ is perverse, since $\bar{p}(r-j) = p(j)$.

Assume now that $\bar{F}$ is perverse. Let $V \in S_i - S_{i-1}$. Given $W' \not\in S'_i$, we have $\mathrm{Hom}(\bar{F}^{-1}(P_{W'}), V[n]) = 0$ for all n, hence the composition factors of $H^n F(V)$ are in S'_i. If $W' \in S'_i - S'_{i-1}$, we have

$\mathrm{Hom}(\bar{F}^{-1}(P_{W'}), V[n]) = 0$ for $n \neq -p(i)$, hence the composition factors of $H^nF(V)$ are in S'_{i-1} for $n \neq -p(i)$ and $\mathrm{Hom}(\bar{F}^{-1}(P_{W'}), V[-p(i)]) \simeq \delta_{VW}\,\mathrm{End}(V)$, so $H^{-p(i)}F(V)$ has exactly one composition factor outside S'_{i-1}, namely V' occurring with multiplicity 1. We deduce that F is perverse by Lemma 4.19. $\square$

4.2.5. *One-sided filtrations.* Let $\mathcal{A}$ and $\mathcal{A}'$ be two abelian categories all of whose objects have finite composition series. Let S (resp. S') the set of isomorphism classes of simple objects of $\mathcal{A}$ (resp. $\mathcal{A}'$).

Consider a filtration $\emptyset = S_{-1} \subset S_0 \subset \cdots \subset S_r = S$ and a function $p : \{0, \ldots, r\} \to \mathbf{Z}$.

Let $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ be an equivalence. Let S'_i be the set of simple objects that appear as composition factors of $H^*(F(V))$ for some $V \in S_i$.

Definition 4.22. *The equivalence F is* perverse *relative to $(S_\bullet, p)$ if it is perverse relative to $(S_\bullet, S'_\bullet, p)$.*

Define $p_S : S \to \mathbf{Z},\ V \mapsto p(i)$ where $i = \min\{j | V \in S_j\}$. The following lemma is a reformulation of Lemma 4.15.

Lemma 4.23. *Let $\emptyset = \tilde{S}_{-1} \subset \tilde{S}_0 \subset \cdots \subset \tilde{S}_{\tilde{r}} = S$ be a refinement of $S_\bullet$ and consider $\tilde{p} : \{0, \ldots, \tilde{r}\} \to \mathbf{Z}$ such that $p_{\tilde{S}} = p_S$.*

The equivalence F is perverse for $(S_\bullet, p)$ if and only if it is perverse for $(\tilde{S}_\bullet, \tilde{p})$.

Remark 4.24. Lemma 4.23 says we can always assume the filtration is maximal (*i.e.*, $S_i - S_{i-1}$ has one element for all i). It also says we can assume the filtration is minimal (*i.e.*, $p(i) \neq p(i+1)$ for all i).

4.3. **Self-equivalences.** Perverse self-equivalences of triangulated categories are absolute notions, independent of t-structures.

Let $\mathcal{T}$ be a triangulated category with a filtration $\mathcal{T}_\bullet$. Let p be a perversity function.

Definition 4.25. *We say that a self-equivalence F of $\mathcal{T}$ is perverse relative to $(\mathcal{T}_\bullet, p)$ if F restricts to equivalences $\mathcal{T}_i \xrightarrow{\sim} \mathcal{T}_i$ for all i and the equivalence $\mathcal{T}_i/\mathcal{T}_{i-1} \xrightarrow{\sim} \mathcal{T}_i/\mathcal{T}_{i-1}$ induced by $F[-p(i)]$ is isomorphic to the identity.*

The following lemma is clear.

Lemma 4.26. *Let F be a perverse self-equivalence relative to $(\mathcal{T}_\bullet, p)$ and t a t-structure of $\mathcal{T}$. If t is compatible with $\mathcal{T}_\bullet$, then $(t, F^{-1}(t), \mathcal{T}_\bullet, p)$ is a perverse data.*

Remark 4.27. Note that a perverse self-equivalence with $p = 0$ needs not be isomorphic to the identity. Take for A the Kronecker algebra over a field k, *i.e.*, the path algebra of the quiver $\bullet \overset{a}{\underset{b}{\rightrightarrows}} \bullet$ over k. It has a grading with $\deg a = 1$ and $\deg b = 0$. This corresponds to an action of $\mathbf{G}_m$ on A, giving rise to an injection of $\mathbf{G}_m$ in the group of outer automorphisms of A. Let $\alpha \in k - \{0, 1\}$ and let F be the self-equivalence of $D^b(A\text{-mod})$ induced by the corresponding automorphism of A. Let $\mathcal{I}$ the thick subcategory of $D^b(A\text{-mod})$ generated by the projective simple A-module. Then F is a perverse self-equivalence relative to the filtration $0 \subset \mathcal{I} \subset \mathcal{T}$ with $p = 0$, but $F \not\simeq \mathrm{id}$.

5. Symmetric algebras

5.1. **Elementary equivalences.** Let k be a field and A a finite dimensional symmetric k-algebra. Let S be the set of isomorphism classes of simple A-modules. Given $V \in A$-mod, we denote by $\phi_V : A_V \to V$ a projective cover of V.

Let $I \subset S$. Given $M \in A$-mod, we denote by M_I the largest quotient of A_M by a submodule of $\ker \phi_M$ such that all composition factors of the kernel of the induced map $M_I \to M$ are in I. Similarly, let $M \to I_M$ be an injective hull. We denote by M^I the largest submodule of I_M containing M and such that all composition factors of M^I/M are in I.

Lemma 3.31 provides a relation with maximal extensions.

Lemma 5.1. *Let $\mathcal{I}$ be the Serre category of A-*mod *with objects those modules whose composition factors are in $\mathcal{I}$. Let $M \in A$-*mod*.*

If $\mathrm{Hom}(M, L) = 0$ *for all $L \in I$, then M_I is the largest extension of M by $\mathcal{I}$.*

If $\mathrm{Hom}(L, M) = 0$ *for all $L \in I$, then M^I is the largest $\mathcal{I}$-extension by M.*

Let $V \in I$. Let Q_V be a projective cover of the kernel of the canonical map $A_V \to V_I$. We define now a complex

$$T_{A,V}(I) = 0 \to Q_V \to A_V \to 0$$

where A_V is in degree 0.

Given $V \in S - I$, we put

$$T_{A,V}(I) = 0 \to A_V \to 0 \to 0$$

where A_V is in degree -1.

Let $T_A(I) = \bigoplus_{V\in S} T_{A,V}(I)$. It is straightforward to check that this is a tilting complex (cf [Ri2]): $T_A(I)$ generates $\mathrm{Ho}^b(A\text{-proj})$ as a thick subcategory and $\mathrm{Hom}_{D^b(A)}(T_A(I), T_A(I)[n]) = 0$ for $n \neq 0$.

Let $A' = \mathrm{End}_{D^b(A)}(T_A(I))$ and let S' be the set of simple A'-modules, up to isomorphism. We have an equivalence

$$F = \mathrm{Hom}^\bullet_A(T_A(I), -) : D^b(A) \xrightarrow{\sim} D^b(A').$$

There is a bijection $S \xrightarrow{\sim} S',\ V \mapsto V'$ such that $F(T_{A,V}(I)) = A'_{V'}$.

The images of simple modules under the equivalence F are described by following lemma [Ok, Lemma 2.1].

Lemma 5.2. *Given $V \in S$, we have*

$$F^{-1}(V') = \begin{cases} V & \text{if } V \in I \\ V^I[1] & \text{otherwise.} \end{cases} \quad \text{and } F(V) = \begin{cases} V' & \text{if } V \in I \\ V'_{I'}[-1] & \text{otherwise.} \end{cases}$$

Proof. Note that V' is, up to isomorphism, the unique object of $D^b(A')$ such that

$$\mathrm{Hom}_{D^b(A')}(A'_{W'}, V'[j]) = \delta_{0j}\delta_{VW} K$$

for all $W \in S$, for some skewfield K.

Assume $V \in I$. If $W \notin I$, we have

$$\mathrm{Hom}_{D^b(A)}(T_W, V^I[j]) \simeq \mathrm{Hom}_{D^b(A)}(A_W, V^I[j-1]) = 0.$$

If $W \in I$, then $\mathrm{Hom}_{D^b(A)}(T_W, V^I) = \delta_{VW}\,\mathrm{End}_k(V)$. Since Q_W is a direct sum of modules of the form A_U, with $U \notin I$, we deduce that $\mathrm{Hom}_{\mathrm{Ho}^b(A)}(T_W, V^I[1]) = 0$. This shows that $V = F^{-1}(V')$.

Assume $V \notin I$. If $W \notin I$, we have

$$\mathrm{Hom}_{D^b(A)}(T_W, V^I[1+j]) \simeq \mathrm{Hom}_{D^b(A)}(A_W, V^I[j]) = \delta_{0j}\delta_{VW}\,\mathrm{End}_k(V).$$

Assume $W \in I$. Since $H^0(T_W(I)) = W_I$, we have $\mathrm{Hom}_A(H^0(T_W(I)), V^I) = 0$. On the other hand, Q_W is a direct sum of modules of the form A_U, with $U \notin I$, hence a map $f : Q_W \to V^I$ factors through $Q_W/\operatorname{rad} Q_W$. We have an exact sequence

$$0 \to Q_W/\operatorname{rad} Q_W \to A_W/\operatorname{rad} Q_W \to W_I \to 0$$

Since $\mathrm{Ext}^1_A(U, V^I) = 0$ for any $U \in I$, it follows that $\mathrm{Ext}^1_A(W_I, V^I) = 0$, hence f factors through A_W. So, $\mathrm{Hom}_{D^b(A)}(T_W, V^I[1]) = 0$. We deduce that $V^I = F^{-1}(V')$.

The second part of the lemma follows from the first one, by replacing F by the opposite inverse equivalence $D^b(A'^{\mathrm{opp}}) \xrightarrow{\sim} D^b(A^{\mathrm{opp}})$. $\square$

Lemma 5.2 shows we have constructed a perverse equivalence.

Proposition 5.3. *The equivalence F is perverse relative to $(0 \subset I \subset S, {}^{0\mapsto 0}_{1\mapsto -1})$.*

We also have a dual construction yielding a tilting complex $T^A(I) = \bigoplus_{V\in S} T^{A,V}(I)$ with summands defined as follows.

Let $V \in I$. Let J_V be an injective hull of the cokernel of the canonical map $V^I \to I_V$. Define

$$T^-_{A,V}(I) = 0 \to A_V \to J_V \to 0$$

where A_V is in degree 0.

Given $V \in S - I$, we put

$$T^-_{A,V}(I) = 0 \to 0 \to A_V \to 0$$

where A_V is in degree 1.

Let $A'' = \mathrm{End}_{D^b(A)}(T^-_A(I))$. We have an equivalence $G = \mathrm{Hom}^\bullet_A(T^-_A(I), -) : D^b(A) \xrightarrow{\sim} D^b(A'')$.

Proposition 5.4. *The equivalence G is perverse relative to $(0 \subset I \subset S, {}^{0\mapsto 0}_{1\mapsto 1})$.*

Note that $F^{-1}(A'_V) \simeq T^-_{A',V}(I)$.

We put $T^+_{A,V}(I) = T_{A,V}(I)$.

5.2. **Construction of perverse equivalences.** Let $\mathcal{E}$ be the set of isomorphism classes of families $(T_V)_{V\in S}$ where T_V is an indecomposable bounded complex of finitely generated projective A-modules, $T_V \not\simeq T_{V'}$ if $V \not\simeq V'$, and $\bigoplus_{V\in S} T_V$ is a tilting complex. We write A for $(A_V)_V$, the map sending a simple module to a projective cover.

Let $\mathcal{P}'(S)$ be the set of proper subsets of S and let Γ be the quotient of $\mathrm{Free}(\mathcal{P}'(S)) \rtimes \mathfrak{S}(S)$ by the relations $IJ = JI$ when $I \subset J \subset S$.

There is an action of $\mathrm{Free}(\mathcal{P}'(S)) \rtimes \mathfrak{S}(S)$ on $\mathcal{E}$. The action of $\mathfrak{S}(S)$ is given by permutation of indices. The action of $I \subset S$ on $(T_V)_V \in \mathcal{E}$ is $(T'_V)_V$ defined as follows.

Let $B = \mathrm{End}_{D^b(A)}(\bigoplus_V T_V)$ and $F = \mathrm{Hom}^\bullet_A(\bigoplus_V T_V, -) : D^b(A) \xrightarrow{\sim} D^b(B)$. We put $T'_V = F^{-1}(T_{B,V}(I))$.

Let $B' = \mathrm{End}_{D^b(B)}(T_B(I))$.

$$\begin{tikzcd} D^b(A) \arrow[r, "F", "\sim"'] & D^b(B) \arrow[r, "\mathrm{Hom}^\bullet_B(T_B(I),-)", "\sim"'] & D^b(B') \\ T_V \arrow[r, mapsto] & B_V & \\ T'_V \arrow[r, mapsto] & T_{B,V}(I) \arrow[r, mapsto] & B'_V \end{tikzcd}$$

To define the action of I^{-1} we replace $T_B(I)$ by $T^B(I)$.

Note that $\mathcal{E}$ has a canonical element $A = (A_V)_V$. We denote by $\mathcal{E}^0$ its Γ-orbit. Note that $(I\cdot A)_V = (T_{A,V}(I))_V$ and $(I^{-1}\cdot A)_V = (T^-_{A,V}(I))_V$.

Consider a family of symmetric algebras $A\{1\} = A, \ldots, A\{r+1\}$. Let $S\{i\}$ be the set isomorphism classes of simple $A\{i\}$-modules. Consider subsets $I\{i\} \subset S\{i\}$, signs $\varepsilon_i = \pm$ and equivalences $F_i : D^b(A\{i\}) \xrightarrow{\sim} D^b(A\{i+1\})$ perverse for $(0 \subset I\{i\} \subset S\{i\}, {}^{0\mapsto 0}_{1\mapsto -\varepsilon_i})$, given $1 \le i \le r$. The equivalence

F_i provides a bijection $S\{i\} \xrightarrow{\sim} S\{i+1\}$. This provides us with bijections $S \xrightarrow{\sim} S\{i\}$ for all i. Let $T_V = (F_r \cdots F_1)^{-1}(A\{r+1\}_V)$.

$$\begin{tikzcd}
D^b(A) \arrow[r, "F_1"] & D^b(A_2) \arrow[r, "F_2"] & D^b(A_3) \arrow[r, "F_3"] & \cdots \arrow[r, "F_{r-1}"] & D^b(A_r) \arrow[r, "F_r"] & D^b(A_{r+1}) \\
T^{\varepsilon_1}_{A,V}(I_1) \arrow[r, mapsto] & A\{2\}_V & & & & \\
& T^{\varepsilon_2}_{A\{2\},V}(I_2) \arrow[r, mapsto] & A\{3\}_V & & & \\
& & & & T^{\varepsilon_r}_{A\{r\},V}(I_r) \arrow[r, mapsto] & A\{r+1\}_V \\
T_V \arrow[rrrrr, mapsto] & & & & & A\{r+1\}_V
\end{tikzcd}$$

We have

$$(T_V)_V = I_r^{\varepsilon_r} \cdots I_1^{\varepsilon_1}((A_V)_V).$$

Proposition 5.5. *The action of* $\mathrm{Free}(\mathcal{P}'(S)) \rtimes \mathfrak{S}(S)$ *on* $\mathcal{E}$ *factors through an action of* Γ.

Consider a filtration $\emptyset \subset I_0 \subset \cdots \subset I_r = S$ *and a map* $p : \{0,\ldots,r\} \to \mathbf{Z}$. *Let*

$$(T_V)_V = I_{r-1}^{p(r-1)-p(r)} \cdots I_0^{p(0)-p(1)} \emptyset^{-p(0)}((A_V)_V)$$

and $B = \mathrm{End}_{D^b(A)}(\bigoplus_V T_V)$. *Then,* $\mathrm{Hom}^\bullet_A(\bigoplus T_V, -) : D^b(A) \xrightarrow{\sim} D^b(B)$ *is perverse with respect to* $I_\bullet$ *and* p.

Proof. Let $I \subset J \subset S$. Let $B_1 = \mathrm{End}_A(\bigoplus_V (I \cdot A)_V)$, $B_2 = \mathrm{End}_A(\bigoplus_V (JI \cdot A)_V)$, $B_1' = \mathrm{End}_A(\bigoplus_V (J \cdot A)_V)$ and $B_2' = \mathrm{End}_A(\bigoplus_V (IJ \cdot A)_V)$. The composite canonical equivalences $D^b(A) \xrightarrow{\sim} D^b(B_1) \xrightarrow{\sim} D^b(B_2)$ and $D^b(A) \xrightarrow{\sim} D^b(B_1') \xrightarrow{\sim} D^b(B_2')$ are perverse relative to $(\emptyset \subset I \subset J \subset S, p)$, where $p(0) = 0$, $p(1) = -1$ and $p(2) = -2$ (Lemma 4.4). We deduce from Proposition 4.17 that the induced equivalence $D^b(B_2) \xrightarrow{\sim} D^b(B_2')$ restricts to an equivalence B_2-mod $\xrightarrow{\sim} B_2'$-mod and $IJ \cdot A = JI \cdot A$ (note that actually $B_2 \xrightarrow{\sim} B_2'$, since both are basic algebras).

$$\begin{tikzcd}
(JIA)_V \arrow[r, mapsto] & (JB_1)_V \arrow[r, mapsto] & (B_2)_V \arrow[ddd, bend left] \\
D^b(A) \arrow[r, "\sim"] \arrow[d, "\mathrm{id}"'] & D^b(B_1) \arrow[r, "\sim"] & D^b(B_2) \arrow[d, "\sim"] \\
D^b(A) \arrow[r, "\sim"'] & D^b(B_1') \arrow[r, "\sim"'] & D^b(B_2') \\
(IJA)_V \arrow[r, mapsto] & (IB_1')_V \arrow[r, mapsto] & (B_2')_V
\end{tikzcd}$$

Consider now $T \in \mathcal{E}$. Let $B = \mathrm{End}_{D^b(A)}(\bigoplus_V T_V)$. The discussion above shows that $IJ(B) = JI(B)$, hence $IJ(T) = JI(T)$.

The second part of the proposition follows from the construction preceding the proposition. $\square$

Note that the action of $\mathrm{Aut}(D^b(A))$ on $\mathcal{E}$ commutes with the action of Γ. It would be very interesting to understand the structure of those actions. Of particular interest is the orbit of the canonical element $A \in \mathcal{E}$. We might hope, for particular classes of algebras, to obtain Garside-type structures.

Remark 5.6. All the constructions and results of §5.1–5.2 hold for selfinjective algebras, under the assumption that the filtrations of the set of simple modules are stable under the Nakayama automorphism.

5.3. **Decreasing perversities.** Consider a filtration $\emptyset = I_{-1} \subset I_0 \subset \cdots\cdots \subset I_{r-1} \subset I_r = S$.

Given $i \in \{0,\ldots,r\}$ and $V \in I_i - I_{i-1}$, we construct T_V as a complex with nonzero terms in degrees $-r,\ldots,-i$, as follows. Put $T_V^{-i} = A_V$. Having constructed T_V^{-j}, let M be the smallest submodule of $K = \ker(d : T_V^{-j} \to T_V^{1-j})$ such that all composition factors of K/M lie in I_j. Define $d : T_V^{-j-1} \to T_V^{-j}$ be the composition of a projective cover $T_V^{-j-1} \to M$ with the inclusion of M into T_V^{-j}.

Proposition 5.7. *The complex $T = \bigoplus_{V\in S} T_V$ is tilting and the equivalence $F = \mathrm{Hom}^\bullet_A(T,-) : D^b(A) \xrightarrow{\sim} D^b(\mathrm{End}_{\mathrm{Ho}^b(A)}(T))$ is perverse relative to $I_\bullet$ and p given by $p(i) = -i$.*

Proof. Note that by construction

- T^{-j} is a direct sum of modules A_W with $W \notin I_{j-1}$ and
- the composition factors of $H^{-j}T$ are in I_j.

We deduce from Lemma 5.8 below that $\mathrm{Hom}_{\mathrm{Ho}(A)}(T,T[n]) = 0$ for $n > 0$. Since A is a symmetric algebra, the identity functor is a Serre functor for $\mathrm{Ho}^b(A\text{-proj})$, hence $\mathrm{Hom}_{\mathrm{Ho}(A)}(T,T[n])^* \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ho}(A)}(T[n],T)$. As a consequence, $\mathrm{Hom}_{\mathrm{Ho}(A)}(T,T[n]) = 0$ for $n \neq 0$.

Let $\mathcal{T}$ be the full triangulated subcategory of $\mathrm{Ho}^b(A\text{-proj})$ generated by $\{T_V\}_{V\in S}$. We show by descending induction on i that $A_V \in \mathcal{T}$ if $V \in I_i - I_{i-1}$.

Let $V \in I_i - I_{i-1}$. There is a distinguished triangle $A_V \to T_V[-i] \to U \rightsquigarrow$, where U is a bounded complex whose terms are direct sums of modules A_W with $W \in S - I_i$. By induction, $U \in \mathcal{T}$, hence $A_V \in \mathcal{T}$.

We have shown that $\mathcal{T} = \mathrm{Ho}^b(A\text{-proj})$ and we deduce that T is a tilting complex.

Let $\mathcal{E} = \mathrm{Ho}^b(A\text{-proj})$, let $B = \mathrm{End}_{\mathrm{Ho}(A)}(T)$ and $\mathcal{E}' = \mathrm{Ho}^b(B\text{-proj})$. Let $\mathcal{E}_i$ be the additive subcategory of $\mathcal{E}$ generated by the modules A_V with $V \in S - I_{r-i-1}$. Given $V \in S$, then $F(T_V)$ is isomorphic to a projective indecomposable B-module whose simple quotient we denote by V'. This defines a bijection $V \mapsto V'$ from the set of simple A-modules to the set of simple B-modules (taken up to isomorphism).

Let $\mathcal{E}'_i$ be the additive subcategory of $\mathcal{E}'$ generated by the B-modules $B_{V'} \simeq F(T_V)$ for $V \in S - I_{r-i-1}$. Consider $V \in S$ such that $B_{V'} \in \mathcal{E}'_i - \mathcal{E}'_{i-1}$, *i.e.*, $V \in I_{r-i} - I_{r-i-1}$. Given $n \neq \bar{p}(i)$, we have $T_V^n \in \mathcal{E}_{i-1}$, since $\bar{p}(i) = p(r-i) = i - r$. Also, we have $T_V^{i-r} = A_V$. We deduce from Lemma 4.8 that F is perverse. □

The following lemma is classical.

Lemma 5.8. *Let C be a bounded complex of projective A-modules and D a bounded complex of A-modules. If $\mathrm{Hom}_A(C^i, H^i(D)) = 0$ for all i, then $\mathrm{Hom}_{D(A)}(C,D) = 0$.*

Proof. We proceed by induction on the number of n such that $H^n(D) \neq 0$. Fix d maximal such that $H^d(D) \neq 0$. We have a distinguished triangle $\tau_{<d}D \to D \to H^d(D)[d] \rightsquigarrow$, hence an exact sequence

$$\mathrm{Hom}_{D(A)}(C,\tau_{<d}D) \to \mathrm{Hom}_{D(A)}(C,D) \to \mathrm{Hom}_{D(A)}(C,H^d(D)[d])$$

Since $\mathrm{Hom}_{\mathrm{Ho}(A)}(C,H^d(D)) = 0$, we deduce that $\mathrm{Hom}_{D(A)}(C,H^d(D)[d]) = 0$, hence by induction $\mathrm{Hom}_{D(A)}(C,D) = 0$. □

Given $V \in I_i - I_{i-1}$, we construct a complex Y_V with terms in degrees $-i, \ldots, 0$. If $i = 0$, we put $Y_V = V$. Otherwise start by putting $Y_V^{-i} = A_V$. Next, define $d : Y_V^{-i} \to Y_V^{1-i}$ to be the composition of the quotient map $A_V \to A_V/V^{I_{i-1}}$ with an injective hull $A_V/V^{I_{i-1}} \to Y_V^{1-i}$.

Having constructed Y_V^{1-j}, where $1-i \le 1-j \le -2$, let N be the largest quotient of $C = \operatorname{coker}(d : Y_V^{-j} \to Y_V^{1-j})$ such that all composition factors of $\ker(C \to N)$ are in I_{j-2}. Then let $d : Y_V^{1-j} \to Y_V^{2-j}$ be the composition of the projection $Y_V^{1-j} \to N$ with an injective hull $N \to Y_V^{2-j}$. When $1-j = -1$ the construction is the same, except that we do not compose with the injective hull, so that $Y_V^0 = N$. Note that $\ker d^{-j} = (\operatorname{im} d^{-j-1})^{I_{j-1}}$ for $j \neq i$ and $\ker d^{-i} = V^{I_{i-1}}$. Note also that

- $\operatorname{soc} Y_V^{-l}$ has no constituent in I_l for $l < i$
- the composition factors of $H^{-l} Y_V$ are in I_{l-1} for $l < i$.

Lemma 5.9. *We have $Y_V \simeq F^{-1}(V')$ for $V \in S$.*

Proof. The lemma can be deduced from Proposition 3.44. We apply the proposition to $\mathcal{T} = D^b(A)^{\mathrm{opp}}$, t the standard t-structure and t' the image by F^{-1} of the standard t-structure on $D^b(B)^{\mathrm{opp}}$. The perversity function is $-p$. We denote by $\mathcal{A}_i$ (resp. $\mathcal{A}'_i$) the Serre subcategory of A-mod (resp. B-mod) whose object have composition factors in I_i (resp. in $\{V'\}_{V \in I_i}$).

Let $i \in \{1, \ldots, r\}$ and $j = i - 1$. Let $V \in I_i - I_{i-1}$. There is a sequence $U_1 = V, U_2, \ldots, U_{i+1} = F^{-1}(V')$ of objects of $D^b(A)$ such that $U_{l+1}[-1]$ is the maximal $\mathcal{A}_{i-l}$-extension by U_l for $l \ge 1$.

Lemma 3.31 shows that $U_2[-1] \simeq V^{I_{i-1}} \simeq H^{-i} Y_V$. Let us assume that $U_l \simeq (\tau_{\le l-i-2} Y_V)[l-i-1]$ for some l with $2 \le l \le i$. Let $L \in (A\text{-mod})_{i-l}$. We have $H^{l-i-1} Y_V \in \mathcal{A}_{i-l}$. We have

$$\operatorname{Hom}(L, (\tau_{\le l-i-1} Y_V)[l-i-1]) = \operatorname{Hom}(L, (\tau_{\le l-i-1} Y_V)[l-i]) = 0$$

since neither $\operatorname{soc} Y_V^{l-i-1}$ nor $\operatorname{soc} Y_V^{l-i}$ have constituents in I_{i-l}. We deduce from Lemma 3.30 that $U_{l+1} \simeq (\tau_{\le l-i-1} Y_V)[l-i]$. It follows by induction that $Y_V \simeq F^{-1}(V')$.

Let us give now a direct proof of the lemma.

As in the proof of Lemma 5.2, we need to check that given $V, W \in S$, we have $\operatorname{Hom}(T_W, Y_V[n]) = \delta_{n0}\delta_{VW} K$ for some skewfield K.

Assume fist $V \in I_0$. We have $\operatorname{Hom}_A(T_W^{-l}, V) = 0$ for $l \neq 0$. If $W \notin I_0$, then $T_W^0 = 0$. If $W \in I_0$, then $T_W^0 = A_W$. We deduce that $\operatorname{Hom}(T_W, Y_V[n]) = \delta_{n0}\delta_{VW} \operatorname{End}_A(V)$ for all W.

Let $V \in I_i - I_{i-1}$ with $i > 0$. Note that the composition factors of $H^{-l} Y_V$ are in I_l for all l. Since T_W^{-m} is a sum of A_W's with $W \notin I_{m-1}$, we deduce that $\operatorname{Hom}_A(T_W^{-m}, H^{-l}(Y_V)) = 0$ whenever $m > l$. It follows that $\operatorname{Hom}(T_W, Y_V[n]) = 0$ for $n > 0$ (cf Lemma 5.8).

Given $l \in \{1, \ldots, i-1\}$, the A-module Y_V^{-l} is a sum of A_W's with $W \notin I_l$, while $Y_V^{-i} = A_V$. Let $\bar{Y}_V = \sigma_{\le -1} Y_V = 0 \to Y_V^{-i} \to \cdots \to Y_V^{-1} \to 0$ be the stupid truncation of Y_V. Given any l, then $\bar{Y}_V^{-l}$ is a sum of A_W's with $W \notin I_{l-1}$. Lemma 5.8 shows that $\operatorname{Hom}_{\mathrm{Ho}(A)}(\bar{Y}_V, T_W[n]) = 0$ for $n > 0$. Given $n > 0$, the complex $T_W[n]$ has all its terms in negative degrees, hence $\operatorname{Hom}_{\mathrm{Ho}(A)}(Y_V^0, T_W[n]) = 0$. We deduce that $\operatorname{Hom}_{\mathrm{Ho}(A)}(Y_V, T_W[n]) = 0$ for $n > 0$, hence $\operatorname{Hom}_{\mathrm{Ho}(A)}(T_W, Y_V[n]) = 0$ for $n < 0$, since A is symmetric.

The discussion in the first part of the proof shows that $\operatorname{Hom}_{\mathrm{Ho}(A)}(T_W, \tau_{\ge 1-i} Y_V) = 0$, since the composition factors of $H^{-l}(\tau_{\ge 1-i} Y_V)$ are in I_{l-1}. So, any morphism $g : T_W \to Y_V$ factors through $g' : T_W \to (H^{-i} Y_V)[-i]$. Since T_W^{-i} has no quotient in I_{i-1}, we deduce that such a morphism factors as $T_W^{-i} \xrightarrow{\mathrm{can}} \operatorname{coker} d_{T_W}^{-1-i} \xrightarrow{f} V \hookrightarrow V^{I_{i-1}}$.

Assume $W \in I_{i-1}$, so that $T_W^{1-i} \neq 0$. Then $\ker d_{TW}^{-i} \subset J(T_W^{-i})$, since T_W is indecomposable. We deduce that f factors through a map $T_W^{-i}/\ker d_{T_W}^{-i} \to V$. Since Y_V^{-i} is injective, we deduce that g factors through $h : \tau_{\ge 1-i} T_W \to Y_V$.

Let $\tilde{Y}_V$ be the complex with $\tilde{Y}_V^l = Y_V^l$ and $d_{\tilde{Y}_V}^{l-1} = d_{Y_V}^{l-1}$ for $l \neq 0$ and where $\tilde{Y}_V^0 = I_{Y_V^0}$ and $d_{\tilde{Y}_V}^{-1}$ is the composition of $d_{Y_V}^{-1}$ with an injection $Y_V^0 \hookrightarrow I_{Y_V^0}$. The distinguished triangle $I_N/N[-1] \to Y_V \to \tilde{Y}_V \rightsquigarrow$ induces an injective map

$$\mathrm{Hom}_{\mathrm{Ho}(A)}(\tau_{\geq 1-i}T_W, Y_V) \hookrightarrow \mathrm{Hom}_{\mathrm{Ho}(A)}(\tau_{\geq 1-i}T_W, \tilde{Y}_V).$$

The composition factors of $H^{-l}(\tau_{\geq 1-i}T_W)$ are in I_l, while $\tilde{Y}_V^{-l}$ is a sum of A_U's with $U \notin I_l$ if $l \leq 1-i$, hence

$$\mathrm{Hom}_{\mathrm{Ho}(A)}(\tau_{\geq 1-i}T_W, \tilde{Y}_V) \simeq \mathrm{Hom}_{\mathrm{Ho}(A)}(\tilde{Y}_V, \tau_{\geq 1-i}T_W)^* = 0$$

by Lemma 5.8. Consequently, $\mathrm{Hom}_{\mathrm{Ho}(A)}(\tau_{\geq 1-i}T_W, Y_V) = 0$, hence $h = 0$ and finally $g = 0$, *i.e.*, $\mathrm{Hom}_{\mathrm{Ho}(A)}(T_W, Y_V) = 0$.

If $W \notin I_i$, then $T_W^{-l} = 0$ for $l \leq i$, hence $\mathrm{Hom}_{\mathrm{Ho}(A)}(T_W, Y_V) = 0$. Assume finally $W \in I_i - I_{i-1}$. We have $T_W^{-l} = 0$ for $l < i$ and $T_W^{-i} = A_W$. It follows that $\mathrm{Hom}_{\mathrm{Ho}(A)}(T_W, Y_V) = \delta_{VW}\,\mathrm{End}_A(V)$. $\square$

Let $F : D^b(A) \xrightarrow{\sim} D^b(B)$ be a perverse equivalence between finite-dimensional symmetric algebras, and suppose that p is weakly decreasing. We may assume that $p(i) = -i$, replacing F by a shift if necessary (cf Lemma 4.23). Then by Proposition 5.7, we have $T_V = F^{-1}(B_{V'}) = I_{r-1} \cdots I_0 \cdot A_V$.

We have a converse statement.

Proposition 5.10. *Let A be a finite-dimensional symmetric k-algebra. Let X be a bounded complex of finitely generated projective A-modules such that X generates* A-perf *and* $\mathrm{Hom}_A(X^i, H^j(X)) = 0$ *for all $i < j$. Then X is a tilting complex and $G = \mathrm{Hom}^\bullet_A(X, -) : D^b(A) \xrightarrow{\sim} D^b(\mathrm{End}_{\mathrm{Ho}(A)}(X))$ is perverse with respect to a weakly decreasing perversity p.*

Proof. As in the proof of Proposition 5.7, we have $\mathrm{Hom}_{\mathrm{Ho}(A)}(X, X[n]) = 0$ for $n \neq 0$, hence X is a tilting complex. We may assume that $X^i = 0$ for $i > 0$ and $H^0(X) \neq 0$, replacing X by a shift of a homotopy equivalent complex if necessary.

Let I_i be the set of simple A-modules V such that $\mathrm{Hom}_A(X^{-j}, V) = 0$ for all $j \geq i$. Let T be the tilting complex constructed at the beginning of §5.3. The proof of Proposition 5.7 shows that $\mathrm{Hom}_{\mathrm{Ho}^b(A)}(T, X[n]) = 0$ for $n \neq 0$. let $F = \mathrm{Hom}^\bullet_A(T, -) : D^b(A) \xrightarrow{\sim} D^b(B))$ be the corresponding perverse derived equivalence where $B = \mathrm{End}_{\mathrm{Ho}(A)}(T)$ (cf Proposition 5.7). We have $H^n(GF^{-1}(V)) = 0$ for $n \neq 0$, and it follows that GF^{-1} restricts to an equivalence B-mod $\xrightarrow{\sim} \mathrm{End}_{\mathrm{Ho}(A)}(X)$-mod. We deduce that G, like F, is a perverse equivalence with perversity function $p(i) = -i$. $\square$

5.4. **Some relations.** Proposition 5.5 shows that for $I \subset J \subset S$ the relation $IJ \cdot A = JI \cdot A$ holds for all algebras A. Other relations may hold for particular algebras and their existence can be translated into properties of A. The first examples are braid relations.

For any subset $K \subset S$ denote by $\mathcal{E}_K$ be the additive subcategory of $\mathcal{E} = A$-proj generated by A_V for all $V \in S - K$ (c.f. §4.2.4).

Proposition 5.11. *Let I and J be subsets of S. Then the following are equivalent:*

1. *$IJ \cdot A = JI \cdot A$*
2. *For $P \in \mathcal{E}_J$ and $Q \in \mathcal{E}_I$, every homomorphism $P \to Q$ and every homomorphism $Q \to P$ factors through a module in $\mathcal{E}_{I \cup J}$.*

Moreover, if either statement holds, then $IJ \cdot A = JI \cdot A = (I \cup J)(I \cap J) \cdot A$ and in particular the canonical equivalence $D^b(A) \xrightarrow{\sim} D^b(B)$, $B = \mathrm{End}_{D^b(A)}(IJ \cdot A)$ is perverse.

Proof. If $T = \oplus_V T_V$ is a tilting complex of A-modules and $K \subset S$ then, by definition, $(K\cdot T)_V = T_V[1]$ if $V \in S - K$ and $(K \cdot T)_V = \mathrm{cone}(f)$ for some $f \in \mathrm{Hom}_{\mathcal{E}}(X, T_V)$, where $X \in \mathrm{add}(\oplus_{W\in S-K} T_W)$, if $V \in K$. We deduce the following shapes for complexes representing $(I \cdot A)_V$ and $(JI \cdot A)_V$:

$V \in$	$(I \cdot A)_V$	$(JI \cdot A)_V$
$S - (I \cup J)$	$A_V \to 0$	$A_V \to 0 \to 0$
$I \cap (S - J)$	$\mathcal{E}_I \to A_V$	$\mathcal{E}_I \to A_V \to 0$
$J \cap (S - I)$	$A_V \to 0$	$\mathcal{E}_I \to A_V \oplus \mathcal{E}_{J\cup(S-I)} \to 0$
$I \cap J$	$\mathcal{E}_I \to A_V$	$\mathcal{E}_I \to \mathcal{E}_{I\cap J} \to A_V$

All missing terms and arrows are zero, the rightmost written terms are in degree 0 and, abusing notation, $\mathcal{E}_K$ stands for an object in $\mathcal{E}_K$.

Suppose that (1) of holds. Then $((JI \cdot A)_{\min})^{-2} = ((IJ \cdot A)_{\min})^{-2} \in \mathcal{E}_I \cap \mathcal{E}_J = \mathcal{E}_{I\cup J}$. To show that (2) is true, it suffices to consider $Q = A_V$ and $P = A_W$, with $V \in I\cap(S-J)$ and $W \in J\cap(S-I)$ (note that the roles of I and J could be interchanged). Then $(JI \cdot A)_V = T_{A,V}(I)[1]$, and $(T_{A,V}(I))^{-1} = Q_V \in \mathcal{E}_{I\cup J}$. It follows that any map from A_W to A_V factors through $Q_V \in \mathcal{E}_{I\cup J}$.

Conversely suppose that (2) holds. Then for all $V \in I\cap(S-J)$, we have $T_{A,V}(I)^{-1} \in \mathcal{E}_{I\cup J}$. It follows from Lemma 4.8 that the canonical equivalence $\mathrm{Ho}^b(A'\text{-proj}) \xrightarrow{\sim} \mathrm{Ho}^b(A\text{-proj})$, $A' = \mathrm{End}_{\mathrm{Ho}^b(A\text{-proj})}(T_A(I))$, is perverse relative to $(0 \subset \mathcal{E}_{I\cup J} \subset \mathcal{E}_J \subset \mathcal{E}_{I\cap J} \subset \mathcal{E} = A\text{-proj}, q)$, where $q(0) = 1, q(1) = 0, q(2) = 1, q(3) = 0$. So by Proposition 4.21 and Lemma 4.2 the canonical equivalence $D^b(A) \to D^b(A')$ is perverse relative to $(\emptyset \subset I\cap J \subset J \subset I\cup J \subset S, p)$, where $p(0) = 0, p(1) = -1, p(2) = 0, p(3) = -1$. Using Proposition 5.5 we conclude that $I \cdot A = (I \cup J)J^{-1}(I \cap J) \cdot A$ and then that $JI \cdot A = (I \cup J)(I \cap J) \cdot A$. The same argument with the roles of I and J reversed shows that $IJ \cdot A = (I \cup J)(I \cap J) \cdot A$ as well.

$V \in$	$(I \cdot A)_V$	$(JI \cdot A)_V = (IJ \cdot A)_V$
$S - (I \cup J)$	$A_V \to 0$	$A_V \to 0 \to 0$
$I \cap (S - J)$	$\mathcal{E}_{I\cup J} \to A_V$	$\mathcal{E}_{I\cup J} \to A_V \to 0$
$J \cap (S - I)$	$A_V \to 0$	$\mathcal{E}_{I\cup J} \to A_V \to 0$
$I \cap J$	$\mathcal{E}_I \to A_V$	$\mathcal{E}_{I\cup J} \to \mathcal{E}_{I\cap J} \to A_V$

□

Proposition 5.12. *Let I and J be subsets of S. Then the following are equivalent:*

(1) $JIJ \cdot A = IJI \cdot A$

(2) *There exists an involution $\sigma : S \xrightarrow{\sim} S$, fixing $(I \cap J) \cup (S - (I \cup J))$ and inducing a bijection of $I \cap (S - J)$ with $J \cap (S - I)$, and, for each $V \in (I \cup J) - (I \cap J)$, a nonzero morphism $f_V \in \mathrm{Hom}_{\mathcal{E}/\mathcal{E}_{I\cup J}}(A_V, A_{\sigma(V)})$, such that the following property holds: If $V \in I \cap (S - J)$ and $W \in J \cap (S - I)$, or vice versa, then any morphism $A_V \to A_W$ in $\mathcal{E}/\mathcal{E}_{I\cup J}$ factors (1) through $f_{\sigma(W)}$ and (2) through f_V.*

Moreover, if either statement holds, then $JIJ \cdot A = IJI \cdot A = \sigma.(I \cup J)(I \cap J)^2 \cdot A$, and in particular the canonical equivalence $D^b(A) \xrightarrow{\sim} D^b(B)$, $B = \mathrm{End}_{D^b(A)}(IJI \cdot A)$ is perverse.

Proof. We have the following shapes for complexes representing $(I \cdot A)_V$, $(JI \cdot A)_V$ and $(IJI \cdot A)_V$:

$V \in$	$(I \cdot A)_V$	$(JI \cdot A)_V$	$(IJI \cdot A)_V$
$S - (I \cup J)$	$A_V \to 0$	$A_V \to 0 \to 0$	$A_V \to 0 \to 0 \to 0$
$I \cap (S - J)$	$\mathcal{E}_I \to A_V$	$\mathcal{E}_I \to A_V \to 0$	$\mathcal{E}_I \to \mathcal{E}_{I\cap J} \to A_V \to 0$
$J \cap (S - I)$	$A_V \to 0$	$\mathcal{E}_I \to A_V \oplus \mathcal{E}_{J\cup(S-I)} \to 0$	$\mathcal{E}_I \to A_V \oplus \mathcal{E}_{J\cup(S-I)} \to 0 \to 0$
$I \cap J$	$\mathcal{E}_I \to A_V$	$\mathcal{E}_I \to \mathcal{E}_{I\cap J} \to A_V$	$\mathcal{E}_I \to \mathcal{E}_{I\cap J} \to \mathcal{E}_{I\cap J} \to A_V$

Suppose that (1) holds. Then $((IJI \cdot A)_{\min})^{-3} = ((JIJ \cdot A)_{\min})^{-3} \in \mathcal{E}_{I\cup J}$. Let $V \in J \cap (S - I)$. Then $(IJI \cdot A)_{V,\min} = (JI \cdot A)_{V,\min}[1]$, and so $((JI \cdot A)_{V,\min})^{-2} \in \mathcal{E}_{I\cup J}$. Now $(JI \cdot A)_{V,\min}$

is homotopy equivalent to the cone of the universal map $X_V \to T_{A,V}(I)$, from a complex $X_V \in \mathrm{add}(\oplus_{W\in S-J} T_{A,W}(I))$ to $T_{A,V}(I) = A_V[1]$. From the restriction on $((JI\cdot A)_{V,\min})^{-2}$ obtained above, we have that $(X_V)^{-1}$ has at most one indecomposable summand A_U with $U \in J\cap(S-I)$ and the only such summand appearing is A_V. On the other hand, if $W \in I\cap(S-J)$ and $T_{A,W}(I)$ is involved in X_V, then $T_{A,W}(I)^{-1}$ contains at least one such summand, for otherwise $T_{A,W}(I)^{-1} \in \mathcal{E}_{I\cup J}$ and then every map $T_{A,W}(I) \to T_{A,V}(I)$ would factor through $\mathrm{add}(\oplus_{U\in S-(I\cup J)} T_{A,U}(I))$, contradicting the assumption that $T_{A,W}(I)$ is a summand of X_V. We deduce that either

(a) $X_V \in \mathrm{add}(\oplus_{W\in S-(I\cup J)} T_{A,W}(I)) = \mathrm{add}(\oplus_{W\in S-(I\cup J)} A_W[1])$; or

(b) There exists $\sigma_1(V) \in I\cap(S-J)$ such that $X_V = T_{A,\sigma_1(V)}(I)\oplus X'_V$, with $X'_V \in \mathrm{add}(\oplus_{W\in S-(I\cup J)} A_W[1])$, and $T_{A,\sigma_1(V)}(I) = \mathrm{cone}(A_V \oplus P_V \xrightarrow{(\tilde{f}_V, g_V)} A_{\sigma_1(V)})$, for some $P_V \in \mathcal{E}_{I\cup J}$.

In case (a), $((JI\cdot A)_{V,\min})^{-1} = A_V$, whereas in case (b), $((JI\cdot A)_{V,\min})^{-1} = A_{\sigma_1(V)}$.

Let $U \in I\cap(S-J)$. Then $((IJI\cdot A)_{U,\min})^{-1} = A_U$ or 0 and $((JIJ\cdot A)_{U,\min})^{-1} \in \mathcal{E}_{I\cup(S-J)}$, hence $((IJI\cdot A)_{U,\min})^{-1} = ((JIJ\cdot A)_{U,\min})^{-1} = 0$. On the other hand $(IJI\cdot A)_U$ is the cone of a map $Y_U \to (JI\cdot A)_U$, where $Y_U \in \mathrm{add}(\oplus_{W\in S-I}(JI\cdot A)_W)$. Since $((JI\cdot A)_{U,\min})^{-1} = A_U$, there exists $V \in J\cap(S-I)$ such that $\sigma_1(V) = U$. We have $Y_U = (JI\cdot A)_V \oplus Y'_U$, with $Y'_U \in \mathrm{add}(\oplus_{W\in S-(I\cup J)}(JI\cdot A)_W)$, since $((IJI\cdot A)_{U,\min})^{-2} \in \mathcal{E}_I$.

Reversing the roles of I and J in the argument above, we arrive at a partially defined map $\sigma_2 : I\cap(S-J) \to J\cap(S-I)$ in an analogous way. Since both σ_1 and σ_2 are surjective, we see that case (a) never occurs and σ_1 and σ_2 are inverse bijections.

Let $\sigma : S \xrightarrow{\sim} S$ be the automorphism extending σ_1 and σ_2 by the identity on $(I\cap J)\cup(S-(I\cup J))$. We have now a new table giving the shapes of the complexes at the end of the proof, and we use this information for the remainder of the proof of (1)⇒(2).

For $V \in (I\cup J)-(I\cap J)$, let f_V be the image of $\tilde{f}_V$ in $\mathrm{Hom}_{\mathcal{E}/\mathcal{E}_{I\cup J}}(A_V, A_{\sigma(V)})$. Note that $f_V \neq 0$.

Let $U, V \in J\cap(S-I)$. The fact that any morphism $A_U \to A_{\sigma(V)}$ in $\mathcal{E}/\mathcal{E}_{I\cup J}$ factors through f_V follows from $\mathrm{Hom}_{\mathrm{Ho}^b(\mathcal{E})}(T_{A,U}(I)[-1], T_{A,\sigma(V)}(I)) = 0$. Likewise, for any $U \in I\cap(S-J)$ and $V \in I\cap(S-J)$, we have $\mathrm{Hom}_{\mathrm{Ho}^b(\mathcal{E})}((JI\cdot A)_{\sigma(V)}, (JI\cdot A)_U[1]) = 0$, which implies that any morphism $A_V \to A_{\sigma(U)}$ in $\mathcal{E}/\mathcal{E}_{I\cup J}$ factors through f_V. The arguments can be repeated with the roles of I and J interchanged.

Conversely suppose that (2) holds. We claim that $(I\cdot A)_V$ and $(JI\cdot A)_V$ may be represented by complexes with shapes given by the first two columns of the table below.

For the first column, the table gives the shape of $T_{A,V}(I)$, valid for any algebra A, except when $V \in I\cap(S-J)$. In that case the first factorisation property of $f_{\sigma(V)}$ shows that $T_{A,V}(I)$ is the cone of a map

$$g^+_{\sigma(V)} = (\tilde{f}_{\sigma(V)}, g_{\sigma(V)}) : A_{\sigma(V)} \oplus R_{\sigma(V)} \to A_V,$$

where $\tilde{f}_{\sigma(V)}$ is a lift of $f_{\sigma(V)}$ and $R_{\sigma(V)} \in \mathcal{E}_{I\cup J}$.

We now continue with the second column of the table, concerning $(JI\cdot A)_V$. Let $V \in J\cap(S-I)$; the other cases are easy. Consider $T_{A,\sigma(V)}(I\cup J)$. By definition it is the cone of a universal map $h_V : Q_V \to A_{\sigma(V)}$ with $Q_V \in \mathcal{E}_{I\cup J}$. So the map $g_V : R_V \to A_{\sigma(V)}$ constructed above factors through h_V. It follows that the image of

$$g^+_V = (\tilde{f}_V, g_V) : A_V \oplus R_V \to A_{\sigma(V)}$$

is contained in the image of

$$h^+_V = (\tilde{f}_V, h_V) : A_V \oplus Q_V \to A_{\sigma(V)}.$$

In fact the images are equal, as g^+_V is universal for maps from $\mathcal{E}_I$ to $A_{\sigma(V)}$, and we deduce in addition than $\mathrm{cone}(h^+_V) \cong \mathrm{cone}(g^+_V) \oplus E_V[1]$, for some $E_V \in \mathcal{E}_{I\cup J}$. Since $\mathrm{cone}(g^+_V) = T_{A,\sigma(V)}(I)$ and each indecomposable summand of $E_V[1]$ is isomorphic to $T_{A,W}(I)$ for some $W \in S-(I\cup J)$, we have $\mathrm{cone}(h_V^+) \in \mathrm{add}(\oplus_{W\in S-J} T_{A,W}(I))$.

Reinterpreting the cone of h_V^+ as the cone of a map $T_{A,V}(I)[-1] = A_V \to T_{A,\sigma(V)}(I\cup J)$, we obtain a distinguished triangle

$$\mathrm{cone}(h_V^+) \to T_{A,V}(I) \to T_{A,\sigma(V)}(I\cup J)[1] \rightsquigarrow .$$

In addition we see that $\mathrm{Hom}_{D^b(A)}(T_{A,W}(I), T_{A,\sigma(V)}(I\cup J)[1]) = 0$ for all $W \in S-J$, using the second factorisation property of $f_{\sigma(W)}$ for $W \in I\cap(S-J)$. We deduce that $(JI\cdot A)_V$ is represented by the complex $T_{A,\sigma(V)}(I\cup J)[1]$, which has the desired shape.

Having established the validity of the second column of the table below, we are in a position to use Lemma 4.8. It implies that the composition of canonical equivalences $\mathrm{Ho}^b(A''\text{-proj}) \xrightarrow{\sim} \mathrm{Ho}^b(A'\text{-proj}) \xrightarrow{\sim} \mathrm{Ho}^b(A\text{-proj})$, $A'' = \mathrm{End}_{\mathrm{Ho}^b}(JI\cdot A)$, $A' = \mathrm{End}_{\mathrm{Ho}^b}(I\cdot A)$, is perverse relative to $(0 \subset \mathcal{E}_{I\cup J} \subset \mathcal{E}_I \subset \mathcal{E}_{I\cap J} \subset \mathcal{E} = A\text{-proj}, q)$, where $q(0)=2, q(1)=1, q(2)=2, q(3)=0$. So by Lemmas 4.21 and 4.2 the composition of canonical equivalences $D^b(A'') \to D^b(A') \to D^b(A')$ is perverse relative to $(\emptyset \subset I\cap J \subset J \subset I\cup J \subset S, p)$, where $p(0)=0, p(1)=-2, p(2)=-1, p(3)=-2$. Using Proposition 5.5 we conclude that $JI\cdot A = \sigma.(I\cup J)J^{-1}(I\cap J)^2\cdot A$ and then that $IJI\cdot A = I\sigma.(I\cup J)J^{-1}(I\cap J)^2\cdot A = \sigma.(I\cup J)(I\cap J)^2\cdot A$. The same argument with the roles of I and J reversed shows that $JIJ\cdot A = \sigma.(I\cup J)(I\cap J)^2\cdot A$.

$V \in$	$(I\cdot A)_V$	$(JI\cdot A)_V$	$(IJI\cdot A)_V = (JIJ\cdot A)_V$
$S-(I\cup J)$	$A_V \to 0$	$A_V \to 0 \to 0$	$A_V \to 0 \to 0 \to 0$
$I\cap(S-J)$	$A_{\sigma(V)} \oplus \mathcal{E}_{I\cup J} \to A_V$	$A_{\sigma(V)} \oplus \mathcal{E}_{I\cup J} \to A_V \to 0$	$\mathcal{E}_{I\cup J} \to A_{\sigma(V)} \to 0 \to 0$
$J\cap(S-I)$	$A_V \to 0$	$\mathcal{E}_{I\cup J} \to A_{\sigma(V)} \to 0$	$\mathcal{E}_{I\cup J} \to A_{\sigma(V)} \to 0 \to 0$
$I\cap J$	$\mathcal{E}_I \to A_V$	$\mathcal{E}_I \to \mathcal{E}_{I\cap J} \to A_V$	$\mathcal{E}_{I\cup J} \to \mathcal{E}_{I\cap J} \to \mathcal{E}_{I\cap J} \to A_V$

□

Remark 5.13. The proof of Proposition 5.12 shows the following. Let I and J be subsets of S and $\sigma : S \xrightarrow{\sim} S$ an involution fixing $(I\cap J)\cup(S-(I\cup J))$ and inducing a bijection of $I\cap(S-J)$ with $J\cap(S-I)$. Given $V \in J\cap(S-I)$, assume there is a non-zero morphism $f_V \in \mathrm{Hom}_{\mathcal{E}/\mathcal{E}_{I\cup J}}(A_V, A_{\sigma(V)})$ such that

$$\begin{cases} \mathrm{Hom}_{\mathcal{E}/\mathcal{E}_{I\cup J}}(A_W, A_{\sigma(V)}) = f_V\, \mathrm{Hom}_{\mathcal{E}/\mathcal{E}_{I\cup J}}(A_W, A_V) \\ \mathrm{Hom}_{\mathcal{E}/\mathcal{E}_{I\cup J}}(A_V, A_W) = \mathrm{Hom}_{\mathcal{E}/\mathcal{E}_{I\cup J}}(A_{\sigma(V)}, A_W) f_V \end{cases} \quad \text{for all } W \in I\cap(S-J).$$

Then $JI\cdot A = \sigma.(I\cup J)J^{-1}(I\cap J)^2\cdot A$ and $IJI\cdot A = \sigma.(I\cup J)(I\cap J)^2\cdot A$.

6. Calabi-Yau algebras

6.1. **Isolated algebras.** Let k be a field and A a k-algebra. We denote by

- $A\text{-mod}_f$ the category of A-modules that are finite-dimensional over k;
- $A\text{-Mod}_{lf}$ the category of A-modules that are locally finite-dimensional over k (*i.e.*, union of their A-submodules that are in $A\text{-mod}_f$);
- $D^b_f(A)$ the full subcategory of $D(A\text{-Mod})$ of complexes whose total cohomology is finite-dimensional. This is the thick subcategory of $D(A\text{-Mod})$ generated by finite-dimensional simple A-modules;
- $D^b_{lf}(A)$ the full subcategory of $D(A\text{-Mod})$ of objects whose cohomology is locally finite.

When all objects of $A\text{-mod}_f$ are finitely presented, the category $A\text{-Mod}_{lf}$ is closed under extensions.

Lemma 6.1. *Assume that given any $L \in A\text{-mod}_f$, there is a projective A-module P and a surjection $P \twoheadrightarrow L$ whose kernel is a finitely generated A-module.*

Then the category $A\text{-Mod}_{lf}$ is a Serre subcategory of $A\text{-Mod}$.

Proof. Consider an exact sequence of A-modules $0 \to M_1 \to M \to M_2 \to 0$ with M_i locally finite for $i = 1, 2$. Let L be an A-submodule of M_2 that is finite-dimensional (over k) and let N be its inverse image in M. Let $g : P \twoheadrightarrow L$ be a surjective map as in the Lemma. There is a morphism $f : P \to N$ such that g is the composition of f with the canonical map $N \to L$. Since $\ker g$ is finitely generated, it follows that $f(\ker g) \subset M_1$ is finite-dimensional, hence $f(P)$ is finite-dimensional as well. We have $N = M_1 + f(P)$, hence N is locally finite.

We deduce that M is locally finite and the lemma follows. $\square$

We assume that A is noetherian and has finite global dimension, *i.e.*, the canonical functor $\mathrm{Ho}^b(A\text{-Proj}) \xrightarrow{\sim} D^b(A\text{-Mod})$ is an equivalence that restricts to an equivalence $\mathrm{Ho}^b(A\text{-proj}) \xrightarrow{\sim} D^b(A\text{-mod})$. We assume further that A is Calabi-Yau of dimension $d \geq 2$, *i.e.*, there is a bifunctorial isomorphism

$$\mathrm{Hom}_{D(A)}(C, D)^* \xrightarrow{\sim} \mathrm{Hom}_{D(A)}(D, C[d]) \text{ for } C \in A\text{-perf and } D \in D^b_f(A).$$

Note that the canonical functor $A\text{-mod}/A\text{-mod}_f \to A\text{-Mod}/A\text{-Mod}_{lf}$ is fully faithful with image an abelian subcategory and all objects of $A\text{-mod}/A\text{-mod}_f$ are noetherian.

Lemma 6.1 shows that $A\text{-Mod}_{lf}$ is a Serre subcategory of A-Mod, hence $D^b_{lf}(A)$ is a thick subcategory of $D^b(A\text{-Mod})$.

We denote by $Q : D^b(A\text{-Mod}) \to D^b(A\text{-Mod})/D^b_{lf}(A)$ the quotient functor. It follows from Lemma 3.9 that the standard t-structure on $D^b(A\text{-Mod})$ induces a t-structure on $D^b(A\text{-Mod})/D^b_{lf}(A)$ with heart $A\text{-Mod}/A\text{-Mod}_{lf}$. In particular, Q restricts to the quotient functor $A\text{-Mod} \to A\text{-Mod}/A\text{-Mod}_{lf}$. It also restricts to the quotient functor $D^b(A\text{-mod}) \to D^b(A\text{-mod})/D^b_f(A)$ and to the quotient functor $A\text{-mod} \to A\text{-mod}/A\text{-mod}_f$.

Definition 6.2. *We say that A is* isolated *if given M any non-zero submodule of a finitely generated free module, then M generates $D^b(A\text{-mod})/D^b_f(A)$.*

The motivation for the definition is the characterization of orbifolds corresponding to isolated singularities.

Proposition 6.3. *Let V be a finite dimensional vector space over k and G a finite subgroup of $\mathrm{GL}(V)$ with $|G| \in k^\times$. Let X be the complement of 0 in V (or in the formal completion of V at 0). We consider the bounded derived category $D^b_G(X)$ of G-equivariant coherent sheaves on X. The following conditions are equivalent*

(i) *every object of $D^b_G(X)$ with support X is a generator*
(ii) *$\mathcal{O}_X$ generates $D^b_G(X)$*
(iii) *G acts freely on $V - \{0\}$*

In particular, $k[[V]] \rtimes G$ is isolated if and only if G acts freely on $V - \{0\}$.

Proof. It is clear that (i)$\Rightarrow$(ii).

Assume (ii). Let l be a line in V fixed pointwise by a non-trivial subgroup H of G. Let M be a non-trivial simple kH-module. We have

$$\mathrm{Hom}_{D^b_G(X)}(\mathcal{O}, \mathrm{Ind}_H^G(\mathcal{O}_{X\cap l} \otimes M)[i]) \simeq \mathrm{Hom}_{D^b_H(X)}(\mathcal{O}, \mathcal{O}_{X\cap l} \otimes M[i]) = \delta_{0i}\Gamma(\mathcal{O}_{X\cap l} \otimes M)^H = 0$$

for all $i \in \mathbf{Z}$. We deduce from (ii) that $\mathrm{Ind}_H^G(\mathcal{O}_{X\cap l} \otimes M) = 0$, a contradiction. So, (ii)$\Rightarrow$(iii).

Assume (iii). The category of G-equivariant coherent sheaves on X is equivalent to the category of coherent sheaves on X/G. Since X/G is quasi-affine, $\mathcal{O}_{X/G}$ is ample, hence every object of $D^b(X/G)$ with support X/G is a generator [Th, Proposition 3.11 and Theorem 3.15]. This shows (i). $\square$

Lemma 6.4. *Let $P \in A\text{-proj}$ such that $Q(P)$ generates $D^b(A\text{-mod})/D^b_f(A)$.*

- *Given $L \in A\text{-mod}$ such that $\dim \mathrm{Hom}_A(P, L) < \infty$, we have $\dim L < \infty$.*

- *There is $n > 0$ and a surjective morphism $f : P^n \to A$ such that* $\dim \operatorname{coker} f < \infty$ *and* $\operatorname{Hom}_A(P, \operatorname{coker} f) = 0$.
- $Q(P)$ *generates* A-mod $/A$-mod$_f$
- *Consider an exact sequence of A-modules $0 \to N \to M \to L \to 0$ where N is finitely generated. Assume* $\dim \operatorname{Hom}_A(P, L) < \infty$ *and* $\operatorname{Hom}_A(P, V) \neq 0$ *whenever V is a non-zero finite-dimensional submodule of M. Then,* $\dim L < \infty$ *and M is finitely generated.*

Proof. Note that P together with $D^b_f(A)$ generate $D^b(A\text{-mod}) = A\text{-perf}$. As a consequence, A is a direct summand of an object of A-perf that is a finite extension of objects of $D^b_f(A)$ and of shifts of P.

Let $L \in A$-mod such that $\dim \operatorname{Hom}_A(P, L) < \infty$. Since $\operatorname{Hom}_{D(A)}(C, L) \simeq \operatorname{Hom}_{D(A)}(L, C[d])^*$ is finite-dimensional for all $C \in D^b_f(A)$, it follows that $L = \operatorname{Hom}_A(A, L)$ is finite-dimensional.

Let M be the sum of the images of A-module morphisms $P \to A$. Since A is noetherian, there is $n > 0$ and a morphism $f : P^n \to A$ with image M. We have $\operatorname{Hom}_A(P, A/M) = 0$, hence A/M is finite-dimensional. As a consequence, $Q(P)$ generates A-mod $/A$-mod$_f$.

Consider an exact sequence $0 \to N \to M \to L \to 0$ as in the lemma. Fix $f : P^n \to A$ as in the first statement of the lemma. We have $\operatorname{Hom}_A(\operatorname{coker} f, M) = 0$. Since A is d-Calabi-Yau, we have $\operatorname{Ext}^1_A(\operatorname{coker} f, N) \simeq \operatorname{Ext}^{d-1}_A(N, \operatorname{coker} f)^*$, a finite-dimensional k-vector space. It follows that $\operatorname{Hom}_A(\operatorname{coker} f, L)$ is finite-dimensional, hence $L = \operatorname{Hom}_A(A, L)$ is finite-dimensional. □

6.2. **Perverse equivalences.** Let A be a noetherian k-algebra of finite global dimension that is Calabi-Yau of dimension $d \geq 2$ and isolated.

Let Υ be a finite set of non-zero objects of A-proj whose sum is a progenerator. Let Ω be a subset of Υ.

• Let $P \in \Omega$. When $\Omega = \Upsilon$, we put $T_P = P$. Assume now $\Omega \neq \Upsilon$. Lemma 6.4 shows that there exists a finite direct sum P' of objects of $\Upsilon - \Omega$ and a map $f_P : P' \to P$ such that $\dim \operatorname{coker} f_P < \infty$ and $\operatorname{Hom}_A(R, \operatorname{coker} f_P) = 0$ for all $R \in \Upsilon - \Omega$. We put

$$T_P = 0 \to P' \xrightarrow{f_P} P \to 0,$$

a complex of A-proj with P in degree 0.

• Given $P \in \Upsilon - \Omega$, we put $T_P = P[1]$.

Let $T = \bigoplus_{P \in \Upsilon} T_P$ and let $B = \operatorname{End}_{D^b(A)}(T)$. Note that, while T is not unique as P' above is not unique, the algebra B is well defined up to Morita equivalence.

Lemma 6.5. *T is a tilting complex for A and $F = \operatorname{Hom}^\bullet_A(T, -)$ induces an equivalence $D(A\text{-Mod}) \xrightarrow{\sim} D(B\text{-Mod})$. Let $U \in B$-Mod and $X = F^{-1}(U)$. We have*

(i) *$H^i(X) = 0$ for $i \neq -1, 0$*
(ii) *$H^0(X)$ is locally finite and* $\operatorname{Hom}_A(P, H^0(X)) = 0$ *for $P \in \Upsilon - \Omega$*
(iii) *given V a non-zero finite-dimensional submodule of $H^{-1}(X)$, we have* $\operatorname{Hom}_A(P, V) \neq 0$ *for some $P \in \Upsilon - \Omega$.*

If $U \in B$-mod, then $H^0(X) \in A$-mod$_f$ and $H^{-1}(X) \in A$-mod.

Proof. It is immediate that T generates A-perf, since Υ generates A-perf. Let $P \in \Omega$ and $R \in \Upsilon - \Omega$. We have $\operatorname{Hom}_A(\operatorname{coker} f_P, R) \simeq \operatorname{Hom}_{D(A)}(R, \operatorname{coker} f_P[d])^* = 0$. We deduce that $\operatorname{Hom}_{\operatorname{Ho}(A)}(T_P, T_R[-1]) = 0$. Similarly, $\operatorname{Hom}_{\operatorname{Ho}(A)}(T_P, T_R[-1]) = 0$ for $P, R \in \Omega$. It follows easily that T is a tilting complex. Consequently, F induces an equivalence of (bounded or unbounded) derived categories.

Statement (i) is clear.

Fix an exact sequence $0 \to V \to W \to U \to 0$ with $W \in B$-Proj and $V \in B$-Mod. Let $Y = F^{-1}(V)$ and $Z = F^{-1}(W)$, a direct summand of a direct sum of T_P's. We have a distinguished triangle $Y \to Z \to X \rightsquigarrow$, hence an exact sequence

$$0 \to H^{-1}(Y) \to H^{-1}(Z) \to H^{-1}(X) \to H^0(Y) \to H^0(Z) \to H^0(X) \to 0.$$

Since $H^0(Z)$ is locally finite and $\mathrm{Hom}(P, H^0(Z)) = 0$ for $P \in \Upsilon - \Omega$, statement (ii) follows.

Let V be finite-dimensional simple A-module with $\mathrm{Hom}_A(R, V) = 0$ for all $R \in \Upsilon - \Omega$. There is $P \in \Omega$ such that $\mathrm{Hom}_A(\mathrm{coker}\, f_P, V) \neq 0$. We have $\mathrm{Hom}_A(\mathrm{coker}\, f_P, H^{-1}(X)) \simeq \mathrm{Hom}_{D(A)}(T_P[1], X) = 0$, hence $\mathrm{Hom}_A(V, H^{-1}(X)) = 0$. This shows (iii).

Assume $U \in B$-mod and choose $W \in B$-proj. Let $M = H^{-1}(X)$ and let N be the image of $H^{-1}(Z)$ in M. Let $L = M/N \subset H^0(Y)$. By (ii), L is locally finite and $\mathrm{Hom}_A(P, L) = 0$ for $P \in \Upsilon - \Omega$. Since Z is isomorphic to a direct summand of a finite direct sum of copies of T, it follows that $H^{-1}(Z) \in A$-mod, hence $H^{-1}(Y) \in A$-mod and $N \in A$-mod. By (iii), given V a non-zero finite-dimensional submodule of M, we have $\mathrm{Hom}_A(P, V) \neq 0$ for some $P \in \Upsilon - \Omega$. Lemma 6.4 shows that L is finite-dimensional. Since $H^0(Y)/L$ is a submodule of $H^0(Z)$, it is finite-dimensional. It follows that $H^0(Y)$ is finite-dimensional and $M \in A$-mod. Since $H^0(X)$ is a quotient of $H^0(Z)$, it is finite-dimensional. □

Let $\mathcal{L}$ be the thick subcategory of A-perf $= D^b(A\text{-mod})$ generated by $\Upsilon - \Omega$. Let S be the set of isomorphism classes of finite-dimensional simple A-modules and I the subset of S of simple modules V such that $\mathrm{Hom}_A(P, V) = 0$ for all $P \in \Upsilon - \Omega$. Let $\mathcal{I}$ (resp. $\bar{\mathcal{I}}$) be the thick subcategory of $D^b_f(A)$ (resp. $D^b_{lf}(A)$) of complexes C such that the composition factors of finite-dimensional A-submodules of $H^*(C)$ are in I.

Theorem 6.6. *The algebra B is noetherian, it has finite global dimension, it is Calabi-Yau of dimension d and isolated.*

The functor $F = \mathrm{Hom}^\bullet_A(T, -)$ induces perverse equivalences

- $\mathrm{Ho}^b(A\text{-proj}) \xrightarrow{\sim} \mathrm{Ho}^b(B\text{-proj})$ *with respect to the filtration* $0 \subset \mathcal{L} \subset \mathrm{Ho}^b(A\text{-proj})$ *and perversity function* $0 \mapsto -1,\ 1 \mapsto 0$*;*
- $D^b(A\text{-Mod}) \xrightarrow{\sim} D^b(B\text{-Mod})$ *with respect to the filtration* $0 \subset \bar{\mathcal{I}} \subset D^b(A\text{-Mod})$ *and perversity function* $0 \mapsto 0,\ 1 \mapsto -1$*;*
- $D^b(A\text{-mod}) \xrightarrow{\sim} D^b(B\text{-mod})$ *with respect to the filtration* $0 \subset \mathcal{I} \subset D^b(A\text{-mod})$ *and perversity function* $0 \mapsto 0,\ 1 \mapsto -1$*;*
- $D^b_f(A) \xrightarrow{\sim} D^b_f(B)$ *with respect to the filtration* $0 \subset \mathcal{I} \subset D^b_f(A)$ *and perversity function* $0 \mapsto 0,\ 1 \mapsto -1$.

The functor $F[1]$ induces equivalences

$$A\text{-Mod}/A\text{-Mod}_{lf} \xrightarrow{\sim} B\text{-Mod}/B\text{-Mod}_{lf} \ \textit{ and } \ A\text{-mod}/A\text{-mod}_f \xrightarrow{\sim} B\text{-mod}/B\text{-mod}_f\,.$$

Proof. We can assume $\Omega \neq \Upsilon$, otherwise $A = B$ and F is the identity.

The equivalence $F : D(A\text{-Mod}) \xrightarrow{\sim} D(B\text{-Mod})$ restricts to equivalences A-perf $\xrightarrow{\sim} B$-perf, $D^b(A\text{-Mod}) \xrightarrow{\sim} D^b(B\text{-Mod})$ and $\mathrm{Ho}^b(A\text{-Proj}) \xrightarrow{\sim} \mathrm{Ho}^b(B\text{-Proj})$. Since the canonical functor $\mathrm{Ho}^b(A\text{-Proj}) \to D^b(A\text{-Mod})$ is an equivalence, we deduce that the canonical functor $\mathrm{Ho}^b(B\text{-Proj}) \to D^b(B\text{-Mod})$ is an equivalence, so B has finite global dimension. Lemma 6.5 shows that $F^{-1}(M) \in A$-perf for $M \in B$-mod. It follows that B-mod $\subset B$-perf, hence B is noetherian.

Note that $D^b_f(A)$ is the thick subcategory of $D(A)$ of objects C such that given any $D \in A$-perf, the k-module $\bigoplus_{i \in \mathbf{Z}} \mathrm{Hom}_{D(A)}(D, C[i])$ is finite-dimensional. We deduce that F restricts to an equivalence $D^b_f(A) \xrightarrow{\sim} D^b_f(B)$. Consequently, B is Calabi-Yau of dimension d.

The perversity property for the restriction of F to $\mathrm{Ho}^b(A\text{-proj})$ is clear.

Let $L \in A$-Mod such that $\mathrm{Hom}_A(P, L) = 0$ for all $P \in \Upsilon - \Omega$. Let M be a finitely generated A-submodule of L. We have $\mathrm{Hom}_A(P, M) = 0$ for all $P \in \Upsilon - \Omega$. By Lemma 6.4, we have $M \in A\text{-mod}_f$. We deduce that $L \in A\text{-Mod}_{lf}$. It follows that $\bar{\mathcal{I}}$ (resp. $\mathcal{I}$) is the full subcategory of $D^b(A\text{-Mod})$ (resp. $D^b(A\text{-mod})$) of objects C such that $\mathrm{Hom}_{D(A)}(P, C[i]) = 0$ for all $P \in \Upsilon - \Omega$ and $i \in \mathbf{Z}$.

Let $M \in A$-Mod. We have $H^i(F(M)) = 0$ for $i \notin \{0, 1\}$ and $\mathrm{Hom}_B(F(T_P), H^0(F(M))) \simeq \mathrm{Hom}_{D(A)}(T_P, M) = 0$ for $P \in \Upsilon - \Omega$. It follows that $H^0(F(M)) \in F(\bar{\mathcal{I}})$. Lemma 4.14 shows the perversity property for $D^b(A\text{-Mod})$. The perversity assertions for $D^b(A\text{-mod})$ and $D^b_f(A)$ are clear.

Consider the equivalence $A\text{-Mod}/(\mathcal{I} \cap A\text{-Mod}) \xrightarrow{\sim} B\text{-Mod}/(F(\mathcal{I}) \cap B\text{-Mod})$ induced by $F[1]$. Since $F^{\pm 1}$ commutes with direct sums and preserves finite-dimensional modules, it follows that it preserves locally finite modules. Consequently, $F[1]$ induces an equivalence $A\text{-Mod}/A\text{-Mod}_{lf} \xrightarrow{\sim} B\text{-Mod}/B\text{-Mod}_{lf}$.

Let $M \in B$-mod, let $r > 0$ and let $f \in \mathrm{Hom}_B(M, B^r)$ be a non-zero injective map. Since B is Calabi-Yau of positive dimension, it follows that M is not finite-dimensional. Note that $Q(F^{-1}(M)[-1])$ is a non-zero subobject of $Q(F^{-1}(B)^r[-1])$, hence is isomorphic to a subobject of $Q(A)^s$ for some $s > 0$. It follows that $Q(F^{-1}(M)[-1])$ generates $D^b(A\text{-mod})/D^b_f(A)$, hence M generates $D^b(B\text{-mod})/D^b_f(B)$. We deduce that B is isolated. $\square$

6.3. **Iteration of perverse equivalences.** Let A be a noetherian k-algebra of finite global dimension that is Calabi-Yau of dimension $d \geq 2$ and isolated.

We assume in addition that every indecomposable object of A-perf has a local endomorphism ring whose division ring quotient is finite-dimensional over k. In particular, the Krull-Schmidt Theorem holds for A-perf. Equivalently, the endomorphism ring of any perfect complex is semi-perfect [La, Chapter 8] with finite-dimensional semi-simplification. This assumption holds if A is a finitely generated module over a central subalgebra that is a complete local noetherian ring with a residue field that is finite-dimensional over k [BuDr, Proposition A.2].

Let S be the set of isomorphism classes of simple A-modules. Note that all objects of S are finite-dimensional over k. We denote by Υ the set of isomorphism classes of indecomposable projective A-modules. The map sending $P \in \Upsilon$ to its largest simple quotient induces a bijection $h : \Upsilon \xrightarrow{\sim} S$.

Given $I \subset S$, we define $\Omega = h^{-1}(I)$. We proceed as in §6.2 to define a complex $T = T(I)$. Given $P \in \Omega$, we take for P' in the definition of T_P a projective cover of the largest submodule of P whose quotient is in $\mathcal{I}$. This makes the complex T_P unique up to isomorphism.

Let $\mathcal{E}$ be the set of isomorphism classes of families $(T_V)_{V \in S}$ where T_V is an indecomposable bounded complex of finitely generated projective A-modules, $T_V \not\simeq T_{V'}$ if $V \not\simeq V'$, and $\bigoplus_{V \in S} T_V$ is a tilting complex.

Let $\mathcal{P}'(S)$ be the set of proper subsets of S and let Γ be the quotient of $\mathrm{Free}(\mathcal{P}'(S)) \rtimes \mathfrak{S}(S)$ by the relations $IJ = JI$ when $I \subset J \subset S$.

We obtain as in §5.2 an action of $\mathrm{Free}(\mathcal{P}'(S)) \rtimes \mathfrak{S}(S)$ on $\mathcal{E}$, commuting with the action of $\mathrm{Aut}(D(A\text{-Mod}))$.

Example 6.7. Let V be a finite dimensional k-vector space of dimension $d \geq 2$ and G a finite subgroup of $\mathrm{SL}(V)$ with $|G| \in k^\times$. Assume G acts freely on $V - \{0\}$. The algebra $A = k[[V]] \rtimes G$ is a noetherian k-algebra of finite global dimension that is Calabi-Yau of dimension $d \geq 2$ and isolated (cf Proposition 6.3). Furthermore, A is a finitely generated module over the complete local noetherian central subalgebra $k[[V]]^G$.

7. Stable categories

7.1. **Bases for triangulated categories of CY dimension** -1. Let $\mathcal{T}$ be a cocomplete compactly generated triangulated category over a field k. We assume that

- given $C \in \mathcal{T}$, we have $C \in \mathcal{T}^c$ if and only if $\dim \mathrm{Hom}(M, C) < \infty$ for all $M \in \mathcal{T}^c$
- $\mathcal{T}^c$ is Calabi-Yau of dimension -1.

Given $\mathcal{F}$ a subset of $\mathcal{T}$, we say that $M \in \mathcal{T}$ is a *finite extension* of objects of $\mathcal{F}$ of length n if there are objects $M_0 = 0, M_1, \ldots, M_{n-1}, M_n = M$ in $\mathcal{T}^c$, $S_1, \ldots, S_n \in \mathcal{F}$ and distinguished triangles $M_i \to M_{i+1} \to S_{i+1} \rightsquigarrow$ for $0 \le i < n$. We say that M has an $\mathcal{F}$-extension $[S_1, \ldots, S_n]$.

We say that a finite family $\mathcal{F}$ of objects of $\mathcal{T}^c$ is a *basis* if

- $\mathrm{Hom}(S, T) = \delta_{S,T} k$ for all $S, T \in \mathcal{F}$
- every object of $\mathcal{T}^c$ is a finite extension of objects of $\mathcal{F}$.

The terms appearing in an $\mathcal{F}$-filtration of minimal length are unique, as shown by the following lemma.

Lemma 7.1. *Let $\mathcal{F}$ be a basis of $\mathcal{T}^c$ and M a finite extension of objects of $\mathcal{F}$. Let $S \in \mathcal{F}$ such that* $\mathrm{Hom}(S, M) \neq 0$.

Then there is a minimal length $\mathcal{F}$-extension $[S = S_1, S_2, \ldots, S_n]$ of M.

Given an $\mathcal{F}$-extension $[S'_1, \ldots, S'_m]$ of M, the multiset $\{S_i\}_{1 \le i \le n}$ is a subset of $\{S'_i\}_{1 \le i \le m}$.

Proof. We prove the lemma by induction on the minimal length n of an $\mathcal{F}$-extension of M. Consider $M'_0 = 0, M'_1, \ldots, M'_m = M$ and distinguished triangles $M'_i \to M'_{i+1} \to S'_{i+1} \rightsquigarrow$ with $S'_{i+1} \in \mathcal{F}$. Let $d \ge 1$ be minimal such that there is $f : S \to M'_d$ such that the composition $S \xrightarrow{f} M'_d \xrightarrow{\mathrm{can}} M$ is not zero. We deduce that the composition $S \xrightarrow{f} M'_d \to S_d$ is an isomorphism, hence $M'_d \simeq S_d \oplus M'_{d-1}$ and we can construct a new $\mathcal{F}$-filtration $0 = M''_0, M''_1, \ldots, M''_m = M$ where $M''_1 = S$, $M''_i = M'_{i-1} \oplus S$ for $2 \le i < d$ and $M''_i = M'_i$ for $i \ge d$. By induction, the lemma holds for the cone of the canonical map $S \to M$, hence we are done. □

Remark 7.2. Note that not all filtrations have the same length, as shown by the following example. Let $\mathcal{T} = (k[x]/x^2)$-Stab, where k is a field. We have $\mathcal{T}^c = (k[x]/x^2)$-stab and $\mathcal{F} = \{k\}$ is a basis. The socle filtration of $k[x]/x^2$ induces a filtration of length 2 of the object 0.

Let $\mathcal{F}$ be a basis of $\mathcal{T}^c$. Let $I \subset \mathcal{F}$ and $C \in \mathcal{T}$. We define by induction a family of objects and maps $C_0 \to C_1 \to \cdots$. We put $C_0 = C$. Assume C_i has been defined. We define C_{i+1} as the cone of the canonical map $\bigoplus_{S \in I} S[-1] \otimes \mathrm{Hom}(S, C_i[1]) \to C_i$. Finally, we put $C^I = \mathrm{hocolim}\, C_i$.

Lemma 7.3. *Given $S \in \mathcal{F} - I$, the canonical map* $\mathrm{Hom}(S, C) \to \mathrm{Hom}(S, C^I)$ *is an isomorphism. Given $S \in I$, we have* $\mathrm{Hom}(S, C^I[1]) = 0$ *and the canonical map* $\mathrm{Hom}(S, C) \to \mathrm{Hom}(S, C^I)$ *is surjective.*

Assume $C \in \mathcal{T}^c$. Then $C^I \in \mathcal{T}^c$ and the cone of the canonical map $C \to C^I$ is a finite extension of objects of I.

Conversely, given $D \in \mathcal{T}^c$ and $f : C \to D$ such that the cone of f is a finite extension of objects of I and the canonical map $\mathrm{Hom}(S, C) \to \mathrm{Hom}(S, D)$ *is surjective for all $S \in I$, then there is a map $g : D \to C^I$ such that the composition with f is the canonical map $C \to C^I$. If in addition* $\mathrm{Hom}(S, D[1]) = 0$ *for all $S \in I$, then g is an isomorphism.*

Proof. Let $S \in \mathcal{F}$. Since S is compact, the canonical map $\mathrm{colim}(S, C_i) \to \mathrm{Hom}(S, C^I)$ is an isomorphism.

The distinguished triangle $C_i \to C_{i+1} \to \bigoplus_{T \in I} T \otimes \mathrm{Hom}(T, C_i[1]) \rightsquigarrow$ gives an exact sequence

$$\bigoplus_{T \in I} \mathrm{Hom}(S, T[-1]) \otimes \mathrm{Hom}(T, C_i[1]) \to \mathrm{Hom}(S, C_i) \to \mathrm{Hom}(S, C_{i+1}) \to$$

$$\to \bigoplus_{T \in I} \mathrm{Hom}(S, T) \otimes \mathrm{Hom}(T, C_i[1]) \to \mathrm{Hom}(S, C_i[1]) \to \mathrm{Hom}(S, C_{i+1}[1]).$$

We have $\mathrm{Hom}(S,T) = k\delta_{ST}$. Since $\mathcal{T}^c$ is (-1)-Calabi-Yau, we have $\mathrm{Hom}(S,T[-1]) \simeq \mathrm{Hom}(T,S)^* = \delta_{ST}k$. So, if $S \in \mathcal{F} - I$, we have an isomorphism $\mathrm{Hom}(S,C_i) \xrightarrow{\sim} \mathrm{Hom}(S,C_{i+1})$ for all i, hence an isomorphism $\mathrm{Hom}(S,C) \xrightarrow{\sim} \mathrm{Hom}(S,C^I)$.

Assume now $S \in I$. We have an exact sequence

$$\mathrm{Hom}(S,S[-1]) \otimes \mathrm{Hom}(S,C_i[1]) \to \mathrm{Hom}(S,C_i) \to \mathrm{Hom}(S,C_{i+1}) \to$$

$$\to \mathrm{End}(S) \otimes \mathrm{Hom}(S,C_i[1]) \to \mathrm{Hom}(S,C_i[1]) \to \mathrm{Hom}(S,C_{i+1}[1]).$$

The map $\mathrm{End}(S) \otimes \mathrm{Hom}(S,C_i[1]) \to \mathrm{Hom}(S,C_i[1])$ is an isomorphism. So, the canonical map $\mathrm{Hom}(S,C_i) \to \mathrm{Hom}(S,C_{i+1})$ is surjective for all i, hence the canonical map $\mathrm{Hom}(S,C) \to \mathrm{Hom}(S,C^I)$ is surjective. Also, the canonical map $\mathrm{Hom}(S,C_i[1]) \to \mathrm{Hom}(S,C_{i+1}[1])$ vanishes for all i, hence $\mathrm{Hom}(S,C^I[1]) = 0$.

Assume $C \in \mathcal{T}^c$. We have $\dim \mathrm{Hom}(S,C) < \infty$ for all $S \in \mathcal{F}$, hence $\dim \mathrm{Hom}(S,C^I) < \infty$ for all $S \in \mathcal{F}$. The generation property of $\mathcal{F}$ implies that $C^I \in \mathcal{T}^c$.

Let C'_i be the cone of the canonical map $C \to C_i$ and let M be the cone of the canonical map $C \to C^I$. There is a map $C'_i \to C'_{i+1}$ such that the composition $C_i \xrightarrow{\text{can}} C_{i+1} \xrightarrow{\text{can}} C'_{i+1}$ factors as $C_i \xrightarrow{\text{can}} C'_i \to C'_{i+1}$. We have an isomorphism $M \xrightarrow{\sim} \mathrm{hocolim}\, C'_i$. Since $M \in \mathcal{T}^c$, the identity map of M factors through C'_i for some i, *i.e.*, $C'_i \simeq M \oplus N$ for some $N \in \mathcal{T}^c$. Since C'_i is a finite I-extension, it follows from Lemma 7.1 that M is a finite I-extension.

Consider now $f : C \to D$ with cone L such that L is a finite I-extension and the canonical map $\mathrm{Hom}(S,C) \to \mathrm{Hom}(S,D)$ is surjective for all $S \in I$. Since L is a finite I-extension, we have $\mathrm{Hom}(L,C^I[1]) = 0$, hence the canonical map $L \to C[1]$ factors through M. Consequently, we have a morphism $g : D \to C^I$ with cone N making the following diagram commutative

$$\begin{tikzcd}
C \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d, dashed, "g"] & L \arrow[r] \arrow[d, dashed] & C[1] \arrow[d, equal] \\
C \arrow[r] \arrow[dr, "0"'] & C^I \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & C[1] \\
& N \arrow[r, equal] \arrow[d] & N \arrow[d] & \\
& D[1] \arrow[r] & L[1] &
\end{tikzcd}$$

Assume now $\mathrm{Hom}(S,D[1]) = 0$ for all $S \in I$. Let $S \in I$. The canonical map $\mathrm{Hom}(S,C) \to \mathrm{Hom}(S,C^I)$ is onto, hence the canonical map $\mathrm{Hom}(S,N) \to \mathrm{Hom}(S,D[1])$ is injective, hence $\mathrm{Hom}(S,N) = 0$. Since M has a finite I-filtration, we deduce that the canonical map $M \to N$ vanishes. So, there is M' such that $L \simeq M \oplus M'$ and M' has a finite I-filtration by Lemma 7.1. The composite map $M' \to L \to C[1]$ vanishes, hence it factors through D. Since the composition $\mathrm{Hom}(S,D) \to \mathrm{Hom}(S,L)$ vanishes for all $S \in I$, we deduce that $M' = 0$, hence g is an isomorphism. □

Given $S \in \mathcal{F}$, we put $S' = S$ if $S \in I$ and $S' = S^I[1]$ otherwise. We put $\mathcal{F}' = \{S'\}_{S\in\mathcal{F}}$.

Proposition 7.4. *$\mathcal{F}'$ is a basis of $\mathcal{T}^c$.*

Proof. Let $S \in \mathcal{F} - I$ and $T \in I$. We have $\mathrm{Hom}(T,S^I[1]) = 0$ by Lemma 7.3. Since $\mathcal{T}^c$ is (-1)-Calabi-Yau, we have $\mathrm{Hom}(S^I[1],T) \simeq \mathrm{Hom}(T,S^I)^* = 0$ by Lemma 7.3.

Consider now $T \in \mathcal{F} - I$. Let M be the cone of the canonical map $T \to T^I$. We have an exact sequence $\mathrm{Hom}(M,S^I) \to \mathrm{Hom}(T^I,S^I) \to \mathrm{Hom}(T,S^I) \to \mathrm{Hom}(M,S^I[1])$. Lemma 7.3 shows that the first and last term are zero and $\mathrm{Hom}(T, S^I) = \delta_{S,T}k$. So, $\mathrm{Hom}(T^I, S^I) = \delta_{ST}k$. We deduce that $\mathcal{F}'$ satisfy the disjunction part of the basis property.

Let $C \in \mathcal{T}^c$ such that $\mathrm{Hom}(S, C[-1]) = 0$ for all $S \in I$. Let n be the length of a minimal $\mathcal{F}$-extension of $C[-1]$. We show by induction on n that C is an $\mathcal{F}'$-extension. Let $T \in \mathcal{F} - I$ such that $\mathrm{Hom}(T, C[-1]) \neq 0$. Lemma 7.1 shows there are distinguished triangles $M \to C[-1] \to N \rightsquigarrow$ and $T \to M \to M' \rightsquigarrow$ such that M' is an I-extension of length n' and N is an $\mathcal{F}$-extension of length $n - n' - 1$ such that $\mathrm{Hom}(S, N) = 0$ for all $S \in I$. The minimality of n shows that $\mathrm{Hom}(S, M) = 0$ for $S \in I$. Lemma 7.3 shows there is a distinguished triangle $M \to T^I \to M'' \rightsquigarrow$, where M'' is an I-extension. Consequently, $M[1]$ is an $\mathcal{F}'$-extension. By induction, $N[1]$ is an $\mathcal{F}'$-extension. So, every object C such that $\mathrm{Hom}(C, S) = 0$ for all $S \in I$ is an $\mathcal{F}'$-extension.

Let now $C \in \mathcal{T}^c$ be arbitrary. We show that C is an $\mathcal{F}'$-extension by induction on the length m of a minimal $\mathcal{F}$-extension. If there is a non-zero map $f : C \to S$ with $S \in I$, then Lemma 7.1 (applied to the opposite category) shows that the cocone of f is an $\mathcal{F}$-extension of length $m - 1$, hence by induction it is an $\mathcal{F}'$-extension and consequently C is an $\mathcal{F}'$-extension as well. If $\mathrm{Hom}(C, S) = 0$ for all $S \in I$, then the discussion above shows that C is an $\mathcal{F}'$-extension. We have shown that $\mathcal{F}'$ is a basis. $\square$

8. Applications

8.1. **Triangularity and Broué's conjecture.** Let $\mathcal{O}$ be a complete discrete valuation ring with fraction field K and residue field k. Let A be an $\mathcal{O}$-algebra, free over $\mathcal{O}$, of finite rank. For $R \in \{K, k\}$, write RA for the R-algebra $R \otimes_{\mathcal{O}} A$. Denote by $d_A : K_0(KA) \to K_0(kA)$ the decomposition map, defined by $d_A([M]) = [k \otimes_{\mathcal{O}} N]$, where N is any $\mathcal{O}$-free A-module such that $M \cong K \otimes_{\mathcal{O}} N$. The decomposition matrix of A is the matrix of d_A with respect to the bases of classes of simple modules.

Proposition 8.1. *Let A and A' be $\mathcal{O}$-algebras, free and of finite rank over $\mathcal{O}$. Let $F : D^b(A) \xrightarrow{\sim} D^b(A')$ be an equivalence such that $kF : D^b(kA) \xrightarrow{\sim} D^b(kA')$ is perverse. Assume that KA is semisimple and that every simple kA-module lifts to an $\mathcal{O}$-free A-module. Then, the decomposition matrix of A' is lower unitriangular for some orderings of the simple modules of KA' and of kA'.*

Proof. We have an equivalence $KF : D^b(KA) \xrightarrow{\sim} D^b(KA')$ and a commutative diagram

$$\begin{array}{ccc} K_0(KA) & \xrightarrow[\sim]{[KF]} & K_0(KA') \\ {\scriptstyle d_A}\downarrow & & \downarrow{\scriptstyle d_{A'}} \\ K_0(kA) & \xrightarrow[\sim]{[kF]} & K_0(kA') \end{array}$$

Since KA is semisimple, so is KA', and we have a bijection $\hat{S} \xrightarrow{\sim} \hat{S}' : L \mapsto L'$ between the simple modules of KA and of KA' such that $[KF]([L]) = \pm[L']$ for all $L \in \hat{S}$. We have an injective map $S \hookrightarrow \hat{S} : V \to L_V$ such that $d_A([L_V]) = [V]$, determined by $L_V \cong K \otimes_{\mathcal{O}} \tilde{V}$, where $\tilde{V}$ is a chosen $\mathcal{O}$-free A-module lifting V.

Suppose $kF : D^b(kA) \xrightarrow{\sim} D^b(kA')$ is perverse relative to $(S_\bullet, S'_\bullet, p)$, so that for $V \in S_i - S_{i-1}$,

$$[kF(V)] = \pm[V'] + \sum_{W' \in S'_{i-1}} a_{VW}[W']$$

for some integers a_{VW}. Then for $V' \in S'_i - S'_{i-1}$, we have

$$d_{A'}([L'_V]) = \pm[V'] + \sum_{W' \in S'_{i-1}} \pm a_{VW}[W'],$$

from which we deduce the claimed unitriangularity. Note that the coefficient of $[V']$ here must be 1, since the entries of the decomposition matrix are nonnegative. $\square$

Let G be a finite group. Assume now that k has characteristic ℓ and K has characteristic 0; assume also that K and k are large enough for G, so that all simple modules over KG and kG are absolutely simple.

Let e be a block idempotent of $\mathcal{O}G$. Let f be the Brauer correspondent of e, so f is a block idempotent of $\mathcal{O}H$, where D is a defect group of e and $H = N_G(D)$.

Broué's abelian defect group conjecture predicts an equivalence $D^b(\mathcal{O}Hf) \xrightarrow{\sim} D^b(\mathcal{O}Ge)$ whenever D is abelian. When G is a finite group of Lie type in characteristic $p \neq l$ one can conjecture that the complex of cohomology of Deligne-Lusztig varieties will give a perverse equivalence, with increasing perversity function. That function should be given by the degree of cohomology where a given unipotent character occurs.

However for arbitrary G there are counterexamples, because the existence of a perverse equivalence would imply unitriangularity of the decomposition matrix of $\mathcal{O}Ge$. A number of perverse equivalences are constructed for sporadic groups in [CrRou].

Corollary 8.2. *Suppose that there is a perverse equivalence $F : D^b(\mathcal{O}Hf) \xrightarrow{\sim} D^b(\mathcal{O}Ge)$. Then the decomposition matrix of $\mathcal{O}Ge$ is unitriangular with respect to some ordering of the simple modules of KGe and kGe.*

Proof. Let $E = N_G(D, f)/C_G(D)$, an ℓ'-group. Since $\mathcal{O}Hf$ is Morita equivalent to a twisted group algebra $\mathcal{O}_*D \rtimes \hat{E}$ [Kü], all simple kHf-modules lift to $\mathcal{O}$-free $\mathcal{O}Hf$-modules and the corollary follows from Proposition 8.1. $\square$

Example 8.3. Let e be the principal block idempotent of $\mathcal{O}G$ where $G = SL_2(8)$ and k has characteristic 2. Then the decomposition matrix of $\mathcal{O}Ge$ is not unitriangular for any orderings of simple modules of KGe and of kGe. Hence no perverse equivalence $D^b(kHf) \xrightarrow{\sim} D^b(kGe)$ exists.

8.2. **Perverse equivalences from $\mathfrak{sl}_2$-categorifications.** Recall that an $\mathfrak{sl}_2$-categorification is an abelian category $\mathcal{B}$ over a field, all of whose objects have finite composition series, together with a biadjoint pair of exact functors $E, F : \mathcal{B} \to \mathcal{B}$ inducing a locally finite action of $\mathfrak{sl}_2$ on $K_0(\mathcal{B})$ via $e = [E]$ and $f = [F]$, and equipped with compatible actions of (classical, degenerate or nil) affine Hecke algebras on powers of E and F. See [ChRou] for details.

One of the main results of [ChRou] is the existence of an equivalence $\Phi : D^b(\mathcal{B}) \xrightarrow{\sim} D^b(\mathcal{B})$ lifting the action of $\exp(-f)\exp(e)\exp(-f)$ on $K_0(\mathcal{B})$.

Let S be the set of simple objects of $\mathcal{B}$. We define two filtrations of S:

$$S_i = \{V \in S \mid F^{i+1}V = 0\} \quad \text{and} \quad S'_i = \{V \in S \mid E^{i+1}V = 0\}.$$

Proposition 8.4. *The equivalence $\Phi : D^b(\mathcal{B}) \xrightarrow{\sim} D^b(\mathcal{B})$ is perverse with respect to the filtrations $S_\bullet$ and $S'_\bullet$ and the perversity $p(i) = i$.*

Proof. Let us recall some constructions and results of [ChRou]. Let $V = \mathbf{Q} \otimes K_0(\mathcal{B})$. The weight space decomposition $V = \bigoplus_{\lambda \in \mathbf{Z}} V_\lambda$ induces a decomposition $\mathcal{B} = \bigoplus_\lambda \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda$ is the full subcategory of $\mathcal{B}$ of objects M with $[M] \in V_\lambda$.

Fix $\lambda \in \mathbf{Z}$. Let $\mathcal{A} = \mathcal{B}_{-\lambda}$, $\mathcal{A}' = \mathcal{B}_\lambda$, $\mathcal{F}_i = \{E^iL \in \mathcal{A} | FL = 0\}$ and $\mathcal{F}'_i = \{F^iL \in \mathcal{A}' | EL = 0\}$. Let $\mathcal{A}_i = \{M \in \mathcal{A} | F^{i+1}M = 0\}$ and $\mathcal{A}'_i = \{M \in \mathcal{A}' | E^{i+1}M = 0\}$. The functor $\Phi[i]$ restricts to an equivalence $\mathcal{F}_i \xrightarrow{\sim} \mathcal{F}'_i$ by [ChRou, Theorem 5.24 and Theorem 6.6] (note that the assumption "$\lambda \geq 0$" in [ChRou, Theorem 6.6] is not necessary for the theorem or for its proof). On the other hand, $\mathcal{A} = \bigcup_i \mathcal{A}_i$ and $\mathcal{A}' = \bigcup_i \mathcal{A}'_i$ and the conditions of Lemma 4.12 are satisfied. It follows that Φ restricts to a perverse equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}'$. $\square$

Remark 8.5. By [ChRou, §7.1], we deduce that any two blocks of symmetric groups with isomorphic defect groups are related by a sequence of perverse equivalences. Also, a block of a symmetric group with abelian defect is related by a sequence of perverse equivalences to the corresponding block of the normalizer of a defect group. We don't know whether the latter can be achieved by a single perverse equivalence rather than a composition of perverse equivalences.

8.3. **Alvis-Curtis duality.** Let G be a finite group of Lie type and k a field of characteristic different from the defining characteristic of G. Let S be the set of simple reflections. Given $J \subset S$, we have a Levi subgroup L_J of G, and Harish-Chandra induction $R_{L_J}^G$ and restriction ${}^*R_{L_J}^G$ functors between kG-mod and kL_J-mod. Cabanes and Rickard have shown in [CaRi] that there is a complex of functors

$$\Theta = 0 \to R_{L_\emptyset}^G {}^*R_{L_\emptyset}^G \to \bigoplus_{J\subset S, |J|=1} R_{L_J}^G {}^*R_{L_J}^G \to \cdots \to \bigoplus_{J\subset S, |S\setminus J|=1} R_{L_J}^G {}^*R_{L_J}^G \to \cdots \to \mathrm{Id} \to 0$$

with the term Id in degree 0 inducing a self-equivalence Φ of $D^b(kG)$.

Let I_i be the set of simple kG-modules V such that ${}^*R_{L_J}^G V = 0$ for all $J \subset S$ such that $|S \setminus J| \geq i$. Let $p : \{0, \ldots, |S|\} \to \mathbf{Z}$ be given by $p(i) = i$.

Proposition 8.6. *The equivalence Φ is perverse with respect to $(I_\bullet, I_\bullet, p)$.*

Proof. Let $\mathcal{F}_i$ be the full subcategory of kG-mod of modules $R_{L_J}^G(M)$, where $i = |S \setminus J|$ and M is a cuspidal kL_J-module. By [CaRi, Theorem 3.1], $\Theta[-i]$ induces an auto-equivalence of $\mathcal{F}_i$. The proposition follows now from Lemma 4.12. □

Remark 8.7. Note that using Lemma 4.5, one obtains a variant of Cabanes-Enguehard's proof that Φ is an equivalence, from [CaRi, Theorem 3.1].

8.4. **Blocks with cyclic defect groups and Brauer tree algebras.** Let k be a field. Recall that a Brauer tree Γ is a connected tree with a planar embedding, with a multiplicity $m \in \mathbf{Z}_{\geq 1}$ and, if $m > 1$, with a specified vertex v, the exceptional vertex. When $m = 1$, some constructions will rely on the choice of a vertex, which we call the exceptional vertex. We also assume that Γ has at least one edge. Associated to Γ, there is a k-algebra $A = A(\Gamma)$, well-defined up to Morita equivalence [Ben, §4.18].

Let e be the number of edges of Γ. Let B be a basic Brauer tree algebra associated with a star with e edges, exceptional vertex v in the center, and multiplicity m.

Rickard [Ri1, §4] has constructed a derived equivalence between A and B. Let us recall his construction.

We identify S, the set of simple A-modules, with the set of edges of Γ. Given $e \in S$, we denote by $e_0, e_1, \ldots, e_l = e$ a minimal path in Γ from v to e. So, v is a vertex of the edge e_0, and the edges e_l and e_{l+1} have one vertex in common. We define the distance from e to v as $d(e, v) = l$.

There is an indecomposable complex of A-modules

$$T_e = 0 \to A_{e_0} \to \cdots \to A_{e_l} \to 0$$

where A_{e_0} is in degree 0. Such a complex is unique up to isomorphism. Let $T = \bigoplus_{e\in S} T_e$. This is a tilting complex for A, with endomorphism ring isomorphic to B [Ri1, Theorem 4.2]. Let $F = T \otimes_B^{\mathbf{L}} - :$ $D^b(B\text{-mod}) \xrightarrow{\sim} D^b(A\text{-mod})$. Let $r = \max\{d(e, v)\}_{e\in I}$ and define $p : \{0, \ldots, r\} \to \mathbf{Z},\ p(i) = i - r$. Let $S_i = \{e | d(e, v) \geq r - i\}$.

Theorem 8.8. *The equivalence F is perverse relative to $(S_\bullet, p)$.*

Proof. Lemma 4.8 shows that F restricts to a perverse equivalence $\mathrm{Ho}^b(B\text{-proj}) \xrightarrow{\sim} \mathrm{Ho}^b(A\text{-proj})$ relative to $\bar{p}$ given by $\bar{p}(i) = -i$ and the filtration of A-proj given by $I'_i = \{A_e\}_{d(e,v)\leq i}$.

It follows from Lemma 4.21 that F is a perverse equivalence relative to $(S_\bullet, p)$. □

Theorem 8.8 shows that there is a perverse equivalence between a block of a finite group with a cyclic defect group and the corresponding block of the normalizer of a defect group.

Corollary 8.9. *Let k be a field of characteristic $p > 0$, let G be a finite group and A a block of kG with cyclic defect group D. Then there is a perverse equivalence between A and the corresponding block of $kN_G(D)$.*

We provide now a combinatorial description of the way a Brauer tree changes under an elementary perverse equivalence.

Let Γ be a Brauer tree. We denote by Δ the set of vertices and E the set of edges, a subset of the set of pairs of distinct elements of Δ. The planar embedding of Γ corresponds to the data, for every vertex $x \in \Delta$, of a cyclic ordering of the N_x vertices adjacent to x, or, equivalently, the action of an automorphism l_x. Given an edge $\{x, y\}$, the ordered vertices adjacent to x are $y, l_x(y), \dots, l_x^{N_x}(y)$. Let A be the associated Brauer tree algebra. The projective indecomposable modules have a 3-step filtration, with middle layer the direct sum of two uniserial modules:

$$A_{\{x,y\}} = \begin{array}{ccc} & \{x,y\} & \\ \boxed{\begin{array}{c}\{x, l_x(y)\}\\ \vdots \\ \{x, l_x^{c_x-1}a(y)\}\end{array}} & & \boxed{\begin{array}{c}\{y, l_y(x)\}\\ \vdots \\ \{y, l_y^{c_y-1}(x)\}\end{array}} \\ & \{x,y\} & \end{array}$$

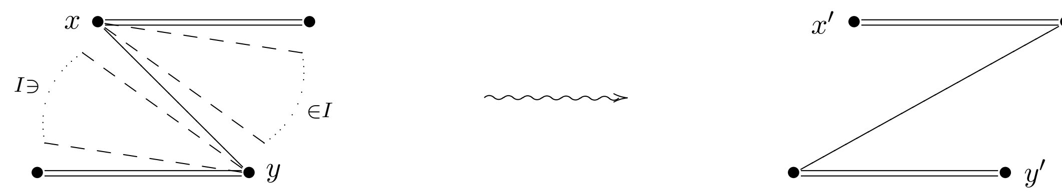
Here, given $v \in \Delta$, we put $c_v = N_v$ if if v is not exceptional and $c_v = mN_v$ if v is exceptional.

Let $I \subset E$. We define a Brauer tree Γ'. Its set of vertices is Δ', with a bijection $\Delta \xrightarrow{\sim} \Delta'$, $x \mapsto x'$. The exceptional vertex is the image of the exceptional vertex of Γ and the multiplicity is that of Γ. There is a bijection $\phi : E \xrightarrow{\sim} E'$ defined as follows.

Let $\{x, y\} \in E$. If $\{x, y\} \notin I$, then $\phi(\{x, y\}) = \{x', y'\}$.

Assume now $\{x, y\} \in I$. We put $\phi(\{x, y\}) = \{a', b'\}$, where a and b are defined below.

- If there is an edge with vertex x that is not contained in I, let $r \geq 1$ be minimal such that $\{x, l_x^r(y)\} \notin I$. We put $a = l_x^r(y)$. If all edges around x are in I, we put $a = x'$.
- If there is an edge with vertex y that is not contained in I, let $s \geq 1$ be minimal such that $\{y, l_y^s(x)\} \notin I$. We put $b = l_y^s(x)$. If all edges around y are in I, we put $b = y'$.



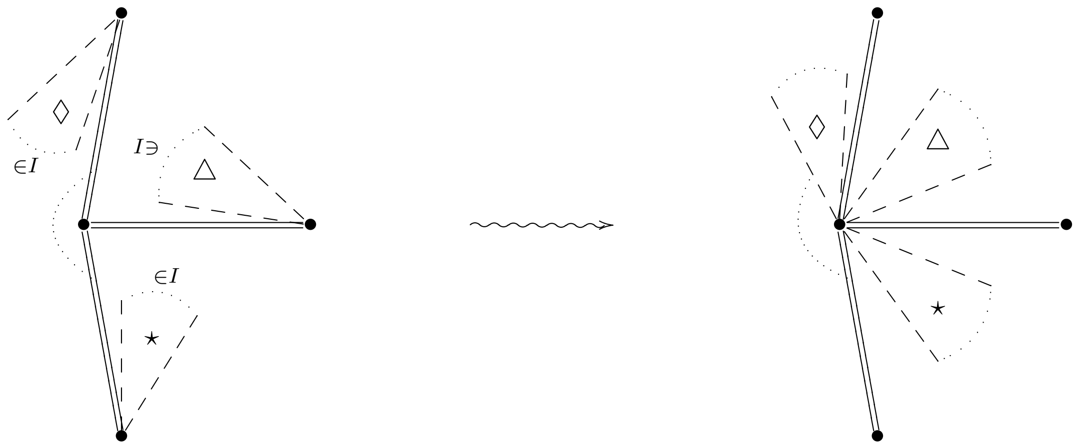
Let us finally describe the cyclic ordering of vertices adjacent to a vertex x'.

• Assume there is a vertex y_0 adjacent to x such that $\{x, y_0\} \notin I$. Let $y_0, y_1, \dots, y_t$ be the ordered sequence of vertices adjacent to x and such that $\{x, y_i\} \notin I$. Let $r_i \geq 0$ be the smallest integer such that $\{y_i, l_{y_i}^{-r_i-1}(x)\} \notin I$. Then, the vertices around x' are ordered as follows:

$$y_0', l_{y_0}^{-1}(x)', \dots, l_{y_0}^{-r_0}(x)', y_1', l_{y_1}^{-1}(x)', \dots, l_{y_1}^{-r_1}(x)', \dots, l_{y_t}^{-r_t}(x)'.$$

• Assume all edges containing x are in I. Let z_0 be a vertex adjacent to x and $z_0, \dots, z_t$ the cyclic ordering of vertices around x. Then, the vertices around x' are ordered as follows:

$$z_0', \dots, z_t'.$$



The above description of Γ' and the following proposition have been provided, for the case $|I| = 1$, by [Ka, §3.5], in the more general setting of Brauer graphs and Brauer graph algebras.

Proposition 8.10. *The algebra* $\mathrm{End}_A(T_A(I))$ *is a Brauer tree algebra with tree* Γ'*.*

Proof. Let $A' = \mathrm{End}_A(T_A(I))$: this is a basic Brauer tree algebra [GaRi, Theorem 2]. Let $F = T_A(I) \otimes_{A'} - : D^b(A') \xrightarrow{\sim} D^b(A)$. Recall (Lemma 5.2) that given $V \in E$, we have

$$F(V') = \begin{cases} V & \text{if } V \in I \\ V^I[1] & \text{otherwise.} \end{cases}$$

The structure of the Brauer tree of A' is determined by $\dim \mathrm{Ext}^1_{A'}(V', W')$ for $V, W \in E$ and by the exceptional vertex and its multiplicity.

Let $V, W \in E$ with $V = \{x, y\}$ and $W = \{z, t\}$. We also put $x_0 = x$, $x_1 = y$, $z_0 = z$ and $z_1 = t$ to simplify some of the statements.

- Assume $V, W \in I$. We have $\mathrm{Ext}^1_{A'}(V', W') \simeq \mathrm{Ext}^1_A(V, W)$, hence

$$\dim \mathrm{Ext}^1_{A'}(V', W') = \begin{cases} 1 & \text{if } \{z, t\} = \{x, l_x(y)\} \text{ or } \{z, t\} = \{y, l_y(x)\} \\ 0 & \text{otherwise.} \end{cases}$$

- Assume $V \notin I$ and $W \in I$. We have $\mathrm{Ext}^1_{A'}(V', W') \simeq \mathrm{Hom}_A(V^I, W)$. We have

$$\{x, y\}^I = \begin{array}{ccc} \{x, l_x^{a_x(y)}(y)\} & & \{y, l_y^{a_y(x)}(x)\} \\ \vdots & & \vdots \\ \{x, l_x^{c_x - 1}(y)\} & & \{y, l_y^{c_y - 1}(x)\} \\ & \{x, y\} & \end{array}$$

where $a_x(y) \le c_x - 1$ is minimal such that $\{x, l_x^i(y)\} \in I$ for $c_x < i \le a_x(y)$ and $\{x, l_x^{a_x(y)-1}(y)\} \notin I$. As a consequence,

$$\dim \mathrm{Ext}^1_{A'}(V', W') = \begin{cases} 1 & \text{if } \{z, t\} = \{x, l_x^a(y)\} \text{ or } \{z, t\} = \{y, l_y^b(x)\} \\ 0 & \text{otherwise.} \end{cases}$$

- Assume $V, W \notin I$. We have $\mathrm{Ext}^1_{A'}(V', W') \simeq \mathrm{Hom}_{A\text{-stab}}(V^I, \Omega^{-1}W^I)$. We have

$$\Omega^{-1}(\{z,t\}^I) = \begin{array}{ccc} & \{z,t\} & \\ \{z, l_z(t)\} & & \{t, l_t(z)\} \\ \vdots & & \vdots \\ \{z, l_z^{a_z(t)-1}(t)\} & & \{t, l_t^{a_t(z)-1}(z)\} \end{array}$$

Note that the simple summands of $\mathrm{soc}\,\Omega^{-1}W^I$ are outside I, while the only simple composition factor of V^I that is not in I is its socle. As a consequence, the canonical map is an isomorphism $\mathrm{Hom}_A(V^I, \Omega^{-1}W^I) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-stab}}(V^I, \Omega^{-1}W^I)$.

Assume these spaces are non-zero, *i.e.*, they are one-dimensional. Up to swapping z and t, we have $x = z$ and $y = l_z^{a_z(t)-1}(t)$. Then x is the only vertex adjacent to y.

Conversely, assume $x = z$, $y = l_z^{a_z(t)-1}(t)$, and x is the only vertex adjacent to y. The sequence $\{x, l_x^{c_x-1}(y)\}, \{x, l_x^{c_x-2}(y)\}, \ldots, \{x, l_x^{a_x(y)}(y)\}$ is the beginning of the sequence $\{z, l_z^{(a_z(t)-1}(t)\}, \ldots, \{z, t\}$, since that sequence finishes with an edge outside I. Since a uniserial module of a Brauer tree algebra is determined up to isomorphism by its socle series, we deduce that there is a non-zero map $V^I \to \Omega^{-1}W^I$. As a consequence,

$$\dim \mathrm{Ext}^1_{A'}(V', W') = \begin{cases} 1 & \text{if } x_i = z_j,\ x_{1-i} = l_{z_j}^{a_{z_j}(z_{j-1})-1}(z_{j-1}) \text{ and } x_i \text{ is the only vertex adjacent} \\ & \text{to } x_{1-i}, \text{ for some } i, j \\ 0 & \text{otherwise.} \end{cases}$$

- Assume $V \in I$ and $W \notin I$. We have $\mathrm{Ext}^1_{A'}(V', W') \simeq \mathrm{Hom}_{A\text{-stab}}(\Omega V, \Omega^{-1}W^I)$. We have

$$\Omega(\{x,y\}) = \begin{array}{ccc} \{x, l_x(y)\} & & \{y, l_y(x)\} \\ \vdots & & \vdots \\ \{x, l_x^{c_x-1}a(y)\} & & \{y, l_y^{c_y-1}(x)\} \\ & \{x,y\} & \end{array}$$

Consider a $f : \Omega V \to \Omega^{-1}W^I$ that doesn't factor through a projective module. Since $\mathrm{soc}\,\Omega^{-1}W^I$ has no summand in I, we deduce that f vanishes on $\mathrm{soc}\,\Omega V \simeq \{x, y\}$. Assume $\mathrm{im}\, f$ is not uniserial. There are i, j such that $\{\{x, l_x(y)\}, \{y, l_y(x)\}\} = \{\{u, l_u^i(v)\}, \{v, l_v^j(u)\}\}$. So, there are four edges, $\{x, y\}, \{x, l_x(y)\}, \{y, l_y(x)\}, \{u, v\}$ involving four vertices: this is impossible, hence $\mathrm{im}\, f$ is uniserial. Up to swapping x and y, we can assume that $\mathrm{hd}\,\mathrm{im}\, f = \{x, l_x(y)\}$.

Assume f is not surjective. Since f is not projective, up to swapping z and t, we have $\{x, l_x(y)\} = \{z, l_z^{a_z(t)-1}(t)\}$ and $\{x, y\} \neq \{z, l_z^{a_z(t)-2}(t)\}$. So, $x \neq z$, *i.e.*, $x = l_z^{a_z(t)-1}(t)$ and $l_x(y) = z$.

If f is surjective, we have $\{x, l_x(y)\} = \{z, t\}$ and z or t has only one adjacent vertex.

Conversely, assume $\{x, l_x(y)\} = \{z, t\}$ and z or t has only one adjacent vertex. Then, there is a surjective map $\Omega V \to \Omega^{-1}W^I$. Note that any non-surjective map $\Omega V \to \Omega^{-1}W^I$ is projective. We deduce that $\dim \mathrm{Hom}_{A\text{-stab}}(\Omega V, \Omega^{-1}W^I) = 1$.

Assume $x = l_z^{a_z(t)-1}(t)$ and $l_x(y) = z$. Since $\mathrm{hd}(\Omega V)$ contains $\{x, l_x(y)\}$ with multiplicity 1 and $\mathrm{soc}(\Omega^{-1}W^I)$ contains $\{x, l_x(y)\}$ with multiplicity 1, we have a non-zero morphism $\Omega V \to \Omega^{-1}W^I$ with image isomorphic to $\{x, l_x(y)\}$. We have $\{x, y\} \neq \{z, l_z^{a_z(t)-2}(t)\}$, hence that morphism is not projective. Furthermore, $\dim \mathrm{Hom}_A(\Omega V, \Omega^{-1}W^I) = 1$.

We have shown that

$$\dim \operatorname{Ext}^1_{A'}(V', W') = \begin{cases} 1 & \text{if } \begin{cases} z_j = x_i,\ z_{1-j} = l_{x_i}(x_{1-i}) \text{ and } z_l \text{ has only one adjacent vertex,} \\ \text{for some } i, j, l \\ \text{or } x_i = l_{z_j}^{a_{z_{1-j}}-1}(z_{1-j}) \text{ and } z_j = l_{x_i}(x_{1-i}) \text{ for some } i, j \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

We deduce that the oriented tree of A' is given by Γ', and $\phi(V) = V'$ for $V \in E$.

Let us determine now the exceptional vertex and its multiplicity. Let v be the exceptional vertex of Γ and m its multiplicity.

• Assume there are two edges $V_0 = \{v, x_0\}$ and $V_1 = \{v, x_1\}$ that are not in I. We have $F(A'_{V'_i}) = A_{V_i}[1]$, hence $\dim \operatorname{Hom}_{A'}(A'_{V_0}, A'_{V_1}) = \dim \operatorname{Hom}_A(A_{V_0}, A_{V_1}) = m$. It follows that v' is the exceptional vertex of A', and its multiplicity is m.

• Assume there is exactly one edge $V = \{v, x\}$ containing v as a vertex and not contained in I. We have

$$\dim \operatorname{End}_{A'}(A'_{V'}) = \dim \operatorname{End}_A(A_V) = m + 1,$$

hence x' or v' is the exceptional vertex, with multiplicity m.

•• Assume there is an edge $W = \{v, y\}$ that is in I. We have $H^{-1}F(A'_{W'}) = \Omega^2 W_I$. Note that $[\Omega W_I : V] = m$ and V occurs in $\operatorname{hd}(\Omega W_I)$. It follows that $[\Omega^2 W_I : V] = 1$. It follows that

$$\dim \operatorname{Hom}_{A'}(A'_{V'}, A'_{W'}) = \dim \operatorname{Hom}_A(A_V, \Omega^2 W_I) = 1.$$

Since $W' = \phi(W)$ contains x', we deduce that v' is the exceptional vertex.

•• Assume V is the only edge containing v. If V is also the only edge containing x (*i.e.*, Γ has only one edge), then v' can be taken to be the exceptional vertex. If there is an edge $W = \{x, y\} \neq V$ that is not in I, then

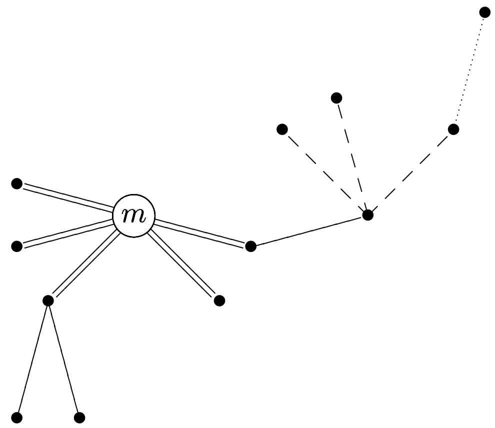
$$\dim \operatorname{Hom}_{A'}(A'_{V'}, A'_{W'}) = \dim \operatorname{Hom}_A(A_V, A_W) = 1,$$

hence x' is not an exceptional vertex. If there is an edge $W = \{x, l_x^{-1}(v)\}$ in I, then $[\Omega W_I : V] = 1$ and V is in the head of ΩW_I, hence $[\Omega^2 W_I : V] = m$. It follows as above that $\dim \operatorname{Hom}_{A'}(A'_{V'}, A'_{W'}) = m$. Since $\phi(W)$ contains v' as a vertex, we deduce that v' is the exceptional vertex.

• Assume finally v is not contained in any edge of I. Note that the edges of Γ' containing v' are precisely the images under ϕ of the edges of Γ containing v. Let $V = \{v, x\}$ be an edge. The maximal uniserial module with head V and composition factors containing v has c_v composition factors. We deduce that the maximal uniserial module with head V' and composition factors containing v' has c_v composition factors. This shows that v' is the exceptional vertex, with multiplicity m. $\square$

The algebra $A(\Gamma)^{\mathrm{opp}}$ is a Brauer tree algebra, with tree obtained from Γ by reversing the orientation. Lemma 4.20 provides now a combinatorial description of the Brauer tree of the algebra $\operatorname{End}_A(T^A(I))$: its tree is obtained by reversing the orientation in the construction $\Gamma \mapsto \Gamma'$. Finally, Proposition 5.5 shows that the effect of any perverse equivalence can be described by a combination of steps $\Gamma \mapsto \Gamma'$ (or their reverse).

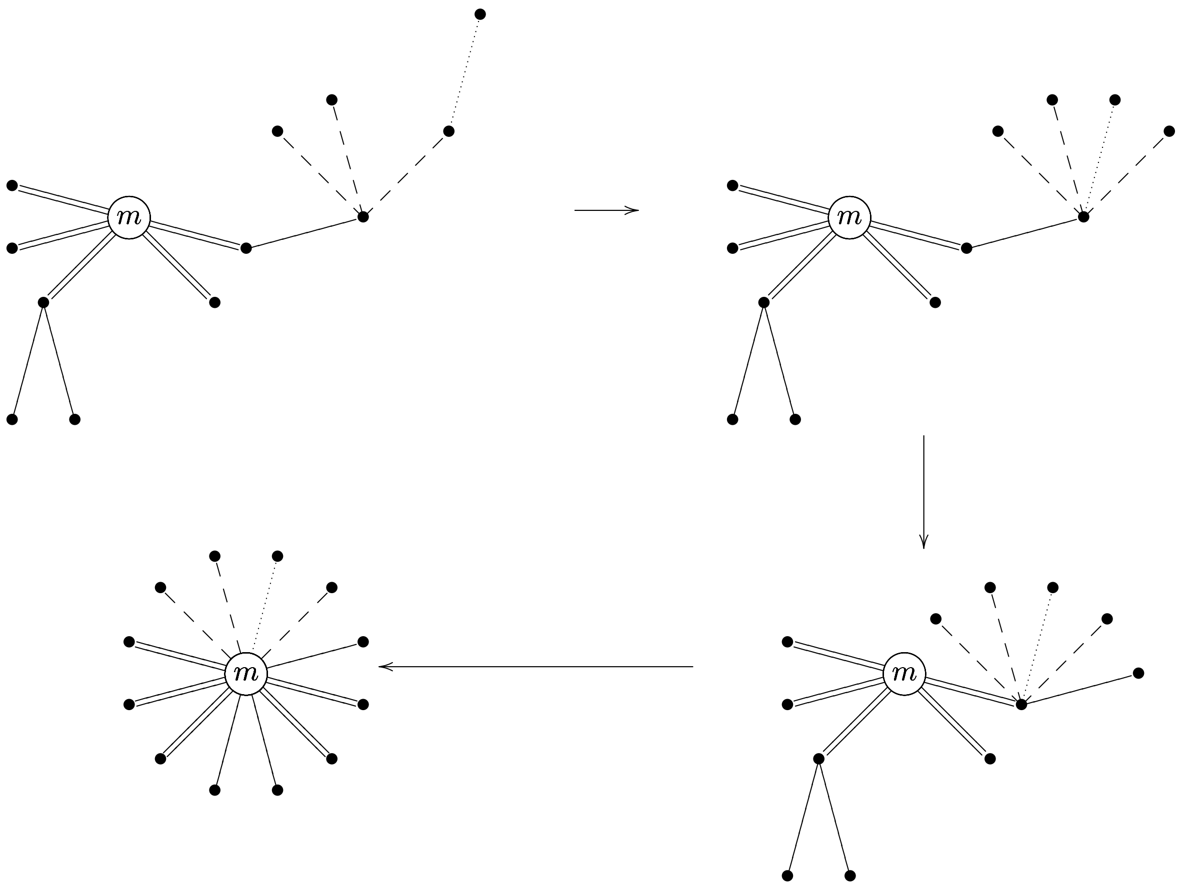
Example 8.11. Let $m \geq 1$ and let

$\Gamma =$ 

Theorem 8.8 provides an equivalence from $D^b(A(\Gamma))$ to the derived category of the Brauer tree algebra associated with a star with exceptional vertex in the center. Up to shift, it is perverse relative to $p : \{0, 1, 2, 3\} \to \mathbf{Z},\ i \mapsto -i$ and

$$S_0 = \{\ \bullet \cdots\cdots \bullet\ \} \subset S_1 = \{\ \bullet \cdots\cdots \bullet\ ,\ \bullet - - - \bullet\ \} \subset S_1 = \{\ \bullet \cdots\cdots \bullet\ ,\ \bullet - - - \bullet\ ,\ \bullet \text{———} \bullet\ \} \subset S_3 = S.$$

The perverse equivalence can be described as a composition of perverse equivalences $S_3 S_2 S_1$. The trees vary according to Proposition 8.10.



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