Chapter 10

The derived category of blocks with cyclic defect groups

by Raphaël Rouquier[1]

10.1 Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$ and field of fractions K of characteristic 0, "big enough" for the groups considered.

Let G be a finite group, $\mathcal{O}Ge$ a block of G with a cyclic defect group D. Let $\mathcal{O}N_G(D)e'$ the block of $N_G(D)$ corresponding to $\mathcal{O}Ge$.

In this work, we give a "self-contained" account of the theory of blocks with cyclic defect groups with aim the proof that the derived category of a block with cyclic defect is (in a strong sense) locally determined — a special case of Broué's conjecture [26] :

Theorem 10.1 *The blocks $\mathcal{O}Ge$ and $\mathcal{O}N_G(D)e'$ are splendidly Rickard equivalent. If $O_p(G) \neq 1$, then these blocks are in addition Morita equivalent.*

Some remarks.

- That the blocks should be not merely Rickard equivalent, but also splendidly Rickard equivalent, is Rickard's refinement of Broué's conjecture.
- When $O_p(G) \neq 1$, there need not be a splendid Morita equivalence !
- The block $\mathcal{O}N_G(D)e'$ has a very simple structure : it is Morita equivalent to $\mathcal{O}D \rtimes E$, where $E = N_G(D, e_D)/C_G(D)$ and (D, e_D) is an e-subpair (Proposition 10.2.15).

Let us outline the organization of the proof of the theorem.

[1]The author thanks for its hospitality the Isaac Newton Institute for Mathematical Sciences, Cambridge, where this paper has been written

In §10.3, we prove that a stable equivalence between $\mathcal{O}Ge$ and $\mathcal{O}D \rtimes E$ (the very local case) can always be lifted in a nice way to a Rickard equivalence. To elucidate part of the structure of $\mathcal{O}Ge$, we follow very closely Thompson's original approach [163], translated to the setting of an abstract stable equivalence of Morita type using Broué's philosophy [28]. Ideas of Green [58], Alperin [2] and Linckelmann [110] enable us to understand well enough $\mathcal{O}Ge$ and the stable equivalence to be left with a problem which is solved with (a simplification of) the ideas of [153].

Theorem 10.1 is then proved in §10.4 by induction on the order of G : it is enough to give a splendid Rickard equivalence between $\mathcal{O}Ge$ and the corresponding block $\mathcal{O}Hf$ for $H = N_G(R)$, where $O_p(G) < R \leq D$ and $[R : O_p(G)] = p$, since by induction $\mathcal{O}Hf$ is splendidly Rickard equivalent to $\mathcal{O}N_G(D)e'$.

First, we consider the case where $O_p(G) = 1$. By induction, we know that $\mathcal{O}Hf$ is Morita equivalent to $\mathcal{O}D \rtimes E$. Now, Green's correspondence gives a stable equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$, hence between $\mathcal{O}Ge$ and $\mathcal{O}D \rtimes E$, which can be lifted to a Rickard equivalence by the results of §10.3. We have then constructed a splendid Rickard equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$.

Assume now $O_p(G) \neq 1$ but $O_p(G)$ is central in G. Then, the blocks have a unique simple module and we have already a Rickard equivalence for the blocks of the groups modulo $O_p(G)$. This equivalence turns out to be nice enough to be lifted to an equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$.

Finally when $O_p(G) \neq 1$ but $O_p(G)$ is not central in G, we extend the known Rickard equivalence between the blocks of $C_G(O_p(G))$ and $C_H(O_p(G))$, following Marcus [118].

In §10.2, we give various general results (i.e., not specific to blocks with cyclic defect) pertaining to §10.3 and §10.4.

In the final part, we illustrate the constructions with an explicit study of $PSL_2(p)$.

As a final remark, let us say that the structure of blocks with cyclic defect has been determined originally by Dade [40, 45]. The existence of a Rickard equivalence is due to Rickard [136] and Linckelmann [107] (the Morita equivalence when $O_p(G) \neq 1$ goes back to Linckelmann and Puig [112, 106]).

My thanks go to M. Broué for many a patient answer to my questions and to G. Robinson for useful discussions.

10.2 Miscellany : stable equivalences, Rickard equivalences and more

10.2.1 Notations

Let G be a finite group, e a block idempotent of $\mathcal{O}G$ (i.e., a primitive idempotent of the center of $\mathcal{O}G$) and $A = \mathcal{O}Ge$ the corresponding block of $\mathcal{O}G$. By an A-module, we mean a finitely generated left A-module. The sign $\otimes$ means $\otimes_{\mathcal{O}}$. For V an $\mathcal{O}$-module, we put $V^* = \mathrm{Hom}(V, \mathcal{O})$. We write KA and kA for the algebras $K \otimes A$ and $k \otimes A$. We denote by A° the $\mathcal{O}$-algebra opposite to A.

Let H be another finite group, f a block idempotent of $\mathcal{O}H$ and $B = \mathcal{O}Hf$.

10.2.2 Stable category, stable equivalences and invariants

We follow here [25, 28] for the main concepts associated with the stable category. We denote by $A-\overline{\mathrm{mod}}$ the stable category of A, whose objects are the A-modules. Let us recall that a relatively $\mathcal{O}$-projective A-module is an A-module of the form $A\otimes_{\mathcal{O}} U$ for some $\mathcal{O}$-module U. Given two A-modules V and W, the set of morphisms $\overline{\mathrm{Hom}}(V,W)$ in $A-\overline{\mathrm{mod}}$ is $\mathrm{Hom}(V,W)$ modulo the submodule of $\mathcal{O}$-projective morphisms, i.e., morphisms which factor through a relatively $\mathcal{O}$-projective module.

Let $\Omega_{A\otimes A^\circ}$ be the kernel of the multiplication map $A\otimes A^\circ \to A$. Let $\Omega^{-1}_{A\otimes A^\circ}$ be the cokernel of the map $\eta_A : A \to A\otimes A^\circ$ dual to the multiplication. These are $(A\otimes A^\circ)$-modules whose restrictions to A and A° are projective. The functors $\Omega_{A\otimes A^\circ}\otimes_A -$ and $\Omega^{-1}_{A\otimes A^\circ}\otimes_A -$ induce inverse self-equivalences of $A-\overline{\mathrm{mod}}$. Given an A-module V, we denote by ΩV an A-module without projective direct summand such that $\Omega_{A\otimes A^\circ}\otimes_A V\simeq \Omega V$ in $A-\overline{\mathrm{mod}}$ — such a module is unique up to isomorphism.

Let $f : V \to W$ be an injection between A-modules. There exists a morphism $\varphi : W \to (A\otimes A^\circ)\otimes_A V$ such that we have a commutative diagram :

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "\eta_A\otimes 1"'] & W \arrow[d, "\varphi"] \\ & (A\otimes A^\circ)\otimes_A V \end{tikzcd}$$

Let δ be the induced map $W/V \to \Omega^{-1}_{A\otimes A^\circ}\otimes_A V$. This gives rise to a "standard distinguished triangle" :

$$V \xrightarrow{f} W \to W/V \overset{\delta}{\rightsquigarrow} \Omega^{-1}_{A\otimes A^\circ}\otimes_A V.$$

The category $A-\overline{\mathrm{mod}}$ becomes a triangulated category, with translation functor $\Omega_{A\otimes A^\circ}\otimes_A -$ and distinguished triangles the triangles isomorphic to the standard distinguished triangles.

We denote by $R^p(A)$ the kernel of the decomposition map $R(KA)\to R(kA)$. We denote by $\bar{R}(A)$ the quotient of $R(KA)$ by the submodule generated by the characters of the projective A-modules. Note that the restriction of the usual bilinear form on $R(KA)\times R(KA)$ to $R^p(KA)\times R(KA)$ factors through $R^p(KA)\times \bar{R}(A)$ and gives a perfect pairing between $R^p(KA)$ and $\times\bar{R}(A)$.

Let M be an $(A\otimes B^\circ)$-module, projective as an A-module and as a B°-module.

The module M induces a *stable equivalence of Morita type* between A and B if

$$M\otimes_B M^* \simeq A\oplus \text{projective modules and}$$

$$M^*\otimes_A M\simeq B\oplus \text{projective modules.}$$

Since we are dealing with blocks, Rickard has noted that it is enough to check one of the properties :

Lemma 10.2.1 *Let M be an $(A\otimes B^\circ)$-module, projective as an A-module and as a B°-module. If A is not a matrix algebra over $\mathcal{O}$ and*

$$M\otimes_B M^*\simeq A\oplus \textit{projective modules},$$

then M induces a stable equivalence between A and B.

Proof. Since A is not a matrix algebra over $\mathcal{O}$, it follows that A is not a projective $(A \otimes A^\circ)$-module, hence M is not projective.

The functors $M \otimes_B -$ and $M^* \otimes_A -$ between $B - \overline{\text{mod}}$ and $A - \overline{\text{mod}}$ are left and right adjoint. Consider the associated natural maps $\eta : B \to M^* \otimes_A M$ and $\varepsilon : M^* \otimes_A M \to B$. By [116, §IV.1, Theorem 1], the map

$$1 \otimes \eta : M \otimes_B B \to M \otimes_B M^* \otimes_A M$$

is a split injection. As $M \otimes_B M^* \otimes_A M$ is isomorphic to M in $(A \otimes B^\circ) - \overline{\text{mod}}$, the map $1 \otimes \eta$ is an isomorphism in $(A \otimes B^\circ) - \overline{\text{mod}}$. Similarly, the map

$$1 \otimes \varepsilon : M \otimes_B M^* \otimes_A M \to M \otimes_B B$$

is an isomorphism in $(A \otimes B^\circ) - \overline{\text{mod}}$. It follows that the composite $(1 \otimes \varepsilon)(1 \otimes \eta) = 1 \otimes (\varepsilon\eta)$ is an automorphism of M in $(A \otimes B^\circ) - \overline{\text{mod}}$. Since B is a block, the algebra $\text{End}_{B \otimes B^\circ}(B)$ is local. If the composite $\varepsilon\eta$ is not invertible, it is in the radical of this algebra, hence gives an element in the radical of $\overline{\text{End}}_{B \otimes B^\circ}(B)$. This can't happen, since after applying $M \otimes_B -$ we get a non-zero isomorphism ! Hence, $\varepsilon\eta$ is invertible and as a consequence, $M^* \otimes_A M \simeq B \oplus P$ for some module P.

Now,

$$M \otimes_B (M^* \otimes_A M) \simeq M \oplus M \otimes_B P$$

$$\text{and } (M \otimes_B M^*) \otimes_A M \simeq M \oplus \text{ projective modules,}$$

hence $M \otimes_B P$ is projective. As P is isomorphic to a direct summand of $M^* \otimes_A M \otimes_B P$, we finally deduce that P is projective. ∎

Assume we have a module M inducing a stable equivalence between A and B.

Then, the functors $M \otimes_B -$ and $M^* \otimes_A -$ induce inverse equivalences of triangulated categories between the stable categories $B - \overline{\text{mod}}$ and $A - \overline{\text{mod}}$. The maps $R(KA) \to R(KB)$ and $R(KB) \to R(KA)$ induced by $M \otimes_B -$ and $M^* \otimes_A -$ give rise to

- inverse isometries between $R^p(A)$ and $R^p(B)$ and
- inverse isomorphisms between $\bar{R}(A)$ and $\bar{R}(B)$.

Lemma 10.2.2 *Let V be an indecomposable non relatively $\mathcal{O}$-projective B-module. Let W be an A-module without projective direct summands such that*

$$M \otimes_B V \simeq W \oplus \textit{ relatively } \mathcal{O}\textit{-projective modules.}$$

Then, W is indecomposable.

If V is free over $\mathcal{O}$ and $V \otimes k$ is indecomposable, then, $W \otimes k$ is indecomposable.

Proof. Assume $W = W_1 \oplus W_2$, with $W_1, W_2 \neq 0$. Then,

$$V \oplus \text{ relatively } \mathcal{O}\text{-projective modules} \simeq M^* \otimes_A W_1 \oplus M^* \otimes_A W_2.$$

As $M^* \otimes_A W_1$ and $M^* \otimes_A W_2$ contain both a non relatively $\mathcal{O}$-projective direct summand, we get a contradiction.

Assume now $V \otimes k$ is indecomposable and $W \otimes k$ has a non-zero projective direct summand. So, there is a projective A-module P and a surjective map $W \twoheadrightarrow P \otimes k$.

As W is free over $\mathcal{O}$, we have $\mathrm{Ext}^1(W, P) = 0$, hence the surjection lifts to a map $W \to P$, which is forced to be surjective by Nakayama's lemma, hence which splits, contradicting the property of W to have no projective direct summand. ∎

The following very useful result comes from [110] :

Lemma 10.2.3 *The $(A \otimes B^\circ)$-module M is the direct sum of a non-projective indecomposable module and of a projective module.*

If M is indecomposable, then for any simple B-module V, the A-module $M \otimes_B V$ is indecomposable.

Proof. Let $M = M_1 \oplus M_2$. Since $M^* \otimes_A M \simeq B \oplus$ projective modules, we have

$$M^* \otimes_A M_1 \oplus M^* \otimes_A M_2 \simeq B \oplus \text{ projective modules.}$$

As B is indecomposable (as a $(B \otimes B^\circ)$-module), there exists $i \in \{1, 2\}$ such that $M^* \otimes_A M_i$ is projective, so $M \otimes_B M^* \otimes_A M_i$ is projective. Now, $(M \otimes_B M^*) \otimes_A M_i \simeq M_i \oplus$ projective modules, hence M_i is projective.

Let us assume now that M is indecomposable. Denote by $\mathrm{soc}(kA)$ the largest semi-simple kA-submodule of kA. Recall that a kA-module V has no projective direct summand if and only if $\mathrm{soc}(kA)V = 0$. We have $\mathrm{soc}(kA \otimes kB^\circ) = \mathrm{soc}(kA) \otimes \mathrm{soc}(kB^\circ)$. Since M has no projective direct summand, $\mathrm{soc}(kA \otimes kB^\circ)M = 0$, hence $\mathrm{soc}(kA)(M \otimes_B \mathrm{soc}(kB)) = 0$, which means that $M \otimes_B \mathrm{soc}(kB)$ has no projective direct summand. But, if V is a simple B-module, it is a direct summand of $\mathrm{soc}(kB)$, so $M \otimes_B V$ has no projective direct summand : as M induces a stable equivalence, $M \otimes_B V$ is the direct sum of an indecomposable non-projective module and of a projective module and the lemma follows. ∎

10.2.3 Derived category and Rickard equivalences

The definition of a splendid Rickard equivalence follows [144]. We denote by $\mathcal{D}^b(A)$ the bounded derived category of A and by $K^b(A)$ the homotopy category of bounded complex of A-modules. Viewing a module as a complex concentrated in degree 0, we have a fully faithful functor from the category of A-modules to $\mathcal{D}^b(A)$ and to $K^b(A)$.

Let C be a bounded complex of $(A \otimes B^\circ)$-modules, all of which are projective as A-modules and as B°-modules.

The complex C induces a *Rickard equivalence* between A and B (or is a *Rickard complex*) if

$$C \otimes_B C^* \simeq A \oplus \text{ complex homotopy equivalent to } 0$$

$$C^* \otimes_A C \simeq B \oplus \text{ complex homotopy equivalent to } 0.$$

Assume we have such a complex C. Then, the functors $C \otimes_B -$ and $C^* \otimes_A -$ induce inverse equivalences of triangulated categories between the categories $\mathcal{D}^b(B)$ and $\mathcal{D}^b(A)$ and between the categories $K^b(A)$ and $K^b(B)$.

They give rise to inverse "perfect" isometries between $R(KA)$ and $R(KB)$, to inverse isomorphisms between the centers of A and B,... [26] and sections 6.3.1, 6.3.3 as well as 6.3.2).

Suppose the complex C has homology only in one degree, isomorphic to M. If C is a Rickard complex, then M induces a Morita equivalence between A and B.

The converse doesn't hold : if M induces a Morita equivalence between A and B, we deduce that $C \otimes_B C^* \simeq A \oplus R$, where R has zero homology, but might be non homotopy equivalent to 0. Nevertheless, if C has only one non-zero term, M, then C induces a Rickard equivalence if and only if M induces a Morita equivalence.

As for stable equivalences, Rickard [144] has proved the following :

Lemma 10.2.4 *Let C be a bounded complex of $(A \otimes B^\circ)$-modules, all of which are projective as A-modules and as B°-modules. If*

$$C \otimes_B C^* \text{ is homotopy equivalent to } A,$$

then C induces a Rickard equivalence between A and B.

(The proof is similar to the proof of Lemma 10.2.1)

In $\mathcal{D}^b(A \otimes B^\circ)$, the complex C is isomorphic to a bounded complex of modules which are all projective except the degree n term N, for some integer n (see lemma 6.3.14). The module $M = \Omega^n N$ induces a stable equivalence between A and B [139, Corollary 5.5].

Lemma 10.2.5 *Let C be a bounded complex of $(A \otimes B^\circ)$-modules, all of whose terms are projective but the degree 0 term M, which is projective as an A-module and as a B°-module. Assume M induces a stable equivalence between A and B. Then, A is a direct summand of $C \otimes_B C^*$ and B is a direct summand of $C^* \otimes_A C$.*

Proof. The functors $C \otimes_B -$ and $C^* \otimes_A -$ between $K^b(B)$ and $K^b(A)$ are left and right adjoint and induce natural maps $\eta : B \to C^* \otimes_A C$ and $\varepsilon : C^* \otimes_A C \to B$. Let $\eta' : B \to M^* \otimes_A M$ and $\varepsilon' : M^* \otimes_A M \to B$ be the natural maps induced by the functors $M \otimes_B -$ and $M^* \otimes_A -$. The composite maps $\varepsilon\eta$ and $\varepsilon'\eta'$ have the same image in $\overline{\mathrm{End}}_{B \otimes B^\circ}(B)$. As M induces a stable equivalence between A and B, the map $\varepsilon'\eta'$ is invertible, hence $\varepsilon\eta$ is invertible. It follows that B is isomorphic to a direct summand of $C^* \otimes_A C$. The second property is obtained by a similar proof. ∎

Assume A and B have a common defect group D. When the modules occurring in C are direct summands of permutation modules induced from $\Delta D = \{(x, x^{-1}) | x \in D\} \subset G \times H^\circ$, the complex C is called *splendid* and the equivalence of derived categories it induces is called a *splendid Rickard equivalence.* Such an equivalence gives rise to derived equivalences for corresponding blocks of centralizers of p-subgroups of D in G and H ([144], [69] ; cf also [130],[111]).

Following a suggestion of M. Harris, we have

Lemma 10.2.6 *Assume $H' \subseteq H$ are two subgroups of G. Let f' be a block idempotent of H'. Assume e, f and f' have defect D. Let C be a splendid Rickard complex for $\mathcal{O}Ge \otimes (\mathcal{O}Hf)^\circ$ and C' a splendid Rickard complex for $\mathcal{O}Hf \otimes (\mathcal{O}H'f')^\circ$. Then, $C \otimes_{\mathcal{O}Hf} C'$ is a splendid Rickard complex for $\mathcal{O}Ge \otimes (\mathcal{O}H'f')^\circ$.*

Proof. Composing Rickard equivalences gives a Rickard equivalence. The point is to check that the modules in $C \otimes_{\mathcal{O}Hf} C'$ have the required properties.

Let $M = \mathrm{Ind}_{\Delta Q}^{G\times H^\circ}\mathcal{O} \simeq \mathcal{O}G\otimes_{\mathcal{O}Q}\mathcal{O}H$ be the trivial module induced from a diagonal subgroup ΔQ to $G\times H^\circ$, where $Q \leq D$. Let $N = \mathrm{Ind}_{\Delta R}^{H\times H'^\circ}\mathcal{O}$, where $R \leq D$. We have

$$M\otimes_{\mathcal{O}H} N \simeq \mathcal{O}G\otimes_{\mathcal{O}Q}\mathcal{O}H\otimes_{\mathcal{O}R}\mathcal{O}H' \simeq \mathrm{Ind}_{Q\times R^\circ}^{G\times H'^\circ}\mathrm{Res}_{Q\times R^\circ}^{H\times H^\circ}\mathrm{Ind}_{\Delta H}^{H\times H^\circ}\mathcal{O}.$$

By Mackey's formula, we obtain

$$\begin{aligned} M\otimes_{\mathcal{O}H} N &\simeq \bigoplus_{(h_1,h_2)\in Q\times R^\circ\backslash H\times H^\circ/\Delta H} \mathrm{Ind}_{(\Delta H)^{(h_1,h_2)}\cap(Q\times R^\circ)}^{G\times H'^\circ}\mathcal{O} \\ &\simeq \bigoplus_{h\in Q\backslash H/R}\mathrm{Ind}_{\Delta(Q^h\cap R)}^{G\times H'^\circ}\mathcal{O} \end{aligned}$$

It follows that $M\otimes_{\mathcal{O}H} N$ is a direct sum of direct summands of permutations modules induced from ΔD to $G\times H'^\circ$. ∎

Homotopy equivalence is detected by a Sylow p-subgroup :

Lemma 10.2.7 *Let G' be a subgroup of G of index prime to p. Let $g : X \to Y$ be a morphism between two bounded complexes of $\mathcal{O}G$-modules. Then, g is a homotopy equivalence if and only if $\mathrm{Res}_{G'}^{G} g$ is a homotopy equivalence.*

Proof. Let Z be the cone of g. Then, g is a homotopy equivalence if and only Z is homotopy equivalent to 0. Assume Z is not homotopy equivalent to 0. Up to homotopy, we can assume $Z = \cdots Z_{r-1} \xrightarrow{\alpha} Z_r \to 0$, where α is not a split surjection. So, α gives a non-zero element in $\mathrm{Ext}^1_{\mathcal{O}G}(Z_r, \mathrm{Ker}\,\alpha)$. As $[G : G']$ is prime to p, the restriction map $\mathrm{Ext}^1_{\mathcal{O}G}(Z_r,\mathrm{Ker}\,\alpha) \to \mathrm{Ext}^1_{\mathcal{O}G'}(\mathrm{Res}_{G'}^G Z_r, \mathrm{Res}_{G'}^G \mathrm{Ker}\,\alpha)$ is injective, hence the restriction to G' of α doesn't split, i.e., the restriction to G' of Z is not homotopy equivalent to 0 and the restriction to G' of g is not a homotopy equivalence. ∎

The following lemma due to A. Marcus [118] shows how to solve (certain) extension problems for Rickard complexes :

Lemma 10.2.8 *Let G' and H' be normal subgroups of G and H with $G/G' = H/H' = E$. Assume e and f are block idempotents of $\mathcal{O}G'$ and $\mathcal{O}H'$. Let C' be a Rickard complex for $e\mathcal{O}G'$ and $f\mathcal{O}H'$. Assume C' extends to a complex $\tilde{C}$ of $\mathcal{O}L$-modules, where $L = \{(g,h)\in G\times H^\circ | (gG', hH'^\circ)\in \Delta E\}$.*

If E has order prime to p or if C has only one non-zero term, then, $C = \mathrm{Ind}_L^{G\times H^\circ}\tilde{C}$ is a Rickard complex for $e\mathcal{O}G$ and $f\mathcal{O}H$.

Proof. We have

$$\mathrm{Res}_{G\times H'^\circ}^{G\times H^\circ} C = \mathrm{Res}_{G\times H'^\circ}^{G\times H^\circ}\mathrm{Ind}_L^{G\times H^\circ}\tilde{C} \simeq \mathrm{Ind}_{G'\times H'^\circ}^{G\times H'^\circ} C'$$

by Mackey's formula. Similarly,

$$\mathrm{Res}_{G'\times H^\circ}^{G\times H^\circ} C \simeq \mathrm{Ind}_{G'\times H'^\circ}^{G'\times H^\circ} C'.$$

Consequently,

$$\mathrm{Res}_{G'\times G^\circ}^{G\times G^\circ} C\otimes_{\mathcal{O}H} C^* \simeq C'\otimes_{\mathcal{O}H'}\mathcal{O}H\otimes_{\mathcal{O}H} C^* \simeq C'\otimes_{\mathcal{O}H'} C'^*\otimes_{\mathcal{O}G'}\mathcal{O}G$$

is homotopy equivalent to $e\mathcal{O}G' \otimes_{\mathcal{O}G'} \mathcal{O}G \simeq e\mathcal{O}G$.

In particular, $C \otimes_{\mathcal{O}H} C^*$ has homology only in degree 0, M, with $\mathrm{Res}^{G\times G^\circ}_{G'\times G^\circ} M \simeq e\mathcal{O}G$. The natural map $g : e\mathcal{O}G \to C \otimes_{\mathcal{O}H} C^*$ induces a map $g_0 = H_0(g) : e\mathcal{O}G \to M$. As the restriction of M to $\mathcal{O}G^\circ$ is isomorphic to $e\mathcal{O}G$, the $\mathcal{O}G^\circ$-submodule of M generated by $g_0(e)$, the unit of $M = \mathrm{End}_{K^b(\mathcal{O}H^\circ)}(C)$, must be isomorphic to $e\mathcal{O}G$, hence g_0 is an isomorphism.

If the index of $G' \times G^\circ$ in $G \times G^\circ$ is prime to p, it follows that g is a homotopy equivalence, since its restriction to $G' \times G^\circ$ is a homotopy equivalence (Lemma 10.2.7). Similarly, one proves that $C^* \otimes_{\mathcal{O}G} C$ is homotopy equivalent to $f\mathcal{O}H$ and the lemma follows. ∎

Let us finally consider normal p-subgroups.

Let R be a normal p-subgroup of G. We put $\bar{G} = G/R$. For M an $\mathcal{O}G$-module, we put $\bar{M} = \mathcal{O}\bar{G} \otimes_{\mathcal{O}G} M$ and for $\varphi : M \to N$ a morphism of $\mathcal{O}G$-modules, we put $\bar{\varphi} = 1 \otimes \varphi : \bar{M} \to \bar{N}$.

Lemma 10.2.9 *Let $\varphi : M \to N$ be a morphism between $\mathcal{O}G$-modules. Then, φ is surjective if and only if $\bar{\varphi}$ is surjective.*

Let N be an $\mathcal{O}G$-module whose restriction to $\mathcal{O}R$ is projective. Then,

- *every projective direct summand of $\bar{N}$ lifts to a projective direct summand of N,*
- *if M is a projective cover of N, then $\bar{M}$ is a projective cover of $\bar{N}$.*

Proof. Assume $\bar{\varphi}$ is surjective. Then, $N = \varphi(M) + \mathrm{rad}(\mathcal{O}R)N$, hence $N = \varphi(M)$ by Nakayama's lemma and φ is surjective.

Note that given a projective indecomposable $\mathcal{O}G$-module N, the projective $\mathcal{O}\bar{G}$-module $\bar{N}$ is indecomposable, and we obtain a bijection between the set of isomorphism classes of projective indecomposable $\mathcal{O}G$-modules and the set of isomorphism classes of projective indecomposable $\mathcal{O}\bar{G}$-modules.

Let M be a projective $\mathcal{O}G$-module and $f : N \to \bar{M}$ a surjection. Let M_0 be the kernel of the canonical surjective map $M \to \bar{M}$. Then, M_0 is a direct summand of $\mathrm{Ind}_R^G \mathrm{Res}_R^G M_0$. In particular,

$$\mathrm{Ext}^1_{\mathcal{O}G}(N, M_0) \leq \mathrm{Ext}^1_{\mathcal{O}G}(N, \mathrm{Ind}_R^G \mathrm{Res}_R^G M_0) \simeq \mathrm{Ext}^1_{\mathcal{O}R}(\mathrm{Res}_R^G N, \mathrm{Res}_R^G M_0).$$

Since $\mathrm{Res}_R^G N$ is projective, this last Ext-group is zero, hence the morphism f lifts to a morphism $\varphi : N \to M$. By the first part of the lemma, this morphism is surjective, hence split surjective, since M is projective.

If $\varphi : M \to N$ is a projective cover of N, then $\bar{\varphi}$ is a surjective map. Assume there is a direct summand M' of $\bar{M}$ such that the restriction of $\bar{\varphi}$ to M' is surjective. Using the second part of the lemma, we can lift M' to a direct summand M'' of M, and the first part of the lemma shows that the restriction of φ to M'' is surjective. So, $M = M''$ and $\bar{M} = M'$, i.e., $\bar{M}$ is a projective cover of $\bar{N}$. ∎

Homotopy equivalence between $\mathcal{O}G$-modules can sometimes be controlled by $\mathcal{O}\bar{G}$:

Lemma 10.2.10 *Let n be an integer and $\varphi : X \to Y$ be a morphism between two bounded complexes of $\mathcal{O}G$-modules whose components are direct sums of indecomposable modules with trivial source and vertices Q such that $|Q \cap R| = n$.*

Then, φ is a homotopy equivalence if and only if $\bar{\varphi}$ is a homotopy equivalence.

Proof. Let P be a Sylow p-subgroup of G. Let Q be a subgroup of P. We have

$$\mathrm{Res}_P^G \mathrm{Ind}_Q^G \mathcal{O} \simeq \bigoplus_{g \in P\backslash G/Q} \mathrm{Ind}_{P\cap Q^g}^P \mathcal{O}$$

and $|(P \cap Q^g) \cap R| = |Q \cap R|$. Since $\mathrm{Ind}_{P\cap Q^g}^P \mathcal{O}$ is indecomposable (it has a unique simple quotient), every direct summand of $\mathrm{Res}_P^G \mathrm{Ind}_Q^G \mathcal{O}$ has a vertex Q' such that $|Q' \cap R| = |Q \cap R|$. Hence, the assumptions of the lemma hold if we restrict the modules from G to P and replace G by P. By Lemma 10.2.7, we can then assume that $G = P$.

As in the proof of Lemma 10.2.7, we are reduced to the following problem. Let $\varphi : M \to N$ be a morphism between $\mathcal{O}G$-modules whose components are direct sums of indecomposable modules with trivial source and vertices Q such that $|Q \cap R| = n$. Assume $\bar{\varphi}$ is a split surjection. Then, we have to show that φ is a split surjection. By Lemma 10.2.9, we know already that φ is a surjection.

We may of course assume that N is indecomposable. But, this implies that N has a unique simple quotient, hence that $\bar{N}$ has a unique simple quotient and is therefore indecomposable. If $M = M_1 \oplus M_2$, then the restriction of $\bar{\varphi}$ to $\bar{M}_1$ or to $\bar{M}_2$ is a split surjection : so, we can assume M, hence $\bar{M}$, are indecomposable as well. Then, φ is a split surjection if and only if it is an isomorphism, i.e., if and only if M and N have the same rank. Let Q and Q' be subgroups of G such that $M \simeq \mathrm{Ind}_Q^G \mathcal{O}$ and $N \simeq \mathrm{Ind}_{Q'}^G \mathcal{O}$. We have $\mathrm{rank}\bar{M} = [G : RQ]$ and $\mathrm{rank}\bar{N} = [G : RQ']$. Since $\bar{\varphi}$ is an isomorphism, we have $|RQ| = |RQ'|$. But, by assumption, $|R \cap Q| = |R \cap Q'|$, hence $|Q| = |Q'|$ and φ is an isomorphism. ∎

Under good circumstances, normal p-subgroups can be factored out, in order to check that a complex induces a Rickard equivalence :

Lemma 10.2.11 *Let R be a common normal p-subgroup of G and H and C a bounded complex of $(\mathcal{O}Ge \otimes (\mathcal{O}Hf)^\circ)$-modules, each of which is a direct sum of indecomposable modules with trivial source and vertices Q such that $Q \cap (1 \times H^\circ) = Q \cap (G \times 1) = 1$ and $R \times R^\circ \leq (R \times 1)Q = (1 \times R^\circ)Q$. Let $\bar{e}$ and $\bar{f}$ be the images of e and f through the canonical morphisms $\mathcal{O}G \to \mathcal{O}\bar{G}$ and $\mathcal{O}H \to \mathcal{O}\bar{H}$ and $\bar{C} = \mathcal{O}\bar{G}\bar{e} \otimes_{\mathcal{O}G} C \otimes_{\mathcal{O}H} \mathcal{O}\bar{H}\bar{f}$.*

Then, C is a Rickard complex for $\mathcal{O}Ge$ and $\mathcal{O}Hf$ if and only $\bar{C}$ is a Rickard complex for $\mathcal{O}\bar{G}\bar{e}$ and $\mathcal{O}\bar{H}\bar{f}$.

Proof. Let Q be a p-subgroup of $G \times H^\circ$ such that $Q \cap (1 \times H^\circ) = Q \cap (G \times 1) = 1$ and $R \times R^\circ \leq (R \times 1)Q = (1 \times R^\circ)Q$. By Mackey's formula, we have

$$\mathrm{Res}_{R\times R^\circ}^{G\times H^\circ} \mathrm{Ind}_Q^{G\times H^\circ} \mathcal{O} \simeq \bigoplus_{g \in R\times R^\circ \backslash G\times H^\circ/Q} \mathrm{Ind}_{(R\times R^\circ)\cap Q^g}^{R\times R^\circ} \mathcal{O}$$

and $Q' = (R \times R^\circ) \cap Q^g$ satisfies $Q' \cap (1 \times H^\circ) = Q' \cap (G \times 1) = 1$ and $R \times R^\circ = (R \times 1)Q' = (1 \times R^\circ)Q'$. Let $N = \mathrm{Ind}_{Q'}^{R\times R^\circ} \mathcal{O}$. We have

$$\mathcal{O} \otimes_{\mathcal{O}R} N \otimes_{\mathcal{O}R} \mathcal{O} \simeq \mathrm{Hom}_{\mathcal{O}R\otimes(\mathcal{O}R)^\circ}(N, \mathcal{O}) \simeq \mathcal{O}.$$

We have

$$\mathrm{Res}_{R\times 1}^{R\times R^\circ} N \simeq \mathrm{Ind}_1^{R\times 1} \mathcal{O}$$

since $R \times R^\circ = (R \times 1)Q'$ and $(R \times 1) \cap Q' = 1$. It follows that $\mathcal{O} \otimes_{\mathcal{O}R} N \simeq \mathcal{O}$. So, the canonical surjective map

$$\mathcal{O} \otimes_{\mathcal{O}R} N \to \mathcal{O} \otimes_{\mathcal{O}R} N \otimes_{\mathcal{O}R} \mathcal{O}$$

is an isomorphism. Similarly, the canonical surjective map

$$N \otimes_{\mathcal{O}R} \mathcal{O} \to \mathcal{O} \otimes_{\mathcal{O}R} N \otimes_{\mathcal{O}R} \mathcal{O}$$

is an isomorphism. So, if M is a direct summand of $\mathrm{Ind}_Q^{G\times H^\circ}\mathcal{O}$, the canonical maps give isomorphisms of $\mathcal{O}(G \times H^\circ)$-modules

$$M \otimes_{\mathcal{O}R} \mathcal{O} \simeq \mathcal{O} \otimes_{\mathcal{O}R} M \simeq \mathcal{O} \otimes_{\mathcal{O}R} M \otimes_{\mathcal{O}R} \mathcal{O}.$$

Consequently, we have

$$\mathcal{O}\bar{G} \otimes_{\mathcal{O}G} C \otimes_{\mathcal{O}H} C^* \otimes_{\mathcal{O}G} \mathcal{O}\bar{G} \simeq \mathcal{O}\bar{G} \otimes_{\mathcal{O}G} C \otimes_{\mathcal{O}H} \mathcal{O}\bar{H} \otimes_{\mathcal{O}H} C^* \otimes_{\mathcal{O}G} \mathcal{O}\bar{G} \simeq \bar{C} \otimes_{\mathcal{O}\bar{H}} \bar{C}^*$$

and

$$\mathcal{O}\bar{H} \otimes_{\mathcal{O}H} C^* \otimes_{\mathcal{O}G} C \otimes_{\mathcal{O}H} \mathcal{O}\bar{H} \simeq \bar{C}^* \otimes_{\mathcal{O}\bar{G}} \bar{C}.$$

The components of $C \otimes_{\mathcal{O}H} C^*$ are direct sums of direct summands of modules

$$\mathrm{Ind}_{Q_1}^{G\times H^\circ}\mathcal{O} \otimes_{\mathcal{O}H} \left(\mathrm{Ind}_{Q_2}^{G\times H^\circ}\mathcal{O}\right)^\circ \simeq \mathcal{O}G \otimes_{\mathcal{O}Q_1} \mathcal{O}H \otimes_{\mathcal{O}Q_2} \mathcal{O}G$$

where $Q_i \cap (G \times 1) = Q_i \cap (1 \times H^\circ) = 1$ and $R \times R^\circ \leq (R \times 1)Q_i = (1 \times R^\circ)Q_i$ for $i \in \{1, 2\}$.

Let $\varphi : G \times H^\circ \to G$ and $\psi : G \times H^\circ \to H^\circ$ be the canonical projections. Then, we have isomorphisms $Q_i \to \varphi(Q_i)$ and $Q_i \to \psi(Q_i)$ and R is contained in $\varphi(Q_i)$ and $\psi(Q_i)$.

We have

$$\mathrm{Res}^{H^\circ\times H}_{\psi(Q_1)\times\psi(Q_2)^\circ}\mathcal{O}H \simeq \bigoplus_{g\in\psi(Q_1)\times\psi(Q_2)^\circ\backslash H\times H^\circ/\Delta H} \mathrm{Ind}^{\psi(Q_1)\times\psi(Q_2)^\circ}_{(\psi(Q_1)\times\psi(Q_2)^\circ)\cap(\Delta H)^g}\mathcal{O}.$$

Note that, for $Q' = (\psi(Q_1) \times \psi(Q_2)^\circ) \cap (\Delta H)^g$, we have $|Q' \cap (R \times R^\circ)| = |\Delta R \cap (\psi(Q_1) \times \psi(Q_2)^\circ)^{g^{-1}}| = |R|$. Hence,

$$\mathrm{Ind}_{Q_1}^{G\times H^\circ}\mathcal{O} \otimes_{\mathcal{O}H} \left(\mathrm{Ind}_{Q_2}^{G\times H^\circ}\mathcal{O}\right)^\circ \simeq \mathrm{Ind}^{G\times G^\circ}_{Q_1\times Q_2^\circ}\mathrm{Res}^{H^\circ\times H}_{\psi(Q_1)\times\psi(Q_2)^\circ}\mathcal{O}H$$

is a direct sum of modules $\mathrm{Ind}_{Q''}^{G\times G^\circ}\mathcal{O}$, where $|Q'' \cap (R \times R^\circ)| = |R|$. Similarly, the components of $C^* \otimes_{\mathcal{O}G} C$ have trivial source and vertices Q'' such that $|Q'' \cap (R \times R^\circ)| = |R|$.

Let $\eta : \mathcal{O}Ge \to C \otimes_{\mathcal{O}H} C^*$ and $\eta' : \mathcal{O}Hf \to C^* \otimes_{\mathcal{O}G} C$ be the natural maps induced by the functors $C \otimes_{\mathcal{O}Hf} -$ and $C^* \otimes_{\mathcal{O}Ge} -$. Then, $\bar{\eta} = 1 \otimes_{\mathcal{O}R} \eta \otimes_{\mathcal{O}R} 1 : \mathcal{O}\bar{G}\bar{e} \to \bar{C} \otimes_{\mathcal{O}\bar{H}} \bar{C}^*$ and $\bar{\eta}' = 1 \otimes_{\mathcal{O}R} \eta' \otimes_{\mathcal{O}R} 1 : \mathcal{O}\bar{H}\bar{f} \to \bar{C}^* \otimes_{\mathcal{O}\bar{G}} \bar{C}$ are the natural maps induced by the functors $\bar{C} \otimes_{\mathcal{O}\bar{H}\bar{f}} -$ and $\bar{C}^* \otimes_{\mathcal{O}\bar{G}\bar{e}} -$. By Lemma 10.2.10, η (resp. η') is a homotopy equivalence if and only if $\bar{\eta}$ (resp. $\bar{\eta}'$) is a homotopy equivalence.

∎

10.2.4 Some more lemmas

For a module V, we denote by P_V a projective cover of V.

Lemma 10.2.12 *Let M be an $(A \otimes B^\circ)$-module. A projective cover of M is*

$$\bigoplus_W P_{M \otimes_B W} \otimes P_W^*$$

where W runs over a complete set of representatives of isomorphism classes of simple B-modules.

Proof. Let V be a simple A-module and W a simple B-module. We have an isomorphism of $(A \otimes A^\circ)$-modules

$$\mathrm{Hom}_{B^\circ}(M, V \otimes W^*) \simeq \mathrm{Hom}_{\mathcal{O}}(M \otimes_B W, V)$$

given by $f \mapsto (m \otimes_B w \mapsto f(m)(w))$ and the lemma follows from the isomorphism

$$\mathrm{Hom}_{A \otimes B^\circ}(M, V \otimes W^*) \simeq \mathrm{Hom}_A(M \otimes_B W, V).$$

■

The following well-known lemma solves the problem of lifting modules through cyclic p'-extensions.

Lemma 10.2.13 *Let H be a normal subgroup of G with $E = G/H$ a cyclic p'-group. Let M be a G-stable $\mathcal{O}H$-module. Then, there exists an $\mathcal{O}G$-module $\tilde{M}$ extending M and for any such module, we have*

$$\mathit{Ind}_H^G M \simeq \mathit{Res}_{\Delta G}^{G \times G^\circ}(\tilde{M} \otimes \mathcal{O}E).$$

Proof. Let $g \in G$ generating G/H. Since M is G-stable, there exists $\varphi \in \mathrm{End}_{\mathcal{O}}(M)$ such that

$$\varphi(g^{-1}hg(m)) = h\varphi(m) \text{ for all } h \in H \text{ and } m \in M.$$

Let $\gamma = \varphi^e g^{-e}$, where $e = |E|$, and R be the subring of $\mathrm{End}_{\mathcal{O}H}(M)$ generated by γ.

Suppose there is $\alpha \in R$ such that $\alpha^e = \gamma$. Let $\psi = \alpha^{-1}\varphi$. Then, ψ^e acts on M as g^e and $\psi(g^{-1}hg(m)) = h\psi(m)$ for $h \in H$ and $m \in M$. It follows that we can extend the action of H on M to an action of G by letting g act as ψ.

The existence of α follows from the fact that R is a finite algebra over the strictly henselian ring $\mathcal{O}$, hence the etale extension $R[X]/(X^e - \gamma)$ of R must be trivial : first, replacing R by one of its blocks, we can assume it is local. Then, the equation $\alpha^e = \gamma$ has e distinct roots in the residue field of R. By Hensel's lemma, these solutions can be lifted to R and we are done.

Let $\tilde{M}$ be an $\mathcal{O}G$-module extending M. Then,

$$\mathrm{Res}_{\Delta G}^{G \times G^\circ} \mathrm{Ind}_{G \times H^\circ}^{G \times G^\circ}(\tilde{M} \otimes \mathcal{O}) \simeq \mathrm{Ind}_H^G \mathrm{Res}_H^G \tilde{M}$$

by Mackey's formula. ■

Let us recall some basic definitions of local block theory and some properties related to Brauer's first main theorem (see for example [2, §IV]).

Assume H is a subgroup of G. We say that the block $\mathcal{O}Ge$ *corresponds* to $\mathcal{O}Hf$ if $\mathcal{O}Ge$ is a direct summand of $\mathrm{Ind}_{H\times H^\circ}^{G\times G^\circ}\mathcal{O}Hf$.

Let D be a defect group of $\mathcal{O}Ge$. When $N_G(D)\le H$, there is a unique block idempotent f of $\mathcal{O}H$ such that $\mathcal{O}Ge$ corresponds to $\mathcal{O}Hf$: $\mathcal{O}Hf$ is called a *Brauer correspondent* of $\mathcal{O}Ge$.

If R is a p-subgroup of D, e_R is a block of $\mathcal{O}RC_G(R)$ (equivalently, a block of $\mathcal{O}C_G(R)$) and $\mathcal{O}Ge$ corresponds to $\mathcal{O}RC_G(R)e_R$, then (R,e_R) is called an *e-subpair.*

Assume furthermore R is normal in G and e_R is G-stable. Then, $e_R=e$, $D\cap RC_G(R)$ is G-conjugate to a defect group of $\mathcal{O}RC_G(R)e_R$ and $p\not|[G:DC_G(R)]$.

The next two lemmas deal with blocks of groups having a normal p-subgroup.

Lemma 10.2.14 *Let R be a normal p-subgroup of G, (R,f) an e-subpair and H its normalizer. Then, $\mathcal{O}Gf$ induces a Morita equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$.*

Proof. We have $ef=f$ and $(g^{-1}fg)f=0$ for $g\in G-H$. The multiplication map $f\mathcal{O}G\otimes_{\mathcal{O}Ge}\mathcal{O}Gf\to f\mathcal{O}Gf$ is an isomorphism and

$$f\mathcal{O}Gf=\sum_{g\in G/H}\mathcal{O}Hgg^{-1}fgf=\mathcal{O}Hf.$$

The lemma is then a consequence of Lemma 10.2.4. ∎

Proposition 10.2.15 *Assume a defect group D of $\mathcal{O}Ge$ is normal in G and e is a block of $\mathcal{O}C_G(D)$. Let $E=G/DC_G(D)$. Assume E is cyclic. Then, $\mathcal{O}Ge$ is Morita equivalent to $\mathcal{O}D\rtimes E$.*

Proof. Let $\bar{e}$ be the image of e in $\mathcal{O}(C_G(D)/Z(D))$. The canonical map from the center of $\mathcal{O}C_G(D)e$ to the center of $\mathcal{O}(C_G(D)/Z(D))\bar{e}$ is onto. Since $\mathcal{O}$ is complete, this forces $\bar{e}$ to be a block of $\mathcal{O}(C_G(D)/Z(D))$. This block has defect zero, hence has a unique simple module. So, $\mathcal{O}C_G(D)e$ has a unique simple module.

Let $H=D\rtimes E$ and $L=N_{G\times H^\circ}(\Delta D)$. The map $C_G(D)\to G\times H^\circ, x\mapsto(x,1)$ factors through L and gives an injection $\varphi: C_G(D)\to L/\Delta D$ with cokernel isomorphic to E.

The G-stable block e of $\mathcal{O}C_G(D)$ has a unique simple module V, which is consequently G-stable. The action of $C_G(D)$ on V lifts to an action of $L/\Delta D$ on V (Lemma 10.2.13). Let P_V be a projective cover of V as an $\mathcal{O}(L/\Delta D)$-module and Q the restriction of P_V to L. Let $M=\mathrm{Ind}_L^{G\times H^\circ}Q$ and $P=\mathrm{Res}^L_{C_G(D)\times Z(D)^\circ}Q$. Then, P has vertex $\Delta Z(D)$ and $\bar{P}=\mathcal{O}(C_G(D)/Z(D))\otimes_{C_G(D)}P\otimes_{Z(D)}\mathcal{O}$ is a projective indecomposable $\mathcal{O}(C_G(D)/Z(D))\bar{e}\otimes\mathcal{O}$-module, hence it induces a Morita equivalence between $\mathcal{O}(C_G(D)/Z(D))\bar{e}$ and $\mathcal{O}$. It follows from Lemma 10.2.11 that P induces a Morita equivalence between $\mathcal{O}C_G(D)e$ and $\mathcal{O}Z(D)$ and from Lemma 10.2.8 that M induces a Morita equivalence between $\mathcal{O}Ge$ and $\mathcal{O}H$. ∎

What this proposition actually determines is a source algebra of the block. Note that instead of assuming E cyclic, one can assume the blocks are principal to get the same result. In general, a similar proof shows that $\mathcal{O}Ge$ is Morita equivalent to a twisted group algebra $\mathcal{O}_*D\rtimes\widehat{E}$ when D is normal in G, as proven by L. Puig [162, §45, theorem 12].

10.3 Blocks stably equivalent to $\mathcal{O}D \rtimes E$

Let G be a finite group, e a block of G with positive defect. Let E be a p'-subgroup of the group of automorphisms of a non-trivial cyclic p-group D. Let $A = \mathcal{O}Ge$ and $B = \mathcal{O}D \rtimes E$. Let M be an indecomposable $(A \otimes B^\circ)$-module which is projective as an A-module and as a B°-module.

This section is devoted to the proof of

Theorem 10.2 *Assume M induces a stable equivalence between A and B. Then, there exists a direct summand N of a projective cover $p : P_M \twoheadrightarrow M$ of M such that the complex $0 \to N \xrightarrow{p_{|N}} M \to 0$ induces a Rickard equivalence between A and B.*

If $E = 1$, then $N = 0$ or $N = P_M$, i.e., M or ΩM induces a Morita equivalence between A and B.

Let $\Psi : R(KB) \to R(KA)$ and $\Theta : R(KA) \to R(KB)$ be the maps induced by the functors $M \otimes_B -$ and $M^* \otimes_A -$.

10.3.1 Exceptional characters

An exceptional character of B is defined to be the character of an irreducible KB-module with a non-trivial D-action. A non-exceptional character of B is an irreducible character which is not exceptional. Let θ_x be the sum of the exceptional characters of B. The simple B-modules are the simple $\mathcal{O}E$-modules. They lift uniquely to $\mathcal{O}$-free B-modules, with projective covers the indecomposable projective B-modules. If V is a one-dimensional $\mathcal{O}$-free $\mathcal{O}E$-module, then its projective cover as a B-module is $\mathrm{Ind}_E^H V$, whose character is the character of V plus θ_x. Recall also that the indecomposable kB-modules are uniserial.

The set $\{\bar{\theta}\}_{\theta \text{ exceptional}}$ is a basis of $\bar{R}(B)$ (we denote by $\bar{x}$ the image of $x \in R(KB)$ under the canonical morphism $R(KB) \to \bar{R}(B)$).

Assume B has at least two exceptional characters. An irreducible character ψ of A is called exceptional if there exists two exceptional characters θ, θ' of B such that $\langle \psi, \Psi(\theta) - \Psi(\theta') \rangle \neq 0$. Note that $\theta - \theta' \in R^p(B)$, hence $||\Psi(\theta) - \Psi(\theta')|| = 2$. So, the number of exceptional characters of A and B are the same. The rank of $R^p(B)$ is this number and $R^p(A)$ and $R^p(B)$ have the same rank. It follows that $R^p(A)$ is not generated by linear combinations of exceptional characters, hence A has at least one non-exceptional character.

We denote by ψ_x the sum of the exceptional characters of A.

When B has a unique exceptional character θ_x, then we pick an irreducible character $\psi_x = \psi_{\theta_x}$ of A such that $\bar{\psi}_x \neq 0$ and we call it exceptional (actually, any irreducible character of A will do).

For the non-exceptional characters, we have

Lemma 10.3.1 *Let ψ be a non-exceptional irreducible character of A. Then, $\overline{\Theta(\psi)}$ is a multiple of $\bar{\theta}_x$.*

Proof. Let θ, θ' be two exceptional characters of B. We have $\langle \Theta(\psi), \theta - \theta' \rangle = \langle \psi, \Psi(\theta - \theta') \rangle = 0$. We are now done since $\bar{\theta} = -\bar{\theta}_x$. ∎

10.3.2 Decomposition numbers

Lemma 10.3.2 *Let V be a non-projective $\mathcal{O}$-free A-module such that $V \otimes k$ is indecomposable. Let W be a non-projective indecomposable B-module such that $M^* \otimes_A V \simeq W \oplus$ projective modules. Let ζ be the character of W. Then,*

(i) ζ is multiplicity free

(ii) ζ contains at most one non-exceptional character

(iii) if ζ contains a non-exceptional character, then it doesn't contain all exceptional characters.

Proof. By Lemma 10.2.2, $W \otimes k$ is indecomposable and non projective. Since $W \otimes k$ is indecomposable, it is the quotient of an indecomposable projective kB-module. Hence, W is the quotient of an indecomposable projective B-module. It follows that the character ζ of W is multiplicity free and contains at most one non-exceptional character. Since $W \otimes k$ is not projective, we know in addition that if ζ contains a non-exceptional character, then it doesn't contain all exceptional characters. ■

Lemma 10.3.3 *Let χ be a character of A and P an indecomposable projective A-module such that χ is contained in the character of P.*

Then, there exists an $\mathcal{O}$-free A-module V with character χ and projective cover P. In particular, $V \otimes k$ is indecomposable.

Proof. Let L' be a KA-submodule of $K \otimes P$ such that $K \otimes P/L'$ has character χ. Let $L = L' \cap P$ and $V = P/L$. Then, V has the required properties. ■

As a consequence of Lemmas 10.3.1, 10.3.2 and 10.3.3, we obtain :

Corollary 10.3.4 *Let ψ be a non-exceptional character of A. Then, $\overline{\Theta(\psi)} = \pm\bar{\theta}_x$.*

If χ and χ' are two characters, we say that χ is contained in χ' if $\chi' - \chi$ is a character.

Proposition 10.3.5 *Let P be an indecomposable projective A-module. Then, its character is the sum of two distinct non-exceptional characters or the sum of a non-exceptional character and of all exceptional characters.*

Proof. Let η be the character of P. Assume there are non-exceptional characters ψ and ψ' such that $\psi + \psi'$ is strictly contained in η. By Lemma 10.3.3, there is an $\mathcal{O}$-free A-module V with character $\psi + \psi'$, such that $V \otimes k$ is indecomposable. By Lemma 10.3.4, $\overline{\Theta(\psi)} + \overline{\Theta(\psi')}$ is zero or $\pm 2\bar{\theta}_x$. By Lemma 10.3.2, the second possibility can't arise. Now, assume $\overline{\Theta(\psi)} + \overline{\Theta(\psi')} = 0$. Let W be a B-module such that $M^* \otimes_A V \simeq W \oplus$ projective modules, with $W \otimes k$ indecomposable. Then, the character of W is 0 in $\bar{R}(B)$. By Lemma 10.3.2, this is impossible.

We assume now that η contains at most one non-exceptional character. Let θ, θ' be two distinct exceptional characters of B. We have

$$\langle \eta, \Psi(\theta) - \Psi(\theta') \rangle = \langle \Theta(\eta), \theta - \theta' \rangle = 0.$$

Let ψ and ψ' be the two distinct exceptional characters of A such that $\Psi(\theta) - \Psi(\theta') = \pm\psi \pm \psi'$. Then, $\langle \eta, \psi \rangle = \langle \eta, \psi' \rangle$, so ψ and ψ' arise with the same multiplicity in η. It follows that there is a positive integer α such that $\chi = \eta - \alpha\psi_x$ is zero or a non-exceptional character.

Assume $\chi = 0$. Then the block A has a projective indecomposable module with character η' different from η such that $\langle \eta, \eta' \rangle \neq 0$. The character η' is a non-zero multiple of ψ_x plus a non-exceptional character ψ. But this implies that $\bar{\psi} = 0$, which is impossible ! Hence, χ is a non-exceptional character.

Now, $\overline{\Theta(\eta)} = 0 = \pm\bar{\theta}_x + \alpha\overline{\Theta(\psi_x)}$ by Lemma 10.3.4. This implies $\alpha = 1$ and $\overline{\Theta(\psi_x)} = \pm\bar{\theta}_x$ and we are done. ∎

10.3.3 The Brauer tree and its walk

Lemma 10.3.6 *Let L be a simple B-module. Then, the module $M \otimes_B L$ has a unique simple quotient V and the correspondence $L \mapsto V$ induces a bijection between the sets of isomorphism classes of simple B-modules and simple A-modules.*

Proof. Let V be a simple A-module and $W = M^* \otimes_A V$. Then, W is indecomposable (Lemma 10.2.3), hence it has a unique simple submodule L_V. So, we have a map h from the set of isomorphism classes of simple A-modules to the set of isomorphisms classes of simple B-modules given by $V \mapsto L_V$.

Let now L be a simple B-module and $U = M \otimes_B L$. Let V be a simple A-module which is a quotient of the indecomposable module U. Then,

$$\overline{\mathrm{Hom}}(L, M^* \otimes_A V) \simeq \overline{\mathrm{Hom}}(U, V) \neq 0.$$

It follows that the map h is surjective. Let now V_1 and V_2 be two simple A-modules such that $W_1 = M^* \otimes_A V_1$ and $W_2 = M^* \otimes_A V_2$ have the same simple submodule L. Since an injective hull of L is uniserial, there is an injection $W_i \to W_j$ for some i, j with $\{i, j\} = \{1, 2\}$. Such an injection between modules with no projective direct summand is not an $\mathcal{O}$-projective morphism, hence $\overline{\mathrm{Hom}}(W_i, W_j) \neq 0$, so $\overline{\mathrm{Hom}}(V_i, V_j) \neq 0$ and V_i and V_j are isomorphic. This proves the injectivity of h.

Since h is bijective, given a simple B-module L, the module $M \otimes_B L$ has a unique simple quotient. ∎

The set $\{\Omega^{2i}k\}_{0 \leq i \leq e-1}$ is a complete set of simple kB-modules (up to isomorphism). By Lemma 10.2.3, the module $M \otimes_B \Omega^{2i}k$ is indecomposable. Hence, $M \otimes_B \Omega^{2i}\mathcal{O}$ is indecomposable as well. On the other hand, $M \otimes_B -$ commutes with Heller translation up to projective modules, i.e., $M \otimes_B \Omega^{2i}\mathcal{O} \simeq \Omega^{2i}S \oplus$ projective modules, where $S = M \otimes_B \mathcal{O}$. It follows that $M \otimes_B \Omega^{2i}\mathcal{O} \simeq \Omega^{2i}S$.

Proposition 10.3.7 *The character of $\Omega^i S$ is a non-exceptional character or the sum of the exceptional characters.*

Proof. Let χ be the character of $\Omega^i S$.

When i is even, $\Omega^i S$ has a unique simple quotient, hence $\Omega^i S$ is a quotient of a projective indecomposable A-module P_i. In particular, χ is contained in the character of P_i.

Assume now i is odd. Then, we have an exact sequence

$$0 \to \Omega^i S \to P_{i-1} \to \Omega^{i-1} S \to 0. \tag{10.1}$$

Again, we see that χ is contained in the character of a projective indecomposable module.

We have $\bar{\chi} = \pm\overline{\Psi(\theta_x)}$ since the character of $\Omega^i\mathcal{O}$ is non-exceptional or equal to θ_x. Hence, $\bar{\chi} = \pm\bar{\psi}_x$ and we get the conclusion from Proposition 10.3.5. ∎

Let us now define the Brauer tree $\mathcal{T}$ of A. The set of vertices is $\{\psi\}_{\psi \text{ non exceptional}} \cup \{\psi_x\}$. The vertices ψ and ψ' are incident if $\psi + \psi'$ is the character of an indecomposable projective module. This defines a graph whose number of edges is the number of isomorphism classes of simple A-modules. By Proposition 10.3.5, this is a tree. The vertex corresponding to ψ_x is called exceptional.

Let v_i be the vertex corresponding to the character of $\Omega^i S$. Then, there is an edge l_i connecting v_i and v_{i+1}, due to the exact sequence (10.1). The set $\{l_{2i}\}$ is the set of all edges of $\mathcal{T}$.

Note that $\Omega^{2e}\mathcal{O} \simeq \mathcal{O}$, where e is the order of $|E|$. It follows that $v_{2e+i} = v_i$ and $l_{2e+i} = l_i$.

For $0 \leq i \leq e-1$, let $v'_i \in \{v_{2i}, v_{2i+1}\}$ be the further vertex of l_{2i} from the exceptional vertex. Let I be the set of non-negative integers $i \leq e-1$ such that $v'_i = v_{2i+1}$.

Note that $\{v'_i\}$ is the set of non-exceptional vertices of $\mathcal{T}$.

10.3.4 Construction of the complex

By Lemma 10.2.12, a projective cover of the $(A \otimes B^\circ)$-module M is

$$\bigoplus_{0 \leq i \leq e-1} P_{\Omega^{2i}S} \otimes P^*_{\Omega^{2i}\mathcal{O}}.$$

Let

$$N = \bigoplus_{i \in I} P_{\Omega^{2i}S} \otimes P^*_{\Omega^{2i}\mathcal{O}}$$

and let

$$C = 0 \to N \xrightarrow{\varphi} M \to 0$$

where M is in degree 0 and φ is the restriction to N of a surjection $\bigoplus_{0 \leq i \leq e-1} P_{\Omega^{2i}S} \otimes P^*_{\Omega^{2i}\mathcal{O}} \to M$.

For $0 \leq i, j \leq e-1$, we have

$$P^*_{\Omega^{2j}\mathcal{O}} \otimes_B \Omega^{2i}\mathcal{O} \simeq \begin{cases} \mathcal{O} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have an isomorphism in $\mathcal{D}^b(A)$

$$C \otimes_B \Omega^{2i}\mathcal{O} \simeq \begin{cases} 0 \to \Omega^{2i+1}S \to 0 \to 0 & \text{if } i \in I, \\ 0 \to 0 \to \Omega^{2i}S \to 0 & \text{otherwise} \end{cases}$$

where $\Omega^{2i}S$ is in degree 0 and $\Omega^{2i+1}S$ in degree -1.

In particular, the (Lefschetz) character of $C \otimes_B \Omega^{2i}\mathcal{O}$ is $\varepsilon_i v'_i$ where $\varepsilon_i = -1$ if $i \in I$ and $\varepsilon_i = 1$ otherwise.

Lemma 10.3.8 *We have* $Hom_{\mathcal{D}^b(A)}(C \otimes_B \Omega^{2i}k, C \otimes_B \Omega^{2j}k[-1]) = 0$ *for all* i, j.

Proof. Let us recall first that $\overline{\mathrm{Hom}}_B(\Omega^n k, k) = 0$ unless $n \equiv 0 \pmod{2e}$. Put now $T = \mathrm{Hom}_{\mathcal{D}^b(A)}(C \otimes_B \Omega^{2i}k, C \otimes_B \Omega^{2j}k[-1])$.

Since $\overline{\mathrm{Hom}}(\Omega^{2i}S \otimes k, \Omega^{2j+1}S \otimes k) \simeq \overline{\mathrm{Hom}}(\Omega^{2i}k, \Omega^{2j+1}k) = 0$, we deduce that $T = 0$ unless $i \notin I$ and $j \in I$, in which case $T \simeq \mathrm{Hom}(\Omega^{2i}S \otimes k, \Omega^{2j+1}S \otimes k)$. Then, we have to prove that there are no k-projective morphisms from $\Omega^{2i}S \otimes k$ to $\Omega^{2j+1}S \otimes k$. As k-projective morphisms $\Omega^{2i}S \otimes k \to \Omega^{2j+1}S \otimes k$ lift to $\mathcal{O}$-projective morphisms $\Omega^{2i}S \to \Omega^{2j+1}S$, we are done, since $\mathrm{Hom}(\Omega^{2i}S, \Omega^{2j+1}S) = 0$ (the character of $\Omega^{2i}S$ is v'_i and the character of $\Omega^{2j+1}S$ is v'_j, hence, these are distinct since $i \neq j$). ∎

Corollary 10.3.9 *The complex* $C^* \otimes_A C$ *is homotopy equivalent to its* 0*-homology.*

Proof. Let L, L' be two simple B-modules. We have

$$\mathrm{Hom}_{\mathcal{D}^b(B \otimes B^\circ)}(C^* \otimes_A C, L' \otimes L^*[-1]) \simeq \mathrm{Hom}_{\mathcal{D}^b(A)}(C \otimes_B L, C \otimes_B L'[-1]) = 0$$

by Lemma 10.3.8. Hence, $C^* \otimes_A C$ has no homology in degree 1. Since the degree 1 component of $C^* \otimes_A C$ is projective, $C^* \otimes_A C$ is homotopy equivalent to a complex with no component in degree 1. Since $C^* \otimes_A C$ is self-dual, it is homotopy equivalent to its 0-homology. ∎

By Lemma 10.2.5, we have $H_0(C^* \otimes_A C) \simeq B \oplus Q$, where Q is a projective $(B \otimes B^\circ)$-module.

Now,

$$\mathrm{Hom}_{\mathcal{D}^b(KB \otimes (KB)^\circ)}(K \otimes (C^* \otimes_A C), K \otimes (\Omega^{2i}\mathcal{O} \otimes (\Omega^{2j}\mathcal{O})^*)) \simeq$$
$$\mathrm{Hom}_{\mathcal{D}^b(KA)}(K \otimes (C \otimes_B \Omega^{2i}\mathcal{O}), K \otimes (C \otimes_B \Omega^{2j}\mathcal{O})) = \delta_{ij}K,$$
$$\text{hence } \mathrm{Hom}(K \otimes Q, K \otimes (\Omega^{2i}\mathcal{O} \otimes (\Omega^{2j}\mathcal{O})^*)) = 0$$

for all i, j. This implies $Q = 0$. So, we have proven that $C^* \otimes_A C$ is homotopy equivalent to B (seen as a complex concentrated in degree 0). By Lemma 10.2.4, we conclude that C is a Rickard complex. This completes the proof of the first part of Theorem 10.2.

Note that, if $E = 1$, then a projective cover of M is indecomposable : it follows that C has homology only in one degree and it is isomorphic to M or $\Omega M[1]$, from which we derive the second part of Theorem 10.2.

10.4 Local study

Let G be a finite group, e a block of G with a non-normal cyclic defect group D.

Let Q be the subgroup of D containing $R = O_p(G)$ as a subgroup of index p. Let $H = N_G(Q)$ and f the block of H corresponding to e.

Theorem 10.1 will follow from the following more precise result :

Theorem 10.3 *Let* M *be an indecomposable direct summand of the* $(\mathcal{O}Ge \otimes (\mathcal{O}Hf)^\circ)$*-module* $e\mathcal{O}Gf$ *with vertex* ΔD. *Then, there is a direct summand* N *of the* $(\mathcal{O}Ge \otimes (\mathcal{O}Hf)^\circ)$*-module* $\mathcal{O}Ge \otimes_{\mathcal{O}R} f\mathcal{O}H$ *such that the complex* $C = 0 \to N \xrightarrow{m} M \to 0$ *induces a splendid Rickard equivalence between* $\mathcal{O}Ge$ *and* $\mathcal{O}Hf$. *Here,* m *is the restriction of the multiplication map* $\mathcal{O}Ge \otimes_{\mathcal{O}R} f\mathcal{O}H \to e\mathcal{O}Gf$.

If $R \neq 1$, *then* C *has homology only in one degree.*

Note that the $(\mathcal{O}Ge \otimes (\mathcal{O}Hf)^\circ)$-module $e\mathcal{O}Gf$ has, up to isomorphism, a unique indecomposable direct summand with vertex ΔD (this can easily be deduced from the forthcoming proof of the theorem).

Let us check that a complex C as defined in the theorem is splendid. Note that $e\mathcal{O}Gf$ is a direct summand of $\mathrm{Ind}_{\Delta D}^{G\otimes H^\circ}\mathcal{O}$, hence is isomorphic to a direct sum of modules with trivial source and vertex contained contained in ΔD. Since $\mathcal{O}Ge \otimes_{\mathcal{O}R} f\mathcal{O}H$ is isomorphic to a direct summand of $\mathrm{Ind}_{\Delta R}^{G\otimes H^\circ}\mathcal{O}$, it follows that N is isomorphic to a direct sum of modules with trivial source and vertex contained in ΔR. Hence, C is splendid.

Let us prove by induction on the order of G that Theorem 10.1 follows from Theorem 10.3.

By induction, we know that Theorem 10.1 holds for $\mathcal{O}Hf$: there is a splendid Rickard equivalence between $\mathcal{O}Hf$ and $\mathcal{O}N_G(D)e'$ given by a complex having homology only in one degree. Now, Theorem 10.3 gives a splendid Rickard equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$. Hence, composing the two equivalences, we get a splendid Rickard equivalence between $\mathcal{O}Ge$ and $\mathcal{O}N_G(D)e'$, by Lemma 10.2.6. Furthermore, when $O_p(G) \neq 1$, the Rickard complex has homology only in one degree. Hence, Theorem 10.1 holds for G and, by induction, the proof of Theorem 10.1 is complete.

We assume now that Theorem 10.3 holds for all finite groups of order strictly less than the order of G. The rest of this section is devoted to proving that the theorem holds then for G.

10.4.1 $O_p(G) = 1$

Let us first consider the case where $R = 1$, i.e., Q has order p.

Following Alperin, we have :

Lemma 10.4.1 *The module $e\mathcal{O}Gf$ induces a stable equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$.*

Proof. We have

$$\mathrm{Res}_{H\otimes H^\circ}^{G\times G^\circ}\mathcal{O}G \simeq \mathrm{Res}_{H\otimes H^\circ}^{G\times G^\circ}\mathrm{Ind}_{\Delta G}^{G\times G^\circ}\mathcal{O} \simeq \bigoplus_{(g_1,g_2)\in H\times H^\circ\backslash G\times G^\circ/\Delta G} \mathrm{Ind}_{(\Delta G)^{(g_1,g_2)}\cap(H\times H^\circ)}^{H\times H^\circ}\mathcal{O}.$$

Now, $(Q \times Q^\circ) \cap (\Delta G)^{(g_1,g_2)} \neq 1$ if and only if $g_1g_2^{-1} \in H$. Since $Q \times Q^\circ$ is the maximal elementary abelian subgroup of $D \times D^\circ$ and is normal in $H \times H^\circ$, it follows that the $\mathcal{O}Hf \otimes (\mathcal{O}Hf)^\circ$-module $f\left(\mathrm{Ind}_{(\Delta G)^{(g_1,g_2)}\cap(H\times H^\circ)}^{H\times H^\circ}\mathcal{O}\right)f$ is projective when $g_1g_2^{-1} \notin H$.

Hence,

$$\mathcal{O}Hf \otimes_{\mathcal{O}H} \mathcal{O}G \otimes_{\mathcal{O}H} f\mathcal{O}H \simeq \mathcal{O}Hf \oplus \text{ projective modules.}$$

Since f is the Brauer correspondent of e, $\mathcal{O}Hf$ is a direct summand of $\mathrm{Res}_{H\times H^\circ}^{G\times G^\circ}\mathcal{O}Ge$. So,

$$f\mathcal{O}Ge\otimes_{\mathcal{O}G}e\mathcal{O}Gf \simeq f\mathcal{O}Gef \simeq \mathcal{O}Hf\otimes_{\mathcal{O}H}\mathcal{O}Ge\otimes_{\mathcal{O}H}f\mathcal{O}H \simeq \mathcal{O}Hf\oplus \text{ projective modules.}$$

The result is now a consequence of Lemma 10.2.1. ∎

Let M be an indecomposable non-projective direct summand of $e\mathcal{O}Gf$: by Lemma 10.2.3, it still induces a stable equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$.

By induction, Theorem 10.1 holds for $\mathcal{O}Hf$. Hence, there is an $(\mathcal{O}Hf \otimes (\mathcal{O}D \rtimes E)^\circ)$-module M' inducing a Morita equivalence between $\mathcal{O}Hf$ and $\mathcal{O}D \rtimes E$. So, the indecomposable module $M_0 = M \otimes_{\mathcal{O}Hf} M'$ induces a stable equivalence between $\mathcal{O}Ge$ and $\mathcal{O}D \rtimes E$. By Theorem 10.2, there exists a direct summand N_0 of a projective cover P_0 of M_0, such that the complex $C_0 = 0 \to N_0 \to M_0 \to 0$ induces a Rickard equivalence between $\mathcal{O}Ge$ and $\mathcal{O}D \rtimes E$. It follows that

$$C_1 = C_0 \otimes_{\mathcal{O}D \rtimes E} M'^* \simeq 0 \to N_0 \otimes_{\mathcal{O}D \rtimes E} M'^* \to M \to 0$$

induces a Rickard equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$. The module $N_0 \otimes_{\mathcal{O}D \rtimes E} M'^*$ is a direct summand of $P_M = P_0 \otimes_{\mathcal{O}D \rtimes E} M'^*$, a projective cover of M, and the map $N_0 \otimes_{\mathcal{O}D \rtimes E} M'^* \to M$ is the restriction of a surjective morphism $P_M \to M$. Now, the multiplication map

$$m : \mathcal{O}Ge \otimes f\mathcal{O}H \to e\mathcal{O}Gf$$

is surjective and $e\mathcal{O}Gf = M \oplus$ projective modules, hence there is a direct summand N of $\mathcal{O}Ge \otimes f\mathcal{O}H$ such that the complex $C = 0 \to N \xrightarrow{m} M \to 0$ is isomorphic to C_1.

So, the theorem holds when $R = 1$.

10.4.2 $O_p(G) \neq 1$

Let us now consider the case where R is non-trivial. Let $\bar{G} = G/R$ and $\bar{e}$ be the image of e through the canonical morphism $\mathcal{O}G \to \mathcal{O}\bar{G}$. Similarly, let $\bar{H} = H/R$ and $\bar{f}$ be the image of f through the canonical morphism $\mathcal{O}H \to \mathcal{O}\bar{H}$. Note that the canonical map $E = N_G(D, e_D)/C_G(D) \to \mathrm{Aut}(D/R)$ is injective, since it factors through the group of p'-automorphisms of D. Hence, $\mathcal{O}\bar{H}\bar{f}$ and $\mathcal{O}\bar{G}\bar{e}$ are blocks with defect D/R.

Let (R, e_R) be an e-subpair. By Lemma 10.2.14, the $(\mathcal{O}Ge \otimes \mathcal{O}N_G(R, e_R)e_R)$-module $e\mathcal{O}Ge_R$ induces a Morita equivalence. Hence, we can assume that G stabilizes e_R, that is, that $e = e_R$.

Let $\bar{C} = C_G(R)/R$. Since $\bar{G}/\bar{C} \simeq G/C_G(R)$ is a cyclic p'-group, a simple $\mathcal{O}\bar{C}\bar{e}$-module extends in exactly $[G : C_G(R)]$ non-isomorphic ways to $\bar{G}$ and every simple $\mathcal{O}\bar{G}\bar{e}$-module is obtained in this way (Lemma 10.2.13). By induction, $\mathcal{O}\bar{G}\bar{e}$ is derived equivalent to $(D/R) \rtimes E$, hence has $|E|$ simple modules. It follows that $[G : C_G(R)] = |E|$, hence the canonical map $N_G(D, e_D)/C_G(D) \to G/C_G(R)$ is an isomorphism. Note that $O_p(C_G(R)) = R$. Similarly, the canonical map $N_H(D, e_D)/C_H(D) \to H/C_H(R)$ is an isomorphism and $O_p(C_H(R)) = R$.

$1 \neq O_p(G) \leq Z(G)$

By assumption, Theorem 10.3 holds for $\mathcal{O}\bar{G}\bar{e}$ and $\mathcal{O}\bar{H}\bar{f}$. The observation above shows that these two blocks have a unique simple module. Let M' be an indecomposable direct summand of the $\mathcal{O}(G \times H^\circ)/\Delta R$-module $e\mathcal{O}Gf$ with vertex $\Delta D/\Delta R$. Since $\bar{e}\mathcal{O}\bar{G}\bar{f}$ is the direct sum of an indecomposable non-projective module and of a projective module, it follows from Lemma 10.2.9 that the $\mathcal{O}(\bar{G} \times \bar{H}^\circ)$-module $M'' = M' \otimes_{\mathcal{O}R} \mathcal{O}$ is an indecomposable direct summand of $\bar{e}\mathcal{O}\bar{G}\bar{f}$. Let $f' : N' \to M'$ be a projective cover of M'. Then, $N'' = N' \otimes_{\mathcal{O}R} \mathcal{O}$ is a projective cover of M'' (Lemma 10.2.9). Let M (resp. N) be the restriction of M' (resp. N') to $G \times H^\circ$.

If M'' induces a Morita equivalence between $\mathcal{O}\bar{G}\bar{e}$ and $\mathcal{O}\bar{H}\bar{f}$, then let C be the complex with only one non-zero term, M, in degree 0. If the kernel of a surjective map $f'' : N'' \to M''$ induces a Morita equivalence between $\mathcal{O}\bar{G}\bar{e}$ and $\mathcal{O}\bar{H}\bar{f}$, then let C be the complex with N in degree -1, M in degree 0 and differential f'.

Then, it follows from Lemma 10.2.11 that C is a Rickard complex. Note that C has the form required.

$O_p(G) \not\leq Z(G)$

Note that $\mathcal{O}C_H(R)f$ is the Brauer correspondent of the block $\mathcal{O}C_G(R)e$.

Let M be an indecomposable direct summand of the $\mathcal{O}(C_G(R) \times C_H(R)^\circ)/\Delta R$-module $e\mathcal{O}C_G(R)f$ with vertex $\Delta(D/R)$. Let $L = N_{G\times H^\circ}(\Delta R)$. The restriction to L of the action of $G \times H^\circ$ on $\mathcal{O}G$ leaves $e\mathcal{O}C_G(R)f$ invariant. This gives a natural extension of the action of $(C_G(R) \times C_H(R))/\Delta R$ on $e\mathcal{O}C_G(R)f$ to $L/\Delta R$. Since $(e\mathcal{O}C_G(R)f)/M$ is a sum of modules with vertices strictly contained in $\Delta(D/R)$, the module M is L-stable. The isomorphisms $G/C_G(R) \simeq E$ and $H/C_H(R) \simeq E$ induce an isomorphism $L/(C_G(R) \times C_H(R)^\circ) \simeq E$. So, by Lemma 10.2.13, there is an indecomposable summand $\tilde{M}$ of $\mathrm{Ind}_{\Delta(H/R)}^{L/\Delta R}\mathcal{O} \simeq \mathcal{O}C_G(R)$ lifting M.

Let N be a projective cover of M. This is an L-stable indecomposable $\mathcal{O}(C_G(R) \times C_H(R)^\circ)/\Delta R$-module, hence it lifts to a projective $L/\Delta R$-module $\tilde{N}$ (Lemma 10.2.13). Since $\mathrm{Ind}_{(C_G(R)\times C_H(R)^\circ)/\Delta R}^{L/\Delta R} N$ is a projective cover of $\mathrm{Ind}_{(C_G(R)\times C_H(R)^\circ)/\Delta R}^{L/\Delta R} M$ and $\tilde{M}$ is isomorphic to a direct summand of the latter, one may choose $\tilde{N}$ to be a projective cover of $\tilde{M}$. We have a natural map

$$\mathrm{Ind}_1^{H/R}\mathrm{Res}_1^{H/R}\mathcal{O} \to \mathcal{O}$$

giving rise to a surjective map

$$f : \mathrm{Ind}_1^{L/\Delta R}\mathcal{O} \to \mathrm{Ind}_{\Delta(H/R)}^{L/\Delta R}\mathcal{O}.$$

So, we may choose $\tilde{N}$ to be a direct summand of $\mathrm{Ind}_1^{L/\Delta R}\mathcal{O}$ with $f(\tilde{N}) = \tilde{M}$.

Let $N' = \mathrm{Ind}_L^{G\times H^\circ}\mathrm{Res}_L^{L/\Delta R}\tilde{N}$ and $M' = \mathrm{Ind}_L^{G\times H^\circ}\mathrm{Res}_L^{L/\Delta R}\tilde{M}$. Then, N' is a direct summand of $\mathcal{O}Ge \otimes_{\mathcal{O}R} f\mathcal{O}H$, M' is a direct summand of $e\mathcal{O}Gf$ and $m'(N') = M'$, where m' is the multiplication map $\mathcal{O}Ge \otimes_{\mathcal{O}R} f\mathcal{O}H \to e\mathcal{O}Gf$.

If $\mathrm{Res}_{C_G(R)\times C_H(R)^\circ}^{(C_G(R)\times C_H(R)^\circ)/\Delta R} M$ induces a Morita equivalence between $\mathcal{O}C_G(R)e$ and $\mathcal{O}C_H(R)f$, then we define C to be M'. Otherwise, let C be the complex with N' in degree -1, M' in degree 0 and differential m'.

Then, Lemma 10.2.8 says that C is a splendid Rickard complex between $e\mathcal{O}G$ and $f\mathcal{O}H$. Note that C has homology only in one degree. Hence, the proof of Theorem 10.3 is complete.

10.5 An example : $PSL_2(p)$

We make the constructions of §10.3 explicit for the group $PSL_2(p)$.

Let V_1 be the natural 2-dimensional representation of $SL_2(p)$. Then, the simple $kSL_2(p)$ modules are the symmetric powers $S^i(V_1)$, where $0 \le i \le p-1$ [2, pp 14–16]. The center of $SL_2(p)$ acts trivially on $S^i(V_1)$ when i is even and we denote by V_i the module $S^{2i}(V_1)$ induced from $SL_2(p)$ to $PSL_2(p)$. Then, the simple modules in the principal block e of $G = PSL_2(p)$ are the V_i, $0 \le i \le \frac{p-3}{2}$. There is only one other block in G, it has defect 0 and its simple module is the Steinberg module $V_{\frac{p-1}{2}}$.

For $i < \frac{p-1}{2}$, the dimension of V_i is $2i+1 < p$. Let B be the normalizer of a Sylow p-subgroup of G. By Proposition 10.4.1, the $(\mathcal{O}Ge \otimes (\mathcal{O}B)^\circ)$-module $\mathcal{O}Ge$ induces a stable equivalence between $\mathcal{O}Ge$ and $\mathcal{O}B$. For V a simple $\mathcal{O}Ge$-module, the module $\mathcal{O}Ge \otimes V$ has a dimension strictly smaller than p, hence it cannot have a projective direct summand. By Proposition 10.2.3, this implies the indecomposabiblity of $\mathcal{O}Ge$ as a $(\mathcal{O}Ge \otimes (\mathcal{O}B)^\circ)$-module.

Since indecomposable kB-modules are uniserial, the restriction of V_i to B has a unique simple quotient W_i, for $0 \le i \le \frac{p-3}{2}$ (this is actually a special case of the general property of simple modules for groups with a (B,N)-pair to have a unique simple quotient when restricted to B). Furthermore, $W_i \simeq W_1^i$.

Let U be the Sylow p-subgroup of B and T a complement to U in B. The group T is cyclic, with order $\frac{p-1}{2}$ and there are two conjugacy classes of non-trivial p-elements in B (hence also in G). Let x_1, x_2 be representatives for these classes. Let T' be a "Coxeter torus" of G, i.e., the centralizer of an element of order $\frac{p+1}{2}$. This is a cyclic group of order $\frac{p+1}{2}$.

Let η be the irreducible character of T which gives the simple module W_1. Let δ be a non-trivial irreducible character of T' which occurs in the character of the restriction of V_1 to T' (the restriction is then $1+\delta+\delta^{-1}$). Let $\varepsilon = \pm 1$ with $\varepsilon \equiv p \pmod 4$.

The character table of the principal block of G is :

	x_1	x_2	$t \in T$	$t' \in T'$
1	1	1	1	1
$(p-1)_j$	-1	-1	0	$-(\delta^j(t') + \delta^{-j}(t'))$
$(p+1)_l$	1	1	$\eta^l(t) + \eta^{-l}(t)$	0
$\left(\frac{p+\varepsilon}{2}\right)_i$	$\frac{1}{2}(\varepsilon + i\sqrt{\varepsilon p})$	$\frac{1}{2}(\varepsilon - i\sqrt{\varepsilon p})$	$\begin{cases} \eta^{\frac{p-1}{4}}(t) & \text{if } \varepsilon = 1 \\ 0 & \text{if } \varepsilon = -1 \end{cases}$	$\begin{cases} 0 & \text{if } \varepsilon = 1 \\ -\delta^{\frac{p+1}{4}}(t') & \text{if } \varepsilon = -1 \end{cases}$

where $j \in \{1, \ldots, \frac{p+\varepsilon-2}{4}\}$, $l \in \{1, \ldots, \frac{p-\varepsilon}{4} - 1\}$ and $i \in \{-1, 1\}$.

The character table of B is :

	x_1	x_2	$t \in T$
1_l	1	1	$\eta^l(t)$
$\left(\frac{p-1}{2}\right)_i$	$\frac{1}{2}(-1 + i\sqrt{-p})$	$\frac{1}{2}(-1 - i\sqrt{-p})$	0

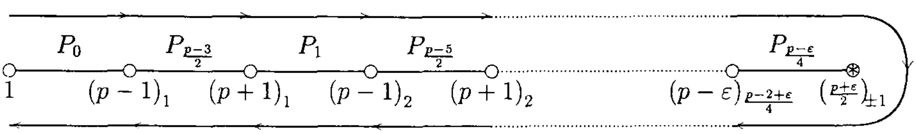
where $l \in \{0, \frac{p-1}{2} - 1\}$ and $i \in \{-1, 1\}$.

The restrictions of the irreducible characters of the principal block of G to B are :

- $\mathrm{Res}_B^G\, 1 = 1$,

- $\mathrm{Res}_B^G\ (p-1)_j = \left(\frac{p-1}{2}\right)_0 + \left(\frac{p-1}{2}\right)_1$,
- $\mathrm{Res}_B^G\ (p+1)_l = \left(\frac{p-1}{2}\right)_0 + \left(\frac{p-1}{2}\right)_1 + 1_l + 1_{\frac{p-1}{2}-l}$.
- $\mathrm{Res}_B^G\ \left(\frac{p+\varepsilon}{2}\right)_i = \begin{cases} \left(\frac{p-1}{2}\right)_i & \text{if } \varepsilon = -1, \\ \left(\frac{p-1}{2}\right)_i + 1_{\frac{q-1}{4}} & \text{if } \varepsilon = 1. \end{cases}$

The Brauer tree of $\mathcal{O}Ge$ is



The arrows describe Green's walk on the tree (i.e., the sequence of vertices $v_0, v_1, \ldots$).

A projective cover of the $(\mathcal{O}Ge \otimes (\mathcal{O}B)^\circ)$-module $\mathcal{O}Ge$ is

$$\bigoplus_{0 \le \lambda \le \frac{p-3}{2}} P_\lambda \otimes Q_\lambda^*$$

where P_λ is a projective cover of V_λ and Q_λ a projective cover of W_λ. The module N constructed in §10.3.4 is

$$N = \bigoplus_{\frac{p+2+\varepsilon}{4} \le \lambda \le \frac{p-3}{2}} P_\lambda \otimes Q_\lambda^*$$

Restricting a surjective map

$$\bigoplus_{0 \le \lambda \le \frac{p-1}{2}} P_\lambda \otimes Q_\lambda^* \twoheadrightarrow \mathcal{O}Ge$$

to N gives a complex

$$C = 0 \to N \to \mathcal{O}Ge$$

which induces a splendid Rickard equivalence between $\mathcal{O}B$ and $\mathcal{O}Ge$.

Let I be the isometry $R(KGe) \to R(KB)$ induced by this equivalence of derived categories. We have

- $I(1) = 1$,
- $I((p+1)_l) = 1_l$,
- $I((p-1)_j) = -1_{\frac{p-1}{2}-j}$,
- $I\left(\left(\frac{p+\varepsilon}{2}\right)_i\right) = -\varepsilon \left(\frac{p-1}{2}\right)_{\varepsilon i}$.