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Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification

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Abstract

We define and study $\mathfrak{sl}_2$-categorifications on abelian categories. We show in particular that there is a self-derived (even homotopy) equivalence categorifying the adjoint action of the simple reflection. We construct categorifications for blocks of symmetric groups and deduce that two blocks are splendidly Rickard equivalent whenever they have isomorphic defect groups and we show that this implies Broué's abelian defect group conjecture for symmetric groups. We give similar results for general linear groups over finite fields. The constructions extend to cyclotomic Hecke algebras. We also construct categorifications for category $\mathcal{O}$ of $\mathfrak{gl}_n(\mathbf{C})$ and for rational representations of general linear groups over $\bar{\mathbf{F}}_p$, where we deduce that two blocks corresponding to weights with the same stabilizer under the dot action of the affine Weyl group have equivalent derived (and homotopy) categories, as conjectured by Rickard.

Contents

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1. Introduction

The aim of this paper is to show that two blocks of symmetric groups with isomorphic defect groups have equivalent derived categories. We deduce in particular that Broué's abelian defect group conjecture holds for symmetric groups. We prove similar results for general linear groups over finite fields and cyclotomic Hecke algebras.

Recall that there is an action of $\hat{\mathfrak{sl}}_p$ on the sum of Grothendieck groups of categories of $k\mathfrak{S}_n$-modules, for $n \geq 0$, where k is a field of characteristic p. The action of the generators e_i and f_i come from exact functors between modules ("i-induction" and "i-restriction"). The adjoint action of the simple reflections of the affine Weyl group can be categorified as functors between derived categories, following Rickard. The key point is to show that these functors are invertible, since two blocks have isomorphic defect groups if and only if they are in the same affine Weyl group orbit. This involves only an $\mathfrak{sl}_2$-action and we solve the problem in a more general framework.

We develop a notion of $\mathfrak{sl}_2$-*categorification* on an abelian category. This involves the data of adjoint exact functors E and F inducing an $\mathfrak{sl}_2$-action on the Grothendieck group and the data of endomorphisms X of E and T of E^2 satisfying the defining relations of (degenerate) affine Hecke algebras.

Our main theorem is a proof that the categorification Θ of the simple reflection is a self-equivalence at the level of derived (and homotopy) categories. We achieve this in two steps. First, we show that there is a minimal categorification of string (=simple) modules coming from certain quotients of (degenerate) affine Hecke algebras: this reduces the proof of invertibility of Θ to the case of the minimal categorification. There, Θ becomes (up to shift) a self-equivalence of the abelian category.

Let us now describe in more detail the structure of this article. The first part (§3) is devoted to the study of (degenerate) affine Hecke algebras of type A completed at a maximal ideal corresponding to a totally ramified central character. We construct (in §3.2) explicit decompositions of tensor products of ideals which we later translate into isomorphisms of functors. In §3.3, we introduce certain quotients, that turn out to be Morita equivalent to cohomology rings of Grassmannians. Section 4 recalls elementary results on adjunctions and on representations of $\mathfrak{sl}_2$.

Section 5 is devoted to the definition and study of $\mathfrak{sl}_2$-categorifications. We first define a weak version (§5.1), with functors E and F satisfying $\mathfrak{sl}_2$-relations in the Grothendieck group. This is enough to get filtrations of the category and to introduce a class of objects that control the abelian category. Then, in §5.2, we introduce the extra data of X and T which give the genuine $\mathfrak{sl}_2$-categorifications. This provides actions of (degenerate) affine Hecke algebras on powers of E and F. This leads immediately to two constructions of divided powers of E and F. In order to study $\mathfrak{sl}_2$-categorifications, we introduce in §5.3 "minimal" categorifications of the simple $\mathfrak{sl}_2$-representations, based on the quotients introduced in §3.3. A key construction (§5.4.2) is a functor from such a minimal categorification to a given categorification, that allows us to reduce part of the study of an arbitrary $\mathfrak{sl}_2$-categorification to this minimal case, where explicit computations can be carried out. This corresponds to the decomposition of the $\mathfrak{sl}_2$-representation on K_0 into a direct sum of irreducible representations. We use this in §5.5 to prove a categorified version of the relation $[e, f] = h$ and deduce a construction of categorifications on the module category of the endomorphism ring of "stable" objects in a given categorification.

Section 6 is devoted to the categorification of the simple reflection of the Weyl group. In §6.1, we construct a complex of functors categorifying this reflection, following Rickard. The main result is Theorem 6.4 in part §6.2, which shows that this complex induces a self-equivalence of the homotopy and of the derived category. The key step in the proof for the derived category is the case of a minimal categorification, where we show that the complex has homology concentrated in one degree (§6.3). The case of the homotopy category is reduced to the derived category thanks to the constructions of §5.5.

In Section 7, we study various examples. We define (in §7.1) $\mathfrak{sl}_2$-categorifications on representations of symmetric groups and deduce derived and even splendid Rickard equivalences. We deduce a proof of Broué's abelian defect group conjecture for blocks of symmetric groups. We give similar constructions for cyclotomic Hecke algebras (§7.2) and for general linear groups over a finite field in the nondefining characteristic case (§7.3) for which we also deduce the validity of Broué's abelian defect group conjecture. We also construct $\mathfrak{sl}_2$-categorifications on category $\mathcal{O}$ for $\mathfrak{gl}_n$ (§7.4) and on rational representations of GL_n over an algebraically closed field of characteristic $p > 0$ (§7.5). This answers in particular the GL case of a conjecture of Rickard on blocks corresponding to weights with the same stabilizers under the dot action of the affine Weyl group. We also explain similar constructions for q-Schur algebras (§7.6) and provide morphisms of categorifications relating the previous constructions. A special role is played by the endomorphism X, which takes various incarnations: the Casimir in the rational representation case and the Jucys-Murphy elements in the Hecke algebra case. In the case of the general linear groups over a finite field, our construction seems to be new. Our last section (§7.7) provides various realizations of minimal categorifications, including one coming from the geometry of Grassmannian varieties.

Our general approach is inspired by [LLT], [Ar1], [Gr], [GrVa], and [BeFreKho] (cf. [Rou3, §3.3]), and our strategy for proving the invertibility of Θ is reminiscent of [DeLu], [CaRi].

In a work in progress, we study the braid relations between the categorifications of the simple reflections, in the more general framework of categorifications of Kac-Moody algebras and in relation to Nakajima's quiver variety constructions.

2. Notation

Given an algebra A, we denote by A^{opp} the opposite algebra. We denote by A-mod the category of finitely generated A-modules. Given an abelian category $\mathcal{A}$, we denote by $\mathcal{A}$-proj the category of projective objects of $\mathcal{A}$.

Let $\mathcal{C}$ be an additive category. We denote by $\mathrm{Comp}(\mathcal{C})$ the category of complexes of objects of $\mathcal{C}$ and by $K(\mathcal{C})$ the corresponding homotopy category.

Given an object M in an abelian category, we denote by $\mathrm{soc}(M)$ (resp. $\mathrm{hd}(M)$) the socle (resp. the head) of M, i.e., the largest semi-simple subobject (resp. quotient) of M, when this exists.

We denote by $K_0(\mathcal{A})$ the Grothendieck group of an exact category $\mathcal{A}$.

Given a functor F, we sometimes write F for the identity endomorphism $\mathbf{1}_F$ of F.

3. Affine Hecke algebras

3.1. *Definitions.* Let k be a field and $q \in k^\times$. We define a k-algebra as $H_n = H_n(q)$.

3.1.1. *The nondegenerate case.* Assume $q \neq 1$. The affine Hecke algebra $H_n(q)$ is the k-algebra with generators

$$T_1, \ldots, T_{n-1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1}$$

subject to the relations

$$
\begin{aligned}
(T_i+1)(T_i-q) &= 0,\\
T_iT_j &= T_jT_i \quad (\text{when } |i-j|>1),\\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1},\\
X_iX_j &= X_jX_i,\\
X_iX_i^{-1} &= X_i^{-1}X_i = 1,\\
X_iT_j &= T_jX_i \quad (\text{when } i-j\neq 0,1),\\
T_iX_iT_i &= qX_{i+1}.
\end{aligned}
$$

We denote by $H_n^f(q)$ the subalgebra of $H_n(q)$ generated by $T_1,\dots,T_{n-1}$. It is the Hecke algebra of the symmetric group $\mathfrak{S}_n$.

Let $P_n = k[X_1^{\pm 1},\dots,X_n^{\pm 1}]$, a subalgebra of $H_n(q)$ of Laurent polynomials. We put also $P_{[i]} = k[X_i^{\pm 1}]$.

3.1.2. *The degenerate case.* Assume $q=1$. The degenerate affine Hecke algebra $H_n(1)$ is the k-algebra with generators

$$T_1,\dots,T_{n-1},X_1,\dots,X_n$$

subject to the relations

$$
\begin{aligned}
T_i^2 &= 1,\\
T_iT_j &= T_jT_i \quad (\text{when } |i-j|>1),\\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1},\\
X_iX_j &= X_jX_i,\\
X_iT_j &= T_jX_i \quad (\text{when } i-j\neq 0,1),\\
X_{i+1}T_i &= T_iX_i+1.
\end{aligned}
$$

Note that the degenerate affine Hecke algebra is not the specialization of the affine Hecke algebra.

We put $P_n = k[X_1,\dots,X_n]$, a polynomial subalgebra of $H_n(1)$. We also put $P_{[i]} = k[X_i]$. The subalgebra $H_n^f(1)$ of $H_n(1)$ generated by $T_1,\dots,T_{n-1}$ is the group algebra $k\mathfrak{S}_n$ of the symmetric group.

3.1.3. We put $H_n = H_n(q)$ and $H_n^f = H_n^f(q)$. There is an isomorphism $H_n \xrightarrow{\sim} H_n^{\mathrm{opp}}, T_i \mapsto T_i,\ X_i \mapsto X_i$. It allows us to switch between right and left H_n-modules. There is an automorphism of H_n defined by $T_i \mapsto T_{n-i},\ X_i \mapsto \tilde{X}_{n-i+1}$, where $\tilde{X}_i = X_i^{-1}$ if $q\neq 1$ and $\tilde{X}_i = -X_i$ if $q=1$.

We denote by $l : \mathfrak{S}_n \to \mathbf{N}$ the length function and put $s_i = (i,i+1) \in \mathfrak{S}_n$. Given $w = s_{i_1}\cdots s_{i_r}$ a reduced decomposition of an element $w \in \mathfrak{S}_n$ (i.e., $r = l(w)$), we put $T_w = T_{s_{i_1}}\cdots T_{s_{i_r}}$.

Now, $H_n = H_n^f \otimes P_n = P_n \otimes H_n^f$. We have an action of $\mathfrak{S}_n$ on P_n by permutation of the variables. Given $p \in P_n$, [Lu, Prop. 3.6],

$$T_i p - s_i(p) T_i = \begin{cases} (q-1)(1-X_iX_{i+1}^{-1})^{-1}(p - s_i(p)) & \text{if } q \neq 1 \\ (X_{i+1}-X_i)^{-1}(p-s_i(p)) & \text{if } q = 1. \end{cases} \tag{1}$$

Note that $(P_n)^{\mathfrak{S}_n} \subset Z(H_n)$ (this is actually an equality, a result of Bernstein).

3.1.4. Let 1 (resp. sgn) be the one-dimensional representation of H_n^f given by $T_{s_i} \mapsto q$ (resp. $T_{s_i} \mapsto -1$). Let $\tau \in \{1, \mathrm{sgn}\}$. Now,

$$c_n^\tau = \sum_{w \in \mathfrak{S}_n} q^{-l(w)} \tau(T_w) T_w$$

and $c_n^\tau \in Z(H_n^f)$. We have $c_n^1 = \sum_{w\in\mathfrak{S}_n} T_w$ and $c_n^{\mathrm{sgn}} = \sum_{w\in\mathfrak{S}_n} (-q)^{-l(w)} T_w$, and $c_n^1 c_n^{\mathrm{sgn}} = c_n^{\mathrm{sgn}} c_n^1 = 0$ for $n \geq 2$.

More generally, given $1 \leq i \leq j \leq n$, we denote by $\mathfrak{S}_{[i,j]}$ the symmetric group on $[i,j] = \{i, i+1, \ldots, j\}$, we define similarly $H_{[i,j]}^f$, $H_{[i,j]}$ and we put $c_{[i,j]}^\tau = \sum_{w \in \mathfrak{S}_{[i,j]}} q^{-l(w)}\tau(T_w)T_w$.

Given I a subset of $\mathfrak{S}_n$ we put $c_I^\tau = \sum_{w\in I} q^{-l(w)}\tau(T_w)T_w$. We have

$$c_n^\tau = c_{[\mathfrak{S}_n/\mathfrak{S}_i]}^\tau c_i^\tau = c_i^\tau c_{[\mathfrak{S}_i \setminus \mathfrak{S}_n]}^\tau$$

where $[\mathfrak{S}_n/\mathfrak{S}_i]$ (resp. $[\mathfrak{S}_i \setminus \mathfrak{S}_n]$) is the set of minimal length representatives of right (resp. left) cosets.

As M is a projective H_n^f-module, $c_n^\tau M = \{m \in M \mid hm = \tau(h)m \text{ for all } h \in H_n^f\}$ and the multiplication map $c_n^\tau H_n^f \otimes_{H_n^f} M \xrightarrow{\sim} c_n^\tau M$ is an isomorphism. Given N an H_n-module, then the canonical map $c_n^\tau H_n^f \otimes_{H_n^f} N \xrightarrow{\sim} c_n^\tau H_n \otimes_{H_n} N$ is an isomorphism.

3.2. *Totally ramified central character.* We gather here a number of properties of (degenerate) affine Hecke algebras after completion at a maximally ramified central character. Compared to classical results, some extra complications arise from the possibility of $n!$ being 0 in k.

3.2.1. We fix $a \in k$, with $a \neq 0$ if $q \neq 1$. We put $x_i = X_i - a$. Let $\mathfrak{m}_n$ be the maximal ideal of P_n generated by $x_1, \ldots, x_n$ and let $\mathfrak{n}_n = (\mathfrak{m}_n)^{\mathfrak{S}_n}$.

Let $e_m(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} x_{i_1} \cdots x_{i_m} \in P_n^{\mathfrak{S}_n}$ be the m-th elementary symmetric function. Then, $x_n^n = \sum_{i=0}^{n-1} (-1)^{n+i+1} x_n^i e_{n-i}(x_1, \ldots, x_n)$. Thus, $x_n^l \in \bigoplus_{i=0}^{n-1} x_n^i \mathfrak{n}_n$ for $l \geq n$. Via Galois theory, we deduce that $P_n^{\mathfrak{S}_{n-1}} = \bigoplus_{i=0}^{n-1} x_n^i P_n^{\mathfrak{S}_n}$. Using that the multiplication map $P_j^{\mathfrak{S}_j} \otimes P_{[j+1,n]} \xrightarrow{\sim} P_n^{\mathfrak{S}_j}$ is an isomorphism, we deduce by induction that

$$P_n^{\mathfrak{S}_r} = \bigoplus_{0 \leq a_i < r+i} x_{r+1}^{a_1} \cdots x_n^{a_{n-r}} P_n^{\mathfrak{S}_n}. \tag{2}$$

3.2.2. We denote by $\widehat{P_n^{\mathfrak{S}_n}}$ the completion of $P_n^{\mathfrak{S}_n}$ at $\mathfrak{n}_n$, and put $\hat{P}_n = P_n \otimes_{P_n^{\mathfrak{S}_n}} \widehat{P_n^{\mathfrak{S}_n}}$ and $\hat{H}_n = H_n \otimes_{P_n^{\mathfrak{S}_n}} \widehat{P_n^{\mathfrak{S}_n}}$. The canonical map $\hat{P}_n^{\mathfrak{S}_n} \xrightarrow{\sim} \widehat{P_n^{\mathfrak{S}_n}}$ is an isomorphism, since $\widehat{P_n^{\mathfrak{S}_n}}$ is flat over $P_n^{\mathfrak{S}_n}$.

We denote by $\mathcal{N}_n$ the category of locally nilpotent $\hat{H}_n$-modules, i.e., the category of H_n-modules on which $\mathfrak{n}_n$ acts locally nilpotently: an H_n-module M is in $\mathcal{N}_n$ if for every $m \in M$, there is $i > 0$ such that $\mathfrak{n}_n^i m = 0$.

We put $\bar{H}_n = H_n/(H_n\mathfrak{n}_n)$ and $\bar{P}_n = P_n/(P_n\mathfrak{n}_n)$. Then multiplication gives an isomorphism $\bar{P}_n \otimes H_n^f \xrightarrow{\sim} \bar{H}_n$. The canonical map

$$\bigoplus_{0\le a_i<i} kx_1^{a_1}\cdots x_n^{a_n} \xrightarrow{\sim} \bar{P}_n$$

is an isomorphism; hence $\dim_k \bar{H}_n = (n!)^2$.

The unique simple object of $\mathcal{N}_n$ is (see [Ka, Th. 2.2])

$$K_n = H_n \otimes_{P_n} P_n/\mathfrak{m}_n \simeq \bar{H}_n c_n^\tau.$$

This has dimension $n!$ over k. It follows that the canonical surjective map $\bar{H}_n \to \operatorname{End}_k(K_n)$ is an isomorphism; hence $\bar{H}_n$ is a simple split k-algebra.

Since K_n is a free module over H_n^f, it follows that any object of $\mathcal{N}_n$ is free by restriction to H_n^f. From §3.1.4, we deduce that for any $M \in \mathcal{N}_n$, the canonical map $c_n^\tau H_n \otimes_{H_n} M \xrightarrow{\sim} c_n^\tau M$ is an isomorphism.

Remark 3.1. We have excluded the case of the affine Weyl group algebra (the affine Hecke algebra at $q = 1$). Indeed, in that case K_n is not simple (when $n \ge 2$) and $\bar{H}_n$ is not a simple algebra. When $n = 2$, we have $\bar{H}_n \simeq \big(k[x]/(x^2)\big) \rtimes \mu_2$, where the group $\mu_2 = \{\pm 1\}$ acts on x by multiplication.

3.2.3. Let $f : M \to N$ be a morphism of finitely generated $\hat{P}_n^{\mathfrak{S}_n}$-modules. Then, f is surjective if and only if $f \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n^{\mathfrak{S}_n}/\hat{\mathfrak{n}}_n$ is surjective.

LEMMA 3.2. *There exist isomorphisms*

$$\hat{H}_n c_n^\tau \otimes_k \bigoplus_{i=0}^{n-1} x_n^i k \xrightarrow[\sim]{\text{can}} \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n^{\mathfrak{S}_{n-1}} \xrightarrow[\sim]{\text{mult}} \hat{H}_n c_{n-1}^\tau.$$

Proof. The first isomorphism follows from the decomposition of $\hat{P}_n^{\mathfrak{S}_{n-1}}$ in (2).

Let us now study the second map. Note that both terms are free $\hat{P}_n^{\mathfrak{S}_n}$-modules of rank $n \cdot n!$, since $\hat{H}_n c_{n-1}^\tau \simeq \hat{P}_n \otimes H_n^f c_{n-1}^\tau$. Consequently, it suffices to show that the map is surjective. Thanks to the remark above, it is enough to check surjectivity after applying $- \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n^{\mathfrak{S}_n}/\hat{\mathfrak{n}}_n$.

Note that the canonical surjective map $k[x_n] \to P_n^{\mathfrak{S}_{n-1}} \otimes_{P_n^{\mathfrak{S}_n}} P_n^{\mathfrak{S}_n}/\mathfrak{n}_n$ factors through $k[x_n]/(x_n^n)$ (cf. §3.2.1). So, we have to show that the multiplication map $f : \bar{H}_n c_n^\tau \otimes k[x_n]/(x_n^n) \to \bar{H}_n c_{n-1}^\tau$ is surjective. This is a morphism of $(\bar{H}_n, k[x_n]/(x_n^n))$-bimodules. The elements $c_n^\tau, c_n^\tau x_n, \ldots, c_n^\tau x_n^{n-1}$ of $\bar{H}_n$ are linearly independent, hence the image of f is a faithful $(k[x_n]/(x_n^n))$-module. It follows that f is injective, since $\bar{H}_n c_n^\tau$ is a simple $\bar{H}_n$-module. Now, $\dim_k \bar{H}_n c_{n-1}^\tau = n \cdot n!$; hence f is an isomorphism. $\square$

Let M be a $k\mathfrak{S}_n$-module. We put $\Lambda^{\mathfrak{S}_n} M = M/(\sum_{0<i<n} M^{s_i})$. If $n! \in k^\times$, then $\Lambda^{\mathfrak{S}_n} M$ is the largest quotient of M on which $\mathfrak{S}_n$ acts via the sign character. Note that given a vector space V, then $\Lambda^{\mathfrak{S}_n}(V^{\otimes n}) = \Lambda^n V$.

Proposition 3.3. *Let $\{\tau, \tau'\} = \{1, \mathrm{sgn}\}$ and $r \leq n$. There exist isomorphisms*

$$\hat{H}_n c_n^\tau \otimes_k \bigoplus_{0 \leq a_i < n-r+i} x_{n-r+1}^{a_1} \cdots x_n^{a_r} k \xrightarrow[\sim]{\text{can}} \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n^{\mathfrak{S}_{[1,n-r]}} \xrightarrow[\sim]{\text{mult}} \hat{H}_n c_{[1,n-r]}^\tau.$$

There is a commutative diagram

$$\begin{tikzcd}
 & \hat{H}_n c_n^\tau \otimes_k \bigoplus_{0 \leq a_1 < \cdots < a_r < n} x_{n-r+1}^{a_1} \cdots x_n^{a_r} k \arrow[d, "\text{can}"', "\sim"] \\
\hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n^{\mathfrak{S}_{[1,n-r]}} \arrow[r, two heads, "\text{can}"] \arrow[dr, two heads, "x \otimes y \mapsto xyc^{\tau'}_{[n-r+1,n]}"'] & \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \Lambda^{\mathfrak{S}_{[n-r+1,n]}} \hat{P}_n^{\mathfrak{S}_{[1,n-r]}} \arrow[d, dotted, "\sim"] \\
 & \hat{H}_n c^\tau_{[1,n-r]} c^{\tau'}_{[n-r+1,n]}.
\end{tikzcd}$$

Proof. The multiplication map $H_n \otimes_{H_{n-i}} H_{n-i} c_{n-i}^\tau \to H_n c_{n-i}^\tau$ is an isomorphism (cf. §3.1.4). It follows from Lemma 3.2 that multiplication is an isomorphism

$$\hat{H}_n c_{n-r+1}^\tau \otimes \bigoplus_{i=0}^{n-r} x_{n-r+1}^i k \xrightarrow{\sim} \hat{H}_n c_{n-r}^\tau$$

and the first statement follows by descending induction on r.

The surjectivity of the diagonal map follows from the first statement of the proposition.

Let $p \in \hat{P}_n^{s_i}$. Then, $c^1_{[i,i+1]} p = p c^1_{[i,i+1]}$. It follows that $c^\tau_{[i,i+1]} p c^{\tau'}_{[i,i+1]} = 0$; hence $c_n^\tau p c^{\tau'}_{[n-r+1,n]} = 0$ whenever $i \geq n-r+1$. This shows the factorization property (existence of the dotted arrow).

Note that $\Lambda^{\mathfrak{S}_{[n-r+1,n]}} \hat{P}_n^{\mathfrak{S}_{n-r}}$ is generated by $\bigoplus_{0 \leq a_1 < \cdots < a_r < n} x_{n-r+1}^{a_1} \cdots x_n^{a_r} k$ as a $\hat{P}_n^{\mathfrak{S}_n}$-module (cf. (2)). It follows that we have surjective maps

$$\begin{aligned}\hat{H}_n c_n^\tau \otimes_k \bigoplus_{0 \leq a_1 < \cdots < a_r < n} x_{n-r+1}^{a_1} \cdots x_n^{a_r} k &\twoheadrightarrow \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \Lambda^{\mathfrak{S}_{[n-r+1,n]}} \hat{P}_n^{\mathfrak{S}_{n-r}} \\ &\twoheadrightarrow \hat{H}_n c_{n-r}^\tau c^{\tau'}_{[n-r+1,n]}.\end{aligned}$$

Now the first and last terms above are free $\hat{P}_n$-modules of rank $\binom{n}{r}$, hence the maps are isomorphisms. □

LEMMA 3.4. *Let $r \leq n$. We have $c_r^\tau \hat{H}_n c_n^\tau = \hat{P}_n^{\mathfrak{S}_r} c_n^\tau$, $c_n^\tau \hat{H}_n c_r^\tau = c_n^\tau \hat{P}_n^{\mathfrak{S}_r}$ and the multiplication maps $c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_r^\tau \xrightarrow{\sim} c_n^\tau \hat{H}_n c_r^\tau$ and $c_r^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_n^\tau \xrightarrow{\sim} c_r^\tau \hat{H}_n c_n^\tau$ are isomorphisms.*

Proof. We have an isomorphism $\hat{P}_n \xrightarrow{\sim} \hat{H}_n c_n^\tau$, $p \mapsto pc_n^\tau$. Let $h \in \hat{H}_n$. We have $c_n^\tau h c_n^\tau = pc_n^\tau$ for some $p \in \hat{P}_n$. Since $T_i c_n^\tau = \tau(T_i)c_n^\tau$, it follows that $T_i p c_n^\tau = \tau(T_i) p c_n^\tau$. So, $(T_i p - s_i(p)T_i)c_n^\tau = \tau(T_i)(p - s_i(p))c_n^\tau$; hence $p - s_i(p) = 0$, by formula (1). It follows that $c_n^\tau \hat{H}_n c_n^\tau \subseteq \hat{P}_n^{\mathfrak{S}_n} c_n^\tau$.

By Proposition 3.3, the multiplication map $\hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n \xrightarrow{\sim} \hat{H}_n$ is an isomorphism. So, the multiplication map $c_n^\tau \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n \xrightarrow{\sim} c_n^\tau \hat{H}_n$ is an isomorphism, hence the canonical map $c_n^\tau \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n \xrightarrow{\sim} \hat{P}_n^{\mathfrak{S}_n} c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n$ is an isomorphism. We deduce that $c_n^\tau \hat{H}_n c_n^\tau = \hat{P}_n^{\mathfrak{S}_n} c_n^\tau$.

Similarly (replacing n by r above), we have $c_n^\tau \hat{P}_r^{\mathfrak{S}_r} c_r^\tau = c_n^\tau \hat{P}_r^{\mathfrak{S}_r}$. Since $P_n^{\mathfrak{S}_r} = P_r^{\mathfrak{S}_r} P_{[r+1,n]}$ (cf. §3.2.1), we deduce that

$$c_n^\tau \hat{H}_n c_r^\tau = c_n^\tau \hat{P}_n c_r^\tau = c_n^\tau \hat{P}_r c_r^\tau \hat{P}_{[r+1,n]} = c_n^\tau \hat{P}_r^{\mathfrak{S}_r} \hat{P}_{[r+1,n]} = c_n^\tau \hat{P}_n^{\mathfrak{S}_r}.$$

By Proposition 3.3, $c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_r^\tau$ is a free $\hat{P}_n^{\mathfrak{S}_r}$-module of rank 1. So, the multiplication map $c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_r^\tau \to c_n^\tau \hat{H}_n c_r^\tau$ is a surjective morphism between free $\hat{P}_n^{\mathfrak{S}_r}$-modules of rank 1, hence it is an isomorphism.

The cases where c_r^τ is on the left are similar. □

PROPOSITION 3.5. *The functors $H_n c_n^\tau \otimes_{P_n^{\mathfrak{S}_n}} -$ and $c_n^\tau H_n \otimes_{H_n} -$ are inverse equivalences of categories between the category of $P_n^{\mathfrak{S}_n}$-modules that are locally nilpotent for $\mathfrak{n}_n$ and $\mathcal{N}_n$.*

Proof. By Proposition 3.3, the multiplication map $\hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} \hat{P}_n \xrightarrow{\sim} \hat{H}_n$ is an isomorphism. It follows that the morphism of $(\hat{H}_n, \hat{H}_n)$-bimodules

$$\hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{\mathfrak{S}_n}} c_n^\tau \hat{H}_n \xrightarrow{\sim} \hat{H}_n,\ hc \otimes ch' \mapsto hch'$$

is an isomorphism.

Since $\hat{P}_n^{\mathfrak{S}_n}$ is commutative, it follows from Lemma 3.4 that the $(\hat{P}_n^{\mathfrak{S}_n}, \hat{P}_n^{\mathfrak{S}_n})$-bimodules $\hat{P}_n^{\mathfrak{S}_n}$ and $c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_n^\tau$ are isomorphic. □

3.3. *Quotients.*

3.3.1. We denote by $\bar{H}_{i,n}$ the image of H_i in $\bar{H}_n$ for $0 \leq i \leq n$ and $\bar{P}_{i,n} = P_i/(P_i \cap (P_n \mathfrak{n}_n))$. Now there is an isomorphism $H_i^f \otimes \bar{P}_{i,n} \xrightarrow[\sim]{\text{mult}} \bar{H}_{i,n}$.

Since $P_n^{\mathfrak{S}_{[i+1,n]}} = \bigoplus_{0 \leq a_l \leq n-l} x_1^{a_1} \cdots x_i^{a_i} P_n^{\mathfrak{S}_n}$ (cf. (2)), we deduce that $P_i = \bigoplus_{0 \leq a_l \leq n-l} x_1^{a_1} \cdots x_n^{a_n} k \oplus (\mathfrak{n}_n P_i \cap P_i)$ and $\mathfrak{n}_n P_i \cap P_i = \mathfrak{n}_n P_n \cap P_i$; hence the canonical map

$$\bigoplus_{0\le a_l\le n-l} x_1^{a_1}\cdots x_n^{a_n}k \xrightarrow{\sim} \bar{P}_{i,n} \tag{3}$$

is an isomorphism. We will identify such a monomial $x_1^{a_1}\cdots x_i^{a_i}$ with its image in $\bar{P}_{i,n}$. Note that $\dim_k \bar{P}_{i,n} = \frac{n!}{(n-i)!}$.

The kernel of the action of $P_i^{\mathfrak{S}_i}$ by right multiplication on $\bar{H}_{i,n}c_i^{\tau}$ is $P_i^{\mathfrak{S}_i}\cap \mathfrak{n}_n P_n$. By Proposition 3.5, we have a Morita equivalence between $\bar{H}_{i,n}$ and $Z_{i,n} = P_i^{\mathfrak{S}_i}/(P_i^{\mathfrak{S}_i}\cap \mathfrak{n}_n P_n)$. Note that $\bar{H}_{i,n}c_i^{\tau}$ is the unique indecomposable projective $\bar{H}_{i,n}$-module and $\dim_k \bar{H}_{i,n} = i!\dim_k \bar{H}_{i,n}c_i^{\tau}$. Thus,

$$\dim_k Z_{i,n} = \frac{1}{(i!)^2}\dim_k \bar{H}_{i,n} = \binom{n}{i}$$

and $Z_{i,n} = Z(\bar{H}_{i,n})$.

We denote by $P(r,s)$ the set of partitions $\mu = (\mu_1 \ge \cdots \ge \mu_r \ge 0)$ with $\mu_1 \le s$. Given $\mu \in P(r,s)$, we denote by m_μ the corresponding monomial symmetric function

$$m_\mu(x_1,\dots,x_r) = \sum_\sigma x_1^{\mu_{\sigma(1)}}\cdots x_r^{\mu_{\sigma(r)}}$$

where σ runs over left coset representatives of $\mathfrak{S}_r$ modulo the stabilizer of $(\mu_1,\dots,\mu_r)$.

The isomorphism (3) shows that the canonical map from

$$\bigoplus_{\mu\in P(i,n-i)} km_\mu(x_1,\dots,x_i)$$

to $\bar{P}_{i,n}$ is injective, with image contained in $Z_{i,n}$. Comparing dimensions, we see that the canonical map

$$\bigoplus_{\mu\in P(i,n-i)} km_\mu(x_1,\dots,x_i) \xrightarrow{\sim} Z_{i,n}$$

is an isomorphism.

Also, comparing dimensions, one sees that the canonical surjective maps

$$P_i \otimes_{P_i^{\mathfrak{S}_i}} Z_{i,n} \xrightarrow{\sim} \bar{P}_{i,n} \text{ and } H_i \otimes_{P_i^{\mathfrak{S}_i}} Z_{i,n} \xrightarrow{\sim} \bar{H}_{i,n}$$

are isomorphisms.

3.3.2. Let $G_{i,n}$ be the Grassmannian variety of i-dimensional subspaces of $\mathbf{C}^n$ and G_n be the variety of complete flags in $\mathbf{C}^n$. The canonical morphism $p: G_n \to G_{i,n}$ induces an injective morphism of algebras $p^*: H^*(G_{i,n}) \to H^*(G_n)$ (cohomology is taken with coefficients in k). We identify G_n with GL_n/B, where B is the stabilizer of the standard flag $(\mathbf{C}(1,0,\dots,0)\subset\cdots\subset \mathbf{C}^n)$. Let L_j be the line bundle associated to the character of B given by the

j-th diagonal coefficient. We have an isomorphism $\bar{P}_n \xrightarrow{\sim} H^*(G_n)$ sending x_j to the first Chern class of L_j. It multiplies degrees by 2. Now, $p^*H^*(G_{i,n})$ coincides with the image of $P_i^{\mathfrak{S}_i}$ in $\bar{P}_n$. So, we have obtained an isomorphism

$$Z_{i,n} \xrightarrow{\sim} H^*(G_{i,n}).$$

Since $G_{i,n}$ is projective, smooth and connected, of dimension $i(n-i)$, Poincaré duality says that the cup product $H^j(G_{i,n}) \times H^{2i(n-i)-j}(G_{i,n}) \to H^{2i(n-i)}(G_{i,n})$ is a perfect pairing. Via the isomorphism $H^{2i(n-i)}(G_{i,n}) \xrightarrow{\sim} k$ given by the fundamental class, this provides $H^*(G_{i,n})$ with the structure of a symmetric algebra.

Note that the algebra $\bar{H}_{i,n}$ is isomorphic to the ring of $i! \times i!$ matrices over $H^*(G_{i,n})$ and it is a symmetric algebra. Up to isomorphism, it is independent of a and q.

3.3.3. Letting $i \leq j$, we have

$$\bar{H}_{j,n} = \bar{H}_{i,n} \otimes \bigoplus_{\substack{w \in [\mathfrak{S}_i \backslash \mathfrak{S}_j] \\ 0 \leq a_l \leq n-l}} k x_{i+1}^{a_{i+1}} \cdots x_j^{a_j} \otimes kT_w;$$

hence $\bar{H}_{j,n}$ is a free $\bar{H}_{i,n}$-module of rank $\frac{(n-i)!j!}{(n-j)!i!}$.

LEMMA 3.6. *The H_i-module $c^\tau_{[i+1,n]}K_n$ has a simple socle and head.*

Proof. By Proposition 3.3, multiplication gives an isomorphism

$$\bigoplus_{0 \leq a_l < l} x_{i+1}^{a_1} \cdots x_n^{a_{n-i}} k \otimes c^\tau_{[i+1,n]} H_{[i+1,n]} \xrightarrow{\sim} H_{[i+1,n]},$$

hence gives an isomorphism of $\bar{H}_{i,n}$-modules

$$\bigoplus_{0 \leq a_l < l} x_{i+1}^{a_1} \cdots x_n^{a_{n-i}} k \otimes c^\tau_{[i+1,n]} \bar{H}_n \xrightarrow{\sim} \bar{H}_n.$$

Since $\bar{H}_n$ is a free $\bar{H}_{i,n}$-module of rank $\frac{(n-i)!n!}{i!}$, it follows that hence $c^\tau_{[i+1,n]}\bar{H}_n$ is a free $\bar{H}_{i,n}$-module of rank $\frac{n!}{i!}$. We have $\bar{H}_{i,n} \simeq i! \cdot M$ as $\bar{H}_{i,n}$-modules, where M has a simple socle and head. Since in addition $\bar{H}_n \simeq n! \cdot K_n$ as $\bar{H}_n$-modules, we deduce that $c^\tau_{[i+1,n]}K_n \simeq M$ has a simple socle and head. $\square$

LEMMA 3.7. *Let $r \leq l \leq n$. We have isomorphisms*

$$\begin{tikzcd}
\displaystyle\bigoplus_{0 \leq a_i \leq n-i} x_1^{a_1} \cdots x_{l-r}^{a_{l-r}} k \otimes \bigoplus_{\mu \in P(r,n-l)} m_\mu(x_{l-r+1}, \ldots, x_l)k \arrow[r, "\sim", "a \otimes b \mapsto abc_l^\tau"'] \arrow[d, "\sim"', "a \otimes b \mapsto ac_n^\tau \otimes b"] & c^\tau_{[l-r+1,l]} \bar{H}_{l,n} c_l^\tau \\
\bar{H}_{l-r,n} c^\tau_{l-r} \otimes \displaystyle\bigoplus_{\mu \in P(r,n-l)} m_\mu(x_{l-r+1}, \ldots, x_l)k & c^\tau_{[l-r+1,l]} \bar{H}_{l,n} \otimes_{\bar{H}_{l,n}} \bar{H}_{l,n} c_l^\tau . \arrow[u, "\mathrm{mult}", "\sim"']
\end{tikzcd}$$

Proof. Let $L = \bigoplus_{\mu\in P(r,n-l), 0\le a_i\le n-i} m_\mu(x_{l-r+1},\ldots,x_l)x_1^{a_1}\cdots x_{l-r}^{a_{l-r}}k$.

We have $L\cap \mathfrak{n}_n P_n = 0$ (cf. (3)); hence the canonical map $f : L \to P_l^{\mathfrak{S}_{[l-r+1,l]}} \otimes_{P_l^{\mathfrak{S}_l}} Z_{l,n}$ is injective. Since $\dim_k Z_{l,n} = \binom{n}{l}$ and $P_l^{\mathfrak{S}_{[l-r+1,l]}}$ is a free $P_l^{\mathfrak{S}_l}$-module of rank $\frac{l!}{r!}$, it follows that f is an isomorphism. Now, we have an isomorphism (Lemma 3.4)

$$\hat{P}_l^{\mathfrak{S}_{[l-r+1,l]}} \xrightarrow{\sim} c^\tau_{[l-r+1,l]}\hat{H}_l c_l^\tau,\ a\mapsto ac_l^\tau.$$

Consequently, the horizontal map of the lemma is an isomorphism.

As seen in §3.3.1, the left vertical map is an isomorphism. By Lemma 3.4, the right vertical map is also an isomorphism. ◻

4. Reminders

4.1. *Adjunctions.*

4.1.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. Let $(G, G^\vee)$ be an adjoint pair of functors, $G : \mathcal{C}\to\mathcal{C}'$ and $G^\vee : \mathcal{C}'\to\mathcal{C}$: these are the data of two morphisms $\eta : \mathrm{Id}_{\mathcal{C}} \to G^\vee G$ (the unit) and $\varepsilon : GG^\vee \to \mathrm{Id}_{\mathcal{C}'}$ (the co-unit), such that $(\varepsilon \mathbf{1}_G)\circ(\mathbf{1}_G\eta) = \mathbf{1}_G$ and $(\mathbf{1}_{G^\vee}\varepsilon)\circ(\eta\mathbf{1}_{G^\vee}) = \mathbf{1}_{G^\vee}$. Here, we have denoted by $\mathbf{1}_G$ the identity map $G\to G$. We have then a canonical isomorphism functorial in $X\in\mathcal{C}$ and $X'\in\mathcal{C}'$:

$$\gamma_G(X,X') : \mathrm{Hom}(GX,X') \xrightarrow{\sim} \mathrm{Hom}(X, G^\vee X'),$$
$$f\mapsto G^\vee(f)\circ\eta(X),\quad \varepsilon(X')\circ G(f') \hookleftarrow f'.$$

Note that the data of such a functorial isomorphism provide a structure of an adjoint pair.

4.1.2. Let $(H, H^\vee)$ be an adjoint pair of functors, with $H : \mathcal{C}\to\mathcal{C}'$. Let $\phi\in\mathrm{Hom}(G,H)$. Then, we define $\phi^\vee : H^\vee\to G^\vee$ as the composition

$$\phi^\vee : H^\vee \xrightarrow{\eta_G \mathbf{1}_{H^\vee}} G^\vee G H^\vee \xrightarrow{\mathbf{1}_{G^\vee}\phi\mathbf{1}_{H^\vee}} G^\vee H H^\vee \xrightarrow{\mathbf{1}_{G^\vee}\varepsilon_H} G^\vee.$$

This is the unique map making the following diagram commutative, for any $X\in\mathcal{C}$ and $X'\in\mathcal{C}'$:

$$\begin{tikzcd}
\mathrm{Hom}(HX,X') \arrow[r, "\mathrm{Hom}(\phi(X),X')"] \arrow[d, "\gamma_H(X,X')"', "\sim"] & \mathrm{Hom}(GX,X') \arrow[d, "\sim"', "\gamma_G(X,X')"] \\
\mathrm{Hom}(X,H^\vee X') \arrow[r, "\mathrm{Hom}(X,\phi^\vee(X'))"'] & \mathrm{Hom}(X,G^\vee X').
\end{tikzcd}$$

We have an isomorphism $\mathrm{Hom}(G,H)\xrightarrow{\sim}\mathrm{Hom}(H^\vee,G^\vee), \phi\mapsto\phi^\vee$. We obtain in particular an isomorphism of monoids $\mathrm{End}(G)\xrightarrow{\sim}\mathrm{End}(G^\vee)^{\mathrm{opp}}$. Given $f\in$

$\mathrm{End}(G)$, then, the following diagrams commute

$$\begin{tikzcd} & G^\vee G \arrow[dr, "\mathbf{1}_{G^\vee} f"] & \\ \mathrm{Id}_{\mathcal{C}} \arrow[ur, "\eta"] \arrow[dr, "\eta"'] & & G^\vee G \\ & G^\vee G \arrow[ur, "f^\vee \mathbf{1}_G"'] & \end{tikzcd} \qquad \begin{tikzcd} & GG^\vee \arrow[dr, "\varepsilon"] & \\ GG^\vee \arrow[ur, "f\mathbf{1}_{G^\vee}"] \arrow[dr, "\mathbf{1}_G f^\vee"'] & & \mathrm{Id}_{\mathcal{C}'} . \\ & GG^\vee \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

4.1.3. Let now $(G_1, G_1^\vee)$ and $(G_2, G_2^\vee)$ be two pairs of adjoint functors, with $G_1 : \mathcal{C}' \to \mathcal{C}''$ and $G_2 : \mathcal{C} \to \mathcal{C}'$. The composite morphisms

$$\mathrm{Id}_{\mathcal{C}} \xrightarrow{\eta_2} G_2^\vee G_2 \xrightarrow{\mathbf{1}_{G_2^\vee}\eta_1\mathbf{1}_{G_2}} G_2^\vee G_1^\vee G_1 G_2 \text{ and } G_1 G_2 G_2^\vee G_1^\vee \xrightarrow{\mathbf{1}_{G_1}\varepsilon_2\mathbf{1}_{G_1^\vee}} G_1 G_1^\vee \xrightarrow{\varepsilon_1} \mathrm{Id}_{\mathcal{C}}$$

give an adjoint pair $(G_1 G_2, G_2^\vee G_1^\vee)$.

4.1.4. Let $F = 0 \to F^r \xrightarrow{d^r} F^{r+1} \to \cdots \to F^s \to 0$ be a complex of functors from $\mathcal{C}$ to $\mathcal{C}'$ (with F^i in degree i). This defines a functor $\mathrm{Comp}(\mathcal{C}) \to \mathrm{Comp}(\mathcal{C}')$ by taking total complexes.

Let $(F^i, F^{i\vee})$ be adjoint pairs for $r \le i \le s$. Let

$$F^\vee = 0 \to F^{s\vee} \xrightarrow{(d^{s-1})^\vee} \cdots \to F^{r\vee} \to 0$$

with $F^{i\vee}$ in degree $-i$. This complex of functors defines a functor $\mathrm{Comp}(\mathcal{C}') \to \mathrm{Comp}(\mathcal{C})$.

There is an adjunction $(F, F^\vee)$ between functors on categories of complexes, uniquely determined by the property that given $X \in \mathcal{C}$ and $X' \in \mathcal{C}'$, then $\gamma_F(X, X') : \mathrm{Hom}_{\mathrm{Comp}(\mathcal{C}')}(FX, X') \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Comp}(\mathcal{C})}(X, F^\vee X')$ is the restriction of

$$\sum_i \gamma_{F^i}(X, X') : \bigoplus_i \mathrm{Hom}_{\mathcal{C}'}(F^i X, X') \xrightarrow{\sim} \bigoplus_i \mathrm{Hom}_{\mathcal{C}}(X, F^{i\vee} X').$$

This extends to the case where F is unbounded, under the assumption that for any $X \in \mathcal{C}$, then $F^r(X) = 0$ for $|r| \gg 0$ and for any $X' \in \mathcal{C}'$, then $F^{r\vee}(X') = 0$ for $|r| \gg 0$.

4.1.5. Assume $\mathcal{C}$ and $\mathcal{C}'$ are abelian categories.

Let $c \in \mathrm{End}(G)$. We put $cG = \mathrm{im}(c)$. We assume the canonical surjection $G \to cG$ splits (i.e., $cG = eG$ for some idempotent $e \in \mathrm{End}(G)$). Then, the canonical injection $c^\vee G^\vee \to G^\vee$ splits as well (indeed, $c^\vee G^\vee = e^\vee G^\vee$).

Let $X \in \mathcal{C}$, $X' \in \mathcal{C}'$ and $\phi \in \operatorname{Hom}(cGX, X')$. There is $\psi \in \operatorname{Hom}(GX, X')$ such that $\phi = \psi_{|cGX}$. We have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\eta"'] & G^\vee GX \arrow[r, "G^\vee c"] & G^\vee GX \arrow[r, "G^\vee \psi"] & G^\vee X' \\ & G^\vee GX \arrow[ur, "c^\vee G"'] \arrow[r, "G^\vee \psi"'] & G^\vee X'. \arrow[ur, "c^\vee"'] & \end{tikzcd}$$

It follows that there is a (unique) map

$$\gamma_{cG}(X, X') : \operatorname{Hom}(cGX, X') \to \operatorname{Hom}(X, c^\vee G^\vee X')$$

making the following diagram commutative

$$\begin{tikzcd} \operatorname{Hom}(GX, X') \arrow[r, "\gamma_G(X,X')", "\sim"'] & \operatorname{Hom}(X, G^\vee X') \\ \operatorname{Hom}(cGX, X') \arrow[u, hook] \arrow[r, dotted, "\sim", "\gamma_{cG}(X,X')"'] & \operatorname{Hom}(X, c^\vee G^\vee X'). \arrow[u, hook] \end{tikzcd}$$

The vertical maps come from the canonical projection $G \to cG$ and injection $c^\vee G^\vee \to G^\vee$.

Similarly, there is a (unique) map $\gamma'_{cG}(X, X') : \operatorname{Hom}(X, c^\vee G^\vee X') \to \operatorname{Hom}(cGX, X')$ making the following diagram commutative

$$\begin{tikzcd} \operatorname{Hom}(GX, X') & \operatorname{Hom}(X, G^\vee X') \arrow[l, "\gamma_G(X,X')^{-1}"', "\sim"] \\ \operatorname{Hom}(cGX, X') \arrow[u, hook] & \operatorname{Hom}(X, c^\vee G^\vee X'). \arrow[u, hook] \arrow[l, dotted, "\sim"', "\gamma'_{cG}(X,X')"] \end{tikzcd}$$

The maps $\gamma_{cG}(X, X')$ and $\gamma'_{cG}(X, X')$ are inverse to each other and they provide $(cG, c^\vee G^\vee)$ with the structure of an adjoint pair. If $p : G \to cG$ denotes the canonical surjection, then $p^\vee : c^\vee G^\vee \to G^\vee$ is the canonical injection.

4.1.6. Let $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{D}$ and $\mathcal{D}'$ be four categories, $G : \mathcal{C} \to \mathcal{C}'$, $G^\vee : \mathcal{C}' \to \mathcal{C}$, $H : \mathcal{D} \to \mathcal{D}'$ and $H^\vee : \mathcal{D}' \to \mathcal{D}$, and $(G, G^\vee)$ and $(H, H^\vee)$ be two adjoint pairs. Let $F : \mathcal{C} \to \mathcal{D}$ and $F' : \mathcal{C}' \to \mathcal{D}'$ be two fully faithful functors and $\phi : F'G \xrightarrow{\sim} HF$ be an isomorphism.

We have isomorphisms

$$\operatorname{Hom}(GG^\vee, \operatorname{Id}_{\mathcal{C}'}) \xrightarrow[\sim]{F'} \operatorname{Hom}(F'GG^\vee, F') \xrightarrow[\sim]{\operatorname{Hom}(\phi^{-1}\mathbf{1}_{G^\vee}, F')} \operatorname{Hom}(HFG^\vee, F')$$
$$\xrightarrow[\sim]{\gamma_H(FG^\vee, F')} \operatorname{Hom}(FG^\vee, H^\vee F')$$

and let $\psi : FG^\vee \to H^\vee F'$ denote the image of ε_G under this sequence of isomorphisms.

Then, ψ is an isomorphism and we have a commutative diagram

$$\begin{tikzcd} F'GG^\vee \arrow[r, "\mathbf{1}_F\varepsilon_G"] \arrow[d, "\phi\mathbf{1}_{G^\vee}"'] & F' \\ HFG^\vee \arrow[r, "\mathbf{1}_H\psi"'] & HH^\vee F'. \arrow[u, "\varepsilon_H\mathbf{1}_{F'}"'] \end{tikzcd}$$

4.2. *Representations of* $\mathfrak{sl}_2$. We put

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ f=\begin{pmatrix}0&0\\1&0\end{pmatrix}\text{ and } h=ef-fe=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

We have

$$s=\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\exp(-f)\exp(e)\exp(-f)$$

$$s^{-1}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\exp(f)\exp(-e)\exp(f).$$

We put $e_+=e$ and $e_-=f$.

Let V be a locally finite representation of $\mathfrak{sl}_2(\mathbf{Q})$ (i.e., a direct sum of finite dimensional representations). Given $\lambda\in\mathbf{Z}$, we denote by V_λ the weight space of V for the weight λ (i.e., the λ-eigenspace of h).

For $v\in V$, let $h_\pm(v)=\max\{i|e_\pm^i v\neq 0\}$ and $d(v)=h_+(v)+h_-(v)+1$.

Lemma 4.1. *Assume V is a direct sum of isomorphic simple $\mathfrak{sl}_2(\mathbf{Q})$-modules of dimension d.*

Let $v\in V_\lambda$. Then,

- $d(v)=d=1+2h_\pm(v)\pm\lambda$
- $e_\mp^{(j)}e_\pm^{(j)}v=\binom{h_\mp(v)+j}{j}\cdot\binom{h_\pm(v)}{j}v$ *for* $0\le j\le h_\pm(v)$.

Lemma 4.2. *Let $\lambda\in\mathbf{Z}$ and $v\in V_{-\lambda}$. Then,*

$$s(v)=\sum_{r=\max(0,-\lambda)}^{h_-(v)}\frac{(-1)^r}{r!(\lambda+r)!}e^{\lambda+r}f^r(v)$$

and

$$s^{-1}(v)=\sum_{r=\max(0,\lambda)}^{h_+(v)}\frac{(-1)^r}{r!(-\lambda+r)!}e^r f^{-\lambda+r}(v).$$

In the following lemma, we investigate bases of weight vectors with positivity properties.

LEMMA 4.3. *Let V be a locally finite $\mathfrak{sl}_2(\mathbf{Q})$-module. Let $\mathcal{B}$ be a basis of V consisting of weight vectors such that $\bigoplus_{b\in\mathcal{B}}\mathbf{Q}_{\geq 0}b$ is stable under the actions of e_+ and e_-. Let $\mathcal{L}_\pm=\{b\in\mathcal{B}|e_\mp b=0\}$ and given $r\geq 0$, let $V^{\leq r}=\bigoplus_{d(b)\leq r}\mathbf{Q}b$.*

Then,

(1) *With $r\geq 0$, then $V^{\leq r}$ is a submodule of V isomorphic to a sum of modules of dimension $\leq r$.*

(2) *With $b\in\mathcal{B}$, there is $e_\pm^{h_\pm(b)}b\in\mathbf{Q}_{\geq 0}\mathcal{L}_\mp$.*

(3) *With $b\in\mathcal{L}_\pm$, there is $\alpha_b\in\mathbf{Q}_{>0}$ such that $\alpha_b^{-1}e_\pm^{h_\pm(b)}b\in\mathcal{L}_\mp$ and the map $b\mapsto\alpha_b^{-1}e_\pm^{h_\pm(b)}b$ is a bijection $\mathcal{L}_\pm\xrightarrow{\sim}\mathcal{L}_\mp$.*

The following assertions are equivalent:

(i) *With $r\geq 0$, then $V^{\leq r}$ is the sum of all the simple submodules of V of dimension $\leq r$.*

(ii) *$\{e_\pm^i b\}_{b\in\mathcal{L}_\pm,0\leq i\leq h_\pm(b)}$ is a basis of V.*

(iii) *$\{e_\pm^i b\}_{b\in\mathcal{L}_\pm,0\leq i\leq h_\pm(b)}$ generates V.*

Proof. Let $b\in\mathcal{B}$. We have $eb=\sum_{c\in\mathcal{B}}u_c c$ with $u_c\geq 0$. Also, $0=e^{h_+(b)}eb=\sum_c u_c e^{h_+(b)}c$ and $e^{h_+(b)}c\in\bigoplus_{b'\in\mathcal{B}}\mathbf{Q}_{\geq 0}b'$; hence $e^{h_+(b)}c=0$ for all $c\in\mathcal{B}$ such that $u_c\neq 0$. So, $h_+(c)\leq h_+(b)$ for all $c\in\mathcal{B}$ such that $u_c\neq 0$. Hence, (1) holds.

We have $e_\pm^{h_\pm(b)}b=\sum_{c\in\mathcal{B}}v_c c$ with $v_c\geq 0$. Since $\sum_{c\in\mathcal{B}}v_c e_\pm c=0$ and $e_\pm c\in\bigoplus_{b'\in\mathcal{B}}\mathbf{Q}_{\geq 0}b'$, it follows that $e_\pm c=0$ for all c such that $v_c\neq 0$; hence (2) holds.

Let $b\in\mathcal{L}_\pm$. We have $e_\pm^{h_\pm(b)}b=\sum_{c\in\mathcal{B}}v_c c$ with $v_c\geq 0$ and $e_\mp^{h_\pm(b)}e_\pm^{h_\pm(b)}b=\beta b$ for some $\beta>0$. So, $\sum_{c\in\mathcal{B}}v_c e_\mp^{h_\pm(b)}c=\beta b$. It follows that given $c\in\mathcal{B}$ with $v_c\neq 0$, there is $\beta_c\geq 0$ with $e_\mp^{h_\pm(b)}c=\beta_c b$. Since $h_\pm(c)=h_\mp(b)$, then $e_\pm^{h_\pm(b)}e_\mp^{h_\pm(b)}c=\beta_c e_\pm^{h_\pm(b)}b$ is a nonzero multiple of c, and it follows that there is a unique c such that $v_c\neq 0$. This shows (3).

Assume (i). We prove by induction on r that $\{e_\pm^i b\}_{b\in\mathcal{L}_\pm,0\leq i\leq h_\pm(b)<r}$ is a basis of $V^{\leq r}$ (this is obvious for $r=0$). Assume it holds for $r=d$. The image of $\{b\in\mathcal{B}|d(b)=d+1\}$ in $V^{\leq d+1}/V^{\leq d}$ is a basis. This module is a multiple of the simple module of dimension $d+1$ and $\{b\in\mathcal{L}_\pm|d(b)=d+1\}$ maps to a basis of the lowest (resp. highest) weight space of $V^{\leq d+1}/V^{\leq d}$ if $\pm=+$ (resp. $\pm=-$). It follows that $\{e_\pm^i b\}_{b\in\mathcal{L}_\pm,0\leq i\leq d=h_\pm(b)}$ maps to a basis of $V^{\leq d+1}/V^{\leq d}$. By induction, then, $\{e_\pm^i b\}_{b\in\mathcal{L}_\pm,0\leq i\leq h_\pm(b)\leq d}$ is a basis of $V^{\leq d+1}$. This proves (ii).

Assuming, (ii), let v be a weight vector with weight λ. We have $v=\sum_{b\in\mathcal{L}_\pm,2i=\lambda\pm h_\pm(b)}u_{b,i}e_\pm^i b$ for some $u_{b,i}\in\mathbf{Q}$. Take s maximal such that there is $b\in\mathcal{L}_\pm$ with $h_\pm(b)=s+i$ and $u_{b,i}\neq 0$. Then, $e_\pm^s v=\sum_{b\in\mathcal{L}_\pm,i=h_{\pm b}-s}u_{b,i}e_\pm^{h_\pm(b)}b$. Since the $e_\pm^{h_\pm(b)}b$ for $b\in\mathcal{L}_\pm$ are linearly independent, it follows that $e_\pm^s v\neq 0$, hence $s \leq h_+(v)$. So, if $d(v) < r$, then $h_\pm(b) < r$ for all b such that $u_{b,i} \neq 0$. We deduce that (i) holds.

The equivalence of (ii) and (iii) is an elementary fact of representation theory of $\mathfrak{sl}_2(\mathbf{Q})$. $\square$

5. $\mathfrak{sl}_2$-categorification

5.1. *Weak categorifications.*

5.1.1. Let $\mathcal{A}$ be an artinian and noetherian k-linear abelian category with the property that the endomorphism ring of any simple object is k (i.e., every object of $\mathcal{A}$ is a successive extension of finitely many simple objects and the endomorphism ring of a simple object is k).

A *weak* $\mathfrak{sl}_2$*-categorification* gives the data of an adjoint pair (E, F) of exact endo-functors of $\mathcal{A}$ such that

- the action of $e = [E]$ and $f = [F]$ on $V = \mathbf{Q} \otimes K_0(\mathcal{A})$ gives a locally finite $\mathfrak{sl}_2$-representation
- the classes of the simple objects of $\mathcal{A}$ are weight vectors
- F is isomorphic to a left adjoint of E.

We denote by $\varepsilon : EF \to \mathrm{Id}$ and $\eta : \mathrm{Id} \to FE$ the (fixed) co-unit and unit of the pair (E, F). We do not fix an adjunction between F and E.

Remark 5.1. Assume $\mathcal{A} = A$-mod for a finite dimensional k-algebra A. The requirement that E and F induce an $\mathfrak{sl}_2$-action on $K_0(\mathcal{A})$ is equivalent to the same condition for $K_0(\mathcal{A}\text{-proj})$. Furthermore, the perfect pairing $K_0(\mathcal{A}\text{-proj}) \times K_0(\mathcal{A}) \to \mathbf{Z}$, $([P], [S]) \mapsto \dim_k \mathrm{Hom}_{\mathcal{A}}(P, S)$ induces an isomorphism of $\mathfrak{sl}_2$-modules between $K_0(\mathcal{A})$ and the dual of $K_0(\mathcal{A}\text{-proj})$.

Remark 5.2. A crucial application will be $\mathcal{A} = A$-mod, where A is a symmetric algebra. In that case, the choice of an adjunction (E, F) determines an adjunction (F, E).

We put $E_+ = E$ and $E_- = F$. By the weight space of an object of $\mathcal{A}$, we will mean the weight space of its class (whenever this is meaningful).

Note that the opposite category $\mathcal{A}^{\mathrm{opp}}$ also carries a weak $\mathfrak{sl}_2$-categorification.

Fixing an isomorphism between F and a left adjoint to E gives another weak categorification, obtained by swapping E and F. We call it the *dual* weak categorification.

The *trivial* weak $\mathfrak{sl}_2$-categorification on $\mathcal{A}$ is the one given by $E = F = 0$.

5.1.2. Let $\mathcal{A}$ and $\mathcal{A}'$ be two weak $\mathfrak{sl}_2$-categorifications. A *morphism of weak* $\mathfrak{sl}_2$*-categorifications* from $\mathcal{A}'$ to $\mathcal{A}$ gives the data of a functor $R : \mathcal{A}' \to \mathcal{A}$ and of isomorphisms of functors $\zeta_\pm : RE'_\pm \xrightarrow{\sim} E_\pm R$ such that the following diagram commutes

(4)
$$\begin{tikzcd} RF' \arrow[r, "\zeta_-"] \arrow[d, "\eta RF'"'] & FR \\ FERF' \arrow[r, "F\zeta_+^{-1}F'"'] & FRE'F'. \arrow[u, "FR\varepsilon'"'] \end{tikzcd}$$

Note that ζ_+ determines ζ_-, and conversely (using a commutative diagram equivalent to the one above).

LEMMA 5.3. *The commutativity of diagram* (4) *is equivalent to the commutativity of either of the following two diagrams*:

$$\begin{tikzcd} & R \arrow[dl, "R\eta'"'] \arrow[dr, "\eta R"] & \\ RF'E' \arrow[r, "\sim", "\zeta_-E'"'] & FRE' \arrow[r, "\sim", "F\zeta_+"'] & FER, \end{tikzcd} \qquad \begin{tikzcd} & R & \\ RE'F' \arrow[ur, "R\varepsilon'"] \arrow[r, "\sim", "\zeta_+F'"'] & ERF' \arrow[r, "\sim", "E\zeta_-"'] & EFR. \arrow[ul, "\varepsilon R"'] \end{tikzcd}$$

Proof. Let us assume diagram (4) is commutative. Now, we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\eta R"] \arrow[d, "R\eta'"'] & FER \arrow[r, "F\zeta_+^{-1}"] \arrow[d, "FER\eta'"'] & FRE' \arrow[d, "FRE'\eta'"'] \arrow[dr, "\mathrm{id}"] & \\ RF'E' \arrow[r, "\eta RF'E'"'] \arrow[rrr, bend right, "\zeta_-E'"'] & FERF'E' \arrow[r, "F\zeta_+^{-1}F'E'"'] & FRE'F'E' \arrow[r, "FR\varepsilon'E'"'] & FRE'. \end{tikzcd}$$

This shows the commutativity of the first diagram of the lemma. The proof of commutativity of the second diagram is similar.

Let us now assume the first diagram of the lemma is commutative. Thus, we have a commutative diagram

$$\begin{tikzcd} RF' \arrow[r, "\mathrm{id}"] \arrow[dr, "R\eta'F'"'] \arrow[dd, "\eta RF'"'] & RF' \arrow[r, "\zeta_-"] & FR \\ & RF'E'F' \arrow[u, "RF'\varepsilon'"'] \arrow[dr, "\zeta_-E'F'"'] & \\ FERF' \arrow[rr, "F\zeta_+^{-1}F'"'] & & FRE'F'. \arrow[uu, "FR\varepsilon'"'] \end{tikzcd}$$

So, diagram (4) is commutative. The case of the second diagram is similar. $\square$

Note that R induces a morphism of $\mathfrak{sl}_2$-modules $K_0(\mathcal{A}'\text{-proj}) \to K_0(\mathcal{A})$.

Remark 5.4. Let $\mathcal{A}'$ be a full abelian subcategory of $\mathcal{A}$ stable under subobjects, quotients, and stable under E and F. Then, the canonical functor $\mathcal{A}' \to \mathcal{A}$ is a morphism of weak $\mathfrak{sl}_2$-categorifications.

5.1.3. We fix now a weak $\mathfrak{sl}_2$-categorification on $\mathcal{A}$ and we investigate the structure of $\mathcal{A}$.

PROPOSITION 5.5. *Let V_λ be a weight space of V. Let $\mathcal{A}_\lambda$ be the full subcategory of $\mathcal{A}$ of objects whose class is in V_λ. Then, $\mathcal{A} = \bigoplus_\lambda \mathcal{A}_\lambda$. So, the class of an indecomposable object of $\mathcal{A}$ is a weight vector.*

Proof. Let M be an object of $\mathcal{A}$ with exactly two composition factors S_1 and S_2. Assume S_1 and S_2 are in different weight spaces. Then, there are $\varepsilon \in \{\pm\}$ and $\{i, j\} = \{1, 2\}$ such that $h_\varepsilon(S_i) > h_\varepsilon(S_j)$. Let $r = h_\varepsilon(S_i)$. We have $E_\varepsilon^r M \xrightarrow{\sim} E_\varepsilon^r S_i \neq 0$; hence all the composition factors of $E_{-\varepsilon}^r E_\varepsilon^r M$ are in the same weight space as S_i. Now,

$$\mathrm{Hom}(E_{-\varepsilon}^r E_\varepsilon^r M, M) \simeq \mathrm{Hom}(E_\varepsilon^r M, E_\varepsilon^r M) \simeq \mathrm{Hom}(M, E_{-\varepsilon}^r E_\varepsilon^r M)$$

and these spaces are not zero. It follows that M has a nonzero simple quotient and a nonzero simple submodule in the same weight space as S_i. Thus, S_i is both a submodule and a quotient of M; hence $M \simeq S_1 \oplus S_2$.

We have shown that $\mathrm{Ext}^1(S, T) = 0$ whenever S and T are simple objects in different weight spaces. The proposition follows. □

Let $\mathcal{B}$ be the set of classes of simple objects of $\mathcal{A}$. This gives a basis of V and we can apply Lemma 4.3.

We have a categorification of the fact that a locally finite $\mathfrak{sl}_2$-module is an increasing union of finite dimensional $\mathfrak{sl}_2$-modules:

PROPOSITION 5.6. *Let M be an object of $\mathcal{A}$. Then, there is a Serre subcategory $\mathcal{A}'$ of $\mathcal{A}$ stable under E and F, containing M and such that $K_0(\mathcal{A}')$ is finite dimensional.*

Proof. Let I be the set of isomorphism classes of simple objects of $\mathcal{A}$ that arise as composition factors of $E^i F^j M$ for some i, j. Since $K_0(\mathcal{A})$ is a locally finite $\mathfrak{sl}_2$-module, $E^i F^j M = 0$ for $i, j \gg 0$; hence I is finite. Now, the Serre subcategory $\mathcal{A}'$ generated by the objects of I satisfies the requirement. □

We have a (weak) generation result for $D^b(\mathcal{A})$:

LEMMA 5.7. *Let $C \in D^b(\mathcal{A})$ such that* $\mathrm{Hom}_{D^b(\mathcal{A})}(E^i T, C[j]) = 0$ *for all $i \geq 0$, $j \in \mathbf{Z}$ and T a simple object of $\mathcal{A}$ such that $FT = 0$. Then, $C = 0$.*

Proof. Assume $C \neq 0$. Take n minimal such that $H^n(C) \neq 0$ and S simple such that $\mathrm{Hom}(S, H^n C) \neq 0$. Let $i = h_-(S)$ and let T be a simple submodule of F^iS. Then,

$$\operatorname{Hom}(E^iT, S) \simeq \operatorname{Hom}(T, F^iS) \neq 0.$$

So, $\operatorname{Hom}_{D(\mathcal{A})}(E^iT, C[n]) \neq 0$ and we are done, since $FT = 0$. ☐

There is an obvious analog of Lemma 5.7 using $\operatorname{Hom}(C[j], F^iT)$ with $ET = 0$. Since E is also a right adjoint of F, there are similar statements with E and F swapped.

PROPOSITION 5.8. *Let $\mathcal{A}'$ be an abelian category and G be a complex of exact functors from $\mathcal{A}$ to $\mathcal{A}'$ that have exact right adjoints. We assume that for any $M \in \mathcal{A}$ (resp. $N \in \mathcal{A}'$), then $G^i(M) = 0$ (resp. $G^{i\vee}(N) = 0$) for $|i| \gg 0$.*

Assume $G(E^iT)$ is acyclic for all $i \geq 0$ and T a simple object of $\mathcal{A}$ such that $FT = 0$. Then, $G(C)$ is acyclic for all $C \in \operatorname{Comp}^b(\mathcal{A})$.

Proof. Consider the right adjoint complex $G^\vee$ to G (cf. §4.1.4). We have an isomorphism

$$\operatorname{Hom}_{D^b(\mathcal{A})}(C, G^\vee G(D)) \simeq \operatorname{Hom}_{D^b(\mathcal{A}')}(G(C), G(D))$$

for any $C, D \in D^b(\mathcal{A})$. These spaces vanish for $C = E^iT$ as in the proposition. By Lemma 5.7, they vanish for all C. The case $C = D$ shows that $G(D)$ is 0 in $D^b(\mathcal{A}')$. ☐

Remark 5.9. Let $\mathcal{F}$ be the smallest full subcategory of $\mathcal{A}$ closed under extensions and direct summands and containing E^iT for all $i \geq 0$ and T a simple object of $\mathcal{A}$ such that $FT = 0$. Then, in general, not every projective object of $\mathcal{A}$ is in $\mathcal{F}$ (cf. the case of $\mathfrak{S}_3$ and $p = 3$ in §7.1). On the other hand, if the representation $K_0(\mathcal{A})$ is isotypic, then every object of $\mathcal{A}$ is a quotient of an object of $\mathcal{F}$ and in particular the projective objects of $\mathcal{A}$ are in $\mathcal{F}$.

Let $V^{\leq d} = \sum_{b \in \mathcal{B}, d(b) \leq d} \mathbf{Q}b$. Let $\mathcal{A}^{\leq d}$ be the full Serre subcategory of $\mathcal{A}$ of objects whose class is in $V^{\leq d}$.

Lemma 4.3(1) gives the following proposition.

PROPOSITION 5.10. *The weak $\mathfrak{sl}_2$-structure on $\mathcal{A}$ restricts to one on $\mathcal{A}^{\leq d}$ and induces one on $\mathcal{A}/\mathcal{A}^{\leq d}$.*

So, we have a filtration of $\mathcal{A}$ as $0 \subseteq \mathcal{A}^{\leq 1} \subseteq \cdots \subseteq \mathcal{A}$ is compatible with the weak $\mathfrak{sl}_2$-structure. It induces the filtration $0 \subseteq V^{\leq 1} \subseteq \cdots \subseteq V$. Some aspects of the study of $\mathcal{A}$ can be reduced to the study of $\mathcal{A}^{\leq r}/\mathcal{A}^{\leq r-1}$. This is particularly interesting when $V^{\leq r}/V^{\leq r-1}$ is a multiple of the r-dimensional simple module.

5.1.4. We now investigate simple objects and the effect of $E_\pm$ on them.

LEMMA 5.11. *Let M be an object of $\mathcal{A}$. Assume that $d(S) \geq r$ whenever S is a simple subobject (resp. quotient) of M. Then, $d(T) \geq r$ whenever T is a simple subobject (resp. quotient) of $E^i_\pm M$.*

Proof. It is enough to consider the case where M lies in a weight space by Proposition 5.5. Let T be a simple subobject of $E^i_\pm M$. Since $\mathrm{Hom}(E^i_\mp T, M) \simeq \mathrm{Hom}(T, E^i_\pm M) \neq 0$, there is S a simple subobject of M that is a composition factor of $E^i_\mp T$. Hence, $d(S) \leq d(E^i_\mp T) \leq d(T)$. The proof for quotients is similar. □

Let $\mathcal{C}_r$ be the full subcategory of $\mathcal{A}^{\leq r}$ with objects M such that whenever S is a simple submodule or a simple quotient of M, then $d(S) = r$.

LEMMA 5.12. *The subcategory $\mathcal{C}_r$ is stable under $E_\pm$.*

Proof. It is enough to consider the case where M lies in a single weight space by Proposition 5.5. Let $M \in \mathcal{C}_r$ lie in a single weight space. Let T be a simple submodule of $E_\pm M$. By Lemma 5.11, we have $d(T) \geq r$. On the other hand, $d(T) \leq d(E_\pm M) \leq d(M)$. Hence, $d(T) = r$. Similarly, one proves the required property for simple quotients. □

5.2. *Categorifications.*

5.2.1. An $\mathfrak{sl}_2$-*categorification* is a weak $\mathfrak{sl}_2$-categorification with the extra data of $q \in k^\times$ and $a \in k$ with $a \neq 0$ if $q \neq 1$ and of $X \in \mathrm{End}(E)$ and $T \in \mathrm{End}(E^2)$ such that

- $(\mathbf{1}_E T) \circ (T\mathbf{1}_E) \circ (\mathbf{1}_E T) = (T\mathbf{1}_E) \circ (\mathbf{1}_E T) \circ (T\mathbf{1}_E)$ in $\mathrm{End}(E^3)$
- $(T + \mathbf{1}_{E^2}) \circ (T - q\mathbf{1}_{E^2}) = 0$ in $\mathrm{End}(E^2)$
- $T \circ (\mathbf{1}_E X) \circ T = \begin{cases} qX\mathbf{1}_E & \text{if } q \neq 1 \\ X\mathbf{1}_E - T & \text{if } q = 1 \end{cases}$ in $\mathrm{End}(E^2)$
- $X - a$ is locally nilpotent.

Let $\mathcal{A}$ and $\mathcal{A}'$ be two $\mathfrak{sl}_2$-categorifications. A *morphism of* $\mathfrak{sl}_2$-*categorifications* from $\mathcal{A}'$ to $\mathcal{A}$ is a morphism of weak $\mathfrak{sl}_2$-categorifications (R, ζ_+, ζ_-) such that $a' = a$, $q' = q$ and the following diagrams commute:

(5)
$$\begin{tikzcd} RE' \arrow[r, "\zeta_+", "\sim"'] \arrow[d, "RX'"'] & ER \arrow[d, "XR"] \\ RE' \arrow[r, "\sim", "\zeta_+"'] & ER, \end{tikzcd} \qquad \begin{tikzcd} RE'E' \arrow[r, "\zeta_+ E'", "\sim"'] \arrow[d, "RT'"'] & ERE' \arrow[r, "E\zeta_+", "\sim"'] & EER \arrow[d, "TR"] \\ RE'E' \arrow[r, "\sim", "\zeta_+ E'"'] & ERE' \arrow[r, "\sim", "E\zeta_+"'] & EER. \end{tikzcd}$$

5.2.2. We define a morphism $\gamma_n : H_n \to \mathrm{End}(E^n)$ by
$$T_i \mapsto \mathbf{1}_{E^{n-i-1}} T \mathbf{1}_{E^{i-1}} \text{ and } X_i \mapsto \mathbf{1}_{E^{n-i}} X \mathbf{1}_{E^{i-1}}.$$

With our assumptions, the H_n-module $\mathrm{End}(E^n)$ (given by left multiplication) is in $\mathcal{N}_n$.

Let $\tau \in \{1, \mathrm{sgn}\}$. We put $E^{(\tau,n)} = E^n c_n^\tau$, the image of $c_n^\tau : E^n \to E^n$. Note that the canonical map $E^n \otimes_{H_n} H_n c_n^\tau \xrightarrow{\sim} E^{(\tau,n)}$ is an isomorphism (cf. §3.2.2).

In the context of symmetric groups, the following lemma is due to Puig. It is an immediate consequence of Proposition 3.5.

LEMMA 5.13. *The canonical map* $E^{(\tau,n)} \otimes_{P_n^{\mathfrak{S}_n}} c_n^\tau H_n \xrightarrow{\sim} E^n$ *is an isomorphism. In particular,* $E^n \simeq n! \cdot E^{(\tau,n)}$ *and the functor* $E^{(\tau,n)}$ *is a direct summand of* E^n.

We denote by $E^{(n)}$ one of the two isomorphic functors $E^{(1,n)}$, $E^{(\mathrm{sgn},n)}$.

Using the adjoint pair (E, F), we obtain a morphism $H_n \to \mathrm{End}(F^n)^{\mathrm{opp}}$. The definitions and results above have counterparts for E replaced by F (cf. §4.1.2).

We obtain a structure of $\mathfrak{sl}_2$-categorification on the dual as follows. Put $\tilde{X} = X^{-1}$ when $q \neq 1$ (resp. $\tilde{X} = -X$ when $q = 1$). We choose an adjoint pair (F, E). Using this adjoint pair, the endomorphisms $\tilde{X}$ of E and T of E^2 provide endomorphisms of F and F^2. We take these as the defining endomorphisms for the dual categorification. We define "a" for the dual categorification as the inverse (resp. the opposite) of a for the original categorification.

Remark 5.14. The scalar a can be shifted: given $\alpha \in k^\times$ when $q \neq 1$ (resp. $\alpha \in k$ when $q = 1$), we can define a new categorification by replacing X by αX (resp. by $X + \alpha \mathbf{1}_E$). This changes a into αa (resp. $\alpha + a$). So, the scalar a can always be adjusted to 1 (resp. to 0).

Remark 5.15. Assume V is a multiple of the simple 2-dimensional $\mathfrak{sl}_2$-module. Then, a weak $\mathfrak{sl}_2$-categorification consists in the data of $\mathcal{A}_{-1}$ and $\mathcal{A}_1$ together with inverse equivalences $E : \mathcal{A}_{-1} \xrightarrow{\sim} \mathcal{A}_1$ and $F : \mathcal{A}_1 \xrightarrow{\sim} \mathcal{A}_{-1}$. An $\mathfrak{sl}_2$-categorification results in the additional data of q, a and $X \in \mathrm{End}(E) \simeq Z(\mathcal{A}_1)$ with $X - a$ nilpotent.

Remark 5.16. As soon as V contains a copy of a simple $\mathfrak{sl}_2$-module of dimension 3 or more, then a and q are determined by X and T.

Example 5.17. Take for V the three dimensional irreducible representation of $\mathfrak{sl}_2$. Let $A_{-2} = A_2 = k$ and $A_0 = k[x]/x^2$. We put $\mathcal{A}_i = A_i$-mod. On $\mathcal{A}_{-2}$, define E to be induction $\mathcal{A}_{-2} \to \mathcal{A}_0$. On $\mathcal{A}_0$, E is restriction $\mathcal{A}_0 \to \mathcal{A}_2$ and F is restriction $\mathcal{A}_0 \to \mathcal{A}_{-2}$. On $\mathcal{A}_2$, then F is induction $\mathcal{A}_2 \to \mathcal{A}_0$.

$$\begin{tikzcd} k \arrow[r, shift left, "\mathrm{Ind}"] & k[x]/x^2 \arrow[l, shift left, "\mathrm{Res}"] \arrow[r, shift left, "\mathrm{Res}"] & k. \arrow[l, shift left, "\mathrm{Ind}"] \end{tikzcd}$$

Let $q = 1$ and $a = 0$. Let X be multiplication by x on $\mathrm{Res} : \mathcal{A}_0 \to \mathcal{A}_2$ and multiplication by $-x$ on $\mathrm{Ind} : \mathcal{A}_{-2} \to \mathcal{A}_0$. Let $T \in \mathrm{End}_k(k[x]/x^2)$ be the automorphism swapping 1 and x. This is an $\mathfrak{sl}_2$-categorification of the adjoint representation of $\mathfrak{sl}_2$. The corresponding weak categorification was constructed in [HueKho].

Remark 5.18. Take for V the three dimensional irreducible representation of $\mathfrak{sl}_2$. Let $A_{-2} = A_2 = k[x]/x^2$ and $A_0 = k$. We put $\mathcal{A}_i = A_i$-mod. On $\mathcal{A}_{-2}$, then E is restriction $\mathcal{A}_{-2} \to \mathcal{A}_0$. On $\mathcal{A}_0$, E is induction $\mathcal{A}_0 \to \mathcal{A}_2$ and F is induction $\mathcal{A}_0 \to \mathcal{A}_{-2}$. On $\mathcal{A}_2$, then F is restriction $\mathcal{A}_2 \to \mathcal{A}_0$.

$$\begin{tikzcd} k[x]/x^2 \arrow[r, shift left, "\mathrm{Res}"] & k \arrow[l, shift left, "\mathrm{Ind}"] \arrow[r, shift left, "\mathrm{Ind}"] & k[x]/x^2. \arrow[l, shift left, "\mathrm{Res}"] \end{tikzcd}$$

This is a weak $\mathfrak{sl}_2$-categorification but not an $\mathfrak{sl}_2$-categorification, since $E^2 : \mathcal{A}_{-2} \to \mathcal{A}_2$ is $(k[x]/x^2) \otimes_k -$, which is an indecomposable functor.

Remark 5.19. Let $A_{-2} = k$, $A_0 = k \times k$ and $A_{-2} = k$. We define E and F as the restriction and induction functors in the same way as in Example 5.17. Then, V is the direct sum of a 3-dimensional simple representation and a 1-dimensional representation. Assume there is $X \in \mathrm{End}(E)$ and $T \in \mathrm{End}(E^2)$ giving an $\mathfrak{sl}_2$-categorification. We have $\mathrm{End}(E^2) = \mathrm{End}_k(k^2)$ and $X1_E = 1_E X = a 1_{E^2}$. But the quotient of $H_2(q)$ by the relation $X_1 = X_2 = a$ is zero! So, we have a contradiction (it is crucial to exclude the affine Hecke algebra at $q = 1$). So, this is a weak $\mathfrak{sl}_2$-categorification but not an $\mathfrak{sl}_2$-categorification (note that we still have $E^2 \simeq E \oplus E$).

5.3. *Minimal categorification.* We introduce here a categorification of the (finite dimensional) simple $\mathfrak{sl}_2$-modules.

We fix $q \in k^\times$ and $a \in k$ with $a \neq 0$ if $q \neq 1$. Let $n \geq 0$ and $B_i = \bar{H}_{i,n}$ for $0 \leq i \leq n$.

We put $\mathcal{A}(n)_\lambda = B_{(\lambda+n)/2}$-mod and $\mathcal{A}(n) = \bigoplus_i B_i$-mod, $E = \bigoplus_{i<n} \mathrm{Ind}_{B_i}^{B_{i+1}}$ and $F = \bigoplus_{i>0} \mathrm{Res}_{B_{i-1}}^{B_i}$. The functors $\mathrm{Ind}_{B_i}^{B_{i+1}} = B_{i+1} \otimes_{B_i} -$ and $\mathrm{Res}_{B_i}^{B_{i+1}} = B_{i+1} \otimes_{B_{i+1}} -$ are left and right adjoint.

We have $EF(B_i) \simeq B_i \otimes_{B_{i-1}} B_i \simeq i(n-i+1)B_i$ and $FE(B_i) \simeq B_{i+1} \simeq (i+1)(n-i)B_i$ as left B_i-modules (cf. §3.3.3). Thus, $(ef - fe)([B_i]) = (2i-n)[B_i]$. Now, $\mathbf{Q} \otimes K_0(\mathcal{A}(n)_\lambda) = \mathbf{Q}[B_{(\lambda+n)/2}]$; hence $ef - fe$ acts on $K_0(\mathcal{A}(n)_\lambda)$ by λ. It follows that e and f induce an action of $\mathfrak{sl}_2$ on $K_0(\mathcal{A}(n))$, hence we have a weak $\mathfrak{sl}_2$-categorification.

The image of X_{i+1} in B_{i+1} gives an endomorphism of $\mathrm{Ind}_{B_i}^{B_{i+1}}$ by right multiplication on B_{i+1}. Taking the sum over all i, we get an endomorphism X of E. Similarly, the image of T_{i+1} in B_{i+2} gives an endomorphism of $\mathrm{Ind}_{B_i}^{B_{i+2}}$ and taking the sum over all i, we get an endomorphism T of E^2.

This provides an $\mathfrak{sl}_2$-categorification. The representation on $K_0(\mathcal{A}(n))$ is the simple $(n+1)$-dimensional $\mathfrak{sl}_2$-module.

5.4. *Link with affine Hecke algebras.*

5.4.1. The following proposition generalizes and strengthens results of Kleshchev [Kl1, Kl2] in the symmetric-group setting and of Grojnowski and Vazirani [GrVa] in the context of cyclotomic Hecke algebras (cf. §7.1 and §7.2).

PROPOSITION 5.20. *Let S be a simple object of $\mathcal{A}$, let $n = h_+(S)$ and $i \le n$.*

(a) *$E^{(n)}S$ is simple.*

(b) *The socle and head of $E^{(i)}S$ are isomorphic to a simple object S' of $\mathcal{A}$. We have isomorphisms of $(\mathcal{A}, H_i)$-bimodules*: $\operatorname{soc} E^iS \simeq \operatorname{hd} E^iS \simeq S' \otimes K_i$.

(c) *The morphism $\gamma_i(S) : H_i \to \operatorname{End}(E^iS)$ factors through $\bar{H}_{i,n}$ and induces an isomorphism $\bar{H}_{i,n} \xrightarrow{\sim} \operatorname{End}(E^iS)$.*

$$\begin{tikzcd} & H_i \arrow[dl, "\text{can}"'] \arrow[dr, "\gamma_i(S)"] & \\ \bar{H}_{i,n} \arrow[rr, dotted, "\sim"'] & & \operatorname{End}(E^iS). \end{tikzcd}$$

(d) *We have $[E^{(i)}S] - \binom{n}{i}[S'] \in V^{\le d(S')-1}$.*

The corresponding statements with E replaced by F and $h_+(S)$ by $h_-(S)$ hold as well.

Proof. • Let us assume (a) holds. We will show that (b), (c), and (d) follow.

We have $E^nS \simeq n! \cdot S''$ for some S'' simple. So, we have $E^nS \simeq S'' \otimes R$ as $(\mathcal{A}, H_n)$-bimodules, where R is a right H_n-module in $\mathcal{N}_n$. Since $\dim R = \dim K_n$, it follows that $R \simeq K_n$.

We have $E^{n-i} \operatorname{soc} E^{(i)}S \subset E^{n-i}E^{(i)}S \simeq S'' \otimes K_n c_i^1$. Since $S'' \otimes K_n c_i^1$ has a simple socle (Lemma 3.6), it follows that $E^{n-i} \operatorname{soc} E^{(i)}S$ is an indecomposable $(\mathcal{A}, H_{n-i})$-bimodule. If S' is a nonzero summand of $\operatorname{soc} E^{(i)}S$, then $E^{n-i}S' \neq 0$ (Lemma 5.12). So, $S' = \operatorname{soc} E^{(i)}S$ is simple. We have $\operatorname{soc} E^iS \simeq S' \otimes R$ for some H_i-module R in $\mathcal{N}_i$. Since $\dim R = i!$, it follows that $R \simeq K_i$. The proof for the head is similar.

The dimension of $\operatorname{End}(E^{(i)}S)$ is at most the multiplicity p of S' as a composition factor of $E^{(i)}S$. Since $E^{(n-i)}S' \neq 0$, it follows that the dimension of $\operatorname{End}(E^{(i)}S)$ is at most the number of composition factors of $E^{(n-i)}E^{(i)}S$. We have $E^{(n-i)}E^{(i)}S \simeq \binom{n}{i} \cdot S''$. So, $\dim \operatorname{End}(E^{(i)}S) \le \binom{n}{i}$ and $\dim \operatorname{End}(E^iS) \le (i!)^2\binom{n}{i} = \dim \bar{H}_{i,n}$.

Since $\ker\gamma_n(S)$ is a proper ideal of H_n, we have $\ker\gamma_n(S) \subset \mathfrak{n}_n H_n$. We have $\ker\gamma_i(S) \subset H_i \cap \ker\gamma_n(S) \subset H_i \cap (\mathfrak{n}_n H_n)$. So, the canonical map $H_i \to \bar{H}_{i,n}$ factors through a surjective map: $\operatorname{im}\gamma_i(S) \twoheadrightarrow \bar{H}_{i,n}$. We deduce that $\gamma_i(S)$ is surjective and $\bar{H}_{i,n} \xrightarrow{\sim} \operatorname{End}(E^i S)$. So, (c) holds. We deduce also that $p = \binom{n}{i}$ and that if L is a composition factor of $E^{(i)}S$ with $E^{(n-i)}L \neq 0$, then $L \simeq S'$. So, (d) holds. Since the simple object $\operatorname{hd} E^{(i)}S$ is not killed by $E^{(n-i)}$ (Lemma 5.12), we deduce that $\operatorname{hd} E^{(i)}S \simeq S'$. We have now shown (b).

• Let us show that (a) (hence (b), (c), and (d)) holds when $FS = 0$. By Lemma 4.3 (3), we have $[E^{(n)}S] = r[S']$ for some simple object S' and $r \geq 1$ integer. Since $[F^{(n)}E^{(n)}S] = [S]$, we have $r = 1$, so (a) holds.

• Let us now show (a) in general. Let L be a simple quotient of $F^{(r)}S$, where $r = h_-(S)$. Since $\operatorname{Hom}(S, E^{(r)}L) \simeq \operatorname{Hom}(F^{(r)}S, L) \neq 0$, we deduce that S is isomorphic to a submodule of $E^{(r)}L$. Since $FL = 0$, we know by (a) that $E^{(n)}E^{(r)}L \simeq \binom{n+r}{r}S'$ for some simple object S'. So, $E^{(n)}S \simeq mS'$ for some positive integer m. We have $\operatorname{Hom}(E^{(n)}S, S') \simeq \operatorname{Hom}(S, F^{(n)}S')$. Since $ES' = 0$, we deduce that $\operatorname{soc} F^{(n)}S'$ is simple (we use (b) in its "F" version). So, $\dim \operatorname{Hom}(S, F^{(n)}S') \leq 1$, hence $m = 1$ and (a) holds. □

Corollary 5.21. *The $\mathfrak{sl}_2(\mathbf{Q})$-module $V^{\leq d}$ is the sum of the simple submodules of V of dimension $\leq d$.*

Proof. Let S be a simple object of $\mathcal{A}$ with $r = h_-(S)$. By Proposition 5.20 (a), $S' = F^{(r)}S$ is simple. We deduce that $S \simeq \operatorname{soc} E^{(r)}S'$ by adjunction. Now, Proposition 5.20 (d) shows that $[E^{(r)}S'] - \binom{d(S)}{r}[S] \in V^{\leq d(S)-1}$.

We deduce by induction on r that $\{[E^r S']\}$ generates V, where S' runs over the isomorphism classes of simple objects killed by F and $0 \leq r \leq h_+(S')$. The corollary follows from Lemma 4.3, (iii)$\Longrightarrow$(i). □

Remark 5.22. Let S be a simple object of $\mathcal{A}$ and $i \leq h_+(S)$. The action of $Z_{i,n} = Z(\bar{H}_{i,n})$ on $E^i S$ restricts to an action on $E^{(i)}S$. Since $E^i S$ is a faithful right $\bar{H}_{i,n}$-module, it follows from Proposition 3.5 that $E^{(i)}S$ is a faithful $Z_{i,n}$-module. Now, $\dim \operatorname{End}_{\mathcal{A}}(E^{(i)}S) = \frac{1}{(i!)^2} \dim \bar{H}_{i,n} = \dim Z_{i,n}$; hence the morphism $Z_{i,n} \to \operatorname{End}_{\mathcal{A}}(E^{(i)}S)$ is an isomorphism.

Let us now continue with the following crucial lemma whose proof uses some of the ideas of the proof of Proposition 5.20.

Lemma 5.23. *Let U be a simple object of $\mathcal{A}$ such that $FU = 0$. Let $n = h_+(U)$, $i < n$, and $B_i = \bar{H}_{i,n}$. The composition of $\eta(E^i U) \otimes 1 : E^i U \otimes_{B_i} B_{i+1} \to FE^{i+1}U \otimes_{B_i} B_{i+1}$ with the action map $FE^{i+1}U \otimes_{B_i} B_{i+1} \to FE^{i+1}U$ is an isomorphism*

$$E^i U \otimes_{B_i} B_{i+1} \xrightarrow{\sim} FE^{i+1}U.$$

Proof. By Proposition 3.5, it is enough to prove that the map becomes an isomorphism after applying $-\otimes_{B_{i+1}} B_{i+1}c_{i+1}^1$. By (3), we have $B_{i+1}c_{i+1}^1 = \bigoplus_{a=0}^{n-i-1} \bar{P}_{i,n}x_{i+1}^a c_{i+1}^1$. Consider the composition

$$\phi = g\circ(f\otimes 1) : E^{(i)}U \otimes \bigoplus_{a=0}^{n-i-1} kx^a \to FE^{(i+1)}U$$

where $f : E^{(i)}U \xrightarrow{\eta(E^{(i)}U)} FEE^{(i)}U \xrightarrow{\mathbf{1}_F c^1_{[\mathfrak{S}_i\setminus\mathfrak{S}_{i+1}]}U} FE^{(i+1)}U$ and $g : FE^{(i+1)}U\otimes \bigoplus_{a=0}^{n-i-1} kx^a \to FE^{(i+1)}U$ are given by the action on F. We have to prove that ϕ is an isomorphism. We have $[FE^{(i+1)}U] = (n-i)[E^{(i)}U]$; hence it suffices to prove that ϕ is injective. In order to do that, one may restrict ϕ to a map between the socles of the objects (viewed in $\mathcal{A}$). Let $\phi_a : \operatorname{soc} E^{(i)}U \to FE^{(i+1)}U$ be the restriction of ϕ to the socle of $E^{(i)}U\otimes kx^a$. Since $\operatorname{soc}(E^{(i)}U)$ is simple (Proposition 5.20), the problem is to prove that the maps ϕ_a for $0\le a\le n-i-1$ are linearly independent. By adjunction, it is equivalent to prove that the maps

$$\psi_a : E\operatorname{soc} E^{(i)}U \xrightarrow{x^a\mathbf{1}_{\operatorname{soc} E^{(i)}U}} E\operatorname{soc} E^{(i)}U \xrightarrow{c^1_{[\mathfrak{S}_i\setminus\mathfrak{S}_{i+1}]}U} E^{(i+1)}U$$

are linearly independent.

We have $\operatorname{soc} E^{i+1}U \simeq S\otimes K_{i+1}$ as $(\mathcal{A},H_{i+1})$-bimodules, where $S = \operatorname{soc} E^{(i+1)}U$ is simple (Proposition 5.20). Consider the right $(k[x_{i+1}]\otimes H_i)$-submodule $L' = \operatorname{Hom}_{\mathcal{A}}(S,\operatorname{soc}(E\operatorname{soc} E^iU)))$ of $L=\operatorname{Hom}_{\mathcal{A}}(S,\operatorname{soc} E^{i+1}U)$. We have $H_{i+1} = (H_i\otimes P_{[i+1]})H_{i+1}^f$, hence $L = L'H_{i+1}^f$ since L is a simple right H_{i+1}-module. So, $L'c_{i+1}^1 = Lc_{i+1}^1$, hence $\operatorname{soc}(E\operatorname{soc} E^iU))c_{i+1}^1 = \operatorname{soc} E^{(i+1)}U$. In particular, the map $E\operatorname{soc} E^{(i)}U \xrightarrow{c^1_{[\mathfrak{S}_{i+1}/\mathfrak{S}_i]}U} E^{(i+1)}U$ is injective, since $E\operatorname{soc} E^{(i)}U$ has a simple socle by Proposition 5.20.

Now, we are left with proving that the maps

$$E\operatorname{soc} E^{(i)}U \xrightarrow{X^a\mathbf{1}_{\operatorname{soc} E^{(i)}U}} E\operatorname{soc} E^{(i)}U$$

are linearly independent; i.e., that the restriction of $\gamma_1(S') : H_1 \to \operatorname{End}_{\mathcal{A}}(ES')$ to $\bigoplus_{a=0}^{n-i-1} kX_1^a$ is injective, where $S' = \operatorname{soc} E^{(i)}U$. Let I be the kernel of $\gamma_{n-i}(S') : H_{n-i} \to \operatorname{End}_{\mathcal{A}}(E^{n-i}S')$. Then, as in the proof of Proposition 5.20, we have $I\subset \mathfrak{n}_{n-i}H_{n-i}$. So, $\ker\gamma_1 \subset H_1\cap \mathfrak{n}_{n-i}H_{n-i}$; hence the canonical map $\bigoplus_{a=0}^{n-i-1} kX_1^a \to \operatorname{End}_{\mathcal{A}}(E^{n-i}S')$ is injective (cf. (3)) and we are done. □

5.4.2. We fix U a simple object of $\mathcal{A}$ such that $FU=0$. Let $n=h_+(U)$. We put $B_i = \bar{H}_{i,n}$ for $0\le i\le n$.

The canonical isomorphisms of functors

$$E(E^iU\otimes_{B_i} -) \xrightarrow{\sim} E^{i+1}U\otimes_{B_i} - \xrightarrow{\sim} E^{i+1}U\otimes_{B_{i+1}} B_{i+1}\otimes_{B_i} -$$

make the following diagram commutative

$$\begin{tikzcd} B_{i+1}\text{-mod} \arrow[r, "E^{i+1}U\otimes_{B_{i+1}}-"] & \mathcal{A} \\ B_i\text{-mod} \arrow[u, "B_{i+1}\otimes_{B_i}-"] \arrow[r, "E^iU\otimes_{B_i}-"'] & \mathcal{A}. \arrow[u, "E"'] \end{tikzcd}$$

The canonical isomorphism of functors from Lemma 5.23

$$E^iU \otimes_{B_i} B_{i+1} \otimes_{B_{i+1}} - \xrightarrow{\sim} F(E^{i+1}U \otimes_{B_{i+1}} -)$$

makes the following diagram commutative:

$$\begin{tikzcd} B_{i+1}\text{-mod} \arrow[r, "E^{i+1}U\otimes_{B_{i+1}}-"] \arrow[d, "B_{i+1}\otimes_{B_{i+1}}-"'] & \mathcal{A} \arrow[d, "F"] \\ B_i\text{-mod} \arrow[r, "E^iU\otimes_{B_i}-"'] & \mathcal{A}. \end{tikzcd}$$

THEOREM 5.24. *The construction above is a morphism of $\mathfrak{sl}_2$-categorifications $R_U : \mathcal{A}(n) \to \mathcal{A}$.*

Proof. The commutativity of diagram (4) (see §5.1.2) follows from the very definition of ζ_- given by Lemma 5.23. The commutativity of the diagram (5) (see §5.2.1) is obvious. □

Remark 5.25. Let I_n be the set of isomorphism classes of simple objects U of $\mathcal{A}$ such that $FU = 0$ and $h_+(U) = n$. We have a morphism of $\mathfrak{sl}_2$-categorifications

$$\sum_{n,U\in I_n} R_U : \bigoplus_{n,U\in I_n} \mathcal{A}(n) \to \mathcal{A}$$

that is not an equivalence in general but that induces an isomorphism

$$\bigoplus_{n,U\in I_n} \mathbf{Q} \otimes K_0(\mathcal{A}(n)\text{-proj}) \xrightarrow{\sim} \mathbf{Q} \otimes K_0(\mathcal{A})$$

giving a canonical decomposition of $\mathbf{Q}\otimes K_0(\mathcal{A})$ into simple summands. In that sense, the categorifications $\mathcal{A}(n)$ are minimal.

The following proposition is clear.

PROPOSITION 5.26. *Assume $\mathbf{Q} \otimes K_0(\mathcal{A})$ is a simple $\mathfrak{sl}_2$-module of dimension $n + 1$. Let U be the unique simple object of $\mathcal{A}$ with $FU = 0$. Then, $R_U : \mathcal{A}(n) \to \mathcal{A}$ is an equivalence of categories if and only if U is projective.*

Note that a categorification corresponding to an isotypic representation need not be isomorphic to a sum of minimal categorifications (take for example a trivial $\mathfrak{sl}_2$-representation).

5.5. *Decomposition of* $[E, F]$.

5.5.1. Let $\sigma : EF \to FE$ be given as the composition

$$EF \xrightarrow{\eta \mathbf{1}_{EF}} FEEF \xrightarrow{\mathbf{1}_F T \mathbf{1}_F} FEEF \xrightarrow{\mathbf{1}_{FE}\varepsilon} FE.$$

The following gives the categorification of the relation $[e, f] = h$.

THEOREM 5.27. *Let* $\lambda \geq 0$. *Then, there are isomorphisms*

$$\sigma + \sum_{j=0}^{\lambda-1} (\mathbf{1}_F X^j) \circ \eta \ : \ EF \,\mathrm{Id}_{\mathcal{A}_{-\lambda}} \oplus \mathrm{Id}_{\mathcal{A}_{-\lambda}}^{\bigoplus \lambda} \xrightarrow{\sim} FE \,\mathrm{Id}_{\mathcal{A}_{-\lambda}}$$

and

$$\sigma + \sum_{j=0}^{\lambda-1} \varepsilon \circ (X^j \mathbf{1}_F) \ : \ EF \,\mathrm{Id}_{\mathcal{A}_{\lambda}} \xrightarrow{\sim} FE \,\mathrm{Id}_{\mathcal{A}_{\lambda}} \oplus \mathrm{Id}_{\mathcal{A}_{\lambda}}^{\bigoplus \lambda}.$$

Proof. By Proposition 5.8, it is enough to check that the maps are isomorphisms after evaluating the functors at $E^i U$, where $i \geq 0$ and U is a simple object of $\mathcal{A}_{-\lambda-2i}$ (resp. of $\mathcal{A}_{\lambda-2i}$) such that $FU = 0$. Thanks to Lemma 5.3 and Theorem 5.24, we can do this with $\mathcal{A}$ replaced by a minimal categorification $\mathcal{A}(n)$ and this is the content of Proposition 5.31 below. $\square$

In the case of cyclotomic Hecke algebras, Vazirani [Va] had shown that the values of the functors on simple objects are isomorphic.

COROLLARY 5.28. *The functors* E *and* F *induce an action of* $\mathfrak{sl}_2$ *on the Grothendieck group of* $\mathcal{A}$, *viewed as an additive category.*

5.5.2. We put $\gamma = \begin{cases} (q-1)a & \text{if } q \neq 1 \\ 1 & \text{if } q = 1 \end{cases}$

and $m_{ij}(c) = \begin{cases} \sum_{j \leq d_1 < \cdots < d_{i-j-c} \leq i-1} T_{d_1} \cdots T_{d_{i-j-c}} & \text{if } c < i-j \\ 1 & \text{if } c = i-j \\ 0 & \text{if } c > i-j. \end{cases}$

LEMMA 5.29. *Let* $j < i$ *and* $c \geq 0$. *We have*

$$T_j T_{j+1} \cdots T_{i-1} x_i^c = \gamma^c m_{ij}(c) \pmod{\mathfrak{m}_i H_i}.$$

In particular, $T_j T_{j+1} \cdots T_{i-1} x_i^c \in \mathfrak{m}_i H_i$ *if* $c > i-j$.

Proof. By (1) (see §3.1.3), we have

$$T_{i-1} x_i^c - x_{i-1}^c T_{i-1} = \begin{cases} (q-1)(x_i + a)(x_{i-1}^{c-1} + x_{i-1}^{c-2} x_i + \cdots + x_i^{c-1}) & \text{if } q \neq 1 \\ x_{i-1}^{c-1} + x_{i-1}^{c-2} x_i + \cdots + x_i^{c-1} & \text{if } q = 1. \end{cases}$$

Hence

$$\begin{aligned}&T_jT_{j+1}\cdots T_{i-1}x_i^c\\&\qquad = T_jT_{j+1}\cdots T_{i-2}x_{i-1}^cT_{i-1}+\gamma T_jT_{j+1}\cdots T_{i-2}x_{i-1}^{c-1}\pmod{\mathfrak{m}_iH_i}.\end{aligned}$$

Since $m_{ij}(c)=m_{i-1,j}(c-1)+m_{i-1,j}(c)T_{i-1}$, the lemma follows by induction. □

LEMMA 5.30. *Let $j\le i$, $c\ge 1$ and $e=\inf(c-1,i-j)$. Then,*

$$\begin{aligned}&T_jT_{j+1}\cdots T_ix_i^c-T_jT_{j+1}\cdots T_{i-1}x_{i+1}^cT_i\\&=\alpha\left(\gamma^e x_{i+1}^{c-e-1}m_{ij}(e)+\gamma^{e-1}x_{i+1}^{c-e}m_{ij}(e-1)+\cdots+x_{i+1}^{c-1}m_{ij}(0)\right)\pmod{\mathfrak{m}_iH_{i+1}}\end{aligned}$$

where $\alpha=\begin{cases}(1-q)(x_{i+1}+a) & \text{if } q\neq 1\\ -1 & \text{if } q=1.\end{cases}$

Proof. We have

$$T_jT_{j+1}\cdots T_ix_i^c-T_jT_{j+1}\cdots T_{i-1}x_{i+1}^cT_i=\alpha T_j\cdots T_{i-1}(x_i^{c-1}+\cdots+x_{i+1}^{c-1})$$

and the result follows from Lemma 5.29. □

The following is a Mackey decomposition for the algebras $B_i=\bar{H}_{i,n}$.

PROPOSITION 5.31. *Let $i\le n/2$. Then, there is an isomorphism of (B_i,B_i)-bimodules*

$$\begin{aligned}B_i\otimes_{B_{i-1}}B_i\oplus B_i^{\oplus n-2i}&\xrightarrow{\sim}B_{i+1}\\(b\otimes b',b_1,\ldots,b_{n-2i})&\mapsto bT_ib'+\sum_{j=1}^{n-2i}b_jX_{i+1}^{j-1}.\end{aligned}$$

Let now $i\ge n/2$. Then, there is an isomorphism of (B_i,B_i)-bimodules

$$\begin{aligned}B_i\otimes_{B_{i-1}}B_i&\xrightarrow{\sim}B_{i+1}\oplus B_i^{\oplus 2i-n}\\b\otimes b'&\mapsto(bT_ib',bb',bX_ib',\ldots,bX_i^{2i-n-1}b').\end{aligned}$$

Proof. Let us consider the first map. We know already that both sides are free B_i-modules of the same rank (cf. §5.3), hence it is enough to show surjectivity.

Let $M=(P_i/\mathfrak{m}_i)\otimes_{P_i}B_{i+1}$. This is a right B_i-module quotient of B_{i+1}. Let L be the right B_i-submodule of M generated by $B_iT_i+\sum_{j=0}^{n-2i-1}X_{i+1}^jk$. The first isomorphism will follow from the proof that $M=L$. From now on, all elements are viewed in M.

We have

$$x_{i+1}^{n-i}=\sum_{j=0}^{n-i-1}(-1)^{n-i-1+j}x_{i+1}^je_{n-i-j}(x_{i+1},\ldots,x_n).$$

Given $r\ge 2$ and $j\le n-i-1$, we have

$$e_{n-i-j}(x_r,\ldots,x_n) = e_{n-i-j}(x_{r-1},x_r,\ldots,x_n) - x_{r-1}e_{n-i-j-1}(x_r,\ldots,x_n).$$

Since $e_{n-i-j}(x_1,\ldots,x_n)=0$, it follows that $e_{n-i-j}(x_{i+1},\ldots,x_n)=0$. So, we have $x_{i+1}^{n-i}=0$.

Take $1\le r\le i$. Then, $r\le n-i$ and by Lemma 5.30,

$$\begin{aligned}&T_{i-r+1}T_{i-r+2}\cdots T_i x_i^{n-i} = x_{i+1}^{n-i}T_{i-r+1}\cdots T_i\\ &+\alpha\left(\gamma^{r-1}x_{i+1}^{n-i-r}+\gamma^{r-2}x_{i+1}^{n-i-r+1}m_{i,i-r+1}(r-2)+\cdots+x_{i+1}^{n-i-1}m_{i,i-r+1}(0)\right).\end{aligned}$$

Thus,

$$T_{i-r+1}T_{i-r+2}\cdots T_i x_i^{n-i}+\alpha\gamma^{r-1}x_{i+1}^{n-i-r}\in\sum_{j\ge 0}x_{i+1}^{n-i-r+1+j}H_i.$$

Since $x_{i+1}^{n-i}=0$, we deduce by induction on r that $x_{i+1}^{n-i-r}\in L$ for $1\le r\le i$. Hence, $x_{i+1}^a\in L$ for all $a\ge 0$. We deduce from Lemma 5.30 that $x_{i+1}^aT_j\cdots T_i\in L$ for all $1\le j\le i$ and $a\ge 0$. Since

$$B_{i+1}=\bigoplus_{0\le a\le n-i-1,w\in[\mathfrak{S}_{i+1}/\mathfrak{S}_i]}\bar{P}_{i,n}x_{i+1}^aT_wH_i^f$$

(cf. §3.3.1), we finally obtain $M=L$ and we are done.

Let us now consider the second isomorphism. We fix an adjunction (F,E) with unit η' and co-unit ε' and consider the dual categorification $\mathcal{A}'$ of $\mathcal{A}(n)$. We denote by X' and T' its defining endomorphisms. Define $\sigma' : FE\xrightarrow{\eta'FE} EFFE\xrightarrow{ET'E} EFFE\xrightarrow{EF\varepsilon'} EF$.

Let $G=FE$ and $H=EF$. There is an adjoint pair (EF,EF) with co-unit $\varepsilon_H : EFEF\xrightarrow{E\varepsilon'F} EF\xrightarrow{\varepsilon}\mathrm{Id}$ and an adjoint pair (FE,FE) with unit $\eta_G:\mathrm{Id}\xrightarrow{\eta} FE\xrightarrow{F\eta'E} FEFE$. Consider the canonical isomorphism

$$\zeta:\mathrm{Hom}(FE,EF)=\mathrm{Hom}(G,H)\xrightarrow{\sim}\mathrm{Hom}(H^\vee,G^\vee)\xrightarrow{\sim}\mathrm{Hom}(EF,FE)$$

corresponding to these adjunctions. The commutativity of the following diagram shows that $\zeta(\sigma')=\sigma$.

$$\begin{tikzcd}
EF \arrow[d, "\eta EF"'] & & & \\
FEEF \arrow[r, "F\eta' EEF"] \arrow[dr, "F\eta' EEF"] \arrow[ddr, "FEEF"'] & FEFEEF \arrow[r, "FE\eta' FEEF"] & FEEFFEEF \arrow[dr, "FEET'EEF"] & \\
& FEFEEF \arrow[r, "FE\eta' FEEF"] \arrow[d, "FE\varepsilon' EF"] & FEEFFEEF \arrow[r, "FTFFEEF"] \arrow[d, "FEEF\varepsilon' EF"] & FEEFFEEF \arrow[d, "FEEF\varepsilon' EF"] \\
& FEEF \arrow[r, "FE\eta' EF"] \arrow[dr, "FEEF"'] & FEEFEF \arrow[d, "FEE\varepsilon' F"] & FEEFEF \arrow[d, "FEE\varepsilon' F"] \\
& & FEEF \arrow[r, "FTF"'] & FEEF \arrow[d, "FE\varepsilon"] \\
& & & FE.
\end{tikzcd}$$

Similarly, using the canonical adjoint pair (Id, Id), we get a canonical isomorphism

$$\zeta' : \operatorname{Hom}(\mathrm{Id}, EF) = \operatorname{Hom}(\mathrm{Id}, H) \xrightarrow{\sim} \operatorname{Hom}(H^\vee, \mathrm{Id}) \xrightarrow{\sim} \operatorname{Hom}(EF, \mathrm{Id}).$$

Now, $\zeta'((\mathbf{1}_E(\tilde{X}')^j) \circ \eta') = \varepsilon \circ (X^j \mathbf{1}_F)$.

We have shown that the adjoint to

$$\sigma + \sum_{j=0}^{\lambda-1} \varepsilon \circ (X^j \mathbf{1}_F) \ : \ EF\,\mathrm{Id}_{\mathcal{A}_\lambda} \xrightarrow{\sim} FE\,\mathrm{Id}_{\mathcal{A}_\lambda} \oplus \mathrm{Id}_{\mathcal{A}_\lambda}^{\bigoplus \lambda}$$

is

$$\sigma' + \sum_{j=0}^{\lambda-1} (\mathbf{1}_{F'}(\tilde{X}')^j) \circ \eta' \ : \ E'F'\,\mathrm{Id}_{\mathcal{A}'_{-\lambda}} \oplus \mathrm{Id}_{\mathcal{A}'_{-\lambda}}^{\bigoplus \lambda} \to F'E'\,\mathrm{Id}_{\mathcal{A}'_{-\lambda}}.$$

One checks easily that the first map of the proposition remains an isomorphism if X_{i+1} is replaced by $\tilde{X}_{i+1}$. Since the categorification $\mathcal{A}'$ is isomorphic to $\mathcal{A}(n)$, this shows that the map $\sigma' + \sum_{j=0}^{\lambda-1} (\mathbf{1}_{F'}(\tilde{X}')^j) \circ \eta'$ is an isomorphism; hence $\sigma + \sum_{j=0}^{\lambda-1} \varepsilon \circ (X^j \mathbf{1}_F)$ is an isomorphism as well. □

5.5.3. We fix a family $\{M_\lambda \in \mathcal{A}_\lambda\}_\lambda$ and let $\mathcal{M}_\lambda$ be the full subcategory of $\mathcal{A}_\lambda$ whose objects are finite direct sums of direct summands of M_λ. We assume that $\mathcal{M} = \bigoplus_\lambda \mathcal{M}_\lambda$ is stable under E and F.

Let $A'_\lambda = \operatorname{End}_{\mathcal{A}}(M_\lambda)$, $\mathcal{A}'_\lambda = A'_\lambda$-mod and $\mathcal{A}' = \bigoplus_\lambda \mathcal{A}'_\lambda$ and put

$$E' = \bigoplus_\lambda \operatorname{Hom}_{\mathcal{A}}(M_{\lambda+2}, EM_\lambda) \otimes_{A'_\lambda} - : \mathcal{A}' \to \mathcal{A}'$$

$$\text{and } F' = \bigoplus_\lambda \operatorname{Hom}_{\mathcal{A}}(M_{\lambda-2}, FM_\lambda) \otimes_{A'_\lambda} - : \mathcal{A}' \to \mathcal{A}'.$$

Now, $\operatorname{Hom}_{\mathcal{A}}(M_{\lambda+2}, EM_\lambda) \simeq \operatorname{Hom}_{\mathcal{A}}(FM_{\lambda+2}, M_\lambda)$ and $FM_{\lambda+2} \in \mathcal{M}_\lambda$. It follows that $\operatorname{Hom}_{\mathcal{A}}(M_{\lambda+2}, EM_\lambda)$ is a projective right A'_λ-module, so that E' is an exact functor. Similarly, F' is an exact functor. Also, they send projectives to projectives.

Consider the functor $R = \bigoplus_\lambda M_\lambda \otimes_{A'_\lambda} - : \mathcal{A}' \to \mathcal{A}$. Its restriction to $\mathcal{A}'$-proj is an equivalence $\mathcal{A}'\text{-proj} \xrightarrow{\sim} \mathcal{M}$. So, the functor $G \mapsto RG$ from the category of exact functors $\mathcal{A}' \to \mathcal{A}'$ sending projectives to projectives to the category of functors $\mathcal{A}' \to \mathcal{A}$ is fully faithful.

The canonical map

$$M_{\lambda+2} \otimes_{A'_{\lambda+2}} \operatorname{Hom}_{\mathcal{A}}(M_{\lambda+2}, EM_\lambda) \xrightarrow{\sim} EM_\lambda, \ m \otimes f \mapsto f(m)$$

is an isomorphism, since $EM_\lambda \in \mathcal{M}_{\lambda+2}$. The induced map

$$M_{\lambda+2} \otimes_{A'_{\lambda+2}} \operatorname{Hom}_{\mathcal{A}}(M_{\lambda+2}, EM_\lambda) \otimes_{A'_\lambda} U \xrightarrow{\sim} E(M_\lambda \otimes_{A'_\lambda} U),$$

$$m \otimes f \otimes u \mapsto E(m' \mapsto m' \otimes u)(f(m))$$

for $U \in A'_\lambda$-mod is an isomorphism, since it is an isomorphism for $U = A'_\lambda$.

We obtain an isomorphism $RE' \xrightarrow{\sim} ER$ and construct similarly an isomorphism $RF' \xrightarrow{\sim} FR$.

Let X' (resp. T') be the inverse image of $X\,\mathrm{id}_R$ (resp. $T\,\mathrm{id}_R$) via the canonical isomorphisms $\mathrm{End}(E') \xrightarrow{\sim} \mathrm{End}(RE') \xrightarrow{\sim} \mathrm{End}(ER)$ (resp. $\mathrm{End}(E'^2) \xrightarrow{\sim} \mathrm{End}(RE'^2) \xrightarrow{\sim} \mathrm{End}(ERE') \xrightarrow{\sim} \mathrm{End}(E^2R)$).

Proceeding similarly, the adjoint pair (E,F) gives an adjoint pair (E',F') and the functor F' is isomorphic to a left adjoint of E'.

THEOREM 5.32. *The data above define an $\mathfrak{sl}_2$-categorification on $\mathcal{A}'$ and a morphism of $\mathfrak{sl}_2$-categorifications $\mathcal{A}' \to \mathcal{A}$.*

Proof. The $\mathfrak{sl}_2$-relations in $K_0(\mathcal{A}'\text{-proj})$ hold thanks to Theorem 5.27 applied to the restriction of functors to $\mathcal{M}$. The local finiteness follows from the case of $\mathcal{A}$. The commutativity of the diagrams of Lemma 5.3 follows immediately from the construction of the adjoint pair (E',F'). This shows that $\mathcal{A}'$ is a weak categorification and that R defines a morphism of weak categorifications.

By construction, this weak categorification is a categorification and the morphism of weak categorifications is actually a morphism of categorifications. □

COROLLARY 5.33. *Let $M \in \mathcal{A}$. Then, there exist a finite dimensional algebra A, an $\mathfrak{sl}_2$-categorification on* A-mod *and a morphism of $\mathfrak{sl}_2$-categorifications $R : A\text{-mod} \to \mathcal{A}$ such that M is a direct summand of $R(A)$.*

Proof. Let $N = \bigoplus_{i,j\geq 0} E^iF^jM$, a finite sum. Let N_λ be the projection of N on $\mathcal{A}_\lambda$. Now, we can apply the constructions and results above, the stability being provided by Corollary 5.28. □

6. Categorification of the reflection

6.1. *Rickard's complexes.* Let $\lambda \in \mathbf{Z}$. We construct a complex of functors

$$\Theta_\lambda : \mathrm{Comp}(\mathcal{A}_{-\lambda}) \to \mathrm{Comp}(\mathcal{A}_\lambda),$$

following Rickard [Ri1] (originally, for blocks of symmetric groups).

We denote by $(\Theta_\lambda)^{-r}$ the restriction of $E^{(\mathrm{sgn},\lambda+r)}F^{(1,r)}$ to $\mathcal{A}_{-\lambda}$ for $r, \lambda+r \geq 0$ and we put $(\Theta_\lambda)^{-r} = 0$ otherwise.

Consider the map

$$f : E^{\lambda+r}F^r = E^{\lambda+r-1}EFF^{r-1} \xrightarrow{\mathbf{1}_{E^{\lambda+r-1}}\varepsilon\mathbf{1}_{F^{r-1}}} E^{\lambda+r-1}F^{r-1}.$$

We have $E^{(\mathrm{sgn},\lambda+r)} = E^{\lambda+r}c^{\mathrm{sgn}}_{[\mathfrak{S}_{\lambda+r}/\mathfrak{S}_{[2,\lambda+r]}]}c^{\mathrm{sgn}}_{[2,\lambda+r]} \subseteq E^{(\mathrm{sgn},\lambda+r-1)}E$ and similarly $F^{(1,r)} \subseteq FF^{(1,r-1)}$; hence f restricts to a map

$$d^{-r} : E^{(\mathrm{sgn},\lambda+r)}F^{(1,r)} \to E^{(\mathrm{sgn},\lambda+r-1)}F^{(1,r-1)}.$$

We put

$$\Theta_\lambda = \cdots \to (\Theta_\lambda)^{-i} \xrightarrow{d^{-i}} (\Theta_\lambda)^{-i+1} \to \cdots .$$

LEMMA 6.1. *Θ_λ is a complex. The map $[\Theta_\lambda] : V_{-\lambda} = K_0(\mathcal{A}_{-\lambda}) \to V_\lambda = K_0(\mathcal{A}_\lambda)$ coincides with the action of s.*

Proof. The map $d^{1-r}d^{-r}$ is the restriction of $\mathbf{1}_{E^{\lambda+r-2}}\varepsilon_2\mathbf{1}_{F^{r-2}}$, where $\varepsilon_2 : EEFF \xrightarrow{\mathbf{1}_E\varepsilon\mathbf{1}_F} EF \xrightarrow{\varepsilon} \mathrm{Id}$. Since $c^{\mathrm{sgn}}_{\lambda+r} = c^{\mathrm{sgn}}_{[\mathfrak{S}_{\lambda+r}/\mathfrak{S}_2]}c^{\mathrm{sgn}}_2$ and $c^1_r = c^1_2 c^1_{[\mathfrak{S}_2\backslash\mathfrak{S}_r]}$, it follows that

$$E^{(\mathrm{sgn},\lambda+r)}F^{(1,r)} \subseteq E^{\lambda+r-2}E^{(\mathrm{sgn},2)}F^{(1,2)}F^{r-2}.$$

So, in order to prove that $d^{1-r}d^{-r} = 0$, it is enough to show that the composition

$$E^2F^2 \xrightarrow{c_2^{\mathrm{sgn}}c_2^1} E^2F^2 \xrightarrow{\varepsilon_2} \mathrm{Id}$$

vanishes, where c_2^{sgn} acts on E^2 and c_2^1 acts on F^2. This composition is equal to the composition $E^2F^2 \xrightarrow{(c_2^{\mathrm{sgn}}c_2^1)\mathbf{1}_{F^2}} E^2F^2 \xrightarrow{\varepsilon_2} \mathrm{Id}$, where $c_2^{\mathrm{sgn}}c_2^1$ acts now on E^2. We are done, since $c_2^{\mathrm{sgn}}c_2^1 = 0$.

The last statement is given by Lemma 4.2. □

Remark 6.2. Let $M \in \mathcal{A}_{-\lambda}$. Let $l = \max\{r \geq 0 | F^rM \neq 0\}$, be a finite integer. Then, $(\Theta_\lambda)^{-i}(M) = 0$ when $i \notin [\max(0, -\lambda), l]$.

6.2. *Derived equivalence from the simple reflection.* Let $\Theta = \bigoplus_\lambda \Theta_\lambda$. The following lemma follows easily from Lemma 5.3.

LEMMA 6.3. *Let $R : \mathcal{A}' \to \mathcal{A}$ be a morphism of $\mathfrak{sl}_2$-categorifications. Then, there is an isomorphism of complexes of functors $\Theta R \xrightarrow{\sim} R\Theta'$.*

We can now state our main theorem (whose proof will be deduced from Theorem 6.6 below).

THEOREM 6.4. *The complex of functors Θ induces a self-equivalence of $K^b(\mathcal{A})$ and of $D^b(\mathcal{A})$ and induces by restriction equivalences $K^b(\mathcal{A}_{-\lambda}) \xrightarrow{\sim} K^b(\mathcal{A}_\lambda)$ and $D^b(\mathcal{A}_{-\lambda}) \xrightarrow{\sim} D^b(\mathcal{A}_\lambda)$. Furthermore, $[\Theta] = s$.*

Remark 6.5. In the context of symmetric groups, the invertibility of Θ_λ when the complex has only one (resp. two) nonzero term is due to Scopes [Sco] (resp. Rickard [Ri1]).

Proof of Theorem 6.4. Since E and F have right adjoints, there is a complex of functors $\Theta_\lambda^\vee$ that gives a right adjoint to Θ_λ (cf. §4.1.4). Let $\varepsilon : \Theta_\lambda\Theta_\lambda^\vee \to \mathrm{Id}$ be the co-unit of adjunction and Z its cone. Thus, Z is a complex of exact functors $\mathcal{A}_{-\lambda} \to \mathcal{A}_\lambda$.

Pick $U \in \mathcal{A}$ with $FU = 0$ and $E^iU \in \mathcal{A}_{-\lambda}$ and put $n = h_+(U)$. The fully faithful functor $R_U : K^b(\mathcal{A}(n)\text{-proj}) \to K^b(\mathcal{A})$ commutes with Θ_λ (Lemma 6.3), hence commutes with $\Theta_\lambda^\vee$ and with Z (cf. §4.1.6). By Theorem 6.6, we have $Z(E^iU) = 0$. Now, Proposition 5.8 shows that $Z(M) = 0$ in $D^b(\mathcal{A}_{-\lambda})$ for all $M \in D^b(\mathcal{A}_{-\lambda})$. So, ε is an isomorphism in $D^b(\mathcal{A}_{-\lambda})$. One shows similarly that Θ_λ has a left inverse in $D^b(\mathcal{A}_{-\lambda})$.

Let us now prove that ε is still an isomorphism in $K^b(\mathcal{A}_{-\lambda})$. Let $M \in \mathrm{Comp}^b(\mathcal{A}_{-\lambda})$. By Corollary 5.33, there are a finite dimensional k-algebra A, an $\mathfrak{sl}_2$-categorification on $\mathcal{A}' = A$-mod and a morphism of $\mathfrak{sl}_2$-categorifications $R : \mathcal{A}' \to \mathcal{A}$ such that the terms of M are direct summands of $R(A)$. The functor R induces a fully faithful triangulated functor $R' : K^b(\mathcal{A}'\text{-proj}) \to K^b(\mathcal{A})$. The derived category case of the theorem shows that ε' is an isomorphism in $K^b(\mathcal{A}'_{-\lambda}\text{-proj}) \xrightarrow{\sim} D^b(\mathcal{A}'_{-\lambda})$. As above, we deduce that ε is an isomorphism in the image of R'; hence $\varepsilon(M)$ is an isomorphism in $K^b(\mathcal{A}_{-\lambda})$. One proceeds similarly to show that Θ_λ has a left inverse in $K^b(\mathcal{A}_{-\lambda})$. □

6.3. *Equivalences for the minimal categorification.*

THEOREM 6.6. *Let $n \geq 0$ and $\mathcal{A} = \mathcal{A}(n)$ be the minimal categorification. Fix $\lambda \geq 0$ and let $l = \frac{n-\lambda}{2}$. The homology of the complex of functors Θ_λ is concentrated in degree $-l$ and $H^{-l}\Theta_\lambda : \mathcal{A}_{-\lambda} \xrightarrow{\sim} \mathcal{A}_\lambda$ is an equivalence.*

Proof. In order to show that the homology of Θ_λ is concentrated in degree $-l$, it suffices to show that $\Theta_\lambda(B_l c_l^1)$ is homotopy equivalent to a complex concentrated in degree $-l$, since $B_l c_l^1$ is a progenerator for B_l-mod. This is equivalent to the property that $H^*(C) = 0$ for $* \neq -l$, where $C = c_{n-l}^{\mathrm{sgn}} \bar{H}_{n-l} \otimes_{B_{n-l}} \Theta_\lambda(B_l c_l^1)$, since $c_{n-l}^{\mathrm{sgn}} \bar{H}_{n-l}$ is the unique simple right B_{n-l}-module and $C^{-r} = 0$ for $r > l$.

We have

$$C^{-r} = c_{n-l}^{\mathrm{sgn}} \bar{H}_{n-l} \otimes_{B_{n-l}} B_{n-l} c_{[l-r+1,n-l]}^{\mathrm{sgn}} \otimes_{B_{l-r}} c_{[l-r+1,l]}^1 B_l \otimes_{B_l} B_l c_l^1.$$

Lemma 3.7 gives an isomorphism

$$C^{-r} \xrightarrow{\sim} c_{n-l}^{\mathrm{sgn}} \bar{H}_{n-l} \otimes_{B_{n-l}} B_{n-l} c_{[l-r+1,n-l]}^{\mathrm{sgn}} c_{[1,l-r]}^1 \otimes_k \bigoplus_{\mu \in P(r,n-l)} m_\mu(x_{l-r+1}, \ldots, x_l) k.$$

Proposition 3.3 and Lemma 3.4 give isomorphisms

$$\bigoplus_{0 \leq a_1 < \cdots < a_{l-r} < n-l} x_1^{a_1} \cdots x_{l-r}^{a_{l-r}} k \xrightarrow[\mathrm{can}]{\sim} \Lambda^{\mathfrak{S}_{l-r}}(P_{n-l}^{\mathfrak{S}_{[l-r+1,n-l]}} \otimes_{P_{n-l}^{\mathfrak{S}_{n-l}}} k),$$

$$\Lambda^{\mathfrak{S}_{l-r}}(P_{n-l}^{\mathfrak{S}_{[l-r+1,n-l]}} \otimes_{P_{n-l}^{\mathfrak{S}_{n-l}}} k) \xrightarrow{\sim} c_{n-l}^{\mathrm{sgn}} \bar{H}_{n-l} \otimes_{B_{n-l}} B_{n-l} c_{[l-r+1,n-l]}^{\mathrm{sgn}} c_{[1,l-r]}^1,$$
$$y \mapsto c_{n-l}^{\mathrm{sgn}} y$$

and these induce isomorphisms $E^{-r} \xrightarrow[\phi^{-r}]{\sim} D^{-r} \xrightarrow[\psi^{-r}]{\sim} C^{-r}$, where

$$E^{-r} = \bigoplus_{0 \leq a_1 < \cdots < a_{l-r} < n-l} x_1^{a_1} \cdots x_{l-r}^{a_{l-r}} k \otimes \bigoplus_{\mu \in P(r,n-l)} m_\mu(x_{l-r+1}, \ldots, x_l)$$

and

$$D^{-r} = \Lambda^{\mathfrak{S}_{l-r}}(P_{n-l}^{\mathfrak{S}_{[l-r+1,n-l]}} \otimes_{P_{n-l}^{\mathfrak{S}_{n-l}}} k) \otimes \bigoplus_{\mu \in P(r,n-l)} m_\mu(x_{l-r+1},\ldots,x_l)k.$$

Let $\mu \in P(r,n-l)$ and $0 \leq a_1 < \cdots < a_{l-r} < n-l$. Given a positive integer b, we write $b \prec \mu$ when b appears in μ and we denote then by $\mu \setminus b$ the partition obtained from μ by removing one instance of b. We have $m_\mu(x_{l-r+1},\ldots,x_l) = \sum_{b\prec\mu} x_{l-r+1}^b m_{\mu\setminus b}(x_{l-r+2},\ldots,x_l)$. It follows that

$$d_C^{-r}\psi^{-r}(x_1^{a_1}\cdots x_{l-r}^{a_{l-r}} \otimes m_\mu) = \psi^{-r+1}\left(\sum_{b\prec\mu} x_1^{a_1}\cdots x_{l-r}^{a_{l-r}} x_{l-r+1}^b \otimes m_{\mu\setminus b}\right).$$

Assume $b = n-l$. Since $x_{l-r+1}^{n-l} \in \mathfrak{n}_{n-l}P_{n-l}$, it follows that $x_1^{a_1}\cdots x_{l-r}^{a_{l-r}} x_{l-r+1}^b$ is 0 in $\Lambda^{\mathfrak{S}_{l-r+1}}(P_{n-l}^{\mathfrak{S}_{[l-r+2,n-l]}} \otimes_{P_{n-l}^{\mathfrak{S}_{n-l}}} k)$. One gets the same conclusion when $b \in \{a_1,\ldots,a_{l-r}\}$. Thus,

$$\begin{aligned} &d_C^{-r}\psi^{-r}\phi^{-r}(x_1^{a_1}\cdots x_{l-r}^{a_{l-r}} \otimes m_\mu) \\ &\qquad = \psi^{-r+1}\phi^{-r+1}\left(\sum_{b\prec\mu,\, b\notin\{a_1,\ldots,a_{l-r},n-l\}} \operatorname{sgn}(\sigma_b) x_1^{a'_1}\cdots x_{l-r+1}^{a'_{l-r+1}} \otimes m_{\mu\setminus b}\right) \end{aligned}$$

where $\sigma_b \in \mathfrak{S}_{l-r+1}$ is the permutation such that, putting $a_{l-r+1} = b$ and $a'_j = a_{\sigma_b(j)}$, we have $a'_1 < a'_2 < \cdots < a'_{l-r+1}$.

Let $L = k^{n-l}$, with canonical basis $\{e_i\}_{1\leq i\leq n-l}$. The Koszul complex K of L is a bigraded k-vector space given by $K^{p,q} = \Lambda^p L \otimes S^q L$, with a differential of bidegree $(-1,1)$ given by

$$(e_{a_1}\cdots e_{a_p}) \otimes x \mapsto \sum_{i=1}^p (-1)^{i+p+1}(e_{a_1}\cdots e_{a_{i-1}}e_{a_{i+1}}\cdots e_{a_p}) \otimes e_{a_i}x.$$

Its dual $\operatorname{Hom}_k(K,k)$ is isomorphic to J defined as follows. We put $J^{p,q} = \Lambda^p(L^*) \otimes S^q(L^*)$. Let $\{f_i\}$ be the dual basis of L^* and $f_\mu = f_{\mu(1)}\cdots f_{\mu(q)} \in S^q(L^*)$ for $\mu \in P(q,n-l)$. Then, the differential $d_J : J^{p,q} \to J^{p+1,q-1}$ is given by

$$\begin{aligned} &(f_{a_1}\cdots f_{a_p}) \otimes f_\mu \\ &\qquad \mapsto \sum_{b\prec\mu,\, a_1<\cdots<a_i<b<a_{i+1}<\cdots<a_p} (-1)^{i+p}(f_{a_1}\cdots f_{a_i}f_b f_{a_{i+1}}\cdots f_{a_p}) \otimes f_{\mu\setminus b}. \end{aligned}$$

The homology of J is concentrated in bidegree $(0,0)$ and isomorphic to k. Note that $J^{\bullet,q}$ is a graded right $\Lambda(L^*)$-module, with action given by right multiplication. This provides J with the structure of a complex of free graded $\Lambda(L^*)$-modules (the degree $-q$ term is $J^{\bullet,q}$), hence of free graded $k[f_{n-l}]/(f_{n-l}^2)$-modules by restriction. So, the $(-q)$-th homology group of $J \otimes_{k[f_{n-l}]/(f_{n-l}^2)} k$ is a one-dimensional graded k-vector space which is in degree q. The complexes of vector spaces $J \otimes_{k[f_{n-l}]/(f_{n-l}^2)} k$ and Jf_{n-l} are isomorphic, with a shift by one in the grading. The complex Jf_{n-l} decomposes as the direct sum (over i) of the complexes $\bigoplus_q \Lambda^{i-q}(L^*)f_{n-l} \otimes S^q(L^*)$ and the cohomology of such a complex is concentrated in degree $-i$.

We have an isomorphism

$$\begin{aligned} E^{-r} &\xrightarrow{\sim} (\Lambda^{l-r}L^*)f_{n-l} \otimes S^r L^* \subseteq J^{l-r+1,r},\ x_1^{a_1}\cdots x_{l-r}^{a_{l-r}} \otimes m_\mu \\ &\mapsto (f_{a_1}\cdots f_{a_{l-r}} f_{n-l}) \otimes f_\mu. \end{aligned}$$

This induces an isomorphism between E and the the subcomplex

$$\bigoplus_r (\Lambda^{l-r}L^*)f_{n-l} \otimes S^r L^*$$

of $J^{l+*+1,-*}$. It follows that the homology of E is concentrated in degree $-l$.

The complex of functors $\Theta_{-\lambda}$ is given by tensor product by a bounded complex of (B_{n-l}, B_l)-bimodules which are projective as B_{n-l}-modules and as B_l-modules. The homology of that complex is concentrated in the lowest degree where the complex has a nonzero component, hence the homology M is still projective as a B_{n-l}-module and as a B_l-module. Lemma 6.1 shows that $M \otimes_{B_l} -$ sends the unique simple B_l-module to the unique simple B_{n-l}-module. By Morita theory, M induces an equivalence. □

7. Examples

In this section, the field k will always be assumed to be big enough so that the simple modules considered are absolutely simple.

In most of our examples, $\mathfrak{sl}_2$-categorifications are constructed in families, using the following recipe. We start with left and right adjoint functors $\hat{E}$ and $\hat{F}$ on an abelian category $\mathcal{A}$, together with $X \in \operatorname{End}(\hat{E})$ and $T \in \operatorname{End}(\hat{E}^2)$ satisfying the defining relations of (possibly degenerate) affine Hecke algebras. We obtain for each $a \in k$ (with $a \neq 0$ if $q \neq 1$) an $\mathfrak{sl}_2$-categorification on $\mathcal{A}$ given by $E = E_a$ and $F = F_a$, the generalised a-eigenspaces of X acting on $\hat{E}$ and $\hat{F}$. While we need to check in each example that E and F do indeed give an action of $\mathfrak{sl}_2$ on $K_0(\mathcal{A})$, it is automatic that X and T restrict to endomorphisms of E and E^2 with the desired properties. That T restricts is a consequence of the identity (a special case of (1))

$$\begin{aligned} &T_1(X_2-a)^N - (X_1-a)^N T_1 \\ &= \begin{cases} (q-1)X_2[(X_1-a)^{N-1} + (X_1-a)^{N-2}(X_2-a) + \cdots + (X_2-a)^{N-1}] & \text{if } q \neq 1 \\ (X_1-a)^{N-1} + (X_1-a)^{N-2}(X_2-a) + \cdots + (X_2-a)^{N-1} & \text{if } q = 1. \end{cases} \end{aligned}$$

in $H_2(q)$.

7.1. *Symmetric groups.*

7.1.1. Let p be a prime number and $k = \mathbf{F}_p$. The quotient of $H_n(1)$ by the ideal generated by X_1 is the group algebra $k\mathfrak{S}_n$. The images of T_i and X_i in $k\mathfrak{S}_n$ are $s_i = (i, i+1)$ and the Jucys-Murphy element $L_i = (1,i) + (2,i) + \cdots + (i-1,i)$.

Let $a \in k$. Given M a $k\mathfrak{S}_n$-module, we denote by $F_{a,n}(M)$ the generalized a-eigenspace of X_n. This is a $k\mathfrak{S}_{n-1}$-module. We have a decomposition $\mathrm{Res}^{k\mathfrak{S}_n}_{k\mathfrak{S}_{n-1}} = \bigoplus_{a\in k} F_{a,n}$. There is a corresponding decomposition $\mathrm{Ind}^{k\mathfrak{S}_n}_{k\mathfrak{S}_{n-1}} = \bigoplus_{a\in k} E_{a,n}$, where $E_{a,n}$ is left and right adjoint to $F_{a,n}$. We put $E_a = \bigoplus_{n\geq 1} E_{a,n}$ and $F_a = \bigoplus_{n\geq 1} F_{a,n}$.

Recall the following classical result [LLT].

THEOREM 7.1. *The functors E_a and F_a for $a \in \mathbf{F}_p$ give rise to an action of the affine Lie algebra $\widehat{\mathfrak{sl}}_p$ on $\bigoplus_{n\geq 0} K_0(k\mathfrak{S}_n\text{-mod})$.*

The decomposition of $K_0(k\mathfrak{S}_n\text{-mod})$ in blocks coincides with its decomposition in weight spaces.

Two blocks of symmetric groups have the same weight if and only if they are in the same orbit under the adjoint action of the affine Weyl group.

In particular for each $a \in \mathbf{F}_p$ the functors E_a and F_a give a weak $\mathfrak{sl}_2$-categorification on $\mathcal{A} = \bigoplus_{n\geq 0} k\mathfrak{S}_n$-mod.

We denote by X the endomorphism of E_a given on $E_{a,n}$ by right multiplication by L_n (on the $(k\mathfrak{S}_n, k\mathfrak{S}_{n-1})$-bimodule $k\mathfrak{S}_n$). We denote by T the endomorphism of E_a^2 given on $E_{a,n}E_{a,n-1}$ by right multiplication by s_{n-1} (on the $(k\mathfrak{S}_n, k\mathfrak{S}_{n-2})$-bimodule $k\mathfrak{S}_n$). This gives an $\mathfrak{sl}_2$-categorification on $\mathcal{A}$ (here, $q = 1$).

7.1.2. Let G and H be two finite groups. Let $R = k$ or $\mathbf{Z}_{(p)}$. Let A (resp. B) be a block of RG (resp. RH). We say that A and B are *splendidly Rickard equivalent* if there is a bounded complex C of finitely generated $(A \otimes B^{\mathrm{opp}})$-modules which are direct summands of permutation modules such that $C \otimes_B C^* \simeq A$ in $K^b(A \otimes A^{\mathrm{opp}})$ and $C^* \otimes_A C \simeq B$ in $K^b(B \otimes B^{\mathrm{opp}})$ (one usually puts some condition on the vertices of the modules involved, but this is actually automatic, as explained in [Rou5]).

THEOREM 7.2. *Let $R = k$ or $\mathbf{Z}_{(p)}$. Let A and B be two blocks of symmetric groups over R with isomorphic defect groups. Then, A and B are splendidly Rickard equivalent* (*in particular, they are derived equivalent*).

Proof. Two blocks of symmetric groups over k have isomorphic defect groups if and only if they have equal weights (cf. §7.1.3 below). By Theorem 7.1, there is a sequence of blocks $A_0 = A, A_1, \ldots, A_r = B$ such that A_j is the image of A_{j-1} by some simple reflection σ_{a_j} of the affine Weyl group.

By Theorem 6.4, the complex of functors Θ associated with $a = a_j$ induces a self-equivalence of $K^b(\mathcal{A})$. It restricts to a splendid Rickard equivalence between A_j and A_{j+1}. By composing these equivalences, we obtain a splendid Rickard equivalence between A and B (note that the composition of splendid equivalences can easily be seen to be splendid; cf. e.g. [Rou2, Lemma 2.6]).

The constructions of E and F lift uniquely to $\mathbf{Z}_{(p)}$: $\mathrm{Ind}_{\mathbf{Z}_{(p)}\mathfrak{S}_{n-1}}^{\mathbf{Z}_{(p)}\mathfrak{S}_n} = \bigoplus_{a\in k} \tilde{E}_a$, $\mathrm{Res}_{\mathbf{Z}_{(p)}\mathfrak{S}_{n-1}}^{\mathbf{Z}_{(p)}\mathfrak{S}_n} = \bigoplus_{a\in k} \tilde{F}_a$, where $\tilde{E}_a \otimes_{\mathbf{Z}_{(p)}} k = E_a$, where $\tilde{F}_a \otimes_{\mathbf{Z}_{(p)}} k = F_a$ and $\tilde{E}_a$ and $\tilde{F}_a$ are left and right adjoint. We denote by $\tilde{T}$ the endomorphism of $\tilde{E}_a^2$ given on $\tilde{E}_{a,n}\tilde{E}_{a,n-1}$ by the action of s_{n-1}.

The construction of Θ in §6.1 lifts to a complex $\tilde{\Theta}$ of functors on $\tilde{\mathcal{A}} = \bigoplus_{n\geq 0} \mathbf{Z}_{(p)}\mathfrak{S}_n$-mod. By [Ri3, end of proof of Theorem 5.2], the lift $\tilde{\Theta}$ of Θ is a splendid self Rickard equivalence of $D^b(\tilde{\mathcal{A}})$ and we conclude as before. □

Remark 7.3. The equivalence depends on the choice of a sequence of simple reflections whose product sends one block to the other. If, as expected, the categorifications of the simple reflections give rise to a braid group action on the derived category of $\bigoplus_{n\geq 0} k\mathfrak{S}_n$-mod, then one can choose the canonical lifting of the affine Weyl group element in the braid group to get a canonical equivalence.

Remark 7.4. Theorem 7.2 gives isomorphisms between Grothendieck groups of the blocks (taken over $\mathbf{Q}$) satisfying certain arithmetical properties (perfect isometries or even isotypes). These arithmetical properties were shown by Enguehard [En, 1990].

Remark 7.5. Two blocks of symmetric groups over k have isomorphic defect groups if and only if they have the same number of simple modules, up to the exception of blocks of weights 0 and 1 for $p = 2$ — note that a block of weight 0 is simple whereas a block of weight 1 is not simple, so two such blocks are not derived equivalent. Now, one can restate Theorem 7.2 as follows:

Let A and B be two blocks of symmetric groups over k. Then, A and B are derived equivalent if and only if they have isomorphic defect groups. Assume A and B are not simple if $p = 2$. Then, A and B are derived equivalent if and only if $\operatorname{rank} K_0(A) = \operatorname{rank} K_0(B)$.

We can now deduce a proof of Broué's abelian defect group conjecture for blocks of symmetric groups:

THEOREM 7.6. *Let A be a block of a symmetric group G over $\mathbf{Z}_{(p)}$, D a defect group and B the corresponding block of $N_G(D)$. If D is abelian, then A and B are splendidly Rickard equivalent.*

Proof. By [ChKe], there is a block A' of a symmetric group which is splendidly Morita equivalent to the principal block of $\mathbf{Z}_{(p)}(\mathfrak{S}_p \wr \mathfrak{S}_w)$, where w is the weight of A. We have a splendid Rickard equivalence between the principal block of $\mathbf{Z}_{(p)}\mathfrak{S}_p$ and $\mathbf{Z}_{(p)}N$, where N is the normalizer of a Sylow p-subgroup of $\mathfrak{S}_p$ by [Rou2, Th. 1.1]. By [Ma, Th. 4.3] (cf. also [Rou2, Lemma 2.8] for the Rickard/derived equivalence part), we deduce a splendid Rickard equivalence between the principal blocks of $\mathbf{Z}_{(p)}(\mathfrak{S}_p \wr \mathfrak{S}_w)$ and $\mathbf{Z}_{(p)}(N \wr \mathfrak{S}_w)$. Now, we have an isomorphism $B \simeq \mathbf{Z}_{(p)}(N \wr \mathfrak{S}_w) \otimes B_0$, where B_0 is a matrix algebra over $\mathbf{Z}_{(p)}$; hence there is a splendid Morita equivalence between B and $\mathbf{Z}_{(p)}(N \wr \mathfrak{S}_w)$. So, we obtain a splendid Rickard equivalence between B and A'.

By Theorem 7.2, we have a splendid Rickard equivalence between A and A' and the theorem follows. □

Remark 7.7. The existence of an isotype between A and B in Theorem 7.6 was known by [Rou1].

7.1.3. Let us analyze more precisely the categorification. Given λ a partition of m, we denote by $|\lambda| = m$ the size of λ. Let κ be a p-core and n an integer such that $p|(n-|\kappa|)$ and $n \geq |\kappa|$. We denote by $b_{\kappa,n}$ the corresponding block of $k\mathfrak{S}_n$ (the irreducible characters of that block are associated to the partitions having κ as their p-core). The integer $\frac{n-|\kappa|}{p}$ is the weight of the block (this notion of weight is not to be confused with the weights relative to Lie algebra actions).

Let λ be a partition with p-core κ and λ' a partition obtained from λ by adding an a-node. Then, the p-core of λ' depends only on κ and a and we denote it by $e_a(\kappa)$. Similarly, we define $f_a(\kappa)$ by removing an a-node.

We will freely identify a functor $M \otimes -$ with the bimodule M. We have

$$E_{a,n+1} = \bigoplus_{\kappa} b_{e_a(\kappa),n+1} k\mathfrak{S}_{n+1} b_{\kappa,n} \tag{6}$$

where κ runs over the p-cores such that $|\kappa| \leq n$, $|\kappa| \equiv n \pmod p$ and $|e_a(\kappa)| \leq n+1$.

Let $b_{\kappa_{-r},l}, b_{\kappa_{-r+2},l+1}, \ldots, b_{\kappa_r,l+r}$ be a chain of blocks with $|f_a(\kappa_{-r})| > l-1$, $|e_a(\kappa_r)| > l+r+1$ and $f_a(\kappa_j) = \kappa_{j-2}$.

Put $n_i = l + (i-r)/2$ and $B_i = k\mathfrak{S}_{n_i} b_{\kappa_i,n_i}$ for $-r \leq i \leq r$ and $i \equiv r \pmod 2$.

Let $\mathcal{A} = \bigoplus_i B_i$-mod. The action of $E = E_a$ and $F = F_a$ on $K_0(\mathcal{A})$ gives a representation of $\mathfrak{sl}_2$. This gives an $\mathfrak{sl}_2$-categorification (here, $q = 1$).

The complex of functors Θ restricts to a splendid Rickard equivalence between B_i and B_{-i}.

Let us recall some results of the local block theory of symmetric groups (cf. [Pu1] or [Br, §2]).

Let P be a p-subgroup of $\mathfrak{S}_n$. Up to conjugacy, we can assume $[1,n]^P = [n_P+1, n]$ for some integer n_P (we call such a P a standard p-subgroup). Then, $C_{\mathfrak{S}_n}(P) = H \times \mathfrak{S}_{[n_P+1,n]}$ where $H = C_{\mathfrak{S}_{n_P}}(P)$. The algebra kH has a unique block.

Given G a finite group and P a p-subgroup of G, we denote by $\mathrm{br}_P : (kG)^P \to kC_G(P)$ the Brauer morphism (restriction of the morphism of k-vector spaces $kG \to kC_G(P)$ which is the identity on $C_G(P)$ and 0 on $G - C_G(P)$). We denote by $\mathrm{Br}_P : kG\text{-mod} \to kC_G(P)\text{-mod}$ the Brauer functor given by $M \mapsto M^P/(\sum_{Q<P} \mathrm{Tr}_Q^P M^Q)$, where $\mathrm{Tr}_Q^P(x) = \sum_{g\in P/Q} g(x)$.

We will use the following result of Puig and Marichal

THEOREM 7.8.
$$\mathrm{br}_P(b_{\kappa,n}) = \begin{cases} 1 \otimes b_{\kappa,n-n_P} & \text{if } \frac{n-n_P-|\kappa|}{p} \in \mathbf{Z}_{\geq 0} \\ 0 & \text{otherwise.} \end{cases}$$

Note in particular that a standard p-subgroup P is a defect group of $b_{\kappa,n}$ if and only if P is a Sylow p-subgroup of $\mathfrak{S}_{n-|\kappa|}$. In particular, two blocks of symmetric groups have isomorphic defect groups if and only if they have equal weights.

So, we deduce from (6) and Theorem 7.8:

LEMMA 7.9. *There is an isomorphism of* $((kH \otimes k\mathfrak{S}_{n-n_P+i}), (kH \otimes k\mathfrak{S}_{n-n_P-1}))$*-bimodules*
$$\mathrm{Br}_{\Delta P}(E_{a,n+i} \cdots E_{a,n+1}E_{a,n}) \xrightarrow{\sim} kH \otimes E_{a,n-n_P+i} \cdots E_{a,n-n_P+1}E_{a,n-n_P}.$$
For $i = 1$, *it is compatible with the action of* T.

Let P *be a nontrivial standard* p*-subgroup of* $\mathfrak{S}_{n_{-i}}$. *If* $\mathrm{br}_P(b_{\kappa_i,n_i})$ *is not* 0, *then*
$$\mathrm{Br}_{\Delta_P}(b_{\kappa_{-i},n_{-i}}\Theta b_{\kappa_i,n_i}) \simeq kH \otimes b_{\kappa_{-i},n_{-i}-n_P}\Theta b_{\kappa_i,n_i-n_P}.$$

Note that this lemma permits us to deduce a proof of the Rickard equivalence in Theorem 7.2 from that of the derived equivalence, by induction on the size of the defect group: By induction, $b_{\kappa_{-i},n_{-i}-n_P}\Theta b_{\kappa_i,n_i-n_P}$ induces a Rickard equivalence. Now, Θ induces a derived equivalence; so, it follows from Theorem 7.10 below that Θ induces a Rickard equivalence between B_i and B_{-i}.

If a splendid complex induces local derived equivalences, then it induces a Rickard equivalence [Rou4, Th. 5.6] (in a more general version, but whose proof extends with no modification):

THEOREM 7.10. *Let* G *be a finite group*, b *a block of* kG *and* D *a defect group of* b. *Asssume* b *is of principal type*, *i.e.*, $\mathrm{br}_D(b)$ *is a block of* $kC_G(D)$. *Let* H *be a subgroup of* G *containing* D *and controlling the fusion of* p*-subgroups of* D. *Let* c *be the block of* kH *corresponding to* b.

Let C *be a bounded complex of* (kGb, kHc)*-bimodules. We assume* C *is splendid*, *i.e.*, *the components* M *of* C *are direct summands of modules* $\mathrm{Ind}_{\Delta D}^{G\times H^\circ} N$, *where* N *is a permutation* ΔD*-module.*

Assume

- $\mathrm{Br}_{\Delta P}(C)$ *induces a Rickard equivalence between* $kC_G(P)\,\mathrm{br}_P(b)$ *and* $kC_H(P)\,\mathrm{br}_P(c)$ *for* P *a nontrivial* p*-subgroup of* D *and*
- C *induces a derived equivalence between* kGb *and* kHc.

Then, C *induces a Rickard equivalence between* kGb *and* kHc.

7.2. *Cyclotomic Hecke algebras.*

7.2.1. We consider here the nondegenerate case $q \neq 1$. We fix $v_1, \ldots, v_d \in k^\times$.

We denote by $\mathcal{H}_n = \mathcal{H}_n(v, q)$ the quotient of $H_n(q)$ by the ideal generated by $(X_1 - v_1) \cdots (X_1 - v_d)$. This is the Hecke algebra of the complex reflection group $G(d, 1, n)$ (cf. e.g. [Ar2, §13.1]).

The algebra $\mathcal{H}_n$ is free over k with basis $\{X_1^{a_1} \cdots X_n^{a_n} T_w\}_{0 \leq a_i < d, w \in \mathfrak{S}_n}$ [ArKo]. In particular $\mathcal{H}_{n-1}$ embeds as a subalgebra of $\mathcal{H}_n$, and $\mathcal{H}_n$ is free as a left and as a right $\mathcal{H}_{n-1}$-module, for the multiplication action. The algebra $\mathcal{H}_n$ is symmetric [MalMat].

7.2.2. Let $a \in k^\times$. Given M an $\mathcal{H}_n$-module, we denote by $F_{a,n}M$ the generalized a-eigenspace of X_n. This is an $\mathcal{H}_{n-1}$-module. We have a decomposition $\mathrm{Res}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} = \bigoplus_{a \in k^\times} F_{a,n}$. There is a corresponding decomposition $\mathrm{Ind}_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} = \bigoplus_{a \in k^\times} E_{a,n}$, where $E_{a,n}$ is left and right adjoint to $F_{a,n}$. We put $E_a = \bigoplus_{n \geq 1} E_{a,n}$ and $F_a = \bigoplus_{n \geq 1} F_{a,n}$.

Now fix $a \in k^\times$. The functors $E = E_a$ and $F = F_a$ give an action of $\mathfrak{sl}_2$ on $\bigoplus_{n \geq 0} K_0(\mathcal{H}_n\text{-mod})$ in which the classes of simple modules are weight vectors [Ar2, Th. 12.5] (only the case where each parameter if a power of q is considered there, but the proof extends immediately to our more general setting). We obtain an $\mathfrak{sl}_2$-categorification on $\bigoplus_{n \geq 0} \mathcal{H}_n$-mod, where the endomorphism X of E is given on $E_{a,n}$ by right multiplication by X_n, and the endomorphism T of E^2 is given on $E_{a,n}E_{a,n-1}$ by right multiplication by T_{n-1}.

Remark 7.11. Let e be the multiplicative order of q in $k^\times$. Fix $a_0 \in k^\times$ and let $I = \{q^m a_0 \mid m \in \mathbf{Z}\}$. Then the functors E_a and F_a for $a \in I$ define an action of $\widehat{\mathfrak{sl}}_e$ on $\bigoplus_{n \geq 0} K_0(\mathcal{H}_n\text{-mod})$.

7.2.3. Consider here the case $d = 1$. Then, $\mathcal{H}_n = \mathcal{H}_n(1, q)$ is the Hecke algebra of $\mathfrak{S}_n$. Let e be the multiplicative order of q in k. We have a notion of weight of a block as in §7.1.1, replacing p by e in the definitions.

We obtain a q-analog of Theorem 7.2:

THEOREM 7.12. *Assume* $d = 1$. *Let* A *be a block of* $\mathcal{H}_n$ *and* B *a block of* $\mathcal{H}_m$. *Then,* A *and* B *are derived equivalent if and only if they are Rickard equivalent if and only if they have the same weight.*

Remark 7.13. All of the constructions and results of §7.2 hold for degenerate cyclotomic Hecke algebras as well, under the assumption that they are symmetric algebras (which should be provable along the lines of [MalMat]). Note that these algebras are known to be self-injective [Kl3, Cor. 7.7.4].

7.3. *General linear groups over a finite field.*

7.3.1. Let q be a prime power, $n \geq 0$ and $G_n = \mathrm{GL}_n(q)$. We assume that k has characteristic $\ell > 0$ and $\ell \not| q(q-1)$. Let $A_n = kG_n b_n$ be the sum of the unipotent blocks of kG_n.

Given H a finite group, we put $e_H = \frac{1}{|H|}\sum_{h\in H} h$. We denote by ${}^t g$ the transpose of a matrix g.

We denote by V_n the subgroup of upper triangular matrices of G_n with diagonal coefficients 1 whose off-diagonal coefficients vanish outside the n-th column. We denote by D_n the subgroup of G_n of diagonal matrices with diagonal entries 1 except the (n,n)-th one.

$$V_n = \begin{pmatrix} 1 & & & * \\ & \ddots & & \vdots \\ & & 1 & * \\ & & & 1 \end{pmatrix}, \quad D_n = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & * \end{pmatrix}.$$

Let $i \leq n$. We view G_i as a subgroup of G_n via the first i coordinates.

We put

$$E_{i,n} = kG_n e_{(V_n \rtimes \cdots \rtimes V_{i+1})\rtimes(D_{i+1}\times\cdots\times D_n)} \otimes_{kG_i} - : A_i\text{-mod} \to A_n\text{-mod}$$
$$\text{and } F_{i,n} = e_{(V_n \rtimes \cdots \rtimes V_{i+1})\rtimes(D_{i+1}\times\cdots\times D_n)} kG_n \otimes_{kG_n} - : A_n\text{-mod} \to A_i\text{-mod}.$$

These functors are canonically left and right adjoint. Furthermore, there are canonical isomorphisms $E_{j,n} \circ E_{i,j} \xrightarrow{\sim} E_{i,n}$ and $F_{i,j} \circ F_{j,n} \xrightarrow{\sim} F_{i,n}$ for $i \leq j \leq n$.

Let $\mathcal{A} = \bigoplus_{n\geq 0} A_n$-mod, $E = \bigoplus_{n\geq 0} E_{n,n+1}$ and $F = \bigoplus_{n\geq 0} F_{n,n+1}$.

We denote by T the endomorphism of E^2 given on $E_{n-2,n}$ by right multiplication by

$$\hat{T}_{n-1} = q e_{V_n V_{n-1} D_{n-1} D_n} (n-1, n) e_{V_n V_{n-1} D_{n-1} D_n}.$$

We denote by X the endomorphism of E given on $E_{n-1,n}$ by right multiplication by

$$\hat{X}_n = q^{n-1} e_{V_n D_n} e_{{}^t V_n} e_{V_n D_n}.$$

LEMMA 7.14.

$$(\mathbf{1}_E T) \circ (T \mathbf{1}_E) \circ (\mathbf{1}_E T) = (T \mathbf{1}_E) \circ (\mathbf{1}_E T) \circ (T \mathbf{1}_E),$$
$$(T + \mathbf{1}_{E^2}) \circ (T - q\mathbf{1}_{E^2}) = 0 \textit{ and } \quad T \circ (\mathbf{1}_E X) \circ T = qX\mathbf{1}_E.$$

Proof. The first statements involving only T's are the classical results of Iwahori.

Let U be the subgroup of G_n with diagonal coefficients 1 and whose off-diagonal coefficients except the $(n,n-1)$-th vanish. We have

$$\begin{aligned}\hat{T}_{n-1}\hat{X}_{n-1}\hat{T}_{n-1} &= q^n e_{V_nV_{n-1}D_{n-1}D_n} e_U (n-1,n) e_{{}^tV_{n-1}} (n-1,n) e_U e_{V_{n-1}V_nD_{n-1}D_n} \\ &= q^n e_{V_nV_{n-1}D_{n-1}D_n} e_{{}^tV_n} e_{V_nV_{n-1}D_{n-1}D_n} = q e_{V_{n-1}D_{n-1}} \hat{X}_n e_{V_{n-1}D_{n-1}}\end{aligned}$$

and this induces the same endomorphism of $E_{n-2,n}$ as $q\hat{X}_n$. $\square$

Lemma 7.14 shows that we have a morphism $H_n(q) \to \mathrm{End}(E_{0,n}) = \mathrm{End}_{kG_n}(kG_n/B_n)$ which sends T_i to the endomorphism given by right multiplication by $qe_{B_n}(i-1,i)e_{B_n}$ and X_1 to the identity, where B_n is the subgroup of G_n of upper triangular matrices (cf. § 5.2.2). The classical result of Iwahori states that the restriction of this morphism to H_n^f is an isomorphism. This gives us a surjective morphism $p : H_n \to H_n^f$ whose restriction to H_n^f is the identity. Since X_1 maps to 1 in $\mathrm{End}(E_{0,n})$ and the quotient of H_n by $X_1 - 1$ is isomorphic to H_n^f, it follows that p is the canonical map $H_n \to H_n^f$. In particular, the image of X_i is (up to an affine transformation) a Jucys-Murphy element:

$$\begin{aligned}p(X_i) &= q^{1-i}T_{i-1}\cdots T_1T_1\cdots T_{i-1} \\ &= 1 + q^{1-i}(q-1)\left(T_{(1,i)} + qT_{(2,i)} + \cdots + q^{i-2}T_{(i-1,i)}\right).\end{aligned}$$

We put $R_n = \mathrm{Hom}_{kG_n}(kG_ne_{B_n}, -) = e_{B_n}kG_n \otimes_{kG_n} - : A_n\text{-mod} \to H_n^f\text{-mod}$. The multiplication maps

$$e_{B_i}kG_i \otimes_{kG_i} e_{V_n\cdots V_{i+1}D_n\cdots D_{i+1}}kG_n \to e_{B_n}kG_n$$

and

$$e_{B_n}kG_ne_{B_n} \otimes_{e_{B_i}kG_ie_{B_i}} e_{B_i}kG_i \to e_{B_n}kG_ne_{V_n\cdots V_{i+1}D_n\cdots D_{i+1}}$$

are isomorphisms. They induce isomorphisms of functors

$$R_iF_{i,n} \xrightarrow{\sim} \mathrm{Res}_{H_i^f}^{H_n^f} R_n \qquad \text{and} \qquad \mathrm{Ind}_{H_i^f}^{H_n^f} R_i \xrightarrow{\sim} R_nE_{i,n}.$$

Remark 7.15. The constructions carried out here make sense more generally for finite groups with a BN-pair and for arbitrary standard parabolic subgroups, the transpose operation corresponding to passing from the unipotent radical of a parabolic subgroup to the unipotent radical of the opposite parabolic subgroup. This produces a very general kind of "Jucys-Murphy element" in Hecke algebras of finite Weyl groups. In type B or C, we should recover the usual Jucys-Murphy elements.

Given $a \in k^\times$, let E_a be the generalized a-eigenspace of X acting on E.

LEMMA 7.16. *The action of $[E_a]$ and $[F_a]$ on $\bigoplus_{n\geq 0} K_0(A_n\text{-mod})$ gives a representation of $\mathfrak{sl}_2$. Furthermore, the classes of simple objects are weight vectors.*

Proof. Let $\mathcal{O}$ be a complete discrete valuation ring with field of fractions K and residue field k. We consider the setting above where k is replaced by K. The functor $\mathrm{Hom}_{KG_n}(KG_n e_{B_n}, -)$ induces an isomorphism from the Grothendieck group L_n of the category of unipotent representations of KG_n to the Grothendieck group of the category of representations of the Hecke algebra of type $\mathfrak{S}_n$ with parameter q over K. This isomorphism is compatible with the actions of E_a and F_a. It follows from §7.2.2 that E_a and F_a give a representation of $\mathfrak{sl}_2$ on $\bigoplus_{n\geq 0} L_n$ and the class of a simple unipotent representation of KG_n is a weight vector. Now, the decomposition map $L_n \to K_0(A_n)$ is an isomorphism [Jam, Th. 16.7] and the result follows. □

So, we have constructed an $\mathfrak{sl}_2$-categorification on $\bigoplus_{n\geq 0} A_n$-mod and a morphism of $\mathfrak{sl}_2$-categorifications $\bigoplus_{n\geq 0} A_n\text{-mod} \to \bigoplus_{n\geq 0} H_n^f\text{-mod}$.

Remark 7.17. Note that we deduce from this that the blocks of A_n correspond to the blocks of H_n^f.

7.3.2. We assume here only that $\ell \not| q$. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbf{Q}_\ell$ and k be the residue field of $\mathcal{O}$.

Let us recall [FoSri] that the ℓ-blocks of $\mathrm{GL}_n(q)$ are parametrized by pairs $((s),(B_1,\dots,B_r))$ where s is a conjugacy class of semi-simple ℓ'-elements of $\mathrm{GL}_n(q)$ and B_i is a block of $\mathcal{H}_{n_i}(q^{d_i})$, where $C_{\mathrm{GL}_n(q)}(s) = \mathrm{GL}_{n_1}(q^{d_1}) \times \cdots \times \mathrm{GL}_{n_r}(q^{d_r})$. Let w_i be the e_i-weight of the block B_i, where e_i is the multiplicative order of q^{d_i} in $k^\times$. We define the weight of the block as the family $\{(w_i,d_i)\}_{1\leq i\leq r}$.

THEOREM 7.18. *Let $R=k$ or $\mathcal{O}$. Two R-blocks of general linear groups* (*defined over the same field $\mathbf{F}_q$*) *with same weights are splendidly Rickard equivalent.*

Proof. The results on the local block theory of symmetric groups generalize to unipotent blocks of general linear groups [Br, §3] and we conclude as in the proof of Theorem 7.2 that the theorem holds for unipotent blocks.

By [BoRou2], a block of a general linear group is splendidly Rickard equivalent to a unipotent block of a product $\mathrm{GL}_{n_1}(q^{d_1}) \times \cdots \times \mathrm{GL}_{n_r}(q^{d_r})$ ([BoRou1, Théorème B] already provides a complex with homology only in one degree inducing a Morita equivalence). Such a block is splendidly Rickard equivalent to the principal block of $\mathrm{GL}_{e_1 w_1}(q^{d_1}) \times \cdots \times \mathrm{GL}_{e_r w_r}(q^{d_r})$ by the unipotent case of the theorem. □

Remark 7.19. Assume $l|(q-1)$. Then, $k\,\mathrm{GL}_n(q)$ has a unique unipotent block, the principal block. The number of simple modules for such a block is the number of partitions of n. Consequently, a unipotent block of $\mathrm{GL}_n(q)$ is not derived equivalent to a unipotent block of $\mathrm{GL}_m(q)$ when $n \neq m$.

THEOREM 7.20. *Let A be a block of a general linear group G over $R = k$ or $\mathcal{O}$, let D be a defect group and B be the corresponding block of $N_G(D)$. If D is abelian, then A and B are splendidly Rickard equivalent.*

Proof. By the result of [BoRou2] stated above, we may assume that A is a unipotent block. Then we proceed as in the proof of Theorem 7.6, using the fact that there is a unipotent block of a general linear group with defect group isomorphic to D that is splendidly Morita equivalent to the principal block of $R(\mathrm{GL}_e(q) \wr \mathfrak{S}_w)$ for some $w \geq 0$, where e is the order of q in $k^\times$ [Pu2], [Mi], [Tu]. $\square$

7.4. *Category $\mathcal{O}$.*

7.4.1. We construct here $\mathfrak{sl}_2$-categorifications on category $\mathcal{O}$ of $\mathfrak{gl}_n$. In particular we show that the weak $\mathfrak{sl}_2$-categorification on singular blocks given by Bernstein, Frenkel and Khovanov [BeFreKho] is an $\mathfrak{sl}_2$-categorification.

We denote by $\mathfrak{h}$ the Cartan subalgebra of diagonal matrices and $\mathfrak{n}$ the nilpotent algebra of strictly upper triangular matrices of the complex Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$. We denote by $\mathcal{O}$ the BGG category of finitely generated $U(\mathfrak{g})$-modules that are diagonalisable for $\mathfrak{h}$ and locally nilpotent for $U(\mathfrak{n})$.

Let $\{e_{ij}\}$ be the standard basis of $\mathfrak{g}$, and let $\varepsilon_1, \ldots, \varepsilon_n$ be the basis of $\mathfrak{h}^*$ dual to $e_{11}, \ldots, e_{nn}$. For each $\lambda \in \mathfrak{h}^*$ we denote by $\lambda_1, \ldots, \lambda_n$ the coefficients of λ with respect to $\varepsilon_1, \ldots, \varepsilon_n$. We write $\lambda \to_a \mu$ if there exists j such that $\lambda_j - j + 1 = a - 1$, $\mu_j - j + 1 = a$ and $\lambda_i = \mu_i$ for $i \neq j$. For each $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ be the Verma module with highest weight λ and let $L(\lambda)$ be its unique irreducible quotient. Recall that $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_\lambda$, where $\mathfrak{b}$ is the subalgebra of upper-triangular matrices and $\mathbf{C}_\lambda$ is the one-dimensional $\mathfrak{b}$-module on which e_{ii} acts as multiplication by λ_i.

Let Θ be the set of maximal ideals of the center Z of $U(\mathfrak{g})$. For each $\theta \in \Theta$ denote by $\mathcal{O}_\theta$ the full subcategory of $\mathcal{O}$ consisting of modules annihilated by some power of θ. The category $\mathcal{O}$ splits as a direct sum of the subcategories $\mathcal{O}_\theta$. Let $\mathrm{pr}_\theta : \mathcal{O} \to \mathcal{O}$ denote the projection onto $\mathcal{O}_\theta$. Each Verma module belongs to some $\mathcal{O}_\theta$, and $M(\lambda)$ and $M(\mu)$ belong to the same subcategory if and only if λ and μ are in the same orbit in the dot action of the Weyl group of $\mathfrak{g}$ on $\mathfrak{h}^*$, i.e., if and only if $(\lambda_1, \lambda_2 - 1, \ldots, \lambda_n - n + 1)$ and $(\mu_1, \mu_2 - 1, \ldots, \mu_n - n + 1)$ are in the same $\mathfrak{S}_n$-orbit. We write $\theta \to_a \theta'$ if there exist $\lambda, \mu \in \mathfrak{h}^*$ such that $M(\lambda) \in \mathcal{O}_\theta$, $M(\mu) \in \mathcal{O}_{\theta'}$ and $\lambda \to_a \mu$.

Let V be the natural n-dimensional representation of $\mathfrak{g}$. The functor $V \otimes - : \mathcal{O} \to \mathcal{O}$ decomposes as a direct sum $\bigoplus_{a \in \mathbf{C}} E_a$, where

$$E_a = \bigoplus_{\substack{\theta, \theta' \in \Theta \\ \theta \to_a \theta'}} \mathrm{pr}_{\theta'} \circ (V \otimes -) \circ \mathrm{pr}_\theta \,.$$

Each summand E_a has a left and right adjoint

$$F_a = \bigoplus_{\substack{\theta,\theta'\in\Theta \\ \theta\to_a\theta'}} \mathrm{pr}_\theta \circ (V^*\otimes -)\circ \mathrm{pr}_{\theta'}\,.$$

Let $\lambda\in\mathfrak{h}^*$. Now, $V\otimes M(\lambda) = V\otimes (U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbf{C}_\lambda) \simeq U(\mathfrak{g})\otimes_{U(\mathfrak{b})}(V\otimes \mathbf{C}_\lambda)$, and therefore $V\otimes M(\lambda)$ has a filtration with quotients isomorphic to the modules $M(\lambda+\varepsilon_i)$, $i=1,\ldots,n$. Similarly $V^*\otimes M(\lambda)$ has a filtration with quotients isomorphic to the modules $M(\lambda-\varepsilon_i)$, $i=1,\ldots,n$. It follows that

$$[E_aM(\lambda)] = \sum_{\substack{\mu\in\mathfrak{h}^*\\ \lambda\to_a\mu}}[M(\mu)],\qquad [F_aM(\lambda)] = \sum_{\substack{\mu\in\mathfrak{h}^*\\ \mu\to_a\lambda}}[M(\mu)]$$

in $K_0(\mathcal{O})$. Hence

$$[E_aF_aM(\lambda)]-[F_aE_aM(\lambda)] = c_{\lambda,a}[M(\lambda)],$$

where $c_{\lambda,a} = \#\{i \mid \lambda_i - i + 1 = a\} - \#\{i \mid \lambda_i - i+1 = a-1\}$. Because the classes of Verma modules are a basis for $K_0(\mathcal{O})$, we deduce that for each $a\in\mathbf{C}$ the functors E_a and F_a give a weak $\mathfrak{sl}_2$-categorification on $\mathcal{O}$ in which the simple module $L(\lambda)$ has weight $c_{\lambda,a}$.

7.4.2. Given M a $\mathfrak{g}$-module, we have an action map $\mathfrak{g}\otimes M\to M$. Let $X_M\in\mathrm{End}_{\mathfrak{g}}(V\otimes M)$ be the corresponding adjoint map. This defines an endomorphism X of the functor $V\otimes -$. Also, $X_M(v\otimes m) = \Omega(v\otimes m)$ where $\Omega = \sum_{i,j=1}^n e_{ij}\otimes e_{ji}\in\mathfrak{g}\otimes\mathfrak{g}$.

Define $T_M\in\mathrm{End}_{\mathfrak{g}}(V\otimes V\otimes M)$ by $T_M(v\otimes v'\otimes m) = v'\otimes v\otimes m$. This defines an endomorphism T of the functor $V\otimes V\otimes -$.

Lemma 7.21. *The following equality in* $\mathrm{End}_{\mathfrak{g}}(V\otimes V\otimes M)$ *gives*

$$T_M\circ(\mathbf{1}_V\otimes X_M) = X_{V\otimes M}\circ T_M - \mathbf{1}_{V\otimes V\otimes M}.$$

Proof. We have

$$\begin{aligned} X_{V\otimes M}T_M(v\otimes v'\otimes m) &= \sum_{i,j=1}^n e_{ij}v'\otimes e_{ji}(v\otimes m)\\ &= \sum_{i,j=1}^n e_{ij}v'\otimes e_{ji}v\otimes m + \sum_{i,j=1}^n e_{ij}v'\otimes v\otimes e_{ji}m\\ &= v\otimes v'\otimes m + T_M(\mathbf{1}_V\otimes X_M)(v\otimes v'\otimes m). \qquad\square\end{aligned}$$

The lemma implies that for each l we can define a morphism $H_l(1)\to\mathrm{End}_{\mathfrak{g}}(V^{\otimes l}\otimes M)$ by $T_i\mapsto \mathbf{1}_V^{\otimes l-i-1}\otimes T_{V^{\otimes i-1}\otimes M}$ and $X_i\mapsto\mathbf{1}_V^{\otimes l-i}\otimes X_{V^{\otimes i-1}\otimes M}$. Jon Brundan has pointed out to us that this coincides up to shift (cf. Remark 5.14) with an action described by Arakawa and Suzuki [ArSu, §2.2].

7.4.3. We shall now show that X and T are restricted to give endomorphisms of the functors E_a and E_a^2 which define $\mathfrak{sl}_2$-categorifications on $\mathcal{O}$. In view of Lemma 7.21, it suffices to identify E_a as the generalised a-eigenspace of X acting on $V \otimes -$.

To this end we observe that $\Omega = \frac{1}{2}(\delta(C) - C \otimes 1 - 1 \otimes C)$, where $C = \sum_{i,j=1}^n e_{ij}e_{ji} \in Z$ is the Casimir element and $\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the co-multiplication. Furthermore $C = \sum_{i=1}^n e_{ii}^2 + \sum_{1\le i<j\le n}(e_{ii} - e_{jj}) + \sum_{1\le i<j\le n} e_{ji}e_{ij}$ acts on the Verma module $M(\lambda)$ as multiplication by $b_\lambda = \sum_{i=1}^n \lambda_i^2 + \sum_{1\le i<j\le n}(\lambda_i - \lambda_j)$. It follows that Ω stabilizes any $\mathfrak{g}$-submodule of $V \otimes M(\lambda) = L(\varepsilon_1) \otimes M(\lambda)$ and that the induced action on any subquotient isomorphic to $M(\lambda+\varepsilon_i)$ is as multiplication by $\frac{1}{2}(b_{\lambda+\varepsilon_i} - b_{\varepsilon_1} - b_\lambda) = \lambda_i - i + 1$. Since $V \otimes M(\lambda) = \bigoplus_{a\in\mathbf{C}} E_aM(\lambda)$, this identifies $E_aM(\lambda)$ as the generalised a-eigenspace of $X_{M(\lambda)}$. We deduce that for any $M \in \mathcal{O}$, the generalized a-eigenspace of X_M is E_aM.

Remark 7.22. The canonical adjunction between $V \otimes -$ and $V^* \otimes -$ is given by the canonical maps $\eta : \mathbf{C} \to V^* \otimes V$ and $\varepsilon : V \otimes V^* \to \mathbf{C}, v \otimes \xi \mapsto \xi(v)$. Let $X_M \in \mathrm{End}_{\mathfrak{g}}(V^* \otimes M)$ and $T_M \in \mathrm{End}_{\mathfrak{g}}(V^* \otimes V^* \otimes M)$ be the induced endomorphisms (cf. §4.1.2). Then $X_M(\varphi \otimes m) = (-\Omega - n)(\varphi \otimes m)$ and $T_M(\varphi \otimes \varphi' \otimes m) = \varphi' \otimes \varphi \otimes m$.

7.5. *Rational representations.*

7.5.1. The construction of $\mathfrak{sl}_2$-categorifications in §7.4 works, more or less in the same way, on the category G-mod of finite-dimensional rational representations of $G = \mathrm{GL}_n(k)$, where k is an algebraically closed field of characteristic $p > 0$.

Denote by $\mathcal{X}$ the character group of the subgroup of diagonal matrices in G. We identify $\mathcal{X}$ with $\mathbf{Z}^n$ via the isomorphism sending $(\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n$ to $\lambda = \sum_i \lambda_i\varepsilon_i \in \mathcal{X}$, where ε_i is defined by $\varepsilon_i(\mathrm{diag}(t_1, \dots, t_n)) = t_i$. This identifies the set $\mathcal{X}_+$ of dominant weights with $\{(\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n \mid \lambda_1 \ge \dots \ge \lambda_n\}$. For each $\lambda \in \mathcal{X}_+$, let $L(\lambda)$ be the unique simple G-module with highest weight λ.

Let B be the Borel subgroup of upper triangular matrices in G. For each $\lambda \in \mathcal{X}$, the cohomology groups $H^i(\lambda)$ of the associated line bundle on G/B are objects of G-mod. The alternating sums $\chi(\lambda) = \sum_{i\ge 0} \mathrm{ch}(H^i(\lambda)) \in \mathbf{Z}[\mathcal{X}]$ span the image of the embedding $\mathrm{ch} : K_0(G\text{-mod}) \to \mathbf{Z}[\mathcal{X}]$.

The Weyl group $W = \mathfrak{S}_n$ of G acts on $\mathcal{X} = \mathbf{Z}^n$ by place permutations. This extends to an action of the affine Weyl group W_p generated by W together with the translations by $p\varepsilon_i - p\varepsilon_{i+1}, 1 \le i \le n-1$. Let Y be the group of permutations of $\mathbf{Z}$ generated by $d, \sigma_0, \dots, \sigma_{p-1}$, where $md = m+1$ and

$$m\sigma_a = \begin{cases} m+1 & \text{if } m \equiv a-1 \pmod p \\ m-1 & \text{if } m \equiv a \pmod p \\ m & \text{otherwise.} \end{cases}$$

The action of W_p on $\mathcal{X} = \mathbf{Z}^n$ commutes with the diagonal action of Y.

LEMMA 7.23. *Two elements $\lambda, \mu \in \mathcal{X}$ have the same stabilizer in W_p if and only if they are in the same Y-orbit.*

Proof. Both conditions are equivalent to the following: for all i, j, and r, we have $\lambda_i - \lambda_j = pr$ if and only if $\mu_i - \mu_j = pr$. □

We shall use the corresponding 'dot actions' obtained by conjugating by the translation by $\rho = (0, -1, \dots, -n+1) \in \mathcal{X}$:

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \qquad \lambda \cdot y = (\lambda + \rho)y - \rho.$$

Let Θ be the set of orbits of the dot action of W_p on $\mathcal{X}$. For each $\theta \in \Theta$, let $\mathcal{M}_\theta$ be the full subcategory of G-mod consisting of modules whose composition factors are all of the form $L(\lambda)$ for $\lambda \in \theta$. The Linkage Principle [CaLu] implies that G-mod decomposes as a direct sum $G\text{-mod} = \bigoplus_{\theta \in \Theta} \mathcal{M}_\theta$. Let $\mathrm{pr}_\theta : G\text{-mod} \to G\text{-mod}$ denote the projection onto $\mathcal{M}_\theta$. Given $\lambda, \mu \in \mathcal{X}$ and $a \in 0, \dots, p-1$, we write $\lambda \to_a \mu$ if there exists j such that $(\lambda_j - j + 1) + 1 = \mu_j - j + 1 \equiv a \pmod p$ and $\lambda_i = \mu_i$ for $i \neq j$. Note that $\lambda \to_a \mu$ implies that $w \cdot \lambda \to_a w \cdot \mu$ for all $w \in W_p$. For $\theta, \theta' \in \Theta$, we write $\theta \to_a \theta'$ if there exist $\lambda \in \theta$ and $\mu \in \theta'$ such that $\lambda \to_a \mu$.

Let V be the natural n-dimensional representation of G. The left and right adjoint functors $V \otimes - : G\text{-mod} \to G\text{-mod}$ and $V^* \otimes - : G\text{-mod} \to G\text{-mod}$ decompose as direct sums $\bigoplus_{0 \le a \le p-1} E_a$ and $\bigoplus_{0 \le a \le p-1} F_a$, where E_a and F_a are sums of translation functors, defined in the same way as in §7.4. The functors E_a and F_a have been studied extensively by Brundan and Kleshchev [BrKl].

Let e_a and f_a be the maps on characters induced by E_a and F_a. For each $\lambda \in \mathcal{X}$, we have (e.g. using [Jan, Prop. 7.8])

$$e_a \chi(\lambda) = \sum_{\substack{\mu \in \mathcal{X} \\ \lambda \to_a \mu}} \chi(\mu), \qquad f_a \chi(\lambda) = \sum_{\substack{\mu \in \mathcal{X} \\ \mu \to_a \lambda}} \chi(\mu) \tag{7}$$

in $\mathbf{Z}[\mathcal{X}]$. Hence

$$e_a f_a \chi(\lambda) - f_a e_a \chi(\lambda) = c_{\lambda,a} \chi(\lambda),$$

where $c_{\lambda,a} = \#\{i \mid \lambda_i - i + 1 \equiv a \pmod p\} - \#\{i \mid \lambda_i - i + 1 \equiv a - 1 \pmod p\}$. We deduce that for each $a \in \{0, \dots, p-1\}$ the functors E_a and F_a give a weak $\mathfrak{sl}_2$-categorification in which the simple module $L(\lambda)$ has weight $c_{\lambda,a}$.

7.5.2. These weak $\mathfrak{sl}_2$-categorifications can be improved to $\mathfrak{sl}_2$-categorifications using the same procedure as in the characteristic zero case §7.4. We first define endomorphisms X of $V \otimes -$ and T of $V \otimes V \otimes -$. Note that to define X, we first pass from G-modules to modules over $\mathrm{Lie}(G) = \mathfrak{gl}_n(k)$. One small modification to the argument is required when $p = 2$: in order to identify E_a with the generalized a-eigenspace of X, we write

$$\Omega = -\delta(Z_2) + 1 \otimes Z_2 + Z_2 \otimes 1 + Z_1 \otimes Z_1 - \frac{n(n+1)}{2},$$

where $Z_1 = \sum_{1 \le i \le n} e_{ii}$ and $Z_2 = \sum_{1 \le i < j \le n} (e_{ii} - i)(e_{jj} - j) - \sum_{1 \le i < j \le n} e_{ji} e_{ij}$ are central elements of $\mathrm{Dist}(G)$ (cf. [CaLu, §2.2]).

By composing the derived (and homotopy) equivalences arising from these $\mathfrak{sl}_2$-categorifications on G-mod, we obtain many equivalences.

THEOREM 7.24. *Let λ and μ be any two weights in $\mathcal{X}$ with the same stabilizer under the dot action of W_p. Then there are equivalences*

$$K^b(\mathcal{M}_{W_p \cdot \lambda}) \xrightarrow{\sim} K^b(\mathcal{M}_{W_p \cdot \mu}) \quad \textit{and} \quad D^b(\mathcal{M}_{W_p \cdot \lambda}) \xrightarrow{\sim} D^b(\mathcal{M}_{W_p \cdot \mu})$$

that induce the map

$$\chi(w \cdot \lambda) \mapsto \chi(w \cdot \mu)$$

on characters.

Remark 7.25. Rickard conjectured the existence of such equivalences for any connected reductive group having a simply connected derived subgroup whose root system has Coxeter number $h < p$ [Ri2, Conj. 4.1]. He proved the truth of his conjecture in the case of trivial stabilizers (under the weaker assumption $h \le p$). We do not place any restriction on p in Theorem 7.24.

Proof. By Lemma 7.23 we may assume that $\mu = \lambda \cdot y$ where $y \in \{d, \sigma_0, \ldots, \sigma_{p-1}\}$. If $\mu = \lambda \cdot d$, then we have an equivalence $L(1, \ldots, 1) \otimes - : \mathcal{M}_{W_p \cdot \lambda} \xrightarrow{\sim} \mathcal{M}_{W_p \cdot \mu}$, given by tensoring with the determinant representation, that induces the desired map on characters.

Suppose that $\mu = \lambda \cdot \sigma_a$. Using the $\mathfrak{sl}_2$-categorification on G-mod provided by $E = E_a$ and $F = F_a$, we obtain a self-equivalence Θ of $K^b(G\text{-mod})$ and of $D^b(G\text{-mod})$ such that $[\Theta] = s$ (Theorem 6.4). We define an $\mathfrak{sl}_2$-module $U = \bigoplus_{i \in \mathbf{Z}} \mathbf{Z} u_i$ by $e u_i = u_{i+1}$ for $i \equiv a - 1 \pmod p$ and $e u_i = 0$ otherwise, and $f u_i = u_{i-1}$ for $i \equiv a \pmod p$ and $f u_i = 0$ otherwise. Then $s u_i = u_{i+1}$ if $i \equiv a - 1 \pmod p$, $s u_i = -u_{i-1}$ if $i \equiv a \pmod p$, and $s u_i = u_i$ otherwise. Thus on the tensor power $U^{\otimes n}$ we have $s u_\nu = (-1)^{h_-(\nu)} u_{\nu \sigma_a}$, where $u_\nu = u_{\nu_1} \otimes \cdots \otimes u_{\nu_n}$ and $h_-(\nu) = \#\{i \mid \nu_i \equiv a \pmod p\}$.

By (7) we have a homomorphism of $\mathfrak{sl}_2$-modules $U^{\otimes n} \to K_0(G\text{-mod})$, $u_{\nu+\rho} \mapsto \chi(\nu)$. It follows that $s\chi(\nu) = (-1)^{h_-(\nu+\rho)} \chi(\nu \cdot \sigma_a)$. Hence $s\chi(w \cdot \lambda) = (-1)^{h_-} \chi(w \cdot \mu)$, where $h_- = h_-(w \cdot \lambda + \rho) = h_-(\lambda + \rho)$. We conclude that $\Theta[-h_-]$ restricts to equivalences $K^b(\mathcal{M}_{W_p \cdot \lambda}) \xrightarrow{\sim} K^b(\mathcal{M}_{W_p \cdot \mu})$ and $D^b(\mathcal{M}_{W_p \cdot \lambda}) \xrightarrow{\sim} D^b(\mathcal{M}_{W_p \cdot \mu})$ that induce the desired map on characters. $\square$

7.6. *q-Schur algebras.* We explain in this part how to obtain $\mathfrak{sl}_2$-categorifications, and hence derived equivalences, for q-Schur algebras.

Let $q \in k^\times$. Let $Y_n = \bigoplus_\lambda \operatorname{Ind}_{H_\lambda^f}^{H_n^f} k$, where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r)$ runs over the partitions of n and $H_\lambda^f = \bigoplus_{w \in \mathfrak{S}_{[1,\lambda_1]} \times \cdots \times \mathfrak{S}_{[n-\lambda_r+1,n]}} T_w k$ is the corresponding parabolic subalgebra of H_n^f and k corresponds to the representation 1. We define the q-Schur algebra $S_n = \operatorname{End}_{H_n^f}(Y_n)$.

Let $\mathcal{Y}_n$ be the full subcategory of H_n^f-mod whose objects are direct sums of direct summands of Y_n ("q-Young modules") and let $\mathcal{Y} = \bigoplus_{n \geq 0} \mathcal{Y}_n$. Mackey's formula shows that $\mathcal{Y}$ is stable under E and F. For each of the $\mathfrak{sl}_2$-categorifications on $\bigoplus_{n \geq 0} H_n^f$-mod constructed in §7.2 we deduce from Theorem 5.32 an $\mathfrak{sl}_2$-categorification on $\bigoplus_{n \geq 0} S_n$-mod and a morphism of $\mathfrak{sl}_2$-categorifications $\bigoplus_{n \geq 0} S_n\text{-mod} \to \bigoplus_{n \geq 0} H_n^f$-mod. This provides a version of Theorem 7.12 for q-Schur algebras.

Remark 7.26. We go back to the setting of §7.3 (in particular, q is a prime power). The canonical map $A_n \to \operatorname{End}_{H_n^{f\,\mathrm{opp}}}(kG_n e_{B_n})^{\mathrm{opp}}$ is surjective and its image S'_n is Morita equivalent to S_n ("double centralizer theorem", cf. [Ta]). This gives by restriction a fully faithful functor $S_n\text{-mod} \xrightarrow{\sim} S'_n\text{-mod} \to A_n$-mod. Since $E(kG_n e_{B_n}) \simeq kG_{n+1} e_{B_{n+1}}$, the fact that $\bigoplus_{n \geq 0} S_n$-mod is stable under E. Mackey's formula shows that it is also stable under F. This gives a morphism of weak $\mathfrak{sl}_2$-categorifications $\bigoplus_{n \geq 0} S_n\text{-mod} \to \bigoplus_{n \geq 0} A_n$-mod and the composition with the morphism $\bigoplus_{n \geq 0} A_n\text{-mod} \to \bigoplus_{n \geq 0} H_n^f$-mod of §7.3.1 is isomorphic to the morphism $\bigoplus_{n \geq 0} S_n\text{-mod} \to \bigoplus_{n \geq 0} H_n^f$-mod constructed above. One deduces that $\bigoplus_{n \geq 0} S_n\text{-mod} \to \bigoplus_{n \geq 0} A_n$-mod is actually a morphism of $\mathfrak{sl}_2$-categorifications.

Note also that we get another proof of Lemma 7.16 using the fact that the canonical map $K_0(S_n\text{-mod}) \xrightarrow{\sim} K_0(A_n\text{-mod})$ is an isomorphism.

Remark 7.27. The interested reader will extend the results of §7.5 to the quantum case and show that the categorification of q-Schur algebras can be realized as a subcategorification of the quantum group case.

7.7. *Realizations of minimal categorifications.*

7.7.1. We now show that the minimal categorification of §5.3 is a special case of the categorification on representations of blocks of cyclotomic Hecke algebras.

Fix $a \in k^\times$ and put $v = (v_1, \ldots, v_n) = (a, \ldots, a)$. Then $\mathcal{H}_i = \mathcal{H}_i(q, v)$ is the quotient of H_i by the ideal generated by x_1^n (where $x_1 = X_1 - a$). The kernel of the action of H_i on the simple module $K_i = H_i \otimes_{P_i} P_i/\mathfrak{m}_i$ contains x_1^n if and only if $i \leq n$ (cf. §3.2.1); let A_i be the block of $\mathcal{H}_i$ containing K_i for $0 \leq i \leq n$. A finitely generated $\mathcal{H}_i$-module M is in A_i if and only if $\mathfrak{n}_i$ acts nilpotently on M (equivalently $\mathfrak{m}_i$ acts nilpotently on M), and K_i is the unique simple module in A_i.

We have $FM = 0$ for $M \in A_0$-mod and $FM = \operatorname{Res}^{\mathcal{H}_i}_{\mathcal{H}_{i-1}} M \in A_{i-1}$-mod for $M \in A_i$-mod and $0 < i \leq n$.

Likewise $EM = 0$ for $M \in A_n$-mod. Let $M \in A_i$-mod with $0 \leq i < n$. Consider N a simple $\mathcal{H}_{i+1}$-quotient of EM. We have $\operatorname{Hom}(EM, N) \simeq \operatorname{Hom}(M, FN) \neq 0$. In particular, FN has a nonzero $\mathcal{H}_i$-submodule M' on which $x_1, \ldots, x_i$ act nilpotently. Let M'' be the $(k[x_{i+1}] \otimes \mathcal{H}_i)$-submodule of FN generated by M'. Then, $x_1, \ldots, x_{i+1}$ act nilpotently on M''. Now, N is a simple $\mathcal{H}_{i+1}$-module, hence it is generated by M'' as an $\mathcal{H}_{i+1}$-module, so that $x_1, \ldots, x_{i+1}$ act nilpotently on N. We deduce that they act nilpotently on EM as well. Thus, $EM \in A_{i+1}$-mod.

Now $\mathcal{A} = \bigoplus_i A_i$-mod is an $\mathfrak{sl}_2$-categorification and $\mathbf{Q} \otimes K_0(\mathcal{A})$ is a simple $\mathfrak{sl}_2$-module of dimension $n+1$. Let $U = K_0 = k$, the simple (projective) module for $A_0 = k$. The morphism of $\mathfrak{sl}_2$-categorifications $R_U : \mathcal{A}(n) \to \mathcal{A}$ is an equivalence (Proposition 5.26). In particular $\bar{H}_{i,n}$ and A_i are isomorphic, as each has an $i!$-dimensional simple module.

7.7.2. We explained in §3.3.2 that $\bar{H}_{i,n}$ is Morita equivalent to its center, which is isomorphic to the cohomology of certain Grasmmannian varieties. We sketch here a realization of the minimal categorification in that setting. We consider only the case $q = 1$; the case $q \neq 1$ can be dealt with similarly, replacing cohomology by $\mathbf{G}_m$-equivariant K-theory.

Let $G_{i,j}$ be the variety of pairs (V_1, V_2) of subspaces of $\mathbf{C}^n$ with $V_1 \subset V_2$, $\dim V_1 = i$ and $\dim V_2 = j$. We put $A_i = H^*(G_i)$. The (A_{i+1}, A_i)-bimodule $H^*(G_{i,i+1})$ defines by tensor product a functor $E_i : A_i$-mod $\to A_{i+1}$-mod and switching sides, a left and right adjoint $F_i : A_{i+1}$-mod $\to A_i$-mod. Let $E = \bigoplus E_i$ and $F = \bigoplus F_i$. This gives a weak $\mathfrak{sl}_2$-categorification that has been considered by Khovanov as a way of categorifying irreducible $\mathfrak{sl}_2$-representations. It is a special case of the construction of irreducible finite dimensional representations of $\mathfrak{sl}_n$ due to Ginzburg [Gi].

We denote by X the endomorphism of E given on $H^*(G_{i,i+1})$ by cup product by $c_1(L_{i+1})$. We have a $\mathbf{P}^1$-fibration $\pi : G_{i,i+1} \times_{G_{i+1}} G_{i+1,i+2} \to G_{i,i+2}$ given by first and last projection. It induces a structure of $H^*(G_{i,i+2})$-module on $H^*(G_{i,i+1} \times_{G_{i+1}} G_{i+1,i+2}) = H^*(G_{i,i+1}) \otimes_{H^*(G_{i+1})} H^*(G_{i+1,i+2})$. There is a unique endomorphism T of $H^*(G_{i,i+2})$-module on $H^*(G_{i,i+1} \times_{G_{i+1}} G_{i+1,i+2})$ satisfying $T(c_1(L_{i+1})) = c_1(L_{i+2}) - 1$. This provides us with an endomorphism of $E_{i+1}E_i$ and taking the sum over all i, we get an endomorphism T of E^2. One checks easily that this gives an $\mathfrak{sl}_2$-categorification (with $a = 0$) that is isomorphic to the minimal categorification.

The functor $E^{(1,r)} : A_i$-mod $\to A_{i+r}$-mod is isomorphic to the functor given by the bimodule $H^*(G_{i,i+r})$.

Take $i \leq n/2$ and let us now consider $\Theta[-i]$, restricted to a functor $D^b(H^*(G_i)\text{-mod}) \xrightarrow{\sim} D^b(H^*(G_{n-i})\text{-mod})$. It is probably isomorphic to the functor given by the cohomology of the subvariety $\{(V, V')|V \cap V' = 0\}$ of $G_i \times G_{n-i}$, the usual kernel for the Grassmannian duality (cf. e.g. [KaScha, Ex. III.15]).

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