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CHEREDNIK ALGEBRAS AND CALOGERO-MOSER CELLS

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CONTENTS

INTRODUCTION

Reconstruction of Lie-theoretic structures from Weyl groups and extension to complex reflection groups

A number of Lie-theoretic questions have their answer in terms of the associated Weyl group. Our work is part of a program to reconstruct combinatorial and categorical structures arising in Lie-theoretic representation theory from rational Cherednik algebras. Such algebras are associated by Etingof and Ginzburg to more general complex reflection groups, and an aspect of the program is to generalize those combinatorial and categorical structures to complex reflection groups, that will not arise from Lie theory in general.

To be more precise, consider a semisimple complex Lie algebra $\mathfrak{g}$ and let W be its Weyl group. Consider also a reductive algebraic group $\mathbf{G}$ over $\mathbb{Z}$, with $\mathfrak{g}$ the Lie algebra of $\mathbf{G}(\mathbb{C})$. Consider the following:

(i) (Parabolic) Blocks of (deformed) category $\mathcal{O}$ for $\mathfrak{g}$, blocks of categories of Harish-Chandra bimodules.
(ii) The set of unipotent characters of $\mathbf{G}(\mathbb{F}_q)$, their generic degrees, Lusztig's Fourier transform matrices.
(iii) Unipotent blocks of modular representations of $\mathbf{G}(\mathbb{F}_q)$ over a field of characteristic prime to q.
(iv) The Hecke algebra of W.
(v) Lattices in the Hecke algebra arising from the Kazhdan-Lusztig basis, Lusztig's asymptotic algebra J.
(vi) Kazhdan-Lusztig cells of W and left cell representations, families of characters of W.
(vii) Lusztig's modular categories associated to two-sided cells.

It is known or conjectured that the structures above depend only on W, viewed as a reflection group. One can hope that (possibly derived) versions of those structures still make sense for W a complex reflection group.

Consider the case where W a real reflection group. A solution to (i) is provided by Soergel's bimodules [**Soe**]. A solution to (ii) was found [**BrMa, Lus3, Mal1**]. The combinatorial theory in (v,vi) extends (partly conjecturally) to that setting. Categories as in (vii) were constructed by Lusztig when W is a dihedral group [**Lus3**].

The structures above might make sense for arbitrary ("unequal") parameters, and this is already an open problem for W a Weyl group. A partly conjectural theory for (v,vi) has been developed by Lusztig [**Lus4**], who is developing an interpretation via character sheaves on disconnected groups [**Lus5**].

Hecke algebras are a starting point: they have a topological definition that makes sense for complex reflection groups [**BrMaRo**], providing a (conjectural) solution to (iv). The Hecke algebras are (conjecturally) deformations of $\mathbb{Z}W$ over the space of class functions on W supported on reflections.

For certain complex reflection groups ("spetsial"), a combinatorial set (a "spets") has been associated by Broué, Malle and Michel, that plays the role of unipotent characters, and providing an answer to (ii) [**Mal2, BrMaMi1, BrMaMi2**]. Generic degrees are associated, building on Fourier transforms generalizing Lusztig's constructions for Weyl groups. There are generalized induction and restriction functors, and a d-Harish-Chandra theory.

When W is a cyclic group and for equal parameters, a solution to (vii) has been constructed in [**BoRou**]. It is a derived version of a modular category. It gives rise to the Fourier transform defined by Malle [**Mal2**].

In this book, we provide a conjectural extension of (vi) to complex reflection groups.

Etingof-Ginzburg's rational Cherednik algebras and Calogero-Moser spaces

Consider a non-trivial finite group W acting on a finite-dimensional complex vector space V and let V^{reg} the complement of the ramification locus of the quotient map $V \to V/W$. Assume W is a reflection group, i.e., W is generated by its set of reflections $\mathrm{Ref}(W)$ (equivalently: V/W is smooth; equivalently: V/W is an affine space). The quotient variety $(V \times V^*)/\Delta W$ by the diagonal action of W is singular. It is a ramified covering of $V/W \times V^*/W$.

Etingof and Ginzburg have constructed a deformation $\Upsilon : \mathscr{Z} \to \mathscr{C} \times V/W \times V^*/W$ of this covering [**EtGi**]. Here, $\mathscr{C}$ is a vector space with basis the quotient of $\mathrm{Ref}(W)$ by the conjugacy action of W. The variety $\mathscr{Z}$ is the Calogero-Moser space. The original covering corresponds to the point $0 \in \mathscr{C}$.

Etingof and Ginzburg define $\mathscr{Z}$ as the spectrum of the center of the rational Cherednik $\mathbf{H}$ associated with W at $t = 0$. It is a remarkable feature of their work that those important but complicated Calogero-Moser spaces have an explicit description based on non-commutative algebra. We will now explain their constructions.

The rational Cherednik algebra $\widetilde{\mathbf{H}}$ associated to W is a flat deformation defined by generators and relations of the algebra $\mathbb{C}[V \times V^*] \rtimes W$ over the space of parameters $(c,t) \in \widetilde{\mathscr{C}} = \mathscr{C} \times \mathbb{A}^1$. Its specialization at $(c = 0, t = 1)$ is the crossed product of the

Weyl algebra of V by W. The Cherednik algebra has a triangular decomposition $\widetilde{\mathbf{H}} = \mathbb{C}[V] \otimes \mathbb{C}[\widetilde{\mathscr{C}}]W \otimes \mathbb{C}[V^*]$. Equivalently, it satisfies a PBW Theorem. The algebra $\widetilde{\mathbf{H}}$ has a faithful representation by Dunkl operators on $\mathbb{C}[\widetilde{\mathscr{C}} \times V^{\mathrm{reg}}]$.

Consider the algebra $\mathbf{H}$ obtained by specializing $\widetilde{\mathbf{H}}$ at $t = 0$ and let Z be its center. It contains $\mathbb{C}[\mathscr{C}] \otimes \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ as a subalgebra. The Calogero-Moser variety is defined as $\mathscr{Z} = \operatorname{Spec} Z$. The Satake Theorem asserts that multiplication by the averaging idempotent $e = \frac{1}{|W|} \sum_{w \in W} w$ defines an isomorphism of algebras $Z \xrightarrow{\sim} e\mathbf{H}e$. As a consequence, the morphism $\Upsilon : \mathscr{Z} \to \mathscr{P} = \mathscr{C} \times V/W \times V^*/W$ is a flat deformation of $(V \times V^*)/\Delta W \to V/W \times V^*/W$ over $\mathscr{C}$.

Galois closure and ramification

The triangular decomposition of the Cherednik algebra at $t = 1$ leads to a construction of representation categories similar to that of enveloping algebras of complex semisimple Lie algebras.

The covering Υ, of degree $|W|$, is not Galois (unless $W = (\mathbb{Z}/2)^n$). Our work is a study of a Galois closure $\mathscr{R}$ of this covering and of the ramification above the closed subvarieties 0×0, $0 \times V^*/W$ and $V/W \times 0$ of $V/W \times V^*/W$.

Let G be the Galois group of $\mathscr{R} \to \mathscr{P}$. At $0 \in \mathscr{C}$, a Galois closure of the covering $(V \times V^*)/\Delta W \to V/W \times V^*/W$ is given by $(V \times V^*)/\Delta Z(W)$. This leads to a realization of G as a group of permutations of W.

This can be reformulated in terms of representations of $\mathbf{H}$: the semisimple $\mathbb{C}(\mathscr{P})$-algebra $\mathbb{C}(\mathscr{P}) \otimes_{\mathbb{C}[\mathscr{P}]} \mathbf{H}$ is not split and $\mathbb{C}(\mathscr{R})$ is a splitting field. The simple $\mathbb{C}(\mathscr{R}) \otimes_{\mathbb{C}[\mathscr{P}]} \mathbf{H}$-modules are in bijection with W. Our work can be viewed as the study of the partition of these modules into blocks corresponding to a given prime ideal of $\mathbb{C}[\mathscr{R}]$.

Calogero-Moser cells

Let X be an irreducible closed subvariety of $\mathscr{R}$. We define the X-cells of W as the orbits of the inertia group of X. Given a parameter $c \in \mathscr{C}$, we study the two-sided c-cells, defined for X an irreducible component of the inverse image of $\bar{X} = c \times 0 \times 0$. We also study the left c-cells (where $\bar{X} = c \times V/W \times 0$) and the right c-cells (where $\bar{X} = c \times 0 \times V/W$).

When W is a Coxeter group, we conjecture that the c-cells coincide with the Kazhdan-Lusztig cells, defined by Kazhdan-Lusztig [KaLu] and Lusztig [Lus1], [Lus3]. This depends on the choice of an appropriate X in a G-orbit.

We analyze in detail the case where W is cyclic (and $\dim V = 1$): this is the only case where we have a complete description of the objects studied in this book. We also provide a detailed study of the case of a Weyl group W of type B_2: the Galois group is a Weyl group of type D_4 and we show that the Calogero-Moser cells coincide with the Kazhdan-Lusztig cells. Our approach for B_2 is based on a detailed study of $\mathscr{X}$ and of the ramification of the covering, without constructing explicitly the variety $\mathscr{R}$.

Etingof and Ginzburg have introduced a deformation of the Euler vector field. We show that G is the Galois group of its minimal polynomial. This element plays an important role in the study of ramification, but is not enough to separate cells in general.

Families and cell representations

We construct a bijection between the set of two-sided c-cells and the set of blocks of the restricted rational Cherednik algebra $\bar{\mathbf{H}}_c$ (the specialization of $\mathbf{H}$ at $(c,0,0) \in \mathscr{C} \times V/W \times V^*/W = \mathscr{P}$). This latter set is in bijection with $\Upsilon^{-1}(c,0,0)$. Given a simple $\mathbb{C}W$-module E, there is an indecomposable representation of Verma type (a "baby-Verma" module) $\bar{\Delta}_c(E)$ of $\bar{\mathbf{H}}$ with a unique simple quotient $L_c(E)$ [Gor1]. The partition into blocks of those modules gives a partition of $\mathrm{Irr}(W)$ into *Calogero-Moser families*, which are conjecturally related to the families of the Hecke algebra of W (cf [Gor2, GoMa, Bel5, Mar1, Mar2]). We show that, in a given Calogero-Moser family, the matrix of multiplicities $[\bar{\Delta}_c(E) : L_c(F)]$ has rank 1, a property conjectured by Ulrich Thiel. Families satisfy a semi-continuity property with respected to specialization of the parameter. We show that families are minimal subsets that are unions of families for a generic parameter and on whose Verma modules the Euler element takes constant values. More precisely, this second statement is related to a set of hyperplanes of $\mathscr{C}$ where the families change. Those hyperplanes are related to the affine hyperplanes where the category $\mathscr{O}$ [**GGOR**] for the specialization at $t = 1$ of $\mathbf{H}$ is not semisimple. These are, in turn, related to the components of the locus of parameters where the Hecke algebra of W is not semisimple.

We introduce a notion of simple cell module associated with a left cell. We conjecture that the multiplicity of such a simple module in a Verma module is the same as the multiplicity of E in the cell representation of Kazhdan-Lusztig, when W is a Coxeter group. We study two-sided cells associated with a smooth point of $\Upsilon^{-1}(c,0,0)$: in that case, Gordon has shown that the corresponding block contains a unique Verma module. We show that the multiplicity of any simple cell module in that Verma module is 1 (for a left cell contained in the given two-sided cell).

There is a unique irreducible representation of W with minimal $\mathbf{b}$-invariant in each Calogero-Moser family. In Lusztig's theory, this is a characterization of special representations (case of equal parameters). We show that in each left cell representation there is also a unique irreducible representation of W with minimal b-invariant, and this can change inside a family. The corresponding result has been proved recently (case by case) in the setting of Lusztig's theory by the first author

[**Bon4**]. It is an instance of the Calogero-Moser theory shedding some light on the Kazhdan-Lusztig and Lusztig theory.

Description of the chapters

We review in Chapter 2 the basic theory of complex reflection groups: invariant theory, rationality of representations, fake degrees. We close that chapter with the particular case of real reflection groups endowed with the choice of a real chamber, i.e., finite Coxeter groups. All along the book, we devote special sections to the case of Coxeter groups when particular features arise in their case.

Chapters 3 and 4 are devoted to the basic structure theory of rational Cherednik algebras, following Etingof and Ginzburg [**EtGi**]. The definition of generic Cherednik algebras is given in Chapter 3, followed by the fundamental PBW Decomposition Theorem. We introduce the polynomial representation via Dunkl operators and prove its faithfulness. We introduce next the spherical algebra and prove some basic properties, including the double endomorphism Theorem. We also introduce the Euler element and gradings, filtrations, and automorphisms.

Chapter 4 is devoted to the Cherednik algebra at $t=0$. An important result is the Satake isomorphism between the center of the Cherednik algebra and the spherical subalgebra. We discuss localizations and cases of Morita equivalence between the Cherednik algebra and its spherical subalgebra. We provide some complements: filtrations, symmetrizing form and Hilbert series.

Our original work starts in Part II: it deals with the covering Υ and its ramification.

We introduce in Chapter 5 some of our basic objects of study, namely the Galois closure of the covering Υ and its Galois group. At parameter 0, the corresponding data is very easily described, and its embedding in the family depends on a choice. We explain this in §5.1.B, and show that this allows an identification of the generic fiber of Υ with W. We show in §5.1.D that the extension $\mathbf{L}/\mathbf{K}$ is generated by the Euler element, and that the corresponding result at the level of rings is true if and only if W is generated by a single reflection. We discuss in §5.2 the decomposition of the $\mathbf{M}$-algebra $\mathbf{M}\mathbf{H}$ as a product of matrix algebras over $\mathbf{M}$. In other parts of §5, we discuss gradings and automorphisms, and construct an order 2 element of G when all reflections of W have order 2 and $-\mathrm{Id}_V \in W$. The last part §5.7 is a geometrical translation of the previous constructions.

We introduce Calogero-Moser cells in Chapter 6. They are defined in §6.1 as orbits of inertia groups on W and shown in §6.2 to coincide with blocks of the Cherednik algebra. We study next the ramification locus and smoothness. We give two more equivalent definitions of Calogero-Moser cells: via irreducible components of the base change by Υ of the Galois cover (§6.5) and via lifting of paths (§6.6).

Part III is the heart of the book. It discusses Calogero-Moser cells associated with the ramification at $\{0\} \times \{0\}$, $V/W \times \{0\}$ and $\{0\} \times V^*/W$, and relations with representations of the Cherednik algebras, as well as (conjectural) relations with Hecke algebras.

We start in Chapter 7 with a discussion of graded representations of Cherednik algebras. They form a highest weight category (in the sense of Appendix §F). We discuss the standard objects, the Verma modules, and the Euler action on them.

Chapter 8 is devoted to Hecke algebras. We recall in §8.1 the definition of Hecke algebras of complex reflection groups and some of its basic (partly conjectural) properties. We introduce a "cyclotomic" version, where the Hecke parameters are powers of a fixed indeterminate. We explain in §8.3 the construction of the Knizhnik-Zamolodchikov functor [**GGOR**] realizing the category of representations of the Hecke algebra as a quotient of a (non-graded) category $\mathcal{O}$ for the Cherednik algebra at $t = 1$. Thanks to the double endomorphism Theorem, the semisimplicity of the Hecke algebra is equivalent to that of the category $\mathcal{O}$. We present in §8.4 Malle's splitting result for irreducible representations of Hecke algebras and we consider central characters. We discuss in §8.5 the notion of Hecke families. We finish in §8.6 with a brief exposition of the theory of Kazhdan-Lusztig cells of elements of W and of families of characters of W and c-cellular characters.

Chapter 9 is devoted to the representation theory of restricted Cherednik algebras and to Calogero-Moser families. We recall in §9.1 and §9.2 some basic results of Gordon [**Gor1**] on representations of restricted Cherednik algebras and Calogero-Moser families. Graded representations give rise to a highest weight category, as noticed by Bellamy and Thiel [**BelTh**], and we follow that approach. We show in §9.4 the existence of a unique representation with minimal **b**-invariant in each family and generalize results of [**Gor1**] on graded dimensions. We discuss in §9.6 the relation between the geometry of Calogero-Moser at $\Upsilon^{-1}(0)$ and the Calogero-Moser families. The final section §9.7 relates Calogero-Moser families with blocks of category $\mathcal{O}$ at $t = 1$ and with blocks of Hecke algebras.

In Chapter 10, we get back to the Galois cover and study two-sided cells. We construct a bijection between the set of two-sided cells and the set of families.

We continue in Chapter 11 with the study of left (and right) cells and we define Calogero-Moser cellular characters. We analyze in §11.2 the choices involved in the definition of left cells, using Verma modules. We study the relevant decomposition groups, and we reinterpret left cells as blocks of a suitable specialization of the Cherednik algebra. We finish in §11.4 with basic properties relating cellular characters and left cells and we give an alternative definition of cellular characters as the socle of the restriction of a projective module. We also show that a cellular character involves a unique irreducible representation with minimal **b**-invariant.

Chapter 12 brings decomposition matrices in the study of cells. We show in §12.4 that the decomposition matrix of baby Verma modules in a block has rank 1, as conjectured by Thiel. In §12.5, we prove that cellular characters are sums with non-negative coefficients of characters of projective modules of the Hecke algebra over appropriate base rings.

Chapter 13 shows, following a suggestion of Etingof, that cells can be interpreted in terms of spectra of certain Gaudin-type operators. This provides a topological approach to cells and cellular characters.

We analyze in Chapter 14 the cells associated to a smooth point of a Calogero-Moser space in $\Upsilon^{-1}(0)$. This is based on the use of the $\mathbb{C}^{\times}$-action and the resulting attracting sets. We show first, without smoothness assumptions, that irreducible components of attracting sets parametrize the cellular characters.

The next two chapters are devoted to conjectures. Chapter 15 discusses the motivation of this book, namely the expected relation between Calogero-Moser cells and Kazhdan-Lusztig cells when W is a Coxeter group. We start in §15.1 with Martino's conjecture that Calogero-Moser families are unions of Hecke families. §15.2 and §15.3 state and discuss our main conjecture. We give some cases where the conjecture on cellular characters does hold and give some evidence for the conjecture on cells.

Chapter 16 gives a conjecture on the cohomology ring and the $\mathbb{C}^{\times}$-equivariant cohomology ring of Calogero-Moser spaces, extending the description of Etingof-Ginzburg in the smooth case. We also conjecture that irreducible components of the fixed points of finite order automorphisms on the Calogero-Moser space are Calogero-Moser spaces for reflection subquotients.

Part IV is based on the study of particular cases. Chapter 17 presents the theory for the parameter $c = 0$.

Chapter 18 is devoted to the case of V of dimension 1. We give a description of the objects introduced earlier, in particular the Galois closure R. We show that generic decomposition groups can be very complicated, for particular values of the parameter.

Chapter 19 analyzes the case of W a finite Coxeter group of type B_2. We determine in §19.3 the ring of diagonal invariants and the minimal polynomial of the Euler element. We continue in §19.4 with the determination of the corresponding deformed objects. The Calogero-Moser families are then easily found. We move next to the determination of the Galois group G. Section §19.7 is the more complicated study of ramification and the determination of the Calogero-Moser cells. We finish in §19.8 with a discussion of fixed points of the action of groups of roots of unity, confirming Conjecture FIX.

We have gathered in six Appendices some general algebraic considerations, few of which are original.

Appendix A is a brief exposition of filtered modules and filtered algebras. We analyze in particular properties of an algebra with a filtration that are consequences of the corresponding properties for the associated graded algebra. We discuss symmetric algebras in §A.4.

Appendix B gathers some basic facts on ramification theory for commutative rings around decomposition and inertia groups. We recollect some properties of Galois groups and discriminants, and close the chapter with a topological version of the ramification theory and its connection with the commutative rings theory.

Appendix C is a discussion of some aspects of the theory of graded rings. We consider general rings in §C.1. We next discuss in §C.2 gradings in the setting of commutative ring extensions. We finally consider gradings and invariants rings in §C.3.

We present in Appendix D some results on blocks and base change for algebras finite and free over a base. We discuss in particular central characters and idempotents, and the locus where the block decomposition changes.

Appendix E deals with finite group actions on rings (commutative or not), and compare the cross-product algebra and the invariant ring. We consider in particular the module categories and the centers.

Appendix F provides a generalized theory of highest weight categories over commutative rings. We discuss in particular base change (§F.1.E), Grothendieck groups (§F.1.F and §F.1.G), decomposition maps (§F.1.H) and blocks (§F.1.I). A particular class of highest weight categories arises from graded algebras with a triangular decomposition (§F.2), generalizing **[GGOR]** to non-inner gradings.

Before the index of notations, we have included in "Prime ideals and geometry" some diagrams summarizing the commutative algebra and geometry studied in this book.

We would like to thank G. Bellamy, P. Etingof, I. Gordon, M. Martino and U. Thiel for their help and their suggestions.

Gunter Malle has suggested many improvements on a preliminary version of this book, and has provided very valuable help with Galois theory questions: we thank him for all of this.

Commentary. — This book contains an earlier version of our work [**BoRou1**]. The structure of the text has changed and the presentation of classical results on Cherednik algebras is now mostly self-contained. There are a number of new results (see for instance Chapters 7 and 9) based on appropriate highest weight category considerations (Appendix F is new), and a new topological approach (Chapter 13 on Gaudin algebras in particular).

PART I

REFLECTION GROUPS AND CHEREDNIK ALGEBRAS

CHAPTER 1

NOTATIONS

1.1. Integers

We put $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$.

1.2. Modules

Let A be a ring. Given L a subset of A, we denote by $< L >$ the two-sided ideal of A generated by L. Given M an A-module, we denote by $\mathrm{Rad}(M)$ the intersection of the maximal proper A-submodules of M. We denote by A-Mod the category of A-modules and by A-mod the category of finitely generated A-modules and we put $G_0(A) = K_0(A\text{-mod})$, where $K_0(\mathscr{C})$ denotes the Grothendieck group of an exact category $\mathscr{C}$. We denote by A-proj the category of finitely generated projective A-modules. Given $\mathscr{A}$ an abelian category, we denote by $\mathrm{Proj}(\mathscr{A})$ its full subcategory of projective objects.

Given $M \in A$-mod, we denote by $[M]_A$ (or simply $[M]$) its class in $G_0(A)$.

We denote by $\mathrm{Irr}(A)$ the set of isomorphism classes of simple A-modules. Assume A is a finite-dimensional algebra over the field $\mathbf{k}$. We have an isomorphism $\mathbb{Z}\,\mathrm{Irr}(A) \xrightarrow{\sim} G_0(A)$, $M \mapsto [M]$. If A is semisimple, we have a bilinear form $\langle -,- \rangle_A$ on $G_0(A)$ given by $\langle [M],[N] \rangle = \dim_{\mathbf{k}} \mathrm{Hom}_A(M,N)$. When A is split semisimple, $\mathrm{Irr}(A)$ provides an orthonormal basis.

Let W be a finite group and assume $\mathbf{k}$ is a field. We denote by $\mathrm{Irr}_{\mathbf{k}}(W)$ (or simply by $\mathrm{Irr}(W)$) the set of irreducible characters of W over $\mathbf{k}$. When $|W| \in \mathbf{k}^{\times}$, there is a bijection $\mathrm{Irr}_{\mathbf{k}}(W) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{k}W)$, $\chi \mapsto E_{\chi}$. The group $\mathrm{Hom}(W, \mathbf{k}^{\times})$ of linear characters of W with values in $\mathbf{k}$ is denoted by $W^{\wedge_{\mathbf{k}}}$ (or $W^{\wedge}$) . We have an embedding $W^{\wedge} \subset \mathrm{Irr}(W)$, and equality holds if and only if W is abelian and $\mathbf{k}$ contains all e-th roots of unity, where e is the exponent of W.

1.3. Gradings

1.3.A. Let $\mathbf{k}$ be a ring and X a set. We denote by $\mathbf{k}X = \mathbf{k}^{(X)}$ the free $\mathbf{k}$-module with basis X. We sometimes denote elements of $\mathbf{k}^X$ as formal sums: $\sum_{x\in X} \alpha_x x$, where $\alpha_x \in \mathbf{k}$.

1.3.B. Let Γ be a monoid. We denote by $\mathbf{k}\Gamma$ (or $\mathbf{k}[\Gamma]$) the monoid algebra of Γ over $\mathbf{k}$. Its basis of elements of Γ is denoted by $\{t^\gamma\}_{\gamma\in\Gamma}$.

A Γ-*graded* $\mathbf{k}$-*module* is a $\mathbf{k}$-module L with a decomposition $L = \bigoplus_{\gamma\in\Gamma} L_\gamma$ (that is the same as a comodule over the coalgebra $\mathbf{k}\Gamma$). Given $\gamma_0 \in \Gamma$, we denote by $L\langle\gamma_0\rangle$ the Γ-graded $\mathbf{k}$-module given by $(L\langle\gamma_0\rangle)_\gamma = L_{\gamma\gamma_0}$. We denote by $\mathbf{k}$-free$^\Gamma$ the additive category of Γ-graded $\mathbf{k}$-modules L such that L_γ is a free $\mathbf{k}$-module of finite rank for all $\gamma\in\Gamma$. Given $L \in \mathbf{k}$-free$^\Gamma$, we put

$$\dim_{\mathbf{k}}^\Gamma(L) = \sum_{\gamma\in\Gamma} \operatorname{rank}_{\mathbf{k}}(L_\gamma)t^\gamma \in \mathbb{Z}^\Gamma.$$

We have defined an isomorphism of abelian groups $\dim_{\mathbf{k}}^\Gamma : K_0(\mathbf{k}\text{-free}^\Gamma) \xrightarrow{\sim} \mathbb{Z}^\Gamma$. This construction provides a bijection from the set of isomorphism classes of objects of $\mathbf{k}$-free$^\Gamma$ to $\mathbb{N}^\Gamma$. Given $P = \sum_{\gamma\in\Gamma} p_\gamma t^\gamma$ with $p_\gamma \in \mathbb{N}$, we define the Γ-graded $\mathbf{k}$-module $\mathbf{k}^P$ by $(\mathbf{k}^P)_\gamma = \mathbf{k}^{p_\gamma}$. We have $\dim_{\mathbf{k}}^\Gamma(\mathbf{k}^P) = P$.

We say that a subset E of a Γ-graded module L is *homogeneous* if every element of E is a sum of elements in $E\cap L_\gamma$ for various elements $\gamma\in\Gamma$.

1.3.C. A *graded* $\mathbf{k}$-*module* L is a $\mathbb{Z}$-graded $\mathbf{k}$-module. We put $L_+ = \bigoplus_{i>0} L_i$. If $L_i = 0$ for $i \ll 0$ (for example, if L is $\mathbb{N}$-graded), then $\dim_{\mathbf{k}}^{\mathbb{Z}}(L)$ is an element of the ring of Laurent power series $\mathbb{Z}((\mathbf{t}))$: this is the Hilbert series of L. Similarly, if $L_i = 0$ for $i \gg 0$, then $\dim_{\mathbf{k}}^{\mathbb{Z}}(L) \in \mathbb{Z}((\mathbf{t}^{-1}))$.

When L has finite rank over $\mathbf{k}$, we define the *weight sequence* of L as the unique sequence of integers $r_1 \leqslant \cdots \leqslant r_m$ such that $\dim_k^{\mathbb{Z}}(L) = t^{r_1} + \cdots + t^{r_m}$.

A *bigraded* $\mathbf{k}$-*module* L is a $(\mathbb{Z}\times\mathbb{Z})$-graded $\mathbf{k}$-module. We put $\mathbf{t} = t^{(1,0)}$ and $\mathbf{u} = t^{(0,1)}$, so that $\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(L) = \sum_{i,j} \dim_{\mathbf{k}}(L_{i,j})\mathbf{t}^i\mathbf{u}^j$ for $L \in \mathbf{k}$-free$^{\mathbb{Z}\times\mathbb{Z}}$. When L is $(\mathbb{N}\times\mathbb{N})$-graded, we have $\dim_{\mathbf{k}}^{\mathbb{N}\times\mathbb{N}}(L) \in \mathbb{Z}[[\mathbf{t},\mathbf{u}]]$.

When A is a graded ring and M is a finitely generated graded A-module, we denote by $[M]_A^{\mathrm{gr}}$ (or simply $[M]^{\mathrm{gr}}$) its class in the Grothendieck group of the category A-modgr of finitely generated graded A-modules. Note that $K_0(A\text{-modgr})$ is a $\mathbb{Z}[\mathbf{t}^{\pm1}]$-module, with $\mathbf{t}[M]^{\mathrm{gr}} = [M\langle -1\rangle]^{\mathrm{gr}}$.

1.3.D. Assume $\mathbf{k}$ is a commutative ring. There is a tensor product of Γ-graded $\mathbf{k}$-modules given by $(L\otimes_{\mathbf{k}} L')_\gamma = \bigoplus_{\gamma'\gamma''=\gamma} L_{\gamma'}\otimes_{\mathbf{k}} L_{\gamma''}$. When the fibers of the multiplication map $\Gamma\times\Gamma\to\Gamma$ are finite, the multiplication in Γ provides $\mathbb{Z}^\Gamma$ with a ring structure, the tensor product preserves $\mathbf{k}$-free$^\Gamma$, and $\dim_{\mathbf{k}}^\Gamma(L\otimes_{\mathbf{k}} L') = \dim_{\mathbf{k}}^\Gamma(L)\dim_{\mathbf{k}}^\Gamma(L')$.

A Γ-*graded* **k**-*algebra* is a **k**-algebra A with a Γ-grading such that $A_\gamma \cdot A_{\gamma'} \subset A_{\gamma\gamma'}$.

CHAPTER 2

REFLECTION GROUPS

All along this book, we consider a fixed characteristic 0 *field* $\mathbf{k}$*, a finite-dimensional* $\mathbf{k}$*-vector space* V *of dimension* n *and a finite subgroup* W *of* $\mathrm{GL}_{\mathbf{k}}(V)$*. We will write* $\otimes$ *for* $\otimes_{\mathbf{k}}$*. We denote by*

$$\mathrm{Ref}(W) = \{s \in W \mid \dim_{\mathbf{k}} \mathrm{Im}(s - \mathrm{Id}_V) = 1\}$$

the set of reflections of W*. We assume that* W *is generated by* $\mathrm{Ref}(W)$*.*

2.1. Determinant, roots, coroots

We denote by ε the determinant representation of W

$$\begin{array}{rccc} \varepsilon: & W & \longrightarrow & \mathbf{k}^\times \\ & w & \longmapsto & \det_V(w). \end{array}$$

We have a perfect pairing between V and its dual V^*

$$\langle , \rangle : V \times V^* \longrightarrow \mathbf{k}.$$

Given $s \in \mathrm{Ref}(W)$, we choose $\alpha_s \in V^*$ and $\alpha_s^\vee \in V$ such that

$$\mathrm{Ker}(s - \mathrm{Id}_V) = \mathrm{Ker}\,\alpha_s \quad \text{and} \quad \mathrm{Im}(s - \mathrm{Id}_V) = \mathbf{k}\alpha_s^\vee$$

or equivalently

$$\mathrm{Ker}(s - \mathrm{Id}_{V^*}) = \mathrm{Ker}\,\alpha_s^\vee \quad \text{and} \quad \mathrm{Im}(s - \mathrm{Id}_{V^*}) = \mathbf{k}\alpha_s.$$

Note that, since $\mathbf{k}$ has characteristic 0, all elements of $\mathrm{Ref}(W)$ are diagonalizable, hence

(2.1.1) $$\langle \alpha_s^\vee, \alpha_s \rangle \neq 0.$$

Given $x \in V^*$ and $y \in V$ we have

(2.1.2) $$s(y) = y - (1 - \varepsilon(s)) \frac{\langle y, \alpha_s \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} \alpha_s^\vee$$

and

$$s(x) = x - (1 - \varepsilon(s)^{-1}) \frac{\langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} \alpha_s. \tag{2.1.3}$$

2.2. Invariants

We denote by $\mathbf{k}[V] = S(V^*)$ (respectively $\mathbf{k}[V^*] = S(V)$) the symmetric algebra of V^* (respectively V). We identify it with the algebra of polynomial functions on V (respectively V^*). The action of W on V induces an action by algebra automorphisms on $\mathbf{k}[V]$ and $\mathbf{k}[V^*]$ and we will consider the graded subalgebras of invariants $\mathbf{k}[V]^W$ and $\mathbf{k}[V^*]^W$. The *coinvariant algebras* $\mathbf{k}[V]^{\mathrm{co}(W)}$ and $\mathbf{k}[V^*]^{\mathrm{co}(W)}$ are the graded finite-dimensional $\mathbf{k}$-algebras

$$\mathbf{k}[V]^{\mathrm{co}(W)} = \mathbf{k}[V]/ < \mathbf{k}[V]_+^W > \quad \text{and} \quad \mathbf{k}[V^*]^{\mathrm{co}(W)} = \mathbf{k}[V^*]/ < \mathbf{k}[V^*]_+^W > .$$

Shephard-Todd-Chevalley's Theorem asserts that the property of W to be generated by reflections is equivalent to structural properties of $\mathbf{k}[V]^W$. We provide here a version augmented with quantitative properties (see for example [**Bro2**, Theorem 4.1]). We state a version with $\mathbf{k}[V]$, while the same statements hold with V replaced by V^*.

Let us define the sequence $d_1 \leqslant \cdots \leqslant d_n$ of *degrees of* W as the weight sequence of $< \mathbf{k}[V]_+^W > / < \mathbf{k}[V]_+^W >^2$ (cf §1.3.C).

***Theorem* 2.2.1 (Shephard-Todd, Chevalley).** — (a) *The algebra $k[V]^W$ is a polynomial algebra generated by homogeneous elements of degrees $d_1, \ldots, d_n$. We have*

$$|W| = d_1 \cdots d_n \quad \textit{and} \quad |\mathrm{Ref}(W)| = \sum_{i=1}^{n} (d_i - 1).$$

(b) *The* $(\mathbf{k}[V]^W[W])$*-module* $\mathbf{k}[V]$ *is free of rank* 1.

(c) *The* $\mathbf{k}W$*-module* $\mathbf{k}[V]^{\mathrm{co}(W)}$ *is free of rank* 1. *So,* $\dim_{\mathbf{k}} \mathbf{k}[V]^{\mathrm{co}(W)} = |W|$.

***Remark* 2.2.2.** — Note that when $\mathbf{k} = \mathbb{C}$, there is a skew-linear isomorphism between the representations V and V^* of W, hence the sequence of degrees for the action of W on V is the same as the one for the action of W on V^*. In general, note that the representation V of W can be defined over a finite extension of $\mathbb{Q}$, which can be embedded in $\mathbb{C}$: so, the equality of degrees for the actions on V and V^* holds for any $\mathbf{k}$.

This equality can also be deduced from Molien's formula [**Bro2**, Lemma 3.28]. ■

Let $N = |\mathrm{Ref}(W)|$. Since $\dim_{\mathbf{k}}^{\mathbb{Z}}(\mathbf{k}[V]^{\mathrm{co}(W)}) = \prod_{i=1}^{n} \frac{1-\mathbf{t}^{d_i}}{1-\mathbf{t}}$, we deduce that $\dim_{\mathbf{k}} \mathbf{k}[V]_N^{\mathrm{co}(W)} = 1$. A generator is given by the image of $\prod_{s\in\mathrm{Ref}(W)} \alpha_s$: this provides an isomorphism $h : \mathbf{k}[V]_N^{\mathrm{co}(W)} \xrightarrow{\sim} \mathbf{k}$.

The composition

$$\mathbf{k}[V]_N \otimes \mathbf{k}[V]^W \xrightarrow{\text{mult}} \mathbf{k}[V] \xrightarrow{\text{can}} \mathbf{k}[V]/(\mathbf{k}[V]^W \mathbf{k}[V]_{<N})$$

factors through an isomorphism $g : \mathbf{k}[V]_N^{\mathrm{co}(W)} \otimes \mathbf{k}[V]^W \xrightarrow{\sim} \mathbf{k}[V]/(\mathbf{k}[V]^W \mathbf{k}[V]_{<N})$. We denote by p_N the composition

$$p_N : \mathbf{k}[V] \xrightarrow{\text{can}} \mathbf{k}[V]/(\mathbf{k}[V]^W \mathbf{k}[V]_{<N}) \xrightarrow{g^{-1}} \mathbf{k}[V]_N^{\mathrm{co}(W)} \otimes \mathbf{k}[V]^W \underset{\sim}{\xrightarrow{h\otimes\mathrm{Id}}} \mathbf{k}[V]^W.$$

We refer to §A.4 for basic facts on symmetric algebras.

Proposition 2.2.3. — *p_N is a symmetrizing form for the $\mathbf{k}[V]^W$-algebra $\mathbf{k}[V]$.*

Proof. — We need to show that the morphism of graded $\mathbf{k}[V]^W$-modules

$$\hat{p}_N : \mathbf{k}[V] \to \mathrm{Hom}_{\mathbf{k}[V]^W}(\mathbf{k}[V], \mathbf{k}[V]^W),\ a \mapsto (b \mapsto p_N(ab))$$

is an isomorphism. By the graded Nakayama lemma, it is enough to do so after applying $-\otimes_{\mathbf{k}[V]^W} \mathbf{k}$. We have $\hat{p}_N \otimes_{\mathbf{k}[V]^W} \mathbf{k} = \hat{\bar{p}}_N$, where $\bar{p}_N : \mathbf{k}[V]^{\mathrm{co}(W)} \to \mathbf{k}[V]_N^{\mathrm{co}(W)} \underset{\sim}{\xrightarrow{h}} \mathbf{k}$ is the projection onto the homogeneous component of degree N. This is a symmetrizing form for $\mathbf{k}[V]^{\mathrm{co}(W)}$ [**Bro2**, Theorem 4.25], hence $\hat{\bar{p}}_N$ is an isomorphism. □

Note that the same statements hold for V replaced by V^*.

2.3. Hyperplanes and parabolic subgroups

> **Notation.** *We fix an embedding of the group of roots of unity of* $\mathbf{k}$ *in* $\mathbb{Q}/\mathbb{Z}$. *When the class of* $\frac{1}{e}$ *is in the image of this embedding, we denote by* ζ_e *the corresponding element of* $\mathbf{k}$.

We denote by $\mathscr{A}$ the set of reflecting hyperplanes of W:

$$\mathscr{A} = \{\mathrm{Ker}(s - \mathrm{Id}_V) \mid s \in \mathrm{Ref}(W)\}.$$

There is a surjective W-equivariant map $\mathrm{Ref}(W) \to \mathscr{A}$, $s \mapsto \mathrm{Ker}(s - \mathrm{Id}_V)$. Given X a subset of V, we denote by W_X the pointwise stabilizer of X:

$$W_X = \{w \in W \mid \forall\ x \in X,\ w(x) = x\}.$$

Given $H \in \mathscr{A}$, we denote by e_H the order of the cyclic subgroup W_H of W. We denote by s_H the generator of W_H with determinant ζ_{e_H}. This is a reflection with hyperplane H. We have

$$\operatorname{Ref}(W) = \{s_H^j \mid H \in \mathscr{A} \text{ and } 1 \leqslant j \leqslant e_H - 1\}.$$

The following lemma is clear.

Lemma 2.3.1. — *s_H^j and $s_{H'}^{j'}$ are conjugate in W if and only if H and H' are in the same W-orbit and $j = j'$.*

Given $\aleph$ a W-orbit of hyperplanes of $\mathscr{A}$, we denote by $e_\aleph$ the common value of the e_H for $H \in \aleph$. Lemma 2.3.1 provides a bijection from $\operatorname{Ref}(W)/W$ to the set $\boldsymbol{\aleph}$ of pairs $(\aleph, j)$ where $\aleph \in \mathscr{A}/W$ and $1 \leqslant j \leqslant e_\aleph - 1$.

We denote by $\boldsymbol{\aleph}^\circ$ the set of pairs $(\aleph, j)$ with $\aleph \in \mathscr{A}/W$ and $0 \leqslant j \leqslant e_\aleph - 1$.

Let $V^{\text{reg}} = \{v \in V \mid \operatorname{Stab}_W(v) = 1\}$. Define the discriminant $\delta = \prod_{H \in \mathscr{A}} \alpha_H^{e_H} \in \mathbf{k}[V]^W$. The following result shows that points outside reflecting hyperplanes have trivial stabilizers [**Bro2**, Theorem 4.7].

Theorem 2.3.2 (Steinberg). — *Given $X \subset V$, the group W_X is generated by its reflections. As a consequence, $V^{\text{reg}} = V \setminus \bigcup_{H \in \mathscr{A}} H$ and $\mathbf{k}[V^{\text{reg}}] = k[V][\delta^{-1}]$.*

2.4. Irreducible characters

The rationality property of the reflection representation of W is classical.

Proposition 2.4.1. — *Let $\mathbf{k}'$ be a subfield of $\mathbf{k}$ containing the traces of the elements of W acting on V. Then there exists a $\mathbf{k}'W$-submodule V' of V such that $V = \mathbf{k} \otimes_{\mathbf{k}'} V'$.*

Proof. — Assume first V is irreducible. Let V'' be a simple $\mathbf{k}'W$-module such that $\mathbf{k} \otimes_{\mathbf{k}'} V'' \simeq V^{\oplus m}$ for some integer $m \geqslant 1$. Let $s \in \operatorname{Ref}(W)$. Since s has only one non-trivial eigenvalue on V, it also has only one non-trivial eigenvalue on V''. Let L be the eigenspace of s acting on V'' for the non-trivial eigenvalue. This is an m-dimensional $\mathbf{k}'$-subspace of V'', stable under the action of the division algebra $\operatorname{End}_{\mathbf{k}'W}(V'')$. Since that division algebra has dimension m^2 over $\mathbf{k}'$ and has a module L that has dimension m over $\mathbf{k}'$, we deduce that $m = 1$. The proposition follows by taking for V' the image of V'' by an isomorphism $\mathbf{k} \otimes_{\mathbf{k}'} V'' \xrightarrow{\sim} V$.

Assume now V is arbitrary. Let $V = V^W \oplus \bigoplus_{i=1}^l V_i$ be a decomposition of the $\mathbf{k}W$-module V, where V_i is irreducible for $1 \leqslant i \leqslant l$. Let W_j be the subgroup of W of elements acting trivially on $\bigoplus_{i \neq j} V_i$. The group W_j is a reflection group on V_j. The discussion above shows there is a $\mathbf{k}'W_j$-submodule V_j' of V_j such that $V_j = \mathbf{k}\otimes_{\mathbf{k}'} V_j'$. Let V'' be a $\mathbf{k}'$-submodule of V^W such that $V^W = \mathbf{k}\otimes_{\mathbf{k}'} V''$. Let $V' = V'' \oplus \bigoplus_{j=1}^{l} V_j'$. We have $W = \prod_{i=1}^{l} W_j$ and $V = \mathbf{k}\otimes_{\mathbf{k}'} V'$: this proves the proposition. $\square$

The following rationality property of all representations of complex reflection groups is proven using the classification of those groups [**Ben, Bes**].

***Theorem* 2.4.2 (Benard, Bessis).** — *Let $\mathbf{k}'$ be a subfield of $\mathbf{k}$ containing the traces of the elements of W acting on V. Then the algebra $\mathbf{k}'W$ is split semisimple. In particular, $\mathbf{k}W$ is split semisimple.*

2.5. Hilbert series

2.5.A. Invariants. — The algebra $\mathbf{k}[V\times V^*] = \mathbf{k}[V]\otimes \mathbf{k}[V^*]$ admits a standard bigrading, by giving to the elements of $V^* \subset \mathbf{k}[V]$ the bidegree $(0,1)$ and to those of $V \subset \mathbf{k}[V^*]$ the bidegree $(1,0)$. We clearly have

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V\times V^*]) = \frac{1}{(1-\mathbf{t})^n(1-\mathbf{u})^n}. \tag{2.5.1}$$

Using the notation of Theorem 2.2.1(a), we get also easily that

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V\times V^*]^{W\times W}) = \prod_{i=1}^{n} \frac{1}{(1-\mathbf{t}^{d_i})(1-\mathbf{u}^{d_i})}. \tag{2.5.2}$$

On the other hand, the bigraded Hilbert series of the diagonal invariant algebra $\mathbf{k}[V\times V^*]^{\Delta W}$ is given by a formula *à la Molien*

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V\times V^*]^{\Delta W}) = \frac{1}{|W|}\sum_{w\in W} \frac{1}{\det(1-w\mathbf{t})\,\det(1-w^{-1}\mathbf{u})}, \tag{2.5.3}$$

whose proof is obtained word by word from the proof of the usual Molien formula.

2.5.B. Fake degrees. — We identify $K_0(\mathbf{k}W\text{-modgr})$ with $G_0(\mathbf{k}W)[\mathbf{t},\mathbf{t}^{-1}]$: given $M = \bigoplus_{i\in\mathbb{Z}} M_i$ a finite dimensional $\mathbb{Z}$-graded $\mathbf{k}W$-module, we make the identification

$$[M]_{\mathbf{k}W}^{\mathrm{gr}} = \sum_{i\in\mathbb{Z}} [M_i]_{\mathbf{k}W}\,\mathbf{t}^i.$$

It is clear that $[M]_{\mathbf{k}W}$ is the evaluation at 1 of $[M]_{\mathbf{k}W}^{\mathrm{gr}}$ and that $[M\langle n\rangle]_{\mathbf{k}W}^{\mathrm{gr}} = \mathbf{t}^{-n}\,[M]_{\mathbf{k}W}^{\mathrm{gr}}$. If M is a bigraded $\mathbf{k}W$-module, we define similarly $[M]_{\mathbf{k}W}^{\mathbb{Z}\times\mathbb{Z}}$: it is an element of $K_0(\mathbf{k}W)[\mathbf{t},\mathbf{u},\mathbf{t}^{-1},\mathbf{u}^{-1}]$.

Let $(f_\chi(\mathbf{t}))_{\chi\in\mathrm{Irr}(W)}$ denote the unique family of elements of $\mathbb{N}[\mathbf{t}]$ such that

$$[\,k[V^*]^{\mathrm{co}(W)}\,]_{\mathbf{k}W}^{\mathbb{Z}\times\mathbb{Z}} = \sum_{\chi\in\mathrm{Irr}(W)} f_\chi(\mathbf{t})\,\chi. \tag{2.5.4}$$

Definition 2.5.5. — *The polynomial $f_\chi(\mathbf{t})$ is called the* ***fake degree*** *of χ. Its $\mathbf{t}$-valuation is denoted by $\mathbf{b}_\chi$ and is called the* ***b-invariant*** *of χ.*

The fake degree of χ satisfies

$$f_\chi(1)=\chi(1). \tag{2.5.6}$$

Note that

$$[\,\mathbf{k}[V]^{\mathrm{co}(W)}\,]_{\mathbf{k}W}^{\mathbb{Z}\times\mathbb{Z}} = \sum_{\chi\in\mathrm{Irr}(W)} f_\chi(\mathbf{u})\,\chi^*, \tag{2.5.7}$$

(here, χ^* denotes the dual character of χ, that is, $\chi^*(w)=\chi(w^{-1})$). Note also that, if $\mathbf{1}_W$ denotes the trivial character of W, then

$$[\,\mathbf{k}[V^*]^{\mathrm{co}(W)}\,]_{\mathbf{k}W}^{\mathbb{Z}\times\mathbb{Z}} \equiv \mathbf{1}_W \mod \mathbf{t}K_0(\mathbf{k}W)[\mathbf{t}]$$

and

$$[\,\mathbf{k}[V]^{\mathrm{co}(W)}\,]_{\mathbf{k}W}^{\mathbb{Z}\times\mathbb{Z}} \equiv \mathbf{1}_W \mod \mathbf{u}K_0(\mathbf{k}W)[\mathbf{u}].$$

We deduce:

Lemma 2.5.8. — *The elements $[\,\mathbf{k}[V]^{\mathrm{co}(W)}\,]_{\mathbf{k}W}^{\mathbb{Z}\times\mathbb{Z}}$ and $[\,\mathbf{k}[V^*]^{\mathrm{co}(W)}\,]_{\mathbf{k}W}^{\mathbb{Z}\times\mathbb{Z}}$ are not zero divisors in $K_0(\mathbf{k}W)[\mathbf{t},\mathbf{u},\mathbf{t}^{-1},\mathbf{u}^{-1}]$.*

Remark 2.5.9. — Note that

$$[\,\mathbf{k}[V]^{\mathrm{co}(W)}\,]_{\mathbf{k}W} = [\,\mathbf{k}[V^*]^{\mathrm{co}(W)}\,]_{\mathbf{k}W} = [\,\mathbf{k}W\,]_{\mathbf{k}W} = \sum_{\chi\in\mathrm{Irr}(W)} \chi(1)\chi$$

is a zero divisor in $K_0(\mathbf{k}W)$ (as soon as $W\neq 1$). ■

We can now give another formula for the Hilbert series $\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V\times V^*]^{\Delta W})$:

Proposition 2.5.10. — $\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V\times V^*]^W) = \dfrac{1}{\prod_{i=1}^n(1-\mathbf{t}^{d_i})(1-\mathbf{u}^{d_i})} \displaystyle\sum_{\chi\in\mathrm{Irr}(W)} f_\chi(\mathbf{t})\,f_\chi(\mathbf{u}).$

Proof. — Let $\mathcal{H}$ be a W-stable graded complement to $<\mathbf{k}[V]_+^W>$ in $\mathbf{k}[V]$. Since $\mathbf{k}[V]$ is a free $\mathbf{k}[V]^W$-module, we have isomorphisms of graded $\mathbf{k}[W]$-modules

$$\mathbf{k}[V] \simeq \mathbf{k}[V]^W \otimes \mathcal{H} \qquad \text{and} \qquad \mathbf{k}[V]^{\mathrm{co}(W)} \simeq \mathcal{H}.$$

Similarly, if $\mathcal{H}'$ is a W-stable graded complement of $<\mathbf{k}[V^*]_+^W>$ in $\mathbf{k}[V^*]$, then we have isomorphisms of graded $\mathbf{k}[W]$-modules

$$\mathbf{k}[V^*] \simeq \mathbf{k}[V^*]^W \otimes \mathcal{H}' \qquad \text{and} \qquad \mathbf{k}[V^*]^{\mathrm{co}(W)} \simeq \mathcal{H}'.$$

In other words, we have isomorphisms of graded $\mathbf{k}[W]$-modules

$$\mathbf{k}[V] \simeq \mathbf{k}[V]^W \otimes \mathbf{k}[V]^{\mathrm{co}(W)} \qquad \text{and} \qquad \mathbf{k}[V^*] \simeq \mathbf{k}[V^*]^W \otimes \mathbf{k}[V^*]^{\mathrm{co}(W)}.$$

We deduce an isomorphism of bigraded $\mathbf{k}$-vector spaces

$$(\mathbf{k}[V] \otimes \mathbf{k}[V^*])^{\Delta W} \simeq (\mathbf{k}[V]^W \otimes \mathbf{k}[V^*]^W) \otimes (\mathbf{k}[V]^{\mathrm{co}(W)} \otimes \mathbf{k}[V^*]^{\mathrm{co}(W)})^{\Delta W}.$$

By (2.5.4) and (2.5.7), we have

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V]^{\mathrm{co}(W)} \otimes \mathbf{k}[V^*]^{\mathrm{co}(W)})^{\Delta W} = \sum_{\chi,\psi \in \mathrm{Irr}(W)} f_\chi(\mathbf{t}) f_\psi(\mathbf{u}) \langle \chi\psi^*, \mathbf{1}_W \rangle_W.$$

So the formula follows from the fact that $\langle \chi\psi^*, \mathbf{1}_W\rangle = \langle \chi, \psi \rangle_W$. ☐

To conclude this section, we gather in a same formula Molien's Formula (2.5.3) and Proposition 2.5.10:

$$\begin{aligned} \dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V \times V^*]^{\Delta W}) &= \frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1 - w\mathbf{t})\ \det(1 - w^{-1}\mathbf{u})} \\ &= \frac{1}{\prod_{i=1}^n (1-\mathbf{t}^{d_i})(1-\mathbf{u}^{d_i})} \sum_{\chi \in \mathrm{Irr}(W)} f_\chi(\mathbf{t})\ f_\chi(\mathbf{u}). \end{aligned}$$

2.6. Coxeter groups

Let us recall the following classical equivalences:

Proposition 2.6.1. — *The following assertions are equivalent:*

(1) *There exists a subset S of* $\mathrm{Ref}(W)$ *such that (W,S) is a Coxeter system.*
(2) $V \simeq V^*$ *as* $\mathbf{k}W$*-modules.*
(3) *There exists a W-invariant non-degenerate symmetric bilinear form $V \times V \to \mathbf{k}$.*
(4) *There exists a subfield $\mathbf{k}_{\mathbb{R}}$ of $\mathbf{k}$ and a W-stable $\mathbf{k}_{\mathbb{R}}$-vector subspace $V_{\mathbf{k}_{\mathbb{R}}}$ of V such that $V = \mathbf{k} \otimes_{\mathbf{k}_{\mathbb{R}}} V_{\mathbf{k}_{\mathbb{R}}}$ and $\mathbf{k}_{\mathbb{R}}$ embeds as a subfield of $\mathbb{R}$.*

Whenever one (or all the) assertion(s) of Proposition 2.6.1 is (are) satisfied, we say that W *is a Coxeter group*. In this case, the text will be followed by a gray line on the left, as below.

> **Assumption, choice.** *From now on, and until the end of §2.6, we assume that W is a Coxeter group. We fix a subfield $\mathbf{k}_{\mathbb{R}}$ of $\mathbf{k}$ that embeds as a subfield of $\mathbb{R}$ and a W-stable $\mathbf{k}_{\mathbb{R}}$-vector subspace $V_{\mathbf{k}_{\mathbb{R}}}$ of V such that $V = \mathbf{k} \otimes_{\mathbf{k}_{\mathbb{R}}} V_{\mathbf{k}_{\mathbb{R}}}$. We also fix a connected component $C_{\mathbb{R}}$ of $\{v \in \mathbb{R} \otimes_{\mathbf{k}_{\mathbb{R}}} V_{\mathbf{k}_{\mathbb{R}}} \mid \mathrm{Stab}_W(v) = 1\}$. We denote by S the set of $s \in \mathrm{Ref}(W)$ such that $\overline{C_{\mathbb{R}}} \cap \ker_{\mathbb{R}\otimes_{\mathbf{k}_{\mathbb{R}}} V_{\mathbf{k}_{\mathbb{R}}}}(s-1)$ has real codimension 1 in $\overline{C_{\mathbb{R}}}$. So, (W,S) is a Coxeter system. This notation will be used all along this book, provided that W is a Coxeter group.*

The following is a particular case of Theorem 2.4.2.

Theorem 2.6.2. — *The $\mathbf{k}_{\mathbb{R}}$-algebra $\mathbf{k}_{\mathbb{R}} W$ is split. In particular, the characters of W are real valued, that is, $\chi = \chi^*$ for all character χ of W.*

Recall also the following.

Lemma 2.6.3. — *If $s \in \mathrm{Ref}(W)$, then s has order 2 and $\varepsilon(s) = -1$.*

Corollary 2.6.4. — *The map $\mathrm{Ref}(W) \to \mathscr{A}$, $s \mapsto \mathrm{Ker}(s - \mathrm{Id}_V)$ is bijective and W-equivariant. In particular, $|\mathscr{A}| = |\mathrm{Ref}(W)| = \sum_{i=1}^n (d_i - 1)$ and $|\mathscr{A}/W| = |\mathrm{Ref}(W)/W|$.*

Let $\ell : W \to \mathbb{N}$ denote the length function with respect to S: given $w \in W$, the integer $\ell(w)$ is minimal such that w is a product of $\ell(w)$ elements of S. When $w = s_1 s_2 \cdots s_l$ with $s_i \in S$ and $l = \ell(w)$, we say that $w = s_1 s_2 \cdots s_l$ is a ***reduced decomposition*** of w. We denote by w_0 the longest element of W: we have $\ell(w_0) = |\mathrm{Ref}(W)| = |\mathscr{A}|$.

Remark 2.6.5. — If $-\mathrm{Id}_V \in W$, then $w_0 = -\mathrm{Id}_V$. Conversely, if w_0 is central and $V^W = 0$, then $w_0 = -\mathrm{Id}_V$. ■

CHAPTER 3

GENERIC CHEREDNIK ALGEBRAS

Let $\mathscr{C}$ be the $\mathbf{k}$-vector space of maps $c : \mathrm{Ref}(W) \to \mathbf{k}$, $s \mapsto c_s$ that are constant on conjugacy classes: this is the *space of parameters*, which we identify with the space of maps $\mathrm{Ref}(W)/W \to \mathbf{k}$.

Given $s \in \mathrm{Ref}(W)$ (or $s \in \mathrm{Ref}(W)/W$), we denote by C_s the linear form on $\mathscr{C}$ given by evaluation at s. The algebra $\mathbf{k}[\mathscr{C}]$ of polynomial functions on $\mathscr{C}$ is the algebra of polynomials on the set of indeterminates $(C_s)_{s\in\mathrm{Ref}(W)/W}$:

$$\mathbf{k}[\mathscr{C}] = \mathbf{k}[(C_s)_{s\in\mathrm{Ref}(W)/W}].$$

We denote by $\widetilde{\mathscr{C}}$ the $\mathbf{k}$-vector space $\mathbf{k}\times\mathscr{C}$ and we introduce $T : \widetilde{\mathscr{C}} \to \mathbf{k}$, $(t, c) \mapsto t$. We have $T \in \widetilde{\mathscr{C}}^*$ and

$$\mathbf{k}[\widetilde{\mathscr{C}}] = \mathbf{k}[T, (C_s)_{s\in\mathrm{Ref}(W)/W}].$$

We will use in this chapter results from Appendices A and E.

3.1. Structure

3.1.A. Symplectic action. — We consider here the action of W on $V \oplus V^*$.

Lemma E.1.1 and Proposition E.2.2 give the following result.

Proposition 3.1.1. — *We have* $Z(\mathbf{k}[V\oplus V^*]\rtimes W) = \mathbf{k}[V\oplus V^*]^W = \mathbf{k}[(V\oplus V^*)/W]$ *and there is an isomorphism*

$$Z(\mathbf{k}[V\oplus V^*]\rtimes W) \xrightarrow{\sim} e(\mathbf{k}[V\oplus V^*]\rtimes W)e,\ z \mapsto ze.$$

The action by left multiplication gives an isomorphism

$$\mathbf{k}[V\oplus V^*]\rtimes W \xrightarrow{\sim} \mathrm{End}_{\mathbf{k}[(V\oplus V^*)/W]^{\mathrm{opp}}}\big((\mathbf{k}[V\oplus V^*]\rtimes W)e\big)^{\mathrm{opp}}.$$

3.1.B. Definition. — The *generic rational Cherednik algebra* (or simply the *generic Cherednik algebra*) is the $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebra $\widetilde{\mathbf{H}}$ defined as the quotient of $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \big(\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W\big)$ by the following relations (here, $\mathrm{T}_{\mathbf{k}}(V \oplus V^*)$ is the tensor algebra of $V \oplus V^*$):

$$(3.1.2) \qquad \begin{cases} [x, x'] = [y, y'] = 0, \\ \\ [y, x] = T\langle y, x\rangle + \displaystyle\sum_{s \in \mathrm{Ref}(W)} (\varepsilon(s) - 1)\, C_s\, \frac{\langle y, \alpha_s\rangle \cdot \langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle}\, s, \end{cases}$$

for x, $x' \in V^*$ and y, $y' \in V$.

Remark 3.1.3. — Thanks to (2.1.2), the second relation is equivalent to

$$(3.1.4) \qquad [y, x] = T\langle y, x\rangle + \sum_{s \in \mathrm{Ref}(W)} C_s \langle s(y) - y, x\rangle\, s$$

and to

$$[y, x] = T\langle y, x\rangle + \sum_{s \in \mathrm{Ref}(W)} C_s \langle y, s^{-1}(x) - x\rangle.\, s$$

This avoids the use of α_s and $\alpha_s^\vee$. ■

3.1.C. PBW Decomposition. — Given the relations (3.1.2), the following assertions are clear:

- There is a unique morphism of $\mathbf{k}$-algebras $\mathbf{k}[V] \to \widetilde{\mathbf{H}}$ sending $y \in V^*$ to the class of $y \in \mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$ in $\widetilde{\mathbf{H}}$.
- There is a unique morphism of $\mathbf{k}$-algebras $\mathbf{k}[V^*] \to \widetilde{\mathbf{H}}$ sending $x \in V$ to the class of $x \in \mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$ in $\widetilde{\mathbf{H}}$.
- There is a unique morphism of $\mathbf{k}$-algebras $\mathbf{k}W \to \widetilde{\mathbf{H}}$ sending $w \in W$ to the class of $w \in \mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$ in $\widetilde{\mathbf{H}}$.
- The $\mathbf{k}$-linear map $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*] \longrightarrow \widetilde{\mathbf{H}}$ induced by the three morphisms defined above and the multiplication map is surjective. Note that it is $\mathbf{k}[\widetilde{\mathscr{C}}]$-linear.

The last statement is strengthened by the following fundamental result [**EtGi**, Theorem 1.3], for which we will provide a proof in Theorem 3.1.11.

***Theorem 3.1.5* (Etingof-Ginzburg)**. — *The multiplication map* $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*] \longrightarrow \widetilde{\mathbf{H}}$ *is an isomorphism of* $\mathbf{k}[\widetilde{\mathscr{C}}]$*-modules.*

3.1.D. Specialization. — Given $(t,c) \in \widetilde{\mathscr{C}}$, we denote by $\widetilde{\mathfrak{C}}_{t,c}$ the maximal ideal of $\mathbf{k}[\widetilde{\mathscr{C}}]$ given by $\widetilde{\mathfrak{C}}_{t,c} = \{f \in \mathbf{k}[\widetilde{\mathscr{C}}] \mid f(t,c) = 0\}$: this is the ideal generated by $T-t$ and $(C_s - c_s)_{s \in \mathrm{Ref}(W)/W}$. We put

$$\widetilde{\mathbf{H}}_{t,c} = \mathbf{k}[\widetilde{\mathscr{C}}]/\widetilde{\mathfrak{C}}_{t,c} \otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]} \widetilde{\mathbf{H}} = \widetilde{\mathbf{H}}/\widetilde{\mathfrak{C}}_{t,c}\widetilde{\mathbf{H}}.$$

The $\mathbf{k}$-algebra $\widetilde{\mathbf{H}}_{t,c}$ is the quotient of $\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$ by the ideal generated by the following relations:

$$\begin{cases} [x,x'] = [y,y'] = 0, \\ [y,x] = t\langle y,x\rangle + \displaystyle\sum_{s \in \mathrm{Ref}(W)} (\varepsilon(s)-1)\, c_s\, \frac{\langle y,\alpha_s\rangle \cdot \langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle}\, s, \end{cases} \tag{3.1.6}$$

for $x,\, x' \in V^*$ and $y,\, y' \in V$.

Example 3.1.7. — We have $\widetilde{\mathbf{H}}_{0,0} = \mathbf{k}[V \oplus V^*] \rtimes W$ and $\widetilde{\mathbf{H}}_{T,0} = \mathscr{D}_T(V) \rtimes W$ (see §A.5). ■

More generally, given $\widetilde{\mathfrak{C}}$ a prime ideal of $\mathbf{k}[\widetilde{\mathscr{C}}]$, we put $\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}}) = \widetilde{\mathbf{H}}/\widetilde{\mathfrak{C}}\widetilde{\mathbf{H}}$.

3.1.E. Filtration. — We endow the $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebra $\widetilde{\mathbf{H}}$ with the filtration defined as follows:

- $\widetilde{\mathbf{H}}^{\leqslant -1} = 0$
- $\widetilde{\mathbf{H}}^{\leqslant 0}$ is the $\mathbf{k}[\widetilde{\mathscr{C}}]$-subalgebra generated by V^* and W
- $\widetilde{\mathbf{H}}^{\leqslant 1} = \widetilde{\mathbf{H}}^{\leqslant 0} V + \widetilde{\mathbf{H}}^{\leqslant 0}$.
- $\widetilde{\mathbf{H}}^{\leqslant i} = (\widetilde{\mathbf{H}}^{\leqslant 1})^i$ for $i \geqslant 2$.

Specializing at $(t,c) \in \widetilde{\mathscr{C}}$, we have an induced filtration of $\widetilde{\mathbf{H}}_{t,c}$.

The canonical maps $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V] \rtimes W \to (\mathrm{gr}\,\widetilde{\mathbf{H}})^0$ and $V \to (\mathrm{gr}\,\widetilde{\mathbf{H}})^1$ induce a surjective morphism of algebras $\rho : \mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V \oplus V^*] \rtimes W \twoheadrightarrow \mathrm{gr}\,\widetilde{\mathbf{H}}$.

3.1.F. Localization at V^{reg}. — Recall that

$$V^{\mathrm{reg}} = V \setminus \bigcup_{H \in \mathscr{A}} H = \{v \in V \mid \mathrm{Stab}_G(v) = 1\} \quad \text{and} \quad \mathbf{k}[V^{\mathrm{reg}}] = k[V][\delta^{-1}].$$

We put $\widetilde{\mathbf{H}}^{\mathrm{reg}} = \widetilde{\mathbf{H}}[\delta^{-1}]$, the non-commutative localization of $\widetilde{\mathbf{H}}$ obtained by adding a two-sided inverse to the image of δ. Note that the filtration of $\widetilde{\mathbf{H}}$ induces a filtration of $\widetilde{\mathbf{H}}^{\mathrm{reg}}$, with $(\widetilde{\mathbf{H}}^{\mathrm{reg}})^{\leqslant i} = \widetilde{\mathbf{H}}^{\leqslant i}[\delta^{-1}]$.

Note that multiplication induces an isomorphism of $\mathbf{k}$-vector spaces $\mathbf{k}[V^{\mathrm{reg}}] \otimes \mathbf{k}[V^*] \xrightarrow{\sim} \mathscr{D}(V^{\mathrm{reg}}) = \mathscr{D}(V)[\delta^{-1}]$ (cf Appendix §A.5).

Lemma 3.1.8. — *We have:*

(a) *There is a Morita equivalence between* $\mathbf{k}[V^{\mathrm{reg}} \times V^*] \rtimes W$ *and* $\mathbf{k}[V^{\mathrm{reg}} \times V^*]^W$ *given by the bimodule* $\mathbf{k}[V^{\mathrm{reg}} \times V^*]$.

(b) *There is a Morita equivalence between* $\mathscr{D}(V^{\mathrm{reg}}) \rtimes W$ *and* $\mathscr{D}(V^{\mathrm{reg}})^W = \mathscr{D}(V^{\mathrm{reg}}/W)$ *given by the bimodule* $\mathscr{D}(V^{\mathrm{reg}})$.

(c) *The action of* $\mathscr{D}(V^{\mathrm{reg}}) \rtimes W$ *on* $\mathbf{k}[V^{\mathrm{reg}}]$ *is faithful.*

Proof. — (a) follows from Corollary E.2.1.

(b) becomes (a) after taking associated graded, hence (b) follows from Lemmas A.3.4 and E.1.1.

(c) It follows from (b) that every two-sided ideal of $\mathscr{D}(V^{\mathrm{reg}}) \rtimes W$ is generated by its intersection with $\mathscr{D}(V^{\mathrm{reg}})^W$. Since $\mathscr{D}(V^{\mathrm{reg}})$ acts faithfully on $\mathbf{k}[V^{\mathrm{reg}}]$ (cf §A.5), we deduce that the kernel of the action of $\mathscr{D}(V^{\mathrm{reg}}) \rtimes W$ vanishes. □

3.1.G. Polynomial representation and Dunkl operators. — Given $y \in V$, we define D_y, a $\mathbf{k}[\widetilde{\mathscr{C}}]$-linear endomorphism of $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V^{\mathrm{reg}}]$ by

$$D_y = T\partial_y - \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) C_s \langle y, \alpha_s \rangle \alpha_s^{-1} s.$$

Note that $D_y \in \mathbf{k}[\mathscr{C}] \otimes \mathscr{D}_T(V^{\mathrm{reg}}) \rtimes W \subset \mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathscr{D}(V^{\mathrm{reg}}) \rtimes W$.

Remark 3.1.9. — The Dunkl operators are traditionally defined as

$$T\partial_y - \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) C_s \langle y, \alpha_s \rangle \alpha_s^{-1}(s-1).$$

With this definition they preserve $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V]$. The results and proofs in this section apply also to those operators. ■

Proposition 3.1.10. — *There is a unique structure of* $\widetilde{\mathbf{H}}$*-module on* $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V^{\mathrm{reg}}]$ *where* $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V^{\mathrm{reg}}]$ *acts by multiplication,* W *acts through its natural action on* V *and* $y \in V$ *acts by* D_y.

Proof. — The following argument is due to Etingof. Let $y \in V$ and $x \in V^*$. We have

$$[\alpha_s^{-1} s, x] = (\varepsilon(s)^{-1} - 1) \frac{\langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s,$$

hence

$$[D_y, x] = T\langle y, x \rangle + \sum_s (\varepsilon(s) - 1) C_s \frac{\langle y, \alpha_s \rangle \cdot \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s.$$

Given $w \in W$, we have $w D_y w^{-1} = D_{w(y)}$.

Consider $y' \in V$. We have

$$[[D_y, D_{y'}], x] = [[D_y, x], D_{y'}] - [[D_{y'}, x], D_y]$$

and

$$
\begin{aligned}
[[D_y,x],D_{y'}] &= \sum_s (\varepsilon(s)-1)C_s \frac{\langle y,\alpha_s\rangle\cdot\langle\alpha_s^\vee,x\rangle}{\langle\alpha_s^\vee,\alpha_s\rangle}[s,D_{y'}] \\
&= \sum_s (\varepsilon(s)-1)^2 C_s \frac{\langle y,\alpha_s\rangle\cdot\langle y',\alpha_s\rangle\cdot\langle\alpha_s^\vee,x\rangle}{\langle\alpha_s^\vee,\alpha_s\rangle^2} D_{\alpha_s^\vee} s \\
&= [[D_{y'},x],D_y].
\end{aligned}
$$

We deduce that $[[D_y,D_{y'}],x]=0$ for all $x\in V^*$. On the other hand, $[D_y,D_{y'}]$ acts by zero on $1\in\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V^{\mathrm{reg}}]$, hence $[D_y,D_{y'}]$ acts by zero on $\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V^{\mathrm{reg}}]$. It follows from Lemma 3.1.8(c) that $[D_y,D_{y'}]=0$. The proposition follows. □

Proposition 3.1.10 provides a morphism of $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebras $\Theta:\widetilde{\mathbf{H}}\to\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})$. We denote by $\Theta^{\mathrm{reg}}:\widetilde{\mathbf{H}}^{\mathrm{reg}}\to\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})$ its extension to $\widetilde{\mathbf{H}}^{\mathrm{reg}}$.

Theorem 3.1.11. — *We have the following statements:*

(a) *The morphism Θ is injective, hence the polynomial representation of $\widetilde{\mathbf{H}}$ is faithful.*
(b) *The multiplication map is an isomorphism $\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W\xrightarrow{\sim}\widetilde{\mathbf{H}}$.*
(c) *We have an isomorphism of algebras $\rho:\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V\oplus V^*]\rtimes W\xrightarrow{\sim}\operatorname{gr}\widetilde{\mathbf{H}}$.*
(d) *The morphism Θ^{reg} is an isomorphism $\widetilde{\mathbf{H}}^{\mathrm{reg}}\xrightarrow{\sim}\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W$.*
(e) *Given $\mathfrak{c}$ a prime ideal of $\mathbf{k}[\widetilde{\mathscr{C}}]$, the morphism $(\mathbf{k}[\widetilde{\mathscr{C}}]/\mathfrak{c})\otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]}\Theta$ is injective. If $T\notin\mathfrak{c}$, then the polynomial representation of $\widetilde{\mathbf{H}}(\mathfrak{c})$ is faithful and $Z(\widetilde{\mathbf{H}}(\mathfrak{c}))=\mathbf{k}[\widetilde{\mathscr{C}}]/\mathfrak{c}$.*

Proof. — Let η be the composition

$$\eta:\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V^{\mathrm{reg}}]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W\xrightarrow{\text{mult}}\widetilde{\mathbf{H}}^{\mathrm{reg}}\xrightarrow{\Theta^{\mathrm{reg}}}\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W.$$

Note that $\operatorname{gr}\eta$ is an isomorphism, since it is equal to the graded map associated to the multiplication isomorphism

$$\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V^{\mathrm{reg}}]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W\xrightarrow{\sim}\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W.$$

We deduce that η is an isomorphism (Lemma A.3.1). Since the multiplication map is surjective, it follows that it is an isomorphism and Θ^{reg} is an isomorphism as well. We deduce also that ρ is injective, hence it is an isomorphism.

Since $\mathbf{k}[T]\otimes\mathbf{k}[V^{\mathrm{reg}}]$ is a faithful representation of $\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W$ (Lemma 3.1.8), we deduce that the polynomial representation induces an injective map

$$\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W\hookrightarrow\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\operatorname{End}_{\mathbf{k}}(\mathbf{k}[V^{\mathrm{reg}}]).$$

There is a commutative diagram

$$\begin{tikzcd}
\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W \arrow[r, two heads, "\text{mult}"] \arrow[d, hook] & \widetilde{\mathbf{H}} \arrow[r, "\text{pol. rep.}"] \arrow[d, "\text{can}"'] & \mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathrm{End}_{\mathbf{k}}(\mathbf{k}[V^{\mathrm{reg}}]) \\
\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V^{\mathrm{reg}}]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W \arrow[r, two heads, "\text{mult}"'] \arrow[rr, bend right, "\eta", "\sim"'] & \widetilde{\mathbf{H}}^{\mathrm{reg}} \arrow[r, "\Theta^{\mathrm{reg}}"'] & \mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W \arrow[u, "\text{pol. rep.}"']
\end{tikzcd}$$

It follows that the multiplication

$$\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W\to\widetilde{\mathbf{H}}$$

is an isomorphism and the polynomial representation of $\widetilde{\mathbf{H}}$ is faithful.

Consider now $\mathfrak{c}$ a prime ideal of $\widetilde{\mathscr{C}}$ and let $A=\mathbf{k}[\widetilde{\mathscr{C}}]/\mathfrak{c}$. There is a commutative diagram

$$\begin{tikzcd}
A\otimes\mathbf{k}[V]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W \arrow[r, two heads, "\text{mult}"] \arrow[d, hook] & \widetilde{\mathbf{H}}(\mathfrak{c}) \arrow[r, "\text{pol. rep.}"] \arrow[d, "\text{can}"'] & A\otimes\mathrm{End}_{\mathbf{k}}(\mathbf{k}[V^{\mathrm{reg}}]) \\
A\otimes\mathbf{k}[V^{\mathrm{reg}}]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W \arrow[r, "\sim", "\text{mult}"'] \arrow[rr, bend right, "\eta", "\sim"'] & \widetilde{\mathbf{H}}^{\mathrm{reg}}(\mathfrak{c}) \arrow[r, "\sim", "\Theta^{\mathrm{reg}}"'] & A\otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]}(\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}}))\rtimes W \arrow[u, "\text{pol. rep.}"']
\end{tikzcd}$$

We deduce as above that $(\mathbf{k}[\widetilde{\mathscr{C}}]/\mathfrak{c})\otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]}\Theta$ is injective. Assume now $T\notin\mathfrak{c}$. Then the polynomial representation of $A\otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]}(\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}}))\rtimes W$ is faithful, hence the polynomial representation of $\widetilde{\mathbf{H}}(\mathfrak{c})$ is faithful as well. Since $Z(\mathscr{D}(V^{\mathrm{reg}}))=\mathbf{k}$, we deduce that $Z(A\otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]}(\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}}))\rtimes W)=A$, hence $Z(\widetilde{\mathbf{H}}(\mathfrak{c}))=\mathbf{k}[\widetilde{\mathscr{C}}]/\mathfrak{c}$. $\square$

Corollary 3.1.12. — *Given $f\in\mathbf{k}[V]$, we have*

$$[y,f]=T\partial_y(f)-\sum_{s\in\mathrm{Ref}(W)}\varepsilon(s)C_s\langle y,\alpha_s\rangle\frac{s(f)-f}{\alpha_s}s.$$

Proof. — The result follows from Proposition 3.1.10 and Theorem 3.1.11. Note that the corollary can also be proven directly by induction on the degree of f. $\square$

3.1.H. Hyperplanes and parameters. — Let $\mathscr{K}$ be the $\mathbf{k}$-vector space of maps $k:\aleph^\circ\to\mathbf{k}$ such that for all $\aleph\in\mathscr{A}/W$, we have $\sum_{j=0}^{e_\aleph-1}k_{\aleph,j}=0$. Let $(K_{\aleph,j})_{(\aleph,j)\in\aleph^\circ}$ be the canonical basis of $\mathbf{k}^{\aleph^\circ}$. We put $K_{H,j}=K_{\aleph,j}$, where $\aleph$ is the W-orbit of H.

There is an isomorphism of $\mathbf{k}$-vector spaces

$$\mathscr{C}^*\xrightarrow{\sim}\mathscr{K}^*,\ C_{s_H{}^i}\mapsto\sum_{j=0}^{e_H-1}\zeta_{e_H}^{i(j-1)}K_{H,j}.$$

The surjectivity is a consequence of the invertibility of the Vandermonde determinant. We denote the dual of that isomorphism by $\kappa : \mathscr{K} \xrightarrow{\sim} \mathscr{C}$. We will often identify $\mathscr{C}$ and $\mathscr{K}$ via the isomorphism κ. Note that the canonical bases of $\mathscr{C}$ and $\mathscr{K}$ provide them with $\mathbb{Q}$-forms (and $\mathbb{Z}$-forms). Unless all reflections of W have order 2, the isomorphism κ is not compatible with the $\mathbb{Q}$-forms.

Note that

$$\sum_{w \in W_H} \varepsilon(w) C_w w = e_H \sum_{j=0}^{e_H-1} \varepsilon_{H,j} K_{H,j}$$

where $\varepsilon_{H,i} = e_H^{-1} \sum_{w \in W_H} \varepsilon(w)^i w$ and

(3.1.13) $$\sum_{s \in \mathrm{Ref}(W)} \varepsilon(s)\, C_s = \sum_{H \in \mathscr{A}} e_H K_{H,0} = -\sum_{H \in \mathscr{A}} \sum_{i=1}^{e_H-1} e_H K_{H,i}.$$

Given $H \in \mathscr{A}$, denote by $\alpha_H \in V^*$ a linear form such that $H = \mathrm{Ker}(\alpha_H)$ and let $\alpha_H^\vee \in V$ such that $V = H \oplus \mathbf{k}\alpha_H^\vee$ and $\mathbf{k}\alpha_H^\vee$ is stable under W_H. Via κ, we can view $\widetilde{\mathbf{H}}$ as a $\mathbf{k}[\mathscr{K}]$-algebra and the second relation in (3.1.2) becomes

(3.1.14) $$[y,x] = T\langle y,x\rangle + \sum_{H \in \mathscr{A}} \sum_{i=0}^{e_H-1} e_H(K_{H,i} - K_{H,i+1}) \frac{\langle y,\alpha_H\rangle \cdot \langle \alpha_H^\vee, x\rangle}{\langle \alpha_H^\vee, \alpha_H\rangle} \varepsilon_{H,i}$$

for $x \in V^*$ and $y \in V$, where $K_{H,e_H} = K_{H,0}$.

Given $y \in V$, we have

$$\Theta(y) = \partial_y - \sum_{H \in \mathscr{A}} \sum_{i=0}^{e_H-1} \frac{\langle y,\alpha_H\rangle}{\alpha_H} e_H K_{H,i} \varepsilon_{H,i}.$$

COMMENT - Our convention for the definition of Cherednik algebras differs from that of [**GGOR**, §3.1]: we have added a coefficient $\varepsilon(s)-1$ in front of the term C_s. On the other hand, our convention is the same as [**EtGi**, §1.15], with $c_s = c_{\alpha_s}$ (when W is a Coxeter group). Note that the $\mathrm{k}_{H,i}$'s from [**GGOR**] are related to the $K_{H,i}$'s above by the relation $\mathrm{k}_{H,i} = K_{H,0} - K_{H,i}$. ■

Remark 3.1.15. — The endomorphism $K_{\aleph,j} \mapsto K_{\aleph,j} - \frac{1}{e_H}\sum_{j'=0}^{e_\aleph-1} K_{\aleph,j'}$ of $\mathbf{k}^{\aleph^\circ}$ induces an injection $\mathscr{K}^* \hookrightarrow \mathbf{k}^{\aleph^\circ}$. The dual map $\mathrm{sec} : \mathbf{k}^{\aleph^\circ} \twoheadrightarrow \mathscr{K}$ provides a section to the inclusion of $\mathscr{K}$ in $\mathbf{k}^{\aleph^\circ}$. It is defined over $\mathbb{Q}$. ■

3.2. Gradings

The algebra $\widetilde{\mathbf{H}}$ admits a natural $(\mathbb{N}\times\mathbb{N})$-grading, thanks to which we can associate, to each morphism of monoids $\mathbb{N}\times\mathbb{N}\to\mathbb{Z}$ (or $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$), a $\mathbb{Z}$-grading (or an $\mathbb{N}$-grading).

We endow the extended tensor algebra $\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\big(\mathrm{T}_{\mathbf{k}}(V\oplus V^*)\rtimes W\big)$ with an $(\mathbb{N}\times\mathbb{N})$-grading by giving to the elements of V the bidegree $(1,0)$, to the elements of V^* the bidegree $(0,1)$, to the elements of $\widetilde{\mathscr{C}}^*$ the bidegree $(1,1)$ and to those of W the bidegree $(0,0)$. The relations (3.1.2) are homogeneous. Hence, $\widetilde{\mathbf{H}}$ inherits an $(\mathbb{N}\times\mathbb{N})$-grading whose homogeneous component of bidegree (i,j) will be denoted by $\widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[i,j]$. We have

$$\widetilde{\mathbf{H}}=\bigoplus_{(i,j)\in\mathbb{N}\times\mathbb{N}}\widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[i,j]\quad\text{and}\quad\widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[0,0]=\mathbf{k}W.$$

Note that all homogeneous components have finite dimension over $\mathbf{k}$.

If $\varphi:\mathbb{N}\times\mathbb{N}\to\mathbb{Z}$ is a morphism of monoids, then $\widetilde{\mathbf{H}}$ inherits a $\mathbb{Z}$-grading whose homogeneous component of degree i will be denoted by $\widetilde{\mathbf{H}}^{\varphi}[i]$:

$$\widetilde{\mathbf{H}}^{\varphi}[i]=\bigoplus_{\varphi(a,b)=i}\widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[a,b].$$

In this grading, the elements of V have degree $\varphi(1,0)$, the elements of V^* have degree $\varphi(0,1)$, the elements of $\widetilde{\mathscr{C}}^*$ have degree $\varphi(1,1)$ and those of W have degree 0.

***Example 3.2.1* ($\mathbb{Z}$-grading).** — The morphism of monoids $\mathbb{N}\times\mathbb{N}\to\mathbb{Z}$, $(i,j)\mapsto j-i$ induces a $\mathbb{Z}$-grading on $\widetilde{\mathbf{H}}$ for which the elements of V have degree -1, the elements of V^* have degree 1 and the elements of $\widetilde{\mathscr{C}}^*$ and W have degree 0. We denote by $\widetilde{\mathbf{H}}^{\mathbb{Z}}[i]$ the homogeneous component of degree i. Then

$$\widetilde{\mathbf{H}}=\bigoplus_{i\in\mathbb{Z}}\widetilde{\mathbf{H}}^{\mathbb{Z}}[i].$$

By specialization at $(t,c)\in\widetilde{\mathscr{C}}$, the algebra $\widetilde{\mathbf{H}}_{t,c}$ inherits a $\mathbb{Z}$-grading whose homogeneous component of degree i will be denoted by $\widetilde{\mathbf{H}}^{\mathbb{Z}}_{t,c}[i]$.

Let w_z be a generator of $W\cap Z(\mathrm{GL}_{\mathbf{k}}(V))$ and let z be its order. We have $w_z=\zeta^{-1}\,\mathrm{Id}_V$ for some root of unity ζ of order z of $\mathbf{k}$. When $\mathbf{k}$ is a subfield of $\mathbb{C}$, we take $w_z=e^{2i\pi/z}\,\mathrm{Id}_V$. Note that $z=\gcd(d_1,\dots,d_n)$ (cf Theorem 2.2.1). Given $h\in\widetilde{\mathbf{H}}^{\mathbb{Z}}[i]$, we have $w_z h w_z^{-1}=\zeta^i h$. So, the $(\mathbb{Z}/d\mathbb{Z})$-grading on $\widetilde{\mathbf{H}}$ deduced from the $\mathbb{Z}$-grading is given by an inner automorphism of $\widetilde{\mathbf{H}}$. ■

Example 3.2.2 (**$\mathbb{N}$-grading**). — The morphism of monoids $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, $(i,j)\mapsto i+j$ induces an $\mathbb{N}$-grading on $\widetilde{\mathbf{H}}$ for which the elements of V or V^* have degree 1, the elements of $\widetilde{\mathscr{C}}^*$ have degree 2 and the elements of W have degree 0. We denote by $\widetilde{\mathbf{H}}^{\mathbb{N}}[i]$ the homogeneous component of degree i. Then

$$\widetilde{\mathbf{H}} = \bigoplus_{i\in\mathbb{N}} \widetilde{\mathbf{H}}^{\mathbb{N}}[i] \quad \text{and} \quad \widetilde{\mathbf{H}}^{\mathbb{N}}[0] = \mathbf{k}W.$$

Note that $\dim_{\mathbf{k}} \widetilde{\mathbf{H}}^{\mathbb{N}}[i] < \infty$ for all i. This grading is not inherited after specialization at $(t,c)\in\widetilde{\mathscr{C}}$, except whenever $(t,c)=(0,0)$: we retrieve the usual $\mathbb{N}$-grading on $\widetilde{\mathbf{H}}_{0,0} = \mathbf{k}[V\times V^*]\rtimes W$ (see Example 3.1.7). ■

3.3. Euler element

Let $(x_1,\dots,x_n)$ be a $\mathbf{k}$-basis of V^* and let $(y_1,\dots,y_n)$ be its dual basis. We define the *generic Euler element* of $\widetilde{\mathbf{H}}$

$$\widetilde{\mathbf{eu}} = -nT + \sum_{i=1}^{n} y_i x_i + \sum_{s\in\mathrm{Ref}(W)} C_s s \in \widetilde{\mathbf{H}}.$$

Note that

$$\widetilde{\mathbf{eu}} = \sum_{i=1}^{n} x_i y_i + \sum_{s\in\mathrm{Ref}(W)} \varepsilon(s) C_s s = \sum_{i=1}^{n} x_i y_i + \sum_{H\in\mathscr{A}} \sum_{j=0}^{e_H-1} e_H \; K_{H,j} \varepsilon_{H,j}.$$

It is easy to check that $\widetilde{\mathbf{eu}}$ does not depend on the choice of the basis $(x_1,\dots,x_n)$ of V^*. Note that

$$\widetilde{\mathbf{eu}} \in \widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[1,1]. \tag{3.3.1}$$

We have

$$\Theta(\widetilde{\mathbf{eu}}) = T\sum_{i=1}^{n} y_i x_i$$

Thanks to Theorem 3.1.11, we deduce the following result [**GGOR**, §3.1(4)].

Proposition 3.3.2. — *If $x\in V^*$, $y\in V$ and $w\in W$, then*

$$[\widetilde{\mathbf{eu}}, x] = Tx, \qquad [\widetilde{\mathbf{eu}}, y] = -Ty \qquad \textit{and} \qquad [\widetilde{\mathbf{eu}}, w] = 0.$$

In [**GGOR**], the Euler element plays a fundamental role in the study of the category $\mathscr{O}$ associated with $\widetilde{\mathbf{H}}_{1,c}$. We will see in this book the role it plays in the theory of Calogero-Moser cells.

Proposition 3.3.3. — *If $h\in\widetilde{\mathbf{H}}^{\mathbb{Z}}[i]$, then $[\widetilde{\mathbf{eu}}, h] = iTh$.*

3.4. Spherical algebra

> **Notation.** *All along this book, we denote by e the primitive central idempotent of $\mathbf{k}W$ defined by*
> $$e = \frac{1}{|W|}\sum_{w\in W} w.$$
> *The $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebra $e\widetilde{\mathbf{H}}e$ will be called the **generic spherical algebra**.*

By specializing at (t, c), and since $e\widetilde{\mathbf{H}}e$ is a direct summand of the $\mathbf{k}[\widetilde{\mathscr{C}}]$-module $\widetilde{\mathbf{H}}$, we get

$$e\widetilde{\mathbf{H}}_{t,c}e = (\mathbf{k}[\widetilde{\mathscr{C}}]/\widetilde{\mathfrak{C}}_{t,c}) \otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]} e\widetilde{\mathbf{H}}e. \tag{3.4.1}$$

Since e has degree 0, the filtration of $\widetilde{\mathbf{H}}$ induces a filtration of the generic spherical algebra given by $(e\widetilde{\mathbf{H}}e)^{\leqslant i} = e(\widetilde{\mathbf{H}}^{\leqslant i})e$. It follows from Theorem 3.1.11 that

$$\operatorname{gr}(e\widetilde{\mathbf{H}}e) = e\operatorname{gr}(\widetilde{\mathbf{H}})e \simeq \mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V \times V^*]^{\Delta W}. \tag{3.4.2}$$

***Theorem 3.4.3* (Etingof-Ginzburg).** — *Let $\widetilde{\mathfrak{C}}$ be a prime ideal of $\mathbf{k}[\widetilde{\mathscr{C}}]$.*

(a) *The algebra $e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$ is a finitely generated $\mathbf{k}$-algebra without zero divisors.*
(b) *$\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$ is a finitely generated right $e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$-module.*
(c) *Left multiplication of $\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})$ on the projective module $\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$ induces an isomorphism $\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}}) \xrightarrow{\sim} \operatorname{End}_{(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}$.*
(d) *There is an isomorphism of algebras $Z(\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})) \xrightarrow{\sim} Z(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)$, $z \mapsto ze$.*

Proof. — The assertion (a) follows from Lemmas A.3.2 and A.2.1. The assertion (b) follows from Lemma A.3.2.

Let $\alpha : \widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}}) \to \operatorname{End}_{(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}$ be the morphism of the theorem. Lemma A.3.3 provides an injective morphism

$$\beta : \operatorname{gr}\operatorname{End}_{(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}} \hookrightarrow \operatorname{End}_{\operatorname{gr}(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\operatorname{gr}\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}.$$

The composition

$$\operatorname{gr}\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}}) \xrightarrow{\operatorname{gr}\alpha} \operatorname{gr}\operatorname{End}_{(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}} \xrightarrow{\beta} \operatorname{End}_{\operatorname{gr}(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\operatorname{gr}\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}$$

is given by the left multiplication action. Via the isomorphism ρ (Theorem 3.1.11), it corresponds to the morphism given by left multiplication

$$\gamma : \mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V \oplus V^*] \rtimes W \to \operatorname{End}_{\mathbf{k}[\widetilde{\mathscr{C}}]\otimes(e(\mathbf{k}[V\oplus V^*]\rtimes W)e)^{\mathrm{opp}}}(\mathbf{k}[\widetilde{\mathscr{C}}] \otimes (\mathbf{k}[V \oplus V^*] \rtimes W)e)^{\mathrm{opp}}.$$

Since the codimension of $(V \times (V^* \setminus (V^*)^{\mathrm{reg}})) \cup ((V \setminus V^{\mathrm{reg}}) \times V^*)$ in $V \times V^*$ is $\geqslant 2$, it follows from Proposition E.2.2 that γ is an isomorphism. So, $\operatorname{gr}\alpha$ is an isomorphism, hence α is an isomorphism by Lemma A.3.1.

The assertion (d) follows from (c) by Lemma E.1.4. □

Remark 3.4.4. — It can actually be shown [**EtGi**, Theorem 1.5] that if $\mathbf{k}[\widetilde{\mathscr{C}}]/\widetilde{\mathfrak{C}}$ is Gorenstein (respectively Cohen-Macaulay), then so is the algebra $e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$ as well as the right $e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$-module $\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$. ■

3.5. Some automorphisms of $\widetilde{\mathbf{H}}$

Let $\mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widetilde{\mathbf{H}})$ denote the group of automorphisms of the $\mathbf{k}$-algebra $\widetilde{\mathbf{H}}$.

3.5.A. Bigrading. — The bigrading on $\widetilde{\mathbf{H}}$ can be seen as an action of the algebraic group $\mathbf{k}^\times \times \mathbf{k}^\times$ on $\widetilde{\mathbf{H}}$. Indeed, if $(\xi,\xi')\in \mathbf{k}^\times\times\mathbf{k}^\times$, we define the automorphism $\mathrm{bigr}_{\xi,\xi'}$ of $\widetilde{\mathbf{H}}$ by the following formula:

$$\forall\ (i,j)\in\mathbb{N}\times\mathbb{N},\ \forall\ h\in\widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[i,j],\ \mathrm{bigr}_{\xi,\xi'}(h)=\xi^i\xi'^j h.$$

Then

$$\mathrm{bigr}:\mathbf{k}^\times\times\mathbf{k}^\times\longrightarrow \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widetilde{\mathbf{H}}) \tag{3.5.1}$$

is a morphism of groups. Concretely,

$$\begin{cases}\forall\ y\in V,\ \mathrm{bigr}_{\xi,\xi'}(y)=\xi y,\\ \forall\ x\in V^*,\ \mathrm{bigr}_{\xi,\xi'}(x)=\xi' x,\\ \forall\ C\in\widetilde{\mathscr{C}}^*,\ \mathrm{bigr}_{\xi,\xi'}(C)=\xi\xi' C,\\ \forall\ w\in W,\ \mathrm{bigr}_{\xi,\xi'}(w)=w.\end{cases}$$

After specialization, for all $\xi\in\mathbf{k}^\times$ and all $(t,c)\in\widetilde{\mathscr{C}}$, the action of $(\xi,1)$ induces an isomorphism of $\mathbf{k}$-algebras

$$\widetilde{\mathbf{H}}_{t,c}\xrightarrow{\sim}\widetilde{\mathbf{H}}_{\xi t,\xi c}. \tag{3.5.2}$$

3.5.B. Linear characters. — Let $\gamma : W\longrightarrow\mathbf{k}^\times$ be a linear character. It provides an automorphism of $\mathscr{C}$ by multiplication: given $c\in\mathscr{C}$, we define $\gamma\cdot c$ as the map $\mathrm{Ref}(W)\to\mathbf{k}$, $s\mapsto\gamma(s)c_s$. This induces an automorphism $\gamma_{\mathscr{C}}:\mathbf{k}[\mathscr{C}]\to\mathbf{k}[\mathscr{C}]$, $f\mapsto(c\mapsto f(\gamma^{-1}\cdot c))$ sending C_s on $\gamma(s)^{-1}C_s$. It extends to an automorphism $\gamma_{\widetilde{\mathscr{C}}}$ of $\mathbf{k}[\widetilde{\mathscr{C}}]$ by setting $\gamma_{\widetilde{\mathscr{C}}}(T)=T$.

On the other hand, γ induces also an automorphism of the group algebra $\mathbf{k}W$ given by $W\ni w\mapsto\gamma(w)w$. Hence, γ induces an automorphism of the $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebra $\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\big(\mathrm{T}_{\mathbf{k}}(V\oplus V^*)\rtimes W\big)$ acting trivially on V and V^*: it will be denoted by γ_{T}. Of course,

$$(\gamma\gamma')_{\mathrm{T}}=\gamma_{\mathrm{T}}\gamma'_{\mathrm{T}}.$$

Since the relations (3.1.2) are stable by the action of γ_{T}, it follows that γ_{T} induces an automorphism γ_* of the $\mathbf{k}$-algebra $\widetilde{\mathbf{H}}$. The map

$$\begin{array}{ccc} W^\wedge & \longrightarrow & \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widetilde{\mathbf{H}}) \\ \gamma & \longmapsto & \gamma_* \end{array} \tag{3.5.3}$$

is an injective morphism of groups. Given $(t,c) \in \widetilde{\mathscr{C}}$ and $\gamma \in W^\wedge$, then γ_* induces an isomorphism of $\mathbf{k}$-algebras

$$\widetilde{\mathbf{H}}_{t,c} \xrightarrow{\sim} \widetilde{\mathbf{H}}_{t,\gamma\cdot c}. \tag{3.5.4}$$

3.5.C. Normalizer. — Let $\mathscr{N}$ denote the normalizer in $\mathrm{GL}_{\mathbf{k}}(V)$ of W. Then:

- $\mathscr{N}$ acts naturally on V and V^*;
- $\mathscr{N}$ acts on W by conjugacy;
- The action of $\mathscr{N}$ on W stabilizes $\mathrm{Ref}(W)$ and so $\mathscr{N}$ acts on $\mathscr{C}$: if $g \in \mathscr{N}$ and $c \in \mathscr{C}$, then ${}^g c : \mathrm{Ref}(W) \to \mathbf{k}$, $s \mapsto c_{g^{-1}sg}$.
- The action of $\mathscr{N}$ on $\mathscr{C}$ induces an action of $\mathscr{N}$ on $\mathscr{C}^*$ (and so on $\mathbf{k}[\mathscr{C}]$) such that, if $g \in \mathscr{N}$ and $s \in \mathrm{Ref}(W)$, then ${}^g C_s = C_{gsg^{-1}}$.
- $\mathscr{N}$ acts trivially on T.

Consequently, $\mathscr{N}$ acts on the $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebra $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \big(\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W\big)$ and it is easily checked, thanks to the relations (3.1.2), that this action induces an action on $\widetilde{\mathbf{H}}$: if $g \in \mathscr{N}$ and $h \in \widetilde{\mathbf{H}}$, we denote by ${}^g h$ the image of h under the action of g. By specialization at $(t,c) \in \widetilde{\mathscr{C}}$, an element $g \in \mathscr{N}$ induces an isomorphism of $\mathbf{k}$-algebras

$$\widetilde{\mathbf{H}}_{t,c} \xrightarrow{\sim} \widetilde{\mathbf{H}}_{t,{}^g c}. \tag{3.5.5}$$

Example 3.5.6. — If $\xi \in \mathbf{k}^\times$, then we can see ξ as an automorphism of V (by scalar multiplication) normalizing (and even centralizing) W. We then recover the automorphism of $\widetilde{\mathbf{H}}$ inducing the $\mathbb{Z}$-grading (up to a sign): if $h \in \widetilde{\mathbf{H}}$, then ${}^\xi h = \mathrm{bigr}_{\xi,\xi^{-1}}(h)$. ■

3.5.D. Compatibilities. — The automorphisms induced by $\mathbf{k}^\times \times \mathbf{k}^\times$ and those induced by $W^\wedge$ commute. On the other hand, the group $\mathscr{N}$ acts on the group $W^\wedge$ and on the $\mathbf{k}$-algebra $\widetilde{\mathbf{H}}$. This induces an action of $W^\wedge \rtimes \mathscr{N}$ on $\widetilde{\mathbf{H}}$ preserving the bigrading, that is, commuting with the action of $\mathbf{k}^\times \times \mathbf{k}^\times$. Given $\gamma \in W^\wedge$ and $g \in \mathscr{N}$, we will denote by $\gamma \rtimes g$ the corresponding element of $W^\wedge \rtimes \mathscr{N}$. We have a morphism of groups

$$\begin{array}{ccc} \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N}) & \longrightarrow & \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widetilde{\mathbf{H}}) \\ (\xi,\xi',\gamma \rtimes g) & \longmapsto & (h \mapsto \mathrm{bigr}_{\xi,\xi'} \circ \gamma_*({}^g h)). \end{array}$$

Given $\tau = (\xi,\xi',\gamma \rtimes g) \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ and $h \in \widetilde{\mathbf{H}}$, we set

$${}^\tau h = \mathrm{bigr}_{\xi,\xi'}\big(\gamma_*({}^g h)\big).$$

The following lemma is immediate.

Lemma 3.5.7. — *Let* $\tau = (\xi, \xi', \gamma \rtimes g) \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathcal{N})$. *Then:*

(a) τ *stabilizes the subalgebras* $\mathbf{k}[\widetilde{\mathscr{C}}]$, $\mathbf{k}[V]$, $\mathbf{k}[V^*]$ *and* $\mathbf{k}W$.

(b) τ *preserves the bigrading.*

(c) ${}^\tau\widetilde{\mathbf{eu}} = \xi\xi'\widetilde{\mathbf{eu}}$.

(d) ${}^\tau e = e$ *if and only if* $\gamma = 1$.

3.6. Special features of Coxeter groups

Assumption. *In this section 3.6, we assume that* W *is a Coxeter group, and we use the notation of §2.6.*

By Proposition 2.6.1, there exists a non-degenerate symmetric bilinear W-invariant form $\boldsymbol{\beta} : V \times V \to \mathbf{k}$. We denote by $\sigma : V \xrightarrow{\sim} V^*$ the isomorphism induced by $\boldsymbol{\beta}$: if y, $y' \in V$, then

$$\langle y, \sigma(y') \rangle = \boldsymbol{\beta}(y, y').$$

The W-invariance of $\boldsymbol{\beta}$ implies that σ is an isomorphism of $\mathbf{k}W$-modules and the symmetry of $\boldsymbol{\beta}$ implies that

$$\langle y, x \rangle = \langle \sigma^{-1}(x), \sigma(y) \rangle \tag{3.6.1}$$

for all $x \in V^*$ and $y \in V$. We denote by $\sigma_{\mathrm{T}} : \mathrm{T}_{\mathbf{k}}(V \oplus V^*) \to \mathrm{T}_{\mathbf{k}}(V \oplus V^*)$ the automorphism of algebras induced by the automorphism of the vector space $V \oplus V^*$ defined by $(y, x) \mapsto (-\sigma^{-1}(x), \sigma(y))$. It is W-invariant, hence extends to an automorphism of $\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$, with trivial action on W. By extension of scalars, we get another automorphism, still denoted by σ_{T}, of $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes (\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W)$. It is easy to check that σ_{T} induces an automorphism $\sigma_{\widetilde{\mathbf{H}}}$ of $\widetilde{\mathbf{H}}$. We have proven the following proposition.

Proposition 3.6.2. — *There exists a unique automorphism* $\sigma_{\widetilde{\mathbf{H}}}$ *of* $\widetilde{\mathbf{H}}$ *such that*

$$\begin{cases} \sigma_{\widetilde{\mathbf{H}}}(y) = \sigma(y) & \textit{if } y \in V, \\ \sigma_{\widetilde{\mathbf{H}}}(x) = -\sigma^{-1}(x) & \textit{if } x \in V^*, \\ \sigma_{\widetilde{\mathbf{H}}}(w) = w & \textit{if } w \in W, \\ \sigma_{\widetilde{\mathbf{H}}}(C) = C & \textit{if } C \in \widetilde{\mathscr{C}}^*. \end{cases}$$

Proposition 3.6.3. — *The following hold:*

(a) $\sigma_{\widetilde{\mathbf{H}}}$ *stabilizes the subalgebras* $\mathbf{k}[\widetilde{\mathscr{C}}]$ *and* $\mathbf{k}W$ *and exchanges the subalgebras* $\mathbf{k}[V]$ *and* $\mathbf{k}[V^*]$.

(b) *If* $h \in \widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[i,j]$, *then* $\sigma_{\widetilde{\mathbf{H}}}(h) \in \widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[j,i]$.

(c) *If* $h \in \widetilde{\mathbf{H}}^{\mathbb{N}}[i]$ *(respectively* $h \in \widetilde{\mathbf{H}}^{\mathbb{Z}}[i]$*), then* $\sigma_{\widetilde{\mathbf{H}}}(h) \in \widetilde{\mathbf{H}}^{\mathbb{N}}[i]$ *(respectively* $\sigma_{\widetilde{\mathbf{H}}}(h) \in \widetilde{\mathbf{H}}^{\mathbb{Z}}[-i]$*).*

(d) $\sigma_{\widetilde{\mathbf{H}}}$ *commutes with the action of* $W^\wedge$ *on* $\widetilde{\mathbf{H}}$.

(e) *If* $(t,c) \in \widetilde{\mathscr{C}}$, *then* $\sigma_{\widetilde{\mathbf{H}}}$ *induces an automorphism of* $\widetilde{\mathbf{H}}_{t,c}$, *still denoted by* $\sigma_{\widetilde{\mathbf{H}}}$ *(or* $\sigma_{\widetilde{\mathbf{H}}_{t,c}}$ *if necessary).*

(f) $\sigma_{\widetilde{\mathbf{H}}}(\widetilde{\mathbf{eu}}) = -nT - \widetilde{\mathbf{eu}}$.

Remark 3.6.4 **(Action of $\mathbf{GL}_2(\mathbf{k})$).** — Let $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbf{k})$. The $\mathbf{k}$-linear map

$$\begin{array}{ccc} V \oplus V^* & \longrightarrow & V \oplus V^* \\ y \oplus x & \longmapsto & ay + b\sigma^{-1}(x) \oplus c\sigma(y) + dx \end{array}$$

is an automorphism of the $\mathbf{k}W$-module $V \oplus V^*$. It extends to an automorphism of the $\mathbf{k}$-algebra $\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$ and to an automorphism ρ_{T} of $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes (\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W)$ by $\rho_{\mathrm{T}}(C) = \det(\rho)C$ for $C \in \widetilde{\mathscr{C}}^*$.

It is easy to check that ρ_{T} induces an automorphism $\rho_{\widetilde{\mathbf{H}}}$ of $\widetilde{\mathbf{H}}$. Moreover, $(\rho\rho')_{\widetilde{\mathbf{H}}} = \rho_{\widetilde{\mathbf{H}}} \circ \rho'_{\widetilde{\mathbf{H}}}$ for all ρ, $\rho' \in \mathbf{GL}_2(\mathbf{k})$. So, we obtain an action of $\mathbf{GL}_2(\mathbf{k})$ on $\widetilde{\mathbf{H}}$. This action preserves the $\mathbb{N}$-grading $\widetilde{\mathbf{H}}^{\mathbb{N}}$.

Finally, note that, for $\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have $\rho_{\widetilde{\mathbf{H}}} = \sigma_{\widetilde{\mathbf{H}}}$ and, if $\rho = \begin{pmatrix} \xi & 0 \\ 0 & \xi' \end{pmatrix}$, then $\rho_{\widetilde{\mathbf{H}}} = \mathrm{bigr}_{\xi,\xi'}$. Hence we have extended the action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ to an action of $\mathbf{GL}_2(\mathbf{k}) \times (W^\wedge \rtimes \mathscr{N})$. ■

CHAPTER 4

CHEREDNIK ALGEBRAS AT $t=0$

Notation. *We put* $\mathbf{H} = \widetilde{\mathbf{H}}/T\widetilde{\mathbf{H}}$. *The* $\mathbf{k}$*-algebra* $\mathbf{H}$ *is called the* ***Cherednik algebra at*** $t=0$.

4.1. Generalities

We gather here those properties that are immediate consequences of results discussed in Chapter 3. We also introduce some notations.

Let us rewrite the defining relations (3.1.2). The algebra $\mathbf{H}$ is the $\mathbf{k}[\mathscr{C}]$-algebra quotient of $\mathbf{k}[\mathscr{C}]\otimes\big(\mathrm{T}_{\mathbf{k}}(V\oplus V^*)\rtimes W\big)$ by the ideal generated by the following relations:

$$(4.1.1)\qquad \begin{cases} [x,x']=[y,y']=0, \\ [y,x]=\displaystyle\sum_{s\in\mathrm{Ref}(W)} (\varepsilon(s)-1)\; C_s\; \frac{\langle y,\alpha_s\rangle\cdot\langle\alpha_s^\vee,x\rangle}{\langle\alpha_s^\vee,\alpha_s\rangle}\; s, \end{cases}$$

for x, $x'\in V^*$ and y, $y'\in V$.

The PBW-decomposition (Theorem 3.1.5) takes the following form.

***Theorem 4.1.2* (Etingof-Ginzburg)**. — *The multiplication map gives an isomorphism of* $\mathbf{k}[\mathscr{C}]$*-modules*

$$\mathbf{k}[\mathscr{C}]\otimes\mathbf{k}[V]\otimes\mathbf{k}W\otimes\mathbf{k}[V^*]\xrightarrow{\sim}\mathbf{H}.$$

Given $c\in\mathscr{C}$, we denote by $\mathfrak{C}_c$ the maximal ideal of $\mathbf{k}[\mathscr{C}]$ defined by $\mathfrak{C}_c=\{f\in\mathbf{k}[\mathscr{C}]\mid f(c)=0\}$: it is the ideal generated by $(C_s-c_s)_{s\in\mathrm{Ref}(W)/W}$. We set

$$\mathbf{H}_c=(\mathbf{k}[\mathscr{C}]/\mathfrak{C}_c)\otimes_{\mathbf{k}[\mathscr{C}]}\mathbf{H}=\mathbf{H}/\mathfrak{C}_c\mathbf{H}=\widetilde{\mathbf{H}}_{0,c}.$$

The $\mathbf{k}$-algebra $\mathbf{H}_c$ is the quotient of the $\mathbf{k}$-algebra $\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$ by the ideal generated by the following relations:

$$\begin{cases} [x, x'] = [y, y'] = 0, \\ \\ [y, x] = \displaystyle\sum_{s \in \mathrm{Ref}(W)} (\varepsilon(s) - 1)\ c_s\ \frac{\langle y, \alpha_s \rangle \cdot \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle}\ s, \end{cases} \tag{4.1.3}$$

for x, $x' \in V^*$ and y, $y' \in V$.

Since T is bi-homogeneous, the $\mathbf{k}$-algebra $\mathbf{H}$ inherits all the gradings, filtrations of the algebra $\widetilde{\mathbf{H}}$: we will use the obvious notation $\mathbf{H}^{\mathbb{N}\times\mathbb{N}}[i, j]$, $\mathbf{H}^{\mathbb{N}}[i]$, $\mathbf{H}^{\mathbb{Z}}[i]$ and $\mathbf{H}^{\leqslant i}$ for the constructions obtained by quotient from $\widetilde{\mathbf{H}}$. We will denote by $\mathbf{eu}$ the image of $\widetilde{\mathbf{eu}}$ in $\mathbf{H}$. This is the *generic Euler element* of $\mathbf{H}$. Note that

$$\mathbf{eu} \in \mathbf{H}^{\mathbb{N}\times\mathbb{N}}[1,1] \subset \mathbf{H}^{\mathbb{Z}}[0] \tag{4.1.4}$$

The ideal generated by T is also stable by the action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$, so $\mathbf{H}$ inherits an action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$. The action of $\tau \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ on $h \in \mathbf{H}$ is still denoted by ${}^\tau h$. The following lemma is immediate from Lemma 3.5.7:

Lemma 4.1.5. — *Let $\tau = (\xi, \xi', \gamma \rtimes g) \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$. Then:*

(a) *τ stabilizes the subalgebras $\mathbf{k}[\mathscr{C}]$, $\mathbf{k}[V]$, $\mathbf{k}[V^*]$ and $\mathbf{k}W$.*
(b) *τ stabilizes the bigrading.*
(c) ${}^\tau\mathbf{eu} = \xi\xi'\ \mathbf{eu}$.

Theorem 3.4.3 implies the following result on the spherical algebra.

***Theorem 4.1.6* (Etingof-Ginzburg)**. — *Let $\mathfrak{C}$ be a prime ideal of $\mathbf{k}[\mathscr{C}]$ and let $\mathbf{H}(\mathfrak{C}) = \mathbf{H}/\mathfrak{C}\mathbf{H}$. Then:*

(a) *The algebra $e\mathbf{H}(\mathfrak{C})e$ is a finitely generated $\mathbf{k}$-algebra without zero divisors.*
(b) *Left multiplication of $\mathbf{H}(\mathfrak{C})$ on the projective module $\mathbf{H}(\mathfrak{C})e$ induces an isomorphism $\mathbf{H}(\mathfrak{C}) \xrightarrow{\sim} \mathrm{End}_{(e\mathbf{H}(\mathfrak{C})e)^{\mathrm{opp}}}(\mathbf{H}(\mathfrak{C})e)^{\mathrm{opp}}$.*

Let $\mathbf{H}^{\mathrm{reg}} = \mathbf{k}[\mathscr{C}] \otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]} \widetilde{\mathbf{H}}^{\mathrm{reg}}$. Theorem 3.1.11 becomes the following result.

***Theorem 4.1.7* (Etingof-Ginzburg)**. — *There exists a unique isomorphism of $\mathbf{k}[\mathscr{C}]$-algebras*

$$\Theta : \mathbf{H}^{\mathrm{reg}} \xrightarrow{\sim} \mathbf{k}[\mathscr{C}] \otimes (\mathbf{k}[V^{\mathrm{reg}} \times V^*] \rtimes W)$$

such that

$$\begin{cases} \Theta(w) = w & \text{for } w \in W, \\ \Theta(y) = y - \displaystyle\sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) C_s\ \frac{\langle y, \alpha_s \rangle}{\alpha_s}\ s & \text{for } y \in V, \\ \Theta(x) = x & \text{for } x \in V^*. \end{cases}$$

Given $\mathfrak{C}$ a prime ideal of $\mathbf{k}[\mathscr{C}]$, the restriction of $(\mathbf{k}[\mathscr{C}]/\mathfrak{C})\otimes_{\mathbf{k}[\mathscr{C}]}\Theta$ to $(\mathbf{k}[\mathscr{C}]/\mathfrak{C})\otimes_{\mathbf{k}[\mathscr{C}]}\mathbf{H}$ is injective.

4.2. Center

Notation. *All along this book, we denote by $Z = \mathrm{Z}(\mathbf{H})$ the center of $\mathbf{H}$. Given $c \in \mathscr{C}$, we set $Z_c = Z/\mathfrak{C}_c Z$. Let P denote the $\mathbf{k}[\mathscr{C}]$-algebra obtained by tensor product of algebras $P = \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V]^W \otimes \mathbf{k}[V^*]^W$. We identify P with a $\mathbf{k}[\mathscr{C}]$-submodule of $\mathbf{H}$ via Theorem 4.1.2.*

4.2.A. A subalgebra of Z. — The first fundamental result about the center Z of $\mathbf{H}$ is the next one [**EtGi**, Proposition 4.15] (we follow [**Gor1**, Proposition 3.6] for the proof).

Lemma 4.2.1. — *P is a subalgebra of Z stable under the action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$. In particular, it is $(\mathbb{N}\times\mathbb{N})$-graded.*

Proof. — The subalgebra $\mathbf{k}[V]^W$ is central in $\mathbf{H}$ by Corollary 3.1.12. Dually, $\mathbf{k}[V^*]^W$ is central as well. The stability property is clear. □

Corollary 4.2.2. — *The PBW-decomposition is an isomorphism of P-modules. In particular, we have isomorphisms of P-modules:*

(a) $\mathbf{H} \simeq \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*]$.
(b) $\mathbf{H}e \simeq \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V] \otimes \mathbf{k}[V^*]$.
(c) $e\mathbf{H}e \simeq \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V \times V^*]^{\Delta W}$.

Hence, $\mathbf{H}$ (respectively $\mathbf{H}e$, respectively $e\mathbf{H}e$) is a free P-module of rank $|W|^3$ (respectively $|W|^2$, respectively $|W|$).

The principal theme of this book is to study the algebra $\mathbf{H}$, viewing it as a P-algebra: given $\mathfrak{p}$ is a prime ideal of P, we will be interested in the finite dimensional $k_P(\mathfrak{p})$-algebra $k_P(\mathfrak{p}) \otimes_P \mathbf{H}$ (splitting, simple modules, blocks, standard modules, decomposition matrix...). Here, $k_P(\mathfrak{p})$ is the fraction field of $P/\mathfrak{p}$, cf Appendix B.

Remark 4.2.3. — Let $(b_i)_{1 \leqslant i \leqslant |W|}$ be a $\mathbf{k}[V]^W$-basis of $\mathbf{k}[V]$ and let $(b_i^*)_{1 \leqslant i \leqslant |W|}$ be a $\mathbf{k}[V^*]^W$-basis of $\mathbf{k}[V^*]$. Corollary 4.2.2 shows that $(b_i\, w\, b_j^*)_{\substack{1 \leqslant i,j \leqslant |W| \\ w \in W}}$ is a P-basis of $\mathbf{H}$ and that $(b_i b_j^* e)_{1 \leqslant i,j \leqslant |W|}$ is a P-basis of $\mathbf{H}e$. ■

Set

$$P_\bullet = \mathbf{k}[V]^W \otimes \mathbf{k}[V^*]^W.$$

If $c \in \mathscr{C}$, then

$$P_\bullet \simeq \mathbf{k}[\mathscr{C}]/\mathfrak{C}_c \otimes_{\mathbf{k}[\mathscr{C}]} P = P/\mathfrak{C}_c P.$$

We deduce from Corollary 4.2.2 the next result:

Corollary 4.2.4. — *We have isomorphisms of $P_\bullet$-modules:*

(a) $\mathbf{H}_c \simeq \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*]$.
(b) $\mathbf{H}_c e \simeq \mathbf{k}[V] \otimes \mathbf{k}[V^*]$.
(c) $e\mathbf{H}_c e \simeq \mathbf{k}[V \times V^*]^{\Delta W}$.

In particular, $\mathbf{H}_c$ (respectively $\mathbf{H}_c e$, respectively $e\mathbf{H}_c e$) is a free $P_\bullet$-module of rank $|W|^3$ (respectively $|W|^2$, respectively $|W|$).

4.2.B. Satake isomorphism. — It follows from Proposition 3.3.2 that

$$\mathbf{eu} \in Z. \tag{4.2.5}$$

Given $c \in \mathscr{C}$, we denote by $\mathbf{eu}_c$ the image of $\mathbf{eu}$ in $\mathbf{H}_c$.

The next structural theorem is a cornerstone of the representation theory of $\mathbf{H}$.

***Theorem 4.2.6* (Etingof-Ginzburg).** — *The morphism of algebras $Z \longrightarrow e\mathbf{H}e$, $z \mapsto ze$ is an isomorphism of $(\mathbb{N} \times \mathbb{N})$-graded algebras. In particular, $e\mathbf{H}e$ is commutative.*

Proof. — Recall (Theorem 3.4.3) that the map $\pi_e : \mathrm{Z}(\mathbf{H}) \to \mathrm{Z}(e\mathbf{H}e)$, $z \mapsto ze$ is an isomorphism of algebras. Theorem 4.1.7 shows that $\Theta(e\mathbf{H}e) = \mathbf{k}[\mathscr{C}] \otimes (e\mathbf{k}[V^{\mathrm{reg}} \times V^*]^W)$ and Θ is injective, hence $e\mathbf{H}e$ is commutative. The theorem follows. ☐

Corollary 4.2.7. — *Let $\mathfrak{C}$ be a prime ideal of $\mathbf{k}[\mathscr{C}]$. Let $Z(\mathfrak{C}) = Z/\mathfrak{C}Z$ and $P(\mathfrak{C}) = P/\mathfrak{C}P$. We have:*

(a) $Z(\mathfrak{C}) = \mathrm{Z}(\mathbf{H}(\mathfrak{C}))$.
(b) *The map $Z(\mathfrak{C}) \to e\mathbf{H}(\mathfrak{C})e$, $z \mapsto ze$ is an isomorphism.*
(c) $\mathrm{End}_{\mathbf{H}(\mathfrak{C})}(\mathbf{H}(\mathfrak{C})e) = Z(\mathfrak{C})$ *and* $\mathrm{End}_{Z(\mathfrak{C})}(\mathbf{H}(\mathfrak{C})e) = \mathbf{H}(\mathfrak{C})$.
(d) $\mathbf{H}(\mathfrak{C}) = Z(\mathfrak{C}) \oplus e\mathbf{H}(\mathfrak{C})(1-e) \oplus (1-e)\mathbf{H}(\mathfrak{C})e \oplus (1-e)\mathbf{H}(\mathfrak{C})(1-e)$. *In particular, $Z(\mathfrak{C})$ is a direct summand of the $Z(\mathfrak{C})$-module $\mathbf{H}(\mathfrak{C})$.*
(e) *$Z(\mathfrak{C})$ is a free $P(\mathfrak{C})$-module of rank $|W|$.*
(f) *If $\mathbf{k}[\mathscr{C}]/\mathfrak{C}$ is integrally closed, then $Z(\mathfrak{C})$ is an integrally closed domain.*

Proof. — Assertion (b) follows from Theorem 4.2.6. We deduce now (c) from Theorem 4.1.6 and (e) from Corollary 4.2.2. We deduce also that $Z(\mathfrak{C})(1-e)\cap Z(\mathfrak{C})=0$. It follows also that $e\mathbf{H}(\mathfrak{C})e=Z(\mathfrak{C})e$, hence $e\mathbf{H}(\mathfrak{C})e\subset Z(\mathfrak{C})+\mathbf{H}(\mathfrak{C})(1-e)$. The decomposition $\mathbf{H}(\mathfrak{C})=e\mathbf{H}(\mathfrak{C})e\oplus e\mathbf{H}(\mathfrak{C})(1-e)\oplus(1-e)\mathbf{H}(\mathfrak{C})e\oplus(1-e)\mathbf{H}(\mathfrak{C})(1-e)$ implies (d).

The canonical map $Z(\mathbf{H}(\mathfrak{C}))\to e\mathbf{H}(\mathfrak{C})e,\ z\mapsto ze$ is injective since $\mathbf{H}(\mathfrak{C})$ acts faithfully on $\mathbf{H}(\mathfrak{C})e$ by (c). Since $Z(\mathfrak{C})$ is a direct summand of $\mathbf{H}(\mathfrak{C})$ contained in $Z(\mathbf{H}(\mathfrak{C}))$, the assertion (a) follows from (b).

The fact that $Z(\mathfrak{C})\simeq e\mathbf{H}(\mathfrak{C})e$ is an integrally domain closed follows from the fact that $\mathrm{gr}(e\mathbf{H}(\mathfrak{C})e)\simeq(\mathbf{k}[\mathscr{C}]/\mathfrak{C})\otimes\mathbf{k}[V\times V^*]^{\Delta W}$ is an integrally closed domain (Lemma A.2.2). ☐

Example 4.2.8. — Recall (Example 3.1.7) that $\mathbf{H}_0=\mathbf{k}[V\oplus V^*]\rtimes W$. It follows from Proposition 3.1.1 and Corollary 4.2.7 (a) that $Z_0=\mathbf{k}[V\oplus V^*]^W=\mathbf{k}[(V\oplus V^*)/W]$. ■

4.2.C. Morphism to the center of $\mathbf{k}[\mathscr{C}]W$. — Let $\mathrm{pbw}:\mathbf{k}[\mathscr{C}]\otimes\mathbf{k}[V]\otimes\mathbf{k}W\otimes\mathbf{k}[V^*]\xrightarrow{\sim}\mathbf{H}$ denote the isomorphism given by the PBW-decomposition (see Theorem 4.1.2) and let $\mathrm{ev}_{0,0}:\mathbf{k}[\mathscr{C}]\otimes\mathbf{k}[V]\otimes\mathbf{k}W\otimes\mathbf{k}[V^*]\to\mathbf{k}[\mathscr{C}]W$ denote the $\mathbf{k}[\mathscr{C}]$-linear map defined by

$$\mathrm{ev}_{0,0}(a\otimes f\otimes w\otimes g)=f(0)g(0)aw.$$

We then set

$$\mathbf{\Omega}^{\mathbf{H}}=\mathrm{ev}_{0,0}\circ\mathrm{pbw}^{-1}:\mathbf{H}\longrightarrow\mathbf{k}[\mathscr{C}]W.$$

It is clearly a W-equivariant morphism of bigraded $\mathbf{k}[\mathscr{C}]$-modules but it is not a morphism of algebras (except if $W=1$). Nevertheless, we have the following result:

Proposition 4.2.9. — *If $z\in Z$ and $h\in\mathbf{H}$, then $\mathbf{\Omega}^{\mathbf{H}}(z)\in \mathrm{Z}(\mathbf{k}[\mathscr{C}]W)$ and*

$$\mathbf{\Omega}^{\mathbf{H}}(zh)=\mathbf{\Omega}^{\mathbf{H}}(z)\mathbf{\Omega}^{\mathbf{H}}(h).$$

Proof. — First, the fact that $\mathbf{\Omega}^{\mathbf{H}}(z)\in\mathrm{Z}(\mathbf{k}[\mathscr{C}]W)$ follows from the W-equivariance of $\mathbf{\Omega}^{\mathbf{H}}$. Now, write $\mathrm{pbw}^{-1}(z)=\sum_{i\in I}a_i\otimes f_i\otimes w_i\otimes g_i$ and $\mathrm{pbw}^{-1}(h)=\sum_{j\in J}a'_j\otimes f'_j\otimes w'_j\otimes g'_j$. We have

$$zh=\sum_{j\in J}za'_jf'_jw'_jg'_j=\sum_{j\in J}a'_jf'_jzw'_jg'_j$$

hence

$$\mathrm{pbw}^{-1}(zh)=\sum_{\substack{i\in I\\ j\in J}}a_ia'_j\otimes f_if'_j\otimes w_iw'_j\otimes {}^{w_j'^{-1}}g_ig'_j.$$

This proves that $\mathbf{\Omega}^{\mathbf{H}}(zh)=\mathbf{\Omega}^{\mathbf{H}}(z)\mathbf{\Omega}^{\mathbf{H}}(h)$. ☐

Let $\mathbf{\Omega} : Z \to \mathrm{Z}(\mathbf{k}[\mathscr{C}]W)$ denote the restriction of $\mathbf{\Omega}^{\mathbf{H}}$ to Z. Since $\mathbf{\Omega}$ respects the bigrading and so respects the $\mathbb{Z}$-grading, we have

$$\mathbf{\Omega}(z) = 0 \tag{4.2.10}$$

if $z \in Z$ is $\mathbb{Z}$-homogeneous of non-zero $\mathbb{Z}$-degree.

Corollary 4.2.11. — *The map $\mathbf{\Omega} : Z \to \mathrm{Z}(\mathbf{K}[\mathscr{C}]W)$ is a morphism of bigraded $\mathbf{k}[\mathscr{C}]$-algebras.*

If $\mathfrak{C}$ is a prime ideal of $\mathbf{k}[\mathscr{C}]$ (resp. if $c \in \mathscr{C}$), the map $\mathbf{\Omega}$ induces a morphism of $\mathbb{Z}$-graded algebras $\mathbf{\Omega}^{\mathfrak{C}} : Z(\mathfrak{C}) \to (\mathbf{k}[\mathscr{C}]/\mathfrak{C}) \otimes \mathrm{Z}(\mathbf{k}W)$ (resp. $\mathbf{\Omega}^{\mathfrak{C}} : Z_c \to \mathrm{Z}(\mathbf{k}W)$).

Remark 4.2.12. — By exchanging the roles of V and V^*, one gets an isomorphism $\mathrm{pbw}_* : \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V^*] \otimes \mathbf{k}W \otimes \mathbf{k}[V] \to \mathbf{H}$ coming from the PBW-decomposition and a map $\mathrm{ev}^*_{0,0} : \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V^*] \otimes \mathbf{k}W \otimes \mathbf{k}[V] \to \mathbf{k}[\mathscr{C}]W$ obtained by evaluating at $(0,0) \in V^* \times V$. One then gets another morphism of bigraded $\mathbf{k}[\mathscr{C}]$-algebras

$$\mathbf{\Omega}^* : Z \longrightarrow \mathrm{Z}(\mathbf{k}[\mathscr{C}]W).$$

It turns out that $\mathbf{\Omega} \neq \mathbf{\Omega}^*$. Indeed, if we still denote by ε the automorphism of the $\mathbf{k}[\mathscr{C}]$-algebra $\mathbf{k}[\mathscr{C}]W$ given by $w \mapsto \varepsilon(w)w$, then it follows from Section 3.3 that

$$\mathbf{\Omega}(\mathbf{eu}) = \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) C_s s = {}^{\varepsilon}\mathbf{\Omega}^*(\mathbf{eu}).$$

We will see in Corollary 9.5.7 that

$$\mathbf{\Omega}(z) = {}^{\varepsilon}\mathbf{\Omega}^*(z)$$

for all $z \in Z$. ■

4.3. Localization

4.3.A. Localization on V^{reg}. — Recall that

$$V^{\mathrm{reg}} = V \setminus \bigcup_{H \in \mathscr{A}} H = \{v \in V \mid \mathrm{Stab}_W(v) = 1\}.$$

Set $P^{\mathrm{reg}} = \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V^{\mathrm{reg}}]^W \times \mathbf{k}[V^*]^W$ and $Z^{\mathrm{reg}} = P^{\mathrm{reg}} \otimes_P Z$, so that $\mathbf{H}^{\mathrm{reg}} = P^{\mathrm{reg}} \otimes_P \mathbf{H} = Z^{\mathrm{reg}} \otimes_Z \mathbf{H}$. Given $s \in \mathrm{Ref}(W)$, let $\alpha_s^W = \prod_{w \in W} w(\alpha_s) \in P$, . The algebra P^{reg} (respectively Z^{reg}) is the localization of P (respectively Z) at the multiplicative subset $(\alpha_s^W)_{s \in \mathrm{Ref}(W)}$. As a consequence,

$$\alpha_s \textit{ is invertible in } \mathbf{H}^{\mathrm{reg}}. \tag{4.3.1}$$

Corollary 4.3.2. — *Θ restricts to an isomorphism of $\mathbf{k}[\mathscr{C}]$-algebras $Z^{\mathrm{reg}} \xrightarrow{\sim} \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V^{\mathrm{reg}} \times V^*]^W$. In particular, Z^{reg} is regular.*

Proof. — The first statement follows from the comparison between the centers of $\mathbf{H}^{\mathrm{reg}}$ and $\mathbf{k}[V^{\mathrm{reg}} \times V^*] \rtimes W$ (Theorem 4.1.7). The second statement follows from the fact that W acts freely on $V^{\mathrm{reg}} \times V^*$. □

Given $c \in \mathscr{C}$, let Z_c^{reg} denote the localization of Z_c at $P_\bullet^{\mathrm{reg}} = \mathbf{k}[V^{\mathrm{reg}}]^W \otimes \mathbf{k}[V^*]^W$. Corollary 4.3.2 shows that

(4.3.3) $$Z_c^{\mathrm{reg}} \simeq \mathbf{k}[V^{\mathrm{reg}} \times V^*]^W \textit{ is a regular ring.}$$

4.3.B. Morita equivalences. — While Z and $\mathbf{H}$ are only related by a double endomorphism theorem, after restricting to a smooth open subset of Z, they become Morita equivalent.

Proposition 4.3.4. — *Let U be a multiplicative subset of Z such that $Z[U^{-1}]$ is regular. Then $\mathbf{H}[U^{-1}]e$ induces a Morita equivalence between the algebras $\mathbf{H}[U^{-1}]$ and $Z[U^{-1}]$.*

Proof. — Let $\mathfrak{m}$ be a maximal ideal of Z such that $Z_{\mathfrak{m}}$ is regular. Let i be maximal such that $\mathrm{Tor}_i^Z(\mathbf{H}e, Z/\mathfrak{m}) \neq 0$. Given any finite length Z-module L with support $\mathfrak{m}$, we have $\mathrm{Tor}_i^Z(\mathbf{H}e, L) \neq 0$.

Let $\mathfrak{n} = P \cap \mathfrak{m}$. We have $\mathrm{Tor}_*^Z(\mathbf{H}e, Z/\mathfrak{n}Z) \simeq \mathrm{Tor}_*^Z(\mathbf{H}e, Z \otimes_P P/\mathfrak{n}) \simeq \mathrm{Tor}_*^P(\mathbf{H}e, P/\mathfrak{n})$ since Z is a free P-module (Corollary 4.2.7). Since $\mathbf{H}e$ is a free P-module (Corollary 4.2.2), it follows that $\mathrm{Tor}_{>0}^Z(\mathbf{H}e, Z/\mathfrak{n}Z) = 0$, hence $\mathrm{Tor}_{>0}^Z(\mathbf{H}e, (Z/\mathfrak{n}Z)_{\mathfrak{m}}) = 0$. We deduce that $i = 0$, hence $(\mathbf{H}e)_{\mathfrak{m}}$ is a free $Z_{\mathfrak{m}}$-module.

We have shown that $\mathbf{H}[U^{-1}]e$ is a projective $Z[U^{-1}]$-module. The Morita equivalence follows from Corollary 4.2.7. □

Corollary 4.3.5. — *The $(\mathbf{H}^{\mathrm{reg}}, Z^{\mathrm{reg}})$-bimodule $\mathbf{H}^{\mathrm{reg}}e$ induces a Morita equivalence between $\mathbf{H}^{\mathrm{reg}}$ and Z^{reg}.*

Proof. — This follows from Proposition 4.3.4 and Corollary 4.3.2. □

4.3.C. Fraction field. — Let $\mathbf{K}$ denote the fraction field of P and let $\mathbf{K}Z = \mathbf{K} \otimes_P Z$. Since Z is a domain and is integral over P, it follows that

(4.3.6) $$\mathbf{K}Z \textit{ is the fraction field of } Z.$$

In particular, $\mathbf{K}Z$ is a regular ring.

Theorem 4.3.7. — *The $\mathbf{K}$-algebras $\mathbf{KH}$ and $\mathbf{K}Z$ are Morita equivalent, the Morita equivalence being induced by $\mathbf{KH}e$. More precisely,*

$$\mathbf{KH} \simeq \mathrm{Mat}_{|W|}(\mathbf{K}Z).$$

Proof. — Proposition 4.3.4 shows the Morita equivalence. Recall that $\mathbf{H}e$ is a free P-module of rank $|W|^2$ and Z is a free P-module of rank $|W|$ (Corollary 4.2.2). It follows that $\mathbf{KH}e$ is a $\mathbf{K}Z$-vector space of dimension $|W|$, whence the result. □

4.4. Complements

4.4.A. Poisson structure. — The PBW-decomposition induces an isomorphism of $\mathbf{k}$*-vector spaces* $\mathbf{k}[T]\otimes\mathbf{H}\xrightarrow{\sim}\widetilde{\mathbf{H}}$. Given $h\in\mathbf{H}$, let $\tilde{h}$ denote the image of $1\otimes h\in\mathbf{k}[T]\otimes\mathbf{H}$ in $\widetilde{\mathbf{H}}$ through this isomorphism. If $z,\ z'\in Z$, then $[z,z']=0$, hence $[\tilde{z},\tilde{z}']\in T\widetilde{\mathbf{H}}$. We denote by $\{z,z'\}$ the image of $[\tilde{z},\tilde{z}']/T\in\widetilde{\mathbf{H}}$ in $\mathbf{H}=\widetilde{\mathbf{H}}/T\widetilde{\mathbf{H}}$. It is easily checked that $\{z,z'\}\in Z$ and that

$$\{-,-\}:Z\times Z\longrightarrow Z \tag{4.4.1}$$

is a $\mathbf{k}[\mathscr{C}]$-linear *Poisson bracket*. Given $c\in\mathscr{C}$, it induces a Poisson bracket

$$\{-,-\}:Z_c\times Z_c\longrightarrow Z_c. \tag{4.4.2}$$

4.4.B. Additional filtrations. — Define a P-algebra filtration of $\mathbf{H}$ by

$$\mathbf{H}^{\preccurlyeq -1}=0,\ \mathbf{H}^{\preccurlyeq 0}=P[W],\ \mathbf{H}^{\preccurlyeq 1}=\mathbf{H}^{\preccurlyeq 0}+\mathbf{H}^{\preccurlyeq 0}V+\mathbf{H}^{\preccurlyeq 0}V^* \text{ and } \mathbf{H}^{\preccurlyeq i}=\mathbf{H}^{\preccurlyeq 1}\mathbf{H}^{\preccurlyeq i-1} \text{ for } i\geqslant 2.$$

Note that $\mathbf{H}^{\preccurlyeq 2N-1}\neq\mathbf{H}$ and $\mathbf{H}^{\preccurlyeq 2N}=\mathbf{H}$.

Let V' be the $\mathbf{k}W$-stable complement to V^W in V. We have an injection of P-modules $P\otimes(V'^*\oplus V')\otimes\mathbf{k}[W]\hookrightarrow\mathbf{H}^{\preccurlyeq 1}$. It extends to a morphism of graded P-algebras

$$f:P\otimes(\mathbf{k}[V']^{\mathrm{co}(W)}\otimes\mathbf{k}[V'^*]^{\mathrm{co}(W)})\rtimes W\to\mathrm{gr}^{\preccurlyeq}\mathbf{H}$$

where P and W are in degree 0 and V'^* and V' are in degree 1.

Proposition 4.4.3. — *The morphism f is an isomorphism of graded P-algebras.*

Proof. — This follows from the PBW decomposition (Corollary 4.2.2). □

Let us define $\dot{\mathbf{H}}=\widetilde{\mathbf{H}}\otimes_{\mathbf{k}[T]}\big(\mathbf{k}[T]/(T-1)\big)$, an algebra over $\mathbf{k}[\widetilde{\mathscr{C}}]/(T-1)$ (we identify that ring with $\mathbf{k}[\mathscr{C}]$). We define a $\mathbf{k}$-algebra filtration of $\dot{\mathbf{H}}$

$$\dot{\mathbf{H}}^{\trianglelefteq -1}=0,\ \dot{\mathbf{H}}^{\trianglelefteq 0}=\mathbf{k}[V]\rtimes W,\ \dot{\mathbf{H}}^{\trianglelefteq 1}=\dot{\mathbf{H}}^{\trianglelefteq 0}V+\dot{\mathbf{H}}^{\trianglelefteq 0}\mathscr{C}^*+\dot{\mathbf{H}}^{\trianglelefteq 0}$$
$$\text{and } \dot{\mathbf{H}}^{\trianglelefteq i}=(\dot{\mathbf{H}}^{\trianglelefteq 1})^i \text{ for } i\geqslant 2.$$

The canonical maps $\mathbf{k}[V]\rtimes W\to(\mathrm{gr}^{\trianglelefteq}\dot{\mathbf{H}})^0$ and $V\oplus\mathscr{C}^*\to(\mathrm{gr}^{\trianglelefteq}\dot{\mathbf{H}})^1$ induce a morphism of $\mathbb{N}$-graded algebras $g:\mathbf{H}\to\mathrm{gr}^{\trianglelefteq}\dot{\mathbf{H}}$

The PBW decomposition (Theorem 3.1.5) shows the following result.

Proposition 4.4.4. — *The morphism g is an isomorphism.*

Note that this proposition shows that $\widetilde{\mathbf{H}}$ is the Rees algebra of $\dot{\mathbf{H}}$ for this filtration.

4.4.C. Symmetrizing form. — Recall (Proposition 2.2.3) that we have symmetrizing forms $p_N : \mathbf{k}[V] \to \mathbf{k}[V]^W$ and $p_N^* : \mathbf{k}[V^*] \to \mathbf{k}[V^*]^W$.

We define a P-linear map

$$\begin{aligned} \tau_{\mathbf{H}} : \mathbf{H} = \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*] &\longrightarrow P \\ a \otimes b \otimes w \otimes c &\longmapsto a\delta_{1w} p_N(b) p_N^*(c). \end{aligned}$$

Theorem 4.4.5 **([BrGoSt**, Theorem 4.4**]).** — *The form $\tau_{\mathbf{H}}$ is symmetrizing for the P-algebra* $\mathbf{H}$.

Proof. — We have an isomorphism

$$\begin{aligned} (\mathrm{gr}^{\preccurlyeq}\mathbf{H})^{2N} &\xrightarrow{\sim} P \\ \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*] \ni a \otimes b \otimes w \otimes c &\longmapsto p_N(b) \otimes a w \otimes p_N^*(c) \end{aligned}$$

Via the isomorphism of Proposition 4.4.3, the P-linear form on $\mathrm{gr}^{\preccurlyeq}\mathbf{H}$ induced by $\tau_{\mathbf{H}}$ is given by

$$P \otimes (\mathbf{k}[V]_W \otimes \mathbf{k}[V^*]_W) \rtimes W \ni a \otimes (b \otimes c) \otimes w \mapsto a\delta_{1w} \langle p_N(b), p_N(c) \rangle.$$

It follows from Lemma A.4.1 that this is a symmetrizing form.

Let $L = V' \oplus V'^*$. We have $S^{N+1}(V') \subset S(V')^{\mathrm{co}(W)}_{>0} \cdot S^{\leqslant N-1}(V')$ (and similarly with V'^*), hence

$$L^{2N+1} \subset (S^{N+1}(V') \otimes S^N(V'^*)) + (S^N(V') \otimes S^{N+1}(V'^*)) + \mathbf{H}^{\preccurlyeq 2N-1} \subset \mathbf{H}^{\preccurlyeq 2N-1}.$$

It follows from Lemma A.4.2 that $\tau_{\mathbf{H}}$ is a trace. We deduce from Proposition A.4.3 that $\tau_{\mathbf{H}}$ is symmetrizing. □

Remark 4.4.6. — Note that while the identification $\mathbf{k}[V]_N^{\mathrm{co}(W)} \xrightarrow{\sim} \mathbf{k}$ is not canonical, there is a canonical choice of isomorphism $\mathbf{k}[V]_N^{\mathrm{co}(W)} \otimes_{\mathbf{k}} \mathbf{k}[V^*]_N^{\mathrm{co}(W)} \xrightarrow{\sim} \mathbf{k}$ obtained by requiring $\langle \alpha_s^\vee, \alpha_s \rangle = 1$ for all $s \in \mathrm{Ref}(W)$. This provides a canonical choice for $\tau_{\mathbf{H}}$. ■

We denote by $\mathrm{cas}_{\mathbf{H}} \in Z$ the central Casimir element of $\mathbf{H}$ (cf §A.4).

4.4.D. Hilbert series. — We compute here the bigraded Hilbert series of $\mathbf{H}$, P, Z and $e\mathbf{H}e$. First of all, note that

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[\mathscr{C}]) = \frac{1}{(1-\mathbf{tu})^{|\mathrm{Ref}(W)/W|}},$$

so that it becomes easy to deduce the Hilbert series for $\mathbf{H}$, using the PBW-decomposition:

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{H}) = \frac{|W|}{(1-\mathbf{t})^n\,(1-\mathbf{u})^n\,(1-\mathbf{tu})^{|\mathrm{Ref}(W)/W|}}. \tag{4.4.7}$$

On the other hand, using the notation of Theorem 2.2.1, we get, thanks to (2.5.2),

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(P) = \frac{1}{(1-\mathbf{tu})^{|\mathrm{Ref}(W)/W|}\prod_{i=1}^{n}(1-\mathbf{t}^{d_i})(1-\mathbf{u}^{d_i})}. \tag{4.4.8}$$

Finally, note that the PBW-decomposition is a W-equivariant isomorphism of bigraded $\mathbf{k}[\mathscr{C}]$-modules, from which we deduce that $\mathbf{H}e \simeq \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V] \otimes \mathbf{k}[V^*]$ as bigraded $\mathbf{k}W$-modules. So

(4.4.9) *the bigraded* $\mathbf{k}$*-vector spaces* Z *and* $\mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V \times V^*]^{\Delta W}$ *are isomorphic.*

We deduce that $\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z) = \dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[\mathscr{C}]) \cdot \dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V \times V^*]^{\Delta W})$. By (2.5.3) and Proposition 2.5.10, we get

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z) = \frac{1}{|W|\,(1-\mathbf{tu})^{|\mathrm{Ref}(W)/W|}} \sum_{w\in W} \frac{1}{\det(1-w\mathbf{t})\,\det(1-w^{-1}\mathbf{u})} \tag{4.4.10}$$

and

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z) = \frac{\displaystyle\sum_{\chi\in\mathrm{Irr}(W)} f_\chi(\mathbf{t})\,f_\chi(\mathbf{u})}{(1-\mathbf{tu})^{|\mathrm{Ref}(W)/W|}\prod_{i=1}^{n}(1-\mathbf{t}^{d_i})(1-\mathbf{u}^{d_i})}. \tag{4.4.11}$$

Example 4.4.12. — Assume here that $n = \dim_{\mathbf{k}}(V) = 1$ and let $d = |W|$. Let $y \in V \setminus \{0\}$ and $x \in V^*$ with $\langle y, x\rangle = 1$. Then $P_\bullet = \mathbf{k}[x^d, y^d]$, $\mathbf{eu}_0 = xy$ and it is easily checked that $Z_0 = \mathbf{k}[x^d, y^d, xy]$, that is, $Z_0 = P_\bullet[\mathbf{eu}_0]$. We will prove here that

$$Z = P[\mathbf{eu}].$$

Indeed, $\mathrm{Irr}(W) = \{\varepsilon^i \mid 0 \leqslant i \leqslant d-1\}$ and $f_{\varepsilon^i}(\mathbf{t}) = \mathbf{t}^i$ for $0 \leqslant i \leqslant d-1$. Consequently, (4.4.11) implies that

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z) = \frac{1+(\mathbf{tu})+\cdots+(\mathbf{tu})^{d-1}}{(1-\mathbf{tu})^{d-1}\,(1-\mathbf{t}^d)\,(1-\mathbf{u}^d)}$$

whereas, since $P[\mathbf{eu}] = P \oplus P\mathbf{eu} \oplus \cdots \oplus P\mathbf{eu}^{d-1}$ by Proposition 5.1.19, we have

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(P[\mathbf{eu}]) = \frac{1+(\mathbf{tu})+\cdots+(\mathbf{tu})^{d-1}}{(1-\mathbf{tu})^{d-1}\,(1-\mathbf{t}^d)\,(1-\mathbf{u}^d)}.$$

Hence, $\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(P[\mathbf{eu}]) = \dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z)$, so $Z = P[\mathbf{eu}]$. ■

In fact, it almost never happens that $Z = P[\mathbf{eu}]$, cf Proposition 5.1.24.

4.5. Special features of Coxeter groups

Assumption. *In this section §4.5, we assume that W is a Coxeter group, and we use the notation of §2.6.*

In relation with the aspects studied in this chapter, one of the features of the situation is that the algebra $\mathbf{H}$ admits another automorphism $\sigma_{\mathbf{H}}$, induced by the isomorphism of W-modules $\sigma : V \xrightarrow{\sim} V^*$. It is the reduction modulo T of the automorphism $\sigma_{\widetilde{\mathbf{H}}}$ de $\widetilde{\mathbf{H}}$ defined in §3.6. Propositions 3.6.2 and 3.6.3 now become:

Proposition 4.5.1. — *There exists a unique automorphism $\sigma_{\mathbf{H}}$ of $\mathbf{H}$ such that*

$$\begin{cases} \sigma_{\mathbf{H}}(y) = \sigma(y) & \text{if } y \in V, \\ \sigma_{\mathbf{H}}(x) = -\sigma^{-1}(x) & \text{if } x \in V^*, \\ \sigma_{\mathbf{H}}(w) = w & \text{if } w \in W, \\ \sigma_{\mathbf{H}}(C) = C & \text{if } C \in \mathscr{C}^*. \end{cases}$$

Proposition 4.5.2. — *We have the following statements:*

(a) *$\sigma_{\mathbf{H}}$ stabilizes the subalgebras $\mathbf{k}[\mathscr{C}]$ and $\mathbf{k}W$ and exchanges the subalgebras $\mathbf{k}[V]$ and $\mathbf{k}[V^*]$.*

(b) *Given $h \in \mathbf{H}^{\mathbb{N}\times\mathbb{N}}[i,j]$, we have $\sigma_{\mathbf{H}}(h) \in \mathbf{H}^{\mathbb{N}\times\mathbb{N}}[j,i]$.*

(c) *Given $h \in \mathbf{H}^{\mathbb{N}}[i]$ (respectively $h \in \mathbf{H}^{\mathbb{Z}}[i]$), we have $\sigma_{\mathbf{H}}(h) \in \mathbf{H}^{\mathbb{N}}[i]$ (respectively $h \in \mathbf{H}^{\mathbb{Z}}[-i]$).*

(d) *$\sigma_{\mathbf{H}}$ commutes with the action of $W^\wedge$ on $\mathbf{H}$.*

(e) *Given $c \in \mathscr{C}$, then $\sigma_{\mathbf{H}}$ induces an automorphism of $\mathbf{H}_c$, still denoted by $\sigma_{\mathbf{H}}$ (or $\sigma_{\mathbf{H}_c}$ if necessary).*

(f) *$\sigma_{\mathbf{H}}(\mathbf{eu}) = -\mathbf{eu}$.*

Similarly, there exists an action of $\mathbf{GL}_2(\mathbf{k})$ on $\mathbf{H}$, which is obtained by reduction modulo T of the action on $\widetilde{\mathbf{H}}$ defined in Remark 3.6.4.

PART II

THE EXTENSION Z/P

Important notation. *All along this book, we fix a copy Q of the P-algebra Z, as well as an isomorphism of P-algebras* $\mathrm{cop} : Z \xrightarrow{\sim} Q$. *This means that P will be seen as a $\mathbf{k}$-subalgebra of both Z and Q, but that Z and Q will be considered as different.*

*We then denote $\mathbf{K} = \mathrm{Frac}(P)$ and $\mathbf{L} = \mathrm{Frac}(Q)$ and we fix a **Galois closure** $\mathbf{M}$ of the extension $\mathbf{L}/\mathbf{K}$. Set $G = \mathrm{Gal}(\mathbf{M}/\mathbf{K})$ and $H = \mathrm{Gal}(\mathbf{M}/\mathbf{L})$. We denote by R the integral closure of P in $\mathbf{M}$. We then have $P \subset Q \subset R$ and, by Corollary 4.2.7, $Q = R^H$ and $P = R^G$. We will use the results of Appendix B.*

Recall that $\mathbf{K}Z = \mathbf{K} \otimes_P Z$ is the fraction field of Z (see (4.3.6)). We still denote by $\mathrm{cop} : \mathrm{Frac}(Z) \xrightarrow{\sim} \mathbf{L}$ *the extension of* cop *to the fraction fields.*

Let $Z^{\mathbb{N}\times\mathbb{N}}$, $Z^{\mathbb{N}}$ and $Z^{\mathbb{Z}}$ denote respectively the $(\mathbb{N}\times\mathbb{N})$-grading, the $\mathbb{N}$-grading, the $\mathbb{Z}$-grading induced by corresponding one of $\widetilde{\mathbf{H}}$ (see §3.2, and the examples 3.2.2 et 3.2.1). Through the isomorphism cop, *we obtain gradings $Q^{\mathbb{N}\times\mathbb{N}}$, $Q^{\mathbb{N}}$ and $Q^{\mathbb{Z}}$ on Q.*

This Galois extension is the main object studied in this book: we shall be particularly interested in the inertia groups of prime ideals of R, and their links with the representation theory of $\mathbf{H}$. Throughout this part, we will use the results of the Appendices B and C, which deal with generality about Galois theory, integral extensions and gradings.

CHAPTER 5

GALOIS THEORY

5.1. Action of G on the set W

Since Q is a free P-module of rank $|W|$, the field extension $\mathbf{L}/\mathbf{K}$ has degree $|W|$:

$$[\mathbf{L}:\mathbf{K}] = |W|. \tag{5.1.1}$$

Recall that the fact that $\mathbf{M}$ is a Galois closure of $\mathbf{L}/\mathbf{K}$ implies that

$$\bigcap_{g\in G} {}^g H = 1. \tag{5.1.2}$$

It follows from (5.1.1) that

$$|G/H| = |W|. \tag{5.1.3}$$

This equality establishes a first link between the pair (G, H) and the group W. We will now construct, using Galois theory, a bijection (depending on some choices) between G/H and W.

5.1.A. Specialization. — We fix here $c \in \mathscr{C}$. Recall that $\mathfrak{C}_c$ is the maximal ideal of $\mathbf{k}[\mathscr{C}]$ whose elements are maps which vanish at c. We set

$$\mathfrak{p}_c = \mathfrak{C}_c P \qquad \text{and} \qquad \mathfrak{q}_c = \mathfrak{C}_c Q = \mathfrak{p}_c Q.$$

Since $P_\bullet = P/\mathfrak{p}_c \simeq \mathbf{k}[V]^W \otimes \mathbf{k}[V^*]^W$ and $Q_c = Q/\mathfrak{q}_c$ are domains (see Corollary 4.2.7(f)), we deduce that $\mathfrak{p}_c$ and $\mathfrak{q}_c$ are prime ideals of P and Q respectively. Fix a prime ideal $\mathfrak{r}_c$ of R lying over $\mathfrak{p}_c$ and let $R_c = R/\mathfrak{r}_c$. Now, let D_c (respectively I_c) be the decomposition (respectively inertia) group $G^D_{\mathfrak{r}_c}$ (respectively $G^I_{\mathfrak{r}_c}$). Let

$$\mathbf{K}_c = \mathrm{Frac}(P_\bullet), \qquad \mathbf{L}_c = \mathrm{Frac}(Q_c) \qquad \text{and} \qquad \mathbf{M}_c = \mathrm{Frac}(R_c).$$

In other words, $\mathbf{K}_c = k_P(\mathfrak{p}_c)$, $\mathbf{L}_c = k_Q(\mathfrak{q}_c)$ and $\mathbf{M}_c = k_R(\mathfrak{r}_c)$.

Remark 5.1.4. — Here, the choice of the ideal $\mathfrak{r}_c$ is relevant. We will meet such issues all along this book. ■

Since $\mathfrak{q}_c = \mathfrak{p}_c Q$ is a prime ideal, we get that $\Upsilon^{-1}(\mathfrak{p}_c)$ contains only one element (here, $\mathfrak{q}_c$), so it follows from Proposition B.3.5 that

$$G = H \cdot D_c = D_c \cdot H. \tag{5.1.5}$$

We also obtain that Q is unramified at P at $\mathfrak{q}_c$ (by definition). Theorem B.2.6 implies that $I_c \subset H$. Since I_c is normal in D_c, we deduce from (5.1.2) and (5.1.5) that $I_c \subset \bigcap_{d \in D_c} {}^d H = \bigcap_{g \in G} {}^g H = 1$, so that

$$I_c = 1. \tag{5.1.6}$$

It follows now from Proposition B.3.10 that

$$\mathbf{M}_c \textit{ is the Galois closure of the extension } \mathbf{L}_c/\mathbf{K}_c. \tag{5.1.7}$$

Finally, by (5.1.6) and Theorem B.2.4, we get

$$\mathrm{Gal}(\mathbf{M}_c/\mathbf{K}_c) = D_c \qquad \text{and} \qquad \mathrm{Gal}(\mathbf{M}_c/\mathbf{L}_c) = D_c \cap H. \tag{5.1.8}$$

We denote by $\mathrm{cop}_c : Z_c \to Q_c$ the specialization of cop at c and we still denote by $\mathrm{cop}_c : \mathrm{Frac}(Z_c) \to \mathbf{L}_c$ the extension of cop_c to the fraction fields.

Remark 5.1.9. — In §5.1.B, we will study the particular case where $c = 0$, and obtain an explicit description of D_0. However, obtaining an explicit description of D_c in general seems to be very difficult, as it will be shown by the examples treated in chapter 18 (case $\dim_{\mathbf{k}}(V) = 1$), see §18.5.C. ■

5.1.B. Specialization at 0. — Recall that $P_0 = P_\bullet = \mathbf{k}[V]^W \otimes \mathbf{k}[V^*]^W$ and $Q_0 \simeq Z_0 = \mathrm{Z}(\mathbf{H}_0) \simeq \mathbf{k}[V \times V^*]^{\Delta W}$, where $\Delta : W \to W \times W$, $w \mapsto (w, w)$ is the diagonal morphism. So,

$$\mathbf{K}_0 = \mathbf{k}(V \times V^*)^{W \times W} \qquad \text{and} \qquad \mathbf{L}_0 = \mathbf{k}(V \times V^*)^{\Delta W},$$

On the other hand, the extension $\mathbf{k}(V \times V^*)/\mathbf{K}_0$ is Galois with group $W \times W$, whereas the extension $\mathbf{k}(V \times V^*)/\mathbf{L}_0$ is Galois with group ΔW. Since $\Delta \mathrm{Z}(W)$ is the biggest normal subgroup of $W \times W$ contained in ΔW, it follows from (5.1.7) that

$$\mathbf{M}_0 \simeq \mathbf{k}(V \times V^*)^{\Delta \mathrm{Z}(W)}.$$

Fundamental choice. *We fix once and for all a prime ideal* $\mathfrak{r}_0$ *of* R *lying over* $\mathfrak{q}_0 = \mathfrak{C}_0 Q$ *as well as a field isomorphism*

$$\mathrm{iso}_0 : \mathbf{k}(V \times V^*)^{\Delta Z(W)} \xrightarrow{\sim} \mathbf{M}_0$$

whose restriction to $\mathbf{k}(V \times V^*)^{\Delta W}$ *is the canonical isomorphism* $\mathbf{k}(V \times V^*)^{\Delta W} \xrightarrow{\sim} \mathrm{Frac}(Z_0) \xrightarrow{\sim} \mathbf{L}_0$. *Here, the isomorphism* $\mathrm{Frac}(Z_0) \xrightarrow{\sim} \mathbf{L}_0$ *is* cop_0.

Convention. *The action of the group* $W \times W$ *on the field* $\mathbf{k}(V \times V^*)$ *is as follows:* $V \times V^*$ *will be seen as a vector subspace of* $\mathbf{k}(V \times V^*)$ *which generates this field, and the action of* (w_1, w_2) *sends* $(y, x) \in V \times V^*$ *to* $(w_1(y), w_2(x))$.

Remark 5.1.10. — The action of $W \times W$ on $\mathbf{k}(V \times V^*)$ described above is not the one obtained by first making $W \times W$ act on the variety $V \times V^*$ and then making it act on the function field $\mathbf{k}(V \times V^*)$ by precomposition: one is deduced from the other thanks to the isomorphism $W \times W \xrightarrow{\sim} W \times W$, $(w_1, w_2) \mapsto (w_2, w_1)$. Nevertheless, this slight difference is important (see Remark 19.7.25). ■

These choices being made, we get a canonical isomorphism $\mathrm{Gal}(\mathbf{M}_0/\mathbf{K}_0) \xrightarrow{\sim} (W \times W)/\Delta Z(W)$, which induces a canonical isomorphism $\mathrm{Gal}(\mathbf{M}_0/\mathbf{L}_0) \xrightarrow{\sim} \Delta W/\Delta Z(W)$. Since $D_0 = \mathrm{Gal}(\mathbf{M}_0/\mathbf{K}_0)$ by (5.1.8), we obtain a group morphism

$$\iota : W \times W \longrightarrow G$$

satisfying the following properties:

Proposition 5.1.11. — (a) $\mathrm{Ker}\,\iota = \Delta Z(W)$.

(b) $\mathrm{Im}\,\iota = D_0$.

(c) $\iota^{-1}(H) = \Delta W$.

Using now (5.1.5), Proposition 5.1.11 provides a bijection

(5.1.12) $$(W \times W)/\Delta W \xleftrightarrow{\sim} G/H.$$

Of course, one can build a bijection between $(W \times W)/\Delta W$ and W using left or right projection. We fix a choice:

Identification. *The morphism* $W \to W \times W$, $w \mapsto (w, 1)$ *composed with the morphism* $\iota : W \times W \to G$ *is injective, and we will identify* W *with its image in* G.

More concretely, $w \in W \subset G$ is the unique automorphism of the P-algebra R such that

$$\big(w(r) \mod \mathfrak{r}_0\big) = (w,1)\big(r \mod \mathfrak{r}_0\big) \quad \text{in } \mathbf{k}(V \times V^*)^{\Delta Z(W)} \tag{5.1.13}$$

for all $r \in R$. Hence, by (5.1.12),

$$G = H \cdot W = W \cdot H \qquad \text{and} \qquad H \cap W = 1. \tag{5.1.14}$$

Corollary 5.1.15. — *Given $c \in \mathscr{C}$, the natural map $D_c \to G/H \xrightarrow{\sim} W$ induces a bijection $D_c/(D_c \cap H) \xrightarrow{\sim} W$.*

Proof. — This follows from (5.1.5) and (5.1.14). □

5.1.C. Action of G on W. — Let $\mathfrak{S}_W$ denote the permutation group of the set W. We identify the group $\mathfrak{S}_{W\setminus\{1\}}$ of permutations of the set $W \setminus \{1\}$ with the stabilizer of 1 in $\mathfrak{S}_W$. The identification $G/H \xleftrightarrow{\sim} W$ and the action of G by left translations on G/H identify G with a subgroup of $\mathfrak{S}_W$. Summarizing, we have

$$G \subseteq \mathfrak{S}_W \qquad \text{and} \qquad H = G \cap \mathfrak{S}_{W\setminus\{1\}}. \tag{5.1.16}$$

Given $g \in G$ and $w \in W$, we denote by $g(w)$ the unique element of W such that $g\iota(w,1)H = \iota(g(w),1)H$. Through this identification of G as a subgroup of $\mathfrak{S}_W$, the map $\iota : W \times W \to G$ is described as follows. Given $(w_1, w_2) \in W \times W$ and $w \in W$, then

$$\iota(w_1, w_2)(w) = w_1 w w_2^{-1}. \tag{5.1.17}$$

This is the action $W \times W$ on the set W by left and right translation. Since ΔW is the stabilizer of $1 \in W$ for this action, we get

$$\iota(\Delta W) = \iota(W \times W) \cap \mathfrak{S}_{W\setminus\{1\}}. \tag{5.1.18}$$

This is of course compatible with Proposition 5.1.11(c) and (5.1.16).

Finally, the choice of the embedding of W in G through $w \mapsto \iota(w,1)$ amounts to identify W with a subgroup of $\mathfrak{S}_W$ through the action on itself by left translation.

5.1.D. Euler element and Galois group. — Let $\mathrm{eu} = \mathrm{cop}(\mathbf{eu}) \in Q$.

Proposition 5.1.19. — *The minimal polynomial of* **eu** *over P has degree $|W|$. Its specialization at c is the minimal polynomial of $\mathbf{eu}_c$ over $P_\bullet$.*

We have $\mathbf{L} = \mathbf{K}[\mathrm{eu}]$.

Proof. — Since $\mathbf{H}_0 = \mathbf{k}[V \oplus V^*] \rtimes W$, we have $\mathrm{Z}(\mathbf{H}_0) = \mathbf{k}[V \times V^*]^{\Delta W}$ and $P_\bullet = \mathbf{k}[V/W \times V^*/W] \subset Z_0$. Moreover, it follows from Theorem 2.2.1 that Z_0 is a free $P_\bullet$-module of rank $|W|$. On the other hand, $\mathbf{eu}_0 = \sum_{i=1}^n x_i y_i$ (using the notation of §3.3). It corresponds to Id_V via the canonical isomorphism $V \otimes V^* \xrightarrow{\sim} \mathrm{End}_{\mathbf{k}}(V)$. Since W acts faithfully on V, it follows that the different elements of W define different elements of $\mathrm{End}_{\mathbf{k}}(V)$. Consequently, the orbit of $\mathbf{eu}_0$ under the action of $W \times W$ has $|W|$ elements. We deduce that the minimal polynomial of $\mathbf{eu}_0$ over $P_\bullet$ has degree $|W|$. As a consequence, the field $\mathbf{k}(V \times V^*)^{\Delta W}$ is generated by $\mathbf{eu}_0$ over $\mathbf{k}(V/W \times V^*/W)$

Let $F_{\mathbf{eu}}(\mathbf{t}) \in P[\mathbf{t}]$ be the minimal polynomial of $\mathbf{eu}$ over P. Since Z is a free P-module of rank $|W|$ (Corollary 4.2.7), we have $\deg F_{\mathbf{eu}} \leqslant |W|$. Since the specialization $\mathbf{eu}_0$ has a minimal polynomial over P_0 of degree $|W|$, it follows that $\deg F_{\mathbf{eu}} = |W|$.

Denote by $\mathfrak{c}$ the prime ideal of $\mathbf{k}[\mathscr{C}]$ corresponding to the line $\mathbf{k}c$. Let F be the minimal polynomial over $P \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{k}[\mathscr{C}]/\mathfrak{c}$ of the image of $\mathbf{eu}$ in the integrally closed domain $Z \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{k}[\mathscr{C}]/\mathfrak{c}$ (Corollary 4.2.7). We have $\deg F \leqslant |W|$. Since $\mathbf{eu}_0$ has a minimal polynomial of degree $|W|$, it follows that $\deg F \geqslant |W|$, hence $\deg F = |W|$, so F is the specialization of $F_{\mathbf{eu}}$.

The Euler element is homogeneous for the $\mathbb{Z}$-grading of Example 3.2.2 (cf Lemma 4.1.5). It follows from Lemma C.2.11 that the specialization of F (hence of $F_{\mathbf{eu}}$) is the minimal polynomial of $\mathbf{eu}_c$ over $P_\bullet$.

The last assertion follows from (5.1.1). $\square$

The computation of the Galois group $G = \mathrm{Gal}(\mathbf{M}/\mathbf{K})$ is now equivalent to the computation of the Galois group of the minimal polynomial of $\mathbf{eu}$ (or eu). Classical methods (reduction modulo a prime ideal, see for instance § B.5) will be useful in small examples.

Let us come back to the computation of the embedding $W \hookrightarrow G \subseteq \mathfrak{S}_W$. Given $w \in W$, let $\mathrm{eu}_w = w(\mathrm{eu}) \in \mathbf{M}$. Recall (see (5.1.16)) that if $g \in G$ and $w \in W$, then $g(w)$ is defined by the equality $g(w)H = gwH$. Since H acts trivially on eu, we deduce that

(5.1.20) $$g(\mathrm{eu}_w) = \mathrm{eu}_{g(w)}$$

and so, given $(w_1, w_2) \in W \times W$, we have

(5.1.21) $$\iota(w_1, w_2)(\mathrm{eu}_w) = \mathrm{eu}_{w_1 w w_2^{-1}}.$$

This extends the equality

(5.1.22) $$w_1(\mathrm{eu}_w) = \mathrm{eu}_{w_1 w}$$

which is an immediate consequence of the definition of eu_w. In particular, by (5.1.14),

(5.1.23) *the minimal polynomial of* eu *over* P *is* $\displaystyle\prod_{w \in W} (\mathbf{t} - \mathrm{eu}_w)$.

Note also that, using (5.1.13) and the convention used for the action of $W \times W$ on $\mathbf{k}(V \times V^*)$, we obtain

$$\mathrm{iso}_0^{-1}(\mathrm{eu}_w \mod \mathfrak{r}_0) = \sum_{i=1}^{n} w(y_i)x_i \in \mathbf{k}[V \times V^*]^{\Delta Z(W)}.$$

The following proposition will not be used in the rest of the book.

Proposition 5.1.24. — *We have* $Z = P[\mathbf{eu}]$ *if and only if* W *is generated by a single reflection.*

Proof. — If W is generated by a single reflection, then an immediate argument allows to reduce to the case where $\dim_{\mathbf{k}} V = 1$. In this case, Example 4.4.12 shows that $Z = P[\mathbf{eu}]$.

Conversely, if $Z = P[\mathbf{eu}]$, then

$$Z = \bigoplus_{j=0}^{|W|-1} P\mathbf{eu}^j$$

since the minimal polynomial of $\mathbf{eu}$ over P has degree $|W|$ (by Proposition 5.1.19). We deduce, using (4.4.8), that

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z) = \frac{\displaystyle\sum_{j=0}^{|W|-1} (\mathbf{tu})^j}{(1-\mathbf{tu})^{|\mathrm{Ref}(W)/W|} \displaystyle\prod_{i=1}^{n}(1-\mathbf{t}^{d_i})(1-\mathbf{u}^{d_i})}.$$

It then follows from (4.4.10) that

$$\frac{1}{|W|}\sum_{w\in W} \frac{(1-\mathbf{t})^n}{\det(1-w\mathbf{t})\,\det(1-w^{-1}\mathbf{u})} = \frac{\displaystyle\sum_{j=0}^{|W|-1} (\mathbf{tu})^j}{\displaystyle\prod_{i=1}^{n}(1+\mathbf{t}+\cdots+\mathbf{t}^{d_i-1})(1-\mathbf{u}^{d_i})}.$$

By specializing $\mathbf{t} \mapsto 1$ in this equality, the left-hand side contributes only whenever $w = 1$. Since $|W| = d_1 \cdots d_n$ by Theorem 2.2.1(a), we obtain

$$\frac{1}{(1-\mathbf{u})^n} = \frac{\displaystyle\sum_{j=0}^{|W|-1} \mathbf{u}^j}{\displaystyle\prod_{i=1}^{n}(1-\mathbf{u}^{d_i})}.$$

In other words,

$$\prod_{i=1}^{n}(1+\mathbf{u}+\cdots+\mathbf{u}^{d_i-1})=\sum_{j=0}^{|W|-1}\mathbf{u}^j.$$

By comparison of the degrees, we get

$$|W|-1=\sum_{i=1}^{n}(d_i-1).$$

But, again by Theorem 2.2.1(a), we have $|\mathrm{Ref}(W)|=\sum_{i=1}^{n}(d_i-1)$, which shows that

$$\mathrm{Ref}(W)=W\setminus\{1\}.$$

Therefore, if $w,\ w'\in W$, then $ww'w^{-1}w'^{-1}$ has determinant 1, so it cannot be a reflection. So $ww'=w'w$ and W is abelian, hence diagonalizable. The proposition follows. □

5.2. Splitting the algebra KH

Recall that Theorem 4.3.7 shows the existence of an isomorphism

$$\mathbf{K}\mathbf{H}\simeq\mathrm{Mat}_{|W|}(\mathbf{K}Z).$$

Recall also that $\mathbf{K}Z$ is the fraction field of Z (see (4.3.6)) and that $\mathrm{cop}:\mathbf{K}Z\xrightarrow{\sim}\mathbf{L}$ denotes the extension of $\mathrm{cop}:Z\xrightarrow{\sim}Q$. The $\mathbf{K}$-algebra $\mathbf{K}\mathbf{H}$ is semisimple, but not $\mathbf{K}$-split in general.

Given $g\in G$, the morphism $\mathbf{K}Z\to\mathbf{M}$, $z\mapsto g(\mathrm{cop}(z))$ obtained by restriction of g to $\mathbf{L}$ (through the isomorphism cop) is $\mathbf{K}$-linear and it extends uniquely to a morphism of $\mathbf{M}$-algebras

$$\begin{array}{rccc} g_Z: & \mathbf{M}\otimes_{\mathbf{K}}\mathbf{K}Z & \longrightarrow & \mathbf{M}\\ & m\otimes_{\mathbf{K}} z & \longmapsto & mg(\mathrm{cop}(z)).\end{array}$$

Of course, $g_Z=(gh)_Z$ for all $h\in H$ and it is a classical fact (see the Proposition B.3.12) that

$$(g_Z)_{gH\in G/H}:\mathbf{M}\otimes_{\mathbf{K}}\mathbf{K}Z\longrightarrow\prod_{gH\in G/H}\mathbf{M}$$

is an isomorphism of $\mathbf{M}$-algebras. Taking (5.1.14) into account, this can be rewritten as follows: there is an isomorphism of $\mathbf{M}$-algebras

$$\begin{array}{ccc} \mathbf{M}\otimes_{\mathbf{K}}\mathbf{K}Z & \xrightarrow{\sim} & \prod_{w\in W}\mathbf{M}\\ x & \longmapsto & (w_Z(x))_{w\in W}\end{array}. \tag{5.2.1}$$

So, the $\mathbf{M}$-algebra $\mathbf{M}\otimes_{\mathbf{K}}\mathbf{K}Z$ is semisimple and split, and its simple representations are the w_Z, for $w\in W$.

Theorem 4.3.7 provides a Morita equivalence between $\mathbf{M}\otimes_{\mathbf{K}}\mathbf{K}Z$ and $\mathbf{M}\mathbf{H}$. We will denote by $\mathscr{L}_w$ the simple $\mathbf{M}\mathbf{H}$-module corresponding to w_Z.

Fix an ordered $\mathbf{K}Z$-basis $\mathscr{B}$ of $\mathbf{KH}e$ (recall that $|\mathscr{B}| = |W|$). This choices provides an isomorphism of $\mathbf{K}$-algebras

$$\rho^{\mathscr{B}} : \mathbf{KH} \xrightarrow{\sim} \mathrm{Mat}_{|W|}(\mathbf{K}Z).$$

Now, given $w \in W$, let $\rho_w^{\mathscr{B}}$ denote the morphism of $\mathbf{M}$-algebras $\mathbf{MH} \longrightarrow \mathrm{Mat}_{|W|}(\mathbf{M})$ defined by

$$\rho_w^{\mathscr{B}}(m \otimes_P h) = m \cdot w(\mathrm{cop}(\rho^{\mathscr{B}}(h)))$$

for all $m \in \mathbf{M}$ and $h \in \mathbf{H}$. Then $\rho_w^{\mathscr{B}}$ is an irreducible representation of $\mathbf{MH}$ corresponding to the simple module $\mathscr{L}_w$.

Let $\mathrm{Irr}(\mathbf{MH})$ denote the set of isomorphism classes of simple $\mathbf{MH}$-modules. We have a bijection

$$\begin{array}{ccc} W & \xrightarrow{\sim} & \mathrm{Irr}\,\mathbf{MH} \\ w & \longmapsto & \mathscr{L}_w \end{array} \tag{5.2.2}$$

and an isomorphism of $\mathbf{M}$-algebras

$$\prod_{w \in W} \rho_w^{\mathscr{B}} : \mathbf{MH} \xrightarrow{\sim} \prod_{w \in W} \mathrm{Mat}_{|W|}(\mathbf{M}). \tag{5.2.3}$$

In particular,

(5.2.4) *the* $\mathbf{M}$-*algebra* $\mathbf{MH}$ *is split semisimple.*

Moreover, the bijection (5.2.2) allows us to identify its Grothendieck group $K_0(\mathbf{MH})$ with the $\mathbb{Z}$-module $\mathbb{Z}W$:

$$K_0(\mathbf{MH}) \xrightarrow{\sim} \mathbb{Z}W, \ [\mathscr{L}_w] \mapsto w. \tag{5.2.5}$$

Since the $\mathbf{M}$-algebra $\mathbf{MH}$ is split semisimple, it follows from [**GePf**, Theorem 7.2.6 and Proposition 7.3.9] that there exists a unique family $(\mathrm{sch}_w)_{w \in W}$ of elements of R such that

$$\tau_{\mathbf{MH}} = \sum_{w \in W} \frac{\mathrm{car}_w}{\mathrm{sch}_w},$$

where $\mathrm{car}_w : \mathbf{MH} \to \mathbf{M}$ denotes the character of the simple $\mathbf{MH}$-module $\mathscr{L}_w$ and $\tau_{\mathbf{MH}} : \mathbf{MH} \to \mathbf{M}$ denotes the extension of the symmetrizing form $\tau_{\mathbf{H}} : \mathbf{H} \to P$. The element sch_w of R is called the *Schur element* associated with the simple module $\mathscr{L}_w$. By [**GePf**, Theorem 7.2.1], $|W| \cdot \mathrm{sch}_w$ is equal to the scalar by which the Casimir element $\mathrm{cas}_{\mathbf{H}} \in Z$ (defined in § 4.4.C) acts on the simple module $\mathscr{L}_w$. Therefore,

$$\mathrm{sch}_w = |W|^{-1} \cdot w(\mathrm{cop}(\mathrm{cas}_{\mathbf{H}})). \tag{5.2.6}$$

Remark 5.2.7. — In the general theory of symmetric algebras, the Schur element sch_w is an important invariant, which can be useful to determine the blocks of a reduction of $R\mathbf{H}$ modulo some prime ideal of R. Here, the formula (5.2.6) shows that this computation is equivalent to the resolution of the following two problems:

(1) Compute the Casimir element $\mathrm{cas}_{\mathbf{H}}$.
(2) Understand the action of W (or G) on the image of $\mathrm{cas}_{\mathbf{H}}$ in $Q \subset R$.

If Problem (1) seems doable (and its solution would be interesting as it would provide, after the Euler element, a new element of the center Z of $\mathbf{H}$), it seems however more difficult to attack Problem (2), as the computation of the ring R (and even of the Galois group G) is for the moment out of reach. ■

5.3. Grading on R

Proposition 5.3.1. — *There exists a unique* $(\mathbb{N}\times\mathbb{N})$*-grading on R extending the one of Q. The Galois group G stabilizes this* $(\mathbb{N}\times\mathbb{N})$*-grading.*

Proof. — The proposition is a consequence of Propositions C.2.8 and C.2.4 and of Proposition C.2.1 and Corollary C.2.2. □

Let $R = \bigoplus_{(i,j)\in\mathbb{N}\times\mathbb{N}} R^{\mathbb{N}\times\mathbb{N}}[i,j]$ denote the $(\mathbb{N}\times\mathbb{N})$-grading extending the one of Q. Similarly, $R = \bigoplus_{i\in\mathbb{N}} R^{\mathbb{N}}[i]$ (respectively $R = \bigoplus_{i\in\mathbb{Z}} R^{\mathbb{Z}}[i]$) will denote the $\mathbb{N}$-grading (respectively $\mathbb{Z}$-grading) extending the one of Q: in other words,

$$R^{\mathbb{N}}[i] = \bigoplus_{i_1+i_2=i} R^{\mathbb{N}\times\mathbb{N}}[i_1,i_2] \quad\text{and}\quad R^{\mathbb{Z}}[i] = \bigoplus_{i_1+i_2=i} R^{\mathbb{N}\times\mathbb{N}}[i_1,i_2].$$

Corollary C.2.10 provides the following stability result.

Corollary 5.3.2. — *The prime ideal $\mathfrak{r}_0$ of R chosen in § 5.1.B is bi-homogeneous (in particular, it is homogeneous).*

Corollary 5.3.3. — *We have* $R^{\mathbb{N}\times\mathbb{N}}[0,0] = \mathbf{k}$.

Proof. — By Corollary 5.3.2, we have $\mathfrak{r}_0 \subset R_+$. Consequently, $R^{\mathbb{N}\times\mathbb{N}}[0,0]$ is isomorphic to the homogeneous component of bidegree $(0,0)$ of $R/\mathfrak{r}_0$. Since $k_R(\mathfrak{r}_0) \simeq \mathbf{k}(V\times V^*)^{\Delta Z(W)}$ and $R/\mathfrak{r}_0$ is integral over $Q_0 = \mathbf{k}[V\times V^*]^{\Delta W}$, it follows that $R/\mathfrak{r}_0 \subset \mathbf{k}[V\times V^*]^{\Delta Z(W)}$, and this inclusion preserves the bigrading, by the uniqueness of the bigrading on $R/\mathfrak{r}_0$ extending the one of $Q_0 = \mathbf{k}[V\times V^*]^{\Delta W}$ (Proposition C.2.1). This shows the result. □

Denote by R_+ the unique maximal bi-homogeneous ideal of R.

Corollary 5.3.4. — *Let D_+ (respectively I_+) be the decomposition (respectively inertia) group of R_+ in G. Then $D_+ = I_+ = G$.*

Proof. — Let $\mathfrak{p}_+ = R_+ \cap P$. Then $k_R(R_+)/k_P(\mathfrak{p}_+)$ is a Galois extension with Galois group D_+/I_+ (see Theorem B.2.4). By Corollary 5.3.3, $k_R(R_+) = \mathbf{k} = k_P(\mathfrak{p}_+)$, so $D_+/I_+ = 1$. Note finally that $D_+ = G$ by Proposition 5.3.1. $\square$

5.4. Action on R of natural automorphisms of $\mathbf{H}$

The previous Section 5.3 was concerned with the extension to R of the automorphisms of Q induced by $\mathbf{k}^\times \times \mathbf{k}^\times$. In Section 3.5, we have introduced an action of $W^\wedge \rtimes \mathscr{N}$ on $\mathbf{H}$ which stabilizes Z (of course), P, but also $\mathfrak{p}_0$ and so $\mathfrak{p}_0 Z$: this action can be transferred to $Q \simeq Z$ and still stabilizes $\mathfrak{q}_0 = \mathfrak{p}_0 Q$. We will show how to extend this action to R, and we will derive some consequences about the Galois group. For this, we will work in a more general framework:

Assumption. *In this section 5.4, we fix a group $\mathscr{G}$ acting both on Z and $\mathbf{k}[V \times V^*]$ and satisfying the following properties:*
(1) *$\mathscr{G}$ stabilizes P and $\mathfrak{p}_0$.*
(2) *The action of $\mathscr{G}$ on $\mathbf{k}[V \times V^*]$ normalizes the action of $W \times W$ and the one of ΔW.*
(3) *The canonical isomorphism of $\mathbf{k}$-algebras $Z_0 \xrightarrow{\sim} \mathbf{k}[V \times V^*]^{\Delta W}$ is $\mathscr{G}$-equivariant.*

The action of $\mathscr{G}$ on Z induces, through the isomorphism cop, an action of $\mathscr{G}$ on Q. If $\tau \in \mathscr{G}$, we denote by τ° the automorphism of $\mathbf{k}[V \times V^*]$ induced by τ: by (2), τ° stabilizes $\mathbf{k}[V \times V^*]^{\Delta Z(W)}$, $\mathbf{k}[V \times V^*]^{\Delta W}$ and $\mathbf{k}[V \times V^*]^{W \times W}$.

Proposition 5.4.1. — *If $\tau \in \mathscr{G}$, then there exists a unique extension $\tilde{\tau}$ of τ to R satisfying the following two properties:*

(1) $\tilde{\tau}(\mathfrak{r}_0) = \mathfrak{r}_0$*;*
(2) *The automorphism of $R/\mathfrak{r}_0$ induced by $\tilde{\tau}$ is equal to τ°, via the identification* $\mathrm{iso}_0 : \mathbf{k}(V \times V^*)^{\Delta Z(W)} \xrightarrow{\sim} \mathbf{M}_0$ *of §5.1.B.*

Proof. — Let us start by showing the existence. First of all, $\mathbf{M}$ being a Galois closure of the extension $\mathbf{L}/\mathbf{K}$, there exists an extension $\tau_{\mathbf{M}}$ of τ to $\mathbf{M}$. Since R is the integral closure of Q in $\mathbf{M}$, it follows that $\tau_{\mathbf{M}}$ stabilizes R. Moreover, since $\tau(\mathfrak{q}_0) = \mathfrak{q}_0$, there exists $h \in H$ such that $\tau_{\mathbf{M}}(\mathfrak{r}_0) = h(\mathfrak{r}_0)$. Let $\tilde{\tau}_{\mathbf{M}} = h^{-1} \circ \tau_{\mathbf{M}}$. Then

$$\tilde{\tau}_{\mathbf{M}}(\mathfrak{r}_0) = \mathfrak{r}_0 \qquad \text{and} \qquad (\tilde{\tau}_{\mathbf{M}})|_{\mathbf{L}} = \tau.$$

Let $\tilde{\tau}_{\mathbf{M},0}$ denote the automorphism of $R/\mathfrak{r}_0$ induced by $\tilde{\tau}_{\mathbf{M}}$.

By construction, the restriction of $\tilde{\tau}_{\mathbf{M},0}$ to $Q/\mathfrak{q}_0$ is equal to the restriction of $\mathrm{iso}_0 \circ \tau^\circ \circ \mathrm{iso}_0^{-1}$. Hence, there exists $d \in D_0 \cap H$ such that $\tilde{\tau}_{\mathbf{M},0} = d \circ (\mathrm{iso}_0 \circ \tau_0 \circ \mathrm{iso}_0^{-1})$. We then set $\tilde{\tau} = d^{-1} \circ \tilde{\tau}_{\mathbf{M}}$: it is clear that $\tilde{\tau}$ satisfies (1) and (2).

Let us now show the uniqueness. Let $\tilde{\tau}_1$ be another extension of τ to R satisfying (1) and (2) and let $\sigma = \tilde{\tau}^{-1}\tilde{\tau}_1$. We have $\sigma \in G$ and, by (1), σ stabilizes $\mathfrak{r}_0$, hence $\sigma \in D_0$. Moreover, by (2), σ induces the identity on $R/\mathfrak{r}_0$, hence $\sigma \in I_0 = 1$ (cf (5.1.6)), and so $\tilde{\tau} = \tilde{\tau}_1$. □

The existence and the uniqueness statements of Proposition 5.4.1 have the following consequences.

Corollary 5.4.2. — *The action of $\mathscr{G}$ on Q extends uniquely to an action of $\mathscr{G}$ on R, which stabilizes $\mathfrak{r}_0$ and is compatible with the isomorphism* iso_0.

In this book, we will denote again by τ the extension $\tilde{\tau}$ of τ defined in Proposition 5.4.1. Since $\mathscr{G}$ stabilizes P, Q, $\mathfrak{p}_0$, $\mathfrak{q}_0 = \mathfrak{p}_0 Q$ and $\mathfrak{r}_0$, we deduce the following.

Corollary 5.4.3. — *The action of $\mathscr{G}$ on R normalizes G, H, $D_0 = \iota(W \times W)$ and $D_0 \cap H = \iota(\Delta W) = W/\mathrm{Z}(W)$.*

From Corollary 5.4.3, we deduce that $\mathscr{G}$ acts on the set $G/H \simeq W$ and that

(5.4.4) *the image of $\mathscr{G}$ in $\mathfrak{S}_W$ normalizes G.*

Example 5.4.5. — The group $\mathscr{G} = \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ acts on $\mathbf{H}$ and stabilizes P and $\mathfrak{p}_0$; by the same formulas, it acts on $\mathbf{k}[V \times V^*]$ and normalizes $W \times W$ and ΔW (in fact, $\mathbf{k}^\times \times \mathbf{k}^\times \times \mathrm{Hom}(W, \mathbf{k}^\times)$ commutes with $W \times W$ and only $\mathscr{N}$ acts non-trivially on $W \times W$).

It follows from the previous results that the action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ on Q extends uniquely to an action on R which stabilizes $\mathfrak{r}_0$ and is compatible with the isomorphism iso_0. By the uniqueness statement, the extension of the action of $\mathbf{k}^\times \times \mathbf{k}^\times \times W^\wedge$ to R commutes with the action of G whereas the one of $\mathscr{N}$ is such that the morphism $G \hookrightarrow \mathfrak{S}_W$ is $\mathscr{N}$-equivariant.

Finally, still by the uniqueness statement, the extension of the action of the subgroup $\mathbf{k}^\times \times \mathbf{k}^\times$ corresponds to the extension to R of the $(\mathbb{N} \times \mathbb{N})$-grading described in Proposition 5.3.1. ■

5.5. A particular situation: reflections of order 2

Assumption and notation. *In this section 5.5, we assume that all the reflections of W have order 2 and that $-\mathrm{Id}_V \in W$. We set $w_0 = -\mathrm{Id}_V$ and $\tau_0 = (-1, 1, \varepsilon) \in \mathbf{k}^\times \times \mathbf{k}^\times \times W^\wedge$.*

By construction, the restriction of τ_0 to $\mathbf{k}[\mathscr{C}]$ is equal to the identity. Since $-\mathrm{Id}_V \in W$, the restriction of τ_0 to $\mathbf{k}[V]^W$ is equal to the identity. Similarly, the restriction of τ_0 to $\mathbf{k}[V^*]^W$ is also equal to the identity. Consequently,

$$\forall\ p \in P,\ \tau_0(p) = p. \tag{5.5.1}$$

Recall that τ_0 denotes also the automorphism of R defined by Proposition 5.4.1. By definition of the Galois group, we have $\tau_0 \in G$. More precisely, we have the following description.

Proposition 5.5.2. — *Assume that all the reflections of W have order 2 and that $w_0 = -\mathrm{Id}_V \in W$. Then τ_0 is a central element of G. Its action on W is given by $\tau_0(w) = w_0 w$ (which means that $\tau_0 = w_0 = \iota(w_0, 1)$, through the canonical embedding $W \hookrightarrow G$) and, through the embedding $G \hookrightarrow \mathfrak{S}_W$, we have*

$$G \subset \{\sigma \in \mathfrak{S}_W \mid \forall\ w \in W,\ \sigma(w_0 w) = w_0 \sigma(w)\}.$$

Moreover, given $w \in W$, we have

$$\tau_0(\mathrm{eu}_w) = -\mathrm{eu}_w = \mathrm{eu}_{w_0 w}.$$

Proof. — By Lemma 3.5.7(c), we have $\tau_0(\mathrm{eu}) = -\mathrm{eu}$. Moreover, by Example 5.4.5, the action of τ_0 on R commutes with the action of G. Therefore, if $w \in W$, then $\tau_0(\mathrm{eu}_w) = -\mathrm{eu}_w$.

On the other hand, there exists $w_1 \in W$ such that $\tau_0(\mathrm{eu}) = \mathrm{eu}_{w_1}$. As $-\mathrm{eu}_0 = w_0(\mathrm{eu}_0)$, it follows from the characterization of the action of W on $\mathbf{L}$ that $\tau_0(\mathrm{eu}) = \mathrm{eu}_{w_0} = -\mathrm{eu}$. Since w_0 is central in W, we have $w_0(\mathrm{eu}_w) = \mathrm{eu}_{w_0 w} = \mathrm{eu}_{w w_0} = w(\mathrm{eu}_{w_0}) = -\mathrm{eu}_w$. So $\tau_0 = w_0$ because $\mathbf{M} = \mathbf{K}[(\mathrm{eu}_w)_{w \in W}]$.

Now, the fact that $G \subset \{\sigma \in \mathfrak{S}_W \mid \forall\ w \in W,\ \sigma(w_0 w) = w_0 \sigma(w)\}$ follows from the fact that $\tau_0 = w_0$ commutes with the action of G. ☐

Notice that $w_0 w = -w$ and so the inclusion of Proposition 5.5.2 can be rewritten

$$G \subset \{\sigma \in \mathfrak{S}_W \mid \forall\ w \in W,\ \sigma(-w) = -\sigma(w)\}. \tag{5.5.3}$$

Viewed like this, it shows that, under the assumption of this section, G is contained in a Weyl group of type $B_{|W|/2}$.

5.6. Special features of Coxeter groups

Assumption. *In this section 5.6, we assume that W is a Coxeter group, and we use the notation of §2.6.*

Recall (Proposition 4.5.1) that the algebra $\mathbf{H}$ admits another automorphism $\sigma_{\mathbf{H}}$ stabilizing P.

Proposition 5.6.1. — *The automorphism $\sigma_{\mathbf{H}}$ of Q extends uniquely to an automorphism $\sigma_{\mathbf{H}}$ of R. Given $g \in G \subset \mathfrak{S}_W$ and $w \in W$, we have $({}^{\sigma_{\mathbf{H}}}g)(w) = g(w^{-1})^{-1}$.*

Proof. — Note that $\sigma_{\mathbf{H}}$ induces an automorphism of $\mathbf{k}[V \times V^*]$ which normalizes $W \times W$ and ΔW. More precisely, consider $\sigma_2 : V \oplus V^* \xrightarrow{\sim} V \oplus V^*, (y, x) \longmapsto (-\sigma^{-1}(x), \sigma(y))$ We have

$$\sigma_2(w, w')\sigma_2^{-1} = (w', w) \tag{5.6.2}$$

for all $(w, w') \in W \times W$. By Proposition 5.4.1, $\sigma_{\mathbf{H}}$ extends uniquely to an automorphism of R which stabilizes $\mathfrak{r}_0$ and which is compatible with iso_0. Since $\sigma_{\mathbf{H}}$ normalizes G and its subgroup $\iota(W \times W)$ (see (5.4.4)), it follows from (5.6.2) that its action on the elements of $W \subset G$ satisfies

$$ {}^{\sigma_{\mathbf{H}}}wH = w^{-1}H \qquad \text{and} \qquad H\,{}^{\sigma_{\mathbf{H}}}w = Hw^{-1} \tag{5.6.3}$$

for all $w \in W$. The proposition follows now from (5.4.4). □

Remark 5.6.4. — Note that if $W \neq 1$, then the action of $\mathbf{GL}_2(\mathbf{k})$ on $\mathbf{H}$ doesn't induce an action on R, since $\mathbf{GL}_2(\mathbf{k})$ doesn't normalize $W \times W$. ■

5.7. Geometry

5.7.A. Extension Z/P. — The $\mathbf{k}$-algebras P, Z, $P_\bullet$ and Z_c being of finite type, we can associate with them algebraic $\mathbf{k}$-varieties that will be denoted by $\mathscr{P}$, $\mathscr{Z}$, $\mathscr{P}_\bullet$ and $\mathscr{Z}_c$ respectively. Notice that

$$\mathscr{P} = \mathscr{C} \times V/W \times V^*/W \quad \text{and} \quad \mathscr{P}_\bullet = V/W \times V^*/W$$

and that

$$\mathscr{Z}_0 = (V \times V^*)/W.$$

It follows from Corollary 4.2.7(f) that

(5.7.1) *the varieties $\mathscr{Z}$ and $\mathscr{Z}_c$ are irreducible and normal.*

Since all the algebraic statements of the previous chapters do not depend on the base field, the statement (5.7.1) can be understood as a "geometric" statement. The inclusions $P \subset Z$ and $P_\bullet \subset Z_c$ induce morphisms of varieties

$$\Upsilon : \mathscr{Z} \longrightarrow \mathscr{P} = \mathscr{C} \times V/W \times V^*/W$$

and $$\Upsilon_c : \mathscr{Z}_c \longrightarrow \mathscr{P}_\bullet = V/W \times V^*/W.$$

The surjective maps $P \to P/\mathfrak{C}_c P \simeq P_\bullet$ and $Z \to Z_c$ induce closed immersions $j_c : \mathscr{Z}_c \hookrightarrow \mathscr{Z}$ and $i_c : \mathscr{P}_\bullet \hookrightarrow \mathscr{P}$, $p \mapsto (c,p)$. Moreover, the canonical injective map $\mathbf{k}[\mathscr{C}] \hookrightarrow P$ induces the canonical projection $\pi : \mathscr{P} \to \mathscr{C}$ and, in the diagram

(5.7.2)
$$\begin{tikzcd}
& \mathscr{Z}_c \arrow[r, hook, "j_c"] \arrow[d, "\Upsilon_c"'] & \mathscr{Z} \arrow[d, "\Upsilon"] & \\
V/W \times V^*/W \arrow[r, equal] & \mathscr{P}_\bullet \arrow[r, hook, "i_c"] \arrow[d] & \mathscr{P} \arrow[r, equal] \arrow[d, "\pi"] & \mathscr{C} \times V/W \times V^*/W \\
& \{c\} \arrow[r, hook] & \mathscr{C} = \mathbf{A}^{\mathrm{Ref}(W)/W} &
\end{tikzcd},$$

all the squares are cartesian. Note also that, by Corollary 4.2.7,

(5.7.3) *the morphisms Υ and Υ_c are finite and flat.*

Moreover,

(5.7.4) *π is smooth,*

since $V/W \times V^*/W$ smooth.

Example 5.7.5. — We have $\mathscr{Z}_0 = (V \times V^*)/W$ and $\Upsilon_0 : (V \times V^*)/W = \mathscr{Z}_0 \to \mathscr{P}_\bullet = V/W \times V^*/W$ is the canonical morphism. ■

Let $\mathscr{Z}^{\mathrm{reg}}$ denote the open subset $\mathrm{Spec}(Z^{\mathrm{reg}})$ of $\mathscr{Z}$. Corollary 4.3.2 shows that

(5.7.6) *$\mathscr{Z}^{\mathrm{reg}} \simeq \mathscr{C} \times (V^{\mathrm{reg}} \times V^*)/W$ is smooth.*

5.7.B. Extension R/P. — Since R and $Q \simeq Z$ are also $\mathbf{k}$-algebras of finite type, they are associated with $\mathbf{k}$-varieties $\mathscr{R}$ and $\mathscr{Q} \simeq \mathscr{Z}$: the isomorphism $\mathrm{cop}^* : \mathscr{Q} \xrightarrow{\sim} \mathscr{Z}$ is induced by $\mathrm{cop} : Z \xrightarrow{\sim} Q$. The inclusion $P \hookrightarrow R$ (respectively $Q \hookrightarrow R$) defines a morphism of varieties $\rho_G : \mathscr{R} \to \mathscr{P}$ (respectively $\rho_H : \mathscr{R} \to \mathscr{Q}$) and the equalities $P = R^G$ and $Q = R^H$ show that ρ_G and ρ_H induce isomorphisms

$$\mathscr{R}/G \xrightarrow{\sim} \mathscr{P} \qquad \text{and} \qquad \mathscr{R}/H \xrightarrow{\sim} \mathscr{Q}. \tag{5.7.7}$$

In this setting, the choice of a prime ideal $\mathfrak{r}_c$ lying over $\mathfrak{q}_c$ is equivalent to the choice of an irreducible component $\mathscr{R}_c$ of $\rho_H^{-1}(\mathscr{Q}_c)$ (whose ideal of definition is $\mathfrak{r}_c$). Similarly, the argument leading to Proposition 5.1.11 implies for instance that the number of irreducible components of $\rho_G^{-1}(\mathscr{Q}_0)$ is equal to $|G|\cdot|\Delta Z(W)|/|W|^2$. It also shows that $\iota(W \times W)$ is the stabilizer of $\mathscr{R}_0$ in G and $\mathscr{R}_0/\iota(W \times W) \simeq \mathscr{P}_0$, that $\iota(\Delta W)$ is the stabilizer of $\mathscr{R}_0$ in H and that $\mathscr{R}_0/\iota(\Delta W) \simeq \mathscr{Q}_0$. We have a commutative diagram

$$\begin{tikzcd}
 & \mathscr{R}_c \arrow[r, hook] \arrow[d] & \mathscr{R} \arrow[d, "\rho_H"] \arrow[dd, bend left=60, "\rho_G"] & \\
 & \mathscr{Q}_c \arrow[r, hook, "j_c"] \arrow[d, "\Upsilon_c"'] & \mathscr{Q} \arrow[d, "\Upsilon"] & \\
V/W \times V^*/W \arrow[r, equal] & \mathscr{P}_\bullet \arrow[r, hook, "i_c"] \arrow[d] & \mathscr{P} \arrow[r, equal] \arrow[d, "\pi"] & \mathscr{C} \times V/W \times V^*/W \\
 & \{c\} \arrow[r, hook] & \mathscr{C} &
\end{tikzcd} \tag{5.7.8}$$

which completes the diagram (5.7.2) (if we identify $\mathscr{Q}$ and $\mathscr{Z}$ through cop^*). Only the two bottom squares of the diagram (5.7.8) are cartesian.

5.7.C. Automorphisms. — The group $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ (which acts on $\mathbf{H}$ through automorphisms of $\mathbf{k}$-algebras) stabilizes the $\mathbf{k}$-subalgebras $\mathbf{k}[\mathscr{C}]$, P and Q of $\mathbf{H}$. Therefore, it acts by automorphisms of the $\mathbf{k}$-varieties on $\mathscr{C}$, $\mathscr{P}$ and $\mathscr{Q}$. The morphisms Υ and π of diagram (5.7.2) are equivariant for this action.

Also, this action extends to an action on $\mathscr{R}$ (see Corollary 5.4.2) which makes ρ_H and ρ_G equivariant.

5.7.D. Irreducible components of $\mathscr{R} \times_{\mathscr{P}} \mathscr{Z}$. — Given $w \in W$, we set

$$\mathscr{R}_w = \{(r, \mathrm{cop}^*(\rho_H(w(r)))) \mid r \in \mathscr{R}\} \subseteq \mathscr{R} \times_{\mathscr{P}} \mathscr{Z}.$$

Lemma 5.7.9. — *If $w \in W$, then $\mathscr{R}_w$ is an irreducible component of $\mathscr{R} \times_{\mathscr{P}} \mathscr{Z}$, isomorphic to $\mathscr{R}$. Moreover,*

$$\mathscr{R} \times_{\mathscr{P}} \mathscr{Z} = \bigcup_{w \in W} \mathscr{R}_w$$

and $\mathscr{R}_w = \mathscr{R}_{w'}$ if and only if $w = w'$.

Proof. — It is only the geometric translation of the fact that the morphism

$$\begin{array}{ccc} R \otimes_P Z & \longrightarrow & \prod_{w \in W} R \\ x & \longmapsto & (w_Z(x))_{w \in W} \end{array}$$

defined by restriction from the morphism (5.2.1) is finite and and becomes an isomorphism after extending scalars to $\mathbf{K}$. □

CHAPTER 6

CALOGERO-MOSER CELLS

> **Notation.** *From now on, and until the end of §6, we fix a prime ideal $\mathfrak{r}$ of R and we set $\mathfrak{q} = \mathfrak{r} \cap Q$ and $\mathfrak{p} = \mathfrak{r} \cap P$. We denote by $D_\mathfrak{r}$ (respectively $I_\mathfrak{r}$) the decomposition (respectively inertia) group of $\mathfrak{r}$ in G.*

6.1. Definition, first properties

Recall that, since we have chosen once and for all a prime ideal $\mathfrak{r}_0$ as well as an isomorphism $k_R(\mathfrak{r}_0) \xrightarrow{\sim} \mathbf{k}(V \times V^*)^{\Delta Z(W)}$, we can identify the sets G/H and W (see §5.1.B). So G acts on the *set* W.

Definition 6.1.1. — *A **Calogero-Moser** $\mathfrak{r}$**-cell** is an orbit of the inertia group $I_\mathfrak{r}$ in the **set** W. We will denote by $\sim_\mathfrak{r}^{\mathrm{CM}}$ the equivalence relation corresponding to the partition of W into Calogero-Moser $\mathfrak{r}$-cells.*

The set of Calogero-Moser $\mathfrak{r}$-cells will be denoted by ${}^{\mathrm{CM}}\mathrm{Cell}_\mathfrak{r}(W)$.

Recall that W can be identified with the set $\mathrm{Hom}_{P-\mathrm{alg}}(Q,R) = \mathrm{Hom}_{\mathbf{K}-\mathrm{alg}}(\mathbf{L},\mathbf{M})$. By Proposition B.3.5, if w and w' are two elements of W, then

(6.1.2) $$w \sim_\mathfrak{r}^{\mathrm{CM}} w' \textit{ if and only if } w(q) \equiv w'(q) \mod \mathfrak{r} \textit{ for all } q \in Q.$$

Remark 6.1.3. — If $\mathfrak{r}$ and $\mathfrak{r}'$ are two prime ideals of R such that $\mathfrak{r} \subset \mathfrak{r}'$, then $I_\mathfrak{r} \subset I_{\mathfrak{r}'}$ and so the Calogero-Moser $\mathfrak{r}'$-cells are unions of Calogero-Moser $\mathfrak{r}$-cells. ■

***Example 6.1.4* (Reflections or order 2)**. — If all the reflections of W have order 2 and if $w_0 = -\mathrm{Id}_V \in W$, then it follows from Proposition 5.5.2 that $G \subset \{\sigma \in \mathfrak{S}_W \mid \forall\, w \in W,\ \sigma(w_0 w) = w_0\sigma(w)\}$. Consequently, if Γ is a Calogero-Moser $\mathfrak{r}$-cell, then $w_0\Gamma = \Gamma w_0$ is a Calogero-Moser $\mathfrak{r}$-cell. ■

The action of G being compatible with the bigrading of R, the following result is not surprising.

Proposition 6.1.5. — *Let Γ be a finitely generated free abelian group and $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ a G-stable Γ-grading on R. Let $\tilde{\mathfrak{r}} = \bigoplus_{\gamma\in\Gamma} \mathfrak{r} \cap R_\gamma$. Then $I_\mathfrak{r} = I_{\tilde{\mathfrak{r}}}$, hence the Calogero-Moser $\tilde{\mathfrak{r}}$-cells and the Calogero-Moser $\mathfrak{r}$-cells coincide.*

Proof. — This follows from Corollary C.2.13. □

6.2. Blocks

Given $w \in W$, we denote by $e_w \in \mathrm{Idem}_{\mathrm{pr}}(Z(\mathbf{MH}))$ the central primitive idempotent of $\mathbf{MH}$ (which is split semisimple by (5.2.4)) associated with the simple module $\mathscr{L}_w$. It is the unique central primitive idempotent of $\mathbf{MH}$ which acts as the identity on the simple $\mathbf{MH}$-module $\mathscr{L}_w$. Given $b \in \mathrm{Idem}_{\mathrm{pr}}(Z(R_\mathfrak{r}\mathbf{H}))$, we denote by $\mathrm{CM}_\mathfrak{r}(b)$ the unique subset of W such that

(6.2.1)
$$b = \sum_{w\in\mathrm{CM}_\mathfrak{r}(b)} e_w.$$

In other words, the bijection $W \stackrel{\sim}{\longleftrightarrow} \mathrm{Irr}\,\mathbf{MH}$ restricts to a bijection $\mathrm{CM}_\mathfrak{r}(b) \stackrel{\sim}{\longleftrightarrow} \mathrm{Irr}\,\mathbf{MH}b$. It is clear that $(\mathrm{CM}_\mathfrak{r}(b))_{b\in\mathrm{Idem}_{\mathrm{pr}}(R_\mathfrak{r}\mathbf{H})}$ is a partition of W. In fact, this partition coincides with the partition into Calogero-Moser $\mathfrak{r}$-cells.

Theorem 6.2.2. — *Let w, $w' \in W$ and let b and b' be the central primitive idempotents of $R_\mathfrak{r}\mathbf{H}$ such that $w \in \mathrm{CM}_\mathfrak{r}(b)$ and $w' \in \mathrm{CM}_\mathfrak{r}(b')$. Then $w \sim_{\mathrm{CM}}^{\mathfrak{r}} w'$ if and only if $b = b'$.*

Proof. — Let $\omega_w : \mathrm{Z}(R\mathbf{H}) = R \otimes_P Z \longrightarrow R$ denote the central character associated with the simple $\mathbf{MH}$-module $\mathscr{L}_w$ (see § D.2.A). By the very definition of $\mathscr{L}_w$, we have

$$\omega_w(r \otimes_P z) = r\, w(\mathrm{cop}(z))$$

for all $r \in R$ and $z \in Z$. Consequently, by (6.1.2), we have $w \sim_{\mathrm{CM}}^{\mathfrak{r}} w'$ if and only if $\omega_w \equiv \omega_{w'} \mod \mathfrak{r}$. The result follows now from Corollary D.2.4. □

Via Proposition D.2.3, we obtain bijections

(6.2.3)
$$^{\mathrm{CM}}\mathrm{Cell}_\mathfrak{r}(W) \xleftarrow[b \mapsfrom \mathrm{CM}_\mathfrak{r}(b)]{\sim} \mathrm{Idem}_{\mathrm{pr}}(R_\mathfrak{r}Z) \xrightarrow[b\mapsto \bar{b}]{\sim} \mathrm{Idem}_{\mathrm{pr}}(k_R(\mathfrak{r})Z)$$

where $\bar{b}$ denote the image of b in $k_R(Z)$.

Since $\mathbf{M}Z$ is the center of $\mathbf{MH}$, the fact that $\mathbf{MH}$ is split semisimple implies immediately that

(6.2.4)
$$\dim_\mathbf{M}(\mathbf{M}Z\,b) = |\mathrm{CM}_\mathfrak{r}(b)|.$$

Recall that, since Z is a direct summand of $\mathbf{H}$, the algebra $k_R(\mathfrak{r})Z$ can be identified with its image in $k_R(\mathfrak{r})\mathbf{H}$. Note however that this image might be different from the center of $k_R(\mathfrak{r})\mathbf{H}$.

Corollary 6.2.5. — *We have* $\dim k_R(\mathfrak{r})Z\bar{b} = |\mathrm{CM}_\mathfrak{r}(b)|$.

Proof. — The $R_\mathfrak{r}$-module $R_\mathfrak{r}Z$ is free (of rank $|W|$), so the $R_\mathfrak{r}$-module $R_\mathfrak{r}Zb$ is projective, hence free since $R_\mathfrak{r}$ is local. By (6.2.4), the $R_\mathfrak{r}$-rank of $R_\mathfrak{r}Zb$ is $|\mathrm{CM}_\mathfrak{r}(b)|$. The corollary follows. □

***Example 6.2.6* (Specialization)**. — Let $c \in \mathscr{C}$. Let $\mathfrak{r}_c$ be a prime ideal of R lying over $\mathfrak{p}_c$ and, as in §5.1.A, define $D_c = G^D_{\mathfrak{r}_c}$ and $I_c = G^I_{\mathfrak{r}_c}$. Then $I_c = 1$ by (5.1.6), hence

the Calogero-Moser $\mathfrak{r}_c$-cells are singletons. ■

6.3. Ramification locus

Let $\mathfrak{r}_{\mathrm{ram}}$ denote the defining ideal of the ramification locus of the finite morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(P)$: in other words, R is étale over P at $\mathfrak{r}$ if and only if $\mathfrak{r}_{\mathrm{ram}} \not\subset \mathfrak{r}$. Recall [**SGA1**, Exposé V, Corollaire 2.4] that the following assertions are equivalent:

- R is étale over P at $\mathfrak{r}$;
- $I_\mathfrak{r} = 1$;
- R is unramified over P at $\mathfrak{r}$.

As G acts faithfully on W, we deduce the following result (taking into account Theorem 6.2.2).

Proposition 6.3.1. — *The following are equivalent:*

(1) $I_\mathfrak{r} \neq 1$.
(2) *R is not étale over P at $\mathfrak{r}$.*
(3) *R is ramified over P at $\mathfrak{r}$.*
(4) $\mathfrak{r}_{\mathrm{ram}} \subset \mathfrak{r}$.
(5) $|\mathrm{Idem}_{\mathrm{pr}}(R_\mathfrak{r}Q)| < |W|$.

Notice that $\mathfrak{r}_{\mathrm{ram}}$ is not necessarily a prime ideal of R. However, the purity theorem [**SGA1**, talk X, Theorem 3.1] implies that $\mathrm{Spec}(R/\mathfrak{r}_{\mathrm{ram}})$ is empty or of pure codimension 1 in $\operatorname{Spec}(R)$ (since R is integrally closed and P is regular). By Corollary 5.3.4 and Proposition 6.3.1, the morphism $\operatorname{Spec}(R) \to \operatorname{Spec}(P)$ is not étale if $W \neq 1$. Hence, if $W \neq 1$, we deduce that

(6.3.2) $\operatorname{Spec}(R/\mathfrak{r}_{\mathrm{ram}})$ *is of pure codimension* 1 *in* $\operatorname{Spec}(R)$.

Of course,

(6.3.3) $\mathfrak{r}_{\mathrm{ram}}$ *is stable under the action of* $\mathbf{k}^\times \times \mathbf{k}^\times \times \big((W^\wedge \times G) \rtimes \mathscr{N}\big)$.

While it is difficult to determine the ideal $\mathfrak{r}_{\mathrm{ram}}$ (we even do not know how to determine the ring R), the ideal $\mathfrak{p}_{\mathrm{ram}} = \mathfrak{r}_{\mathrm{ram}} \cap P$ is determined by the extension Q/P. The following result is classical [**SGA1**, Proposition 4.10].

Lemma 6.3.4. — *Let* $\operatorname{disc}(Q/P)$ *denote the discriminant ideal of* Q *in* P. *Then* $\mathfrak{p}_{\mathrm{ram}} = \sqrt{\operatorname{disc}(Q/P)}$.

Remark 6.3.5. — We proved that $\operatorname{Spec}(R/\mathfrak{r}_{\mathrm{ram}})$ is of pure codimension 1 in $\operatorname{Spec}(R)$ by using the purity theorem. Using the equivalence between (4) and (5) in Lemma 6.3.1, we obtain another proof using Proposition D.2.11. ■

6.4. Smoothness

Let $\mathscr{Z}_{\mathrm{sing}}$ denote the singular locus of $\mathscr{Z} = \operatorname{Spec}(Z)$ and $\mathfrak{z}_{\mathrm{sing}}$ its defining ideal. Since Z is integrally closed, it follows that

(6.4.1) $\mathscr{Z}_{\mathrm{sing}}$ *has codimension* $\geqslant 2$ *in* $\mathscr{Z}$.

Of course, $\mathfrak{z}_{\mathrm{sing}}$ needs not be a prime ideal. Since $\Upsilon : \mathscr{Z} \to \mathscr{P}$ is finite and flat, we deduce that

(6.4.2) $\Upsilon(\mathscr{Z}_{\mathrm{sing}})$ *is closed and of codimension* $\geqslant 2$ *in* $\mathscr{P}$.

The defining ideal of $\Upsilon(\mathscr{Z}_{\mathrm{sing}})$ is $\sqrt{\mathfrak{z}_{\mathrm{sing}} \cap P}$.

Assumption. *In the remainder of §6.4, we assume that* $\operatorname{Spec}(P/\mathfrak{p})$ *is not contained in* $\Upsilon(\mathscr{Z}_{\mathrm{sing}})$, *i.e., we assume* $\mathfrak{z}_{\mathrm{sing}} \cap P \not\subset \mathfrak{p}$.

Proposition 6.4.3. — *The* $(\mathbf{H}_\mathfrak{p}, Z_\mathfrak{p})$*-bimodule* $\mathbf{H}_\mathfrak{p} e$ *is both left and right projective and induces a Morita equivalence between* $\mathbf{H}_\mathfrak{p}$ *and* $Z_\mathfrak{p}$.

Proof. — By assumption, there exists $p \in \mathfrak{z}_{\mathrm{sing}} \cap P$ such that $p \notin \mathfrak{p}$. So $Z[1/p] = P[1/p] \otimes_P Z \subset Z_\mathfrak{p} = P_\mathfrak{p} \otimes_P Z$ and $\operatorname{Spec}(Z[1/p])$ is regular. It follows from Proposition 4.3.4 that $P[1/p] \otimes_P \mathbf{H}$ and $Q[1/p]$ are Morita equivalent via the bimodule $P[1/p] \otimes_P \mathbf{H}e$. The proposition follows by scalar extension. □

By reducing modulo $\mathfrak{p}$, one gets the following consequence.

Corollary 6.4.4. — *The $(k_P(\mathfrak{p})\mathbf{H}, k_P(\mathfrak{p})Z)$-bimodule $k_P(\mathfrak{p})\mathbf{H}e$ is both left and right projective and induces a Morita equivalence between $k_P(\mathfrak{p})\mathbf{H}$ and $k_P(\mathfrak{p})Z$.*

Extending again scalars, we obtain the following result.

Corollary 6.4.5. — *The $(k_R(\mathfrak{r})\mathbf{H}, k_R(\mathfrak{r})Z)$-bimodule $k_R(\mathfrak{r})\mathbf{H}e$ is both left and right projective and induces a Morita equivalence between $k_R(\mathfrak{r})\mathbf{H}$ and $k_R(\mathfrak{r})Z$.*

Theorem 6.4.6. — *The $k_R(\mathfrak{r})$-algebra $k_R(\mathfrak{r})\mathbf{H}$ is split. Every block of $k_R(\mathfrak{r})\mathbf{H}$ admits a unique simple module, which has dimension $|W|$. In particular, the simple $k_R(\mathfrak{r})\mathbf{H}$-modules are parametrized by the Calogero-Moser $\mathfrak{r}$-cells, that is, by the $I_\mathfrak{r}$-orbits in W.*

Proof. — Let us first show that $k_R(\mathfrak{r})Z = k_R(\mathfrak{r}) \otimes_P Z = k_R(\mathfrak{r}) \otimes_{P_\mathfrak{p}} Z_\mathfrak{p}$ is a split $k_R(\mathfrak{r})$-algebra. Let $\mathfrak{z}_1, \dots, \mathfrak{z}_l$ be the prime ideals of Z lying over $\mathfrak{p}$: in other words, $k_P(\mathfrak{p})\mathfrak{z}_1, \dots, k_P(\mathfrak{p})\mathfrak{z}_l$ are the prime (so, maximal) ideals of $k_P(\mathfrak{p})Z = Z_\mathfrak{p}/\mathfrak{p}Z_\mathfrak{p}$. Then $k_P(\mathfrak{p})(\mathfrak{z}_1 \cap \cdots \cap \mathfrak{z}_l)$ is the radical I of $k_P(\mathfrak{p})Z$. Moreover,

$$(k_P(\mathfrak{p})Z)/I \simeq k_Z(\mathfrak{z}_1) \times \cdots \times k_Z(\mathfrak{z}_l).$$

Since $k_R(\mathfrak{r})$ is a Galois extension of $k_P(\mathfrak{p})$ containing (the image through cop^{-1} of) $k_Z(\mathfrak{z}_i)$, for all i, it follows that $k_R(\mathfrak{r}) \otimes_{k_P(\mathfrak{p})} k_Z(\mathfrak{z}_i)$ is a split $k_R(\mathfrak{r})$-algebra (see Proposition B.3.12). As a consequence, $k_R(\mathfrak{r})Z$ is split. We deduce from Corollary 6.4.5 that $k_R(\mathfrak{r})\mathbf{H}$ is also split.

On the other hand, since $k_R(\mathfrak{r})Z$ is commutative, every block of $k_R(\mathfrak{r})Z$ admits a unique simple module. Using again the Morita equivalence, the same property holds for $k_R(\mathfrak{r})\mathbf{H}$. Finally, as the projective $Z_\mathfrak{p}$-module $\mathbf{H}_\mathfrak{p}e$ has rank $|W|$, the same is true for the projective $k_R(\mathfrak{r})Z$-module $k_R(\mathfrak{r})\mathbf{H}e$, and so the simple $k_R(\mathfrak{r})\mathbf{H}$-modules have dimension $|W|$.

The last statement of the theorem is now clear. □

Example 6.4.7. — Taking Corollary 4.3.2 into account, the condition $\mathfrak{z}_{\mathrm{sing}} \cap P \not\subset \mathfrak{p}$ is satisfied if $\operatorname{Spec}(P/\mathfrak{p})$ meets the open subset $\mathscr{P}^{\mathrm{reg}} = \mathscr{C} \times V^{\mathrm{reg}}/W \times V^*/W$ or, by symmetry, the open subset $\mathscr{C} \times V/W \times V^{*\mathrm{reg}}/W$. ■

6.5. Geometry

By Lemma 5.7.9, the irreducible components of $\mathscr{R} \times_{\mathscr{P}} \mathscr{Z}$ are the

$$\mathscr{R}_w = \{(r, \mathrm{cop}^*(\rho_H(w(r)))) \mid r \in \mathscr{R}\},$$

where w runs over W and the morphism $\Upsilon_{\mathscr{R}} : \mathscr{R} \times_{\mathscr{P}} \mathscr{Z} \to \mathscr{R}$ obtained from $\Upsilon : \mathscr{Z} \to \mathscr{P}$ by base change induces an isomorphism between the irreducible component $\mathscr{R}_w$ and $\mathscr{R}$.

Consequently, the inverse image through $\Upsilon_{\mathscr{R}}$ of the closed irreducible subvariety $\mathscr{R}(\mathfrak{r}) = \mathrm{Spec}(R/\mathfrak{r})$ is a union of closed irreducible subvarieties

$$\Upsilon_{\mathscr{R}}^{-1}(\mathscr{R}(\mathfrak{r})) = \bigcup_{w \in W} \mathscr{R}_w(\mathfrak{r}), \tag{6.5.1}$$

where $\mathscr{R}_w(\mathfrak{r}) \simeq \mathscr{R}(\mathfrak{r})$ is the inverse image of $\mathscr{R}(\mathfrak{r})$ in $\mathscr{R}_w$.

Lemma 6.5.2. — *Let w and $w' \in W$. We have $\mathscr{R}_w(\mathfrak{r}) = \mathscr{R}_{w'}(\mathfrak{r})$ if and only if $w \sim_{\mathfrak{r}}^{\mathrm{CM}} w'$.*

Proof. — Indeed, $\mathscr{R}_w(\mathfrak{r}) = \mathscr{R}_{w'}(\mathfrak{r})$ if and only if, for all $r \in \mathscr{R}(\mathfrak{r})$, we have $\rho_H(w(r)) = \rho_H(w'(r))$. Translated at the level of the rings Q and R, this becomes equivalent to say that, for all $q \in Q$, we have $w(q) \equiv w'(q) \mod \mathfrak{r}$. ☐

In other words, Lemma 6.5.2 shows that the Calogero-Moser $\mathfrak{r}$-cells parametrize the irreducible components of the inverse image of $\mathscr{R}(\mathfrak{r})$ in the fiber product $\mathscr{R} \times_{\mathscr{P}} \mathscr{Z}$.

6.6. Topology

We assume in §6.6 that $\mathbf{k} = \mathbb{C}$. We fix for the remainder of the book a pair $(v_{\mathbb{C}}, v_{\mathbb{C}}^*) \in V^{\mathrm{reg}} \times V^{*\mathrm{reg}}$.

Let $Y = \mathscr{R}(\mathbb{C})$, $\bar{Y} = \mathscr{Q}(\mathbb{C})$ and $X = \mathscr{P}(\mathbb{C})$. We denote by Y^{nr} the complement of the ramification locus of $\rho_G : Y \to X$ and by X^{nr} its image.

Let $\gamma : [0,1] \to X$ be a path with $\gamma([0,1)) \subset X^{\mathrm{nr}}$ and such that $\gamma(0) = (0, W \cdot v_{\mathbb{C}}, W \cdot v_{\mathbb{C}}^*)$. Given $w \in W$, there is a unique path $\gamma_w : [0,1] \to \bar{Y}$ lifting γ and such that $\gamma_w(0) = (0, (w(v_{\mathbb{C}}), v_{\mathbb{C}}^*)\Delta W)$ (Lemma B.7.2).

Definition 6.6.1. — *We say that $w, w' \in W$ are in the same **Calogero-Moser γ-cell** if $\gamma_w(1) = \gamma_{w'}(1)$.*

The choice of the prime ideal $\mathfrak{r}_0$ (cf §5.1.B) corresponds to the choice of an irreducible component of $\rho_G^{-1}(\{0\} \times V/W/ \times V^*/W)$. The isomorphism iso_0 extends to an isomorphism $\mathbf{k}[V^{\mathrm{reg}} \times V^{*\mathrm{reg}}]^{\Delta Z(W)} \xrightarrow{\sim} (R/\mathfrak{r}_0) \otimes_{\mathbf{k}[V \times V^*]^{W \times W}} \mathbf{k}[V^{\mathrm{reg}} \times V^{*\mathrm{reg}}]^{W \times W}$. We have a corresponding isomorphism of varieties $\mathrm{Spec}(R/\mathfrak{r}_0) \times_{V/W \times V^*/W} (V^{\mathrm{reg}}/W \times V^{*\mathrm{reg}}/W) \xrightarrow{\sim}$

$(V^{\mathrm{reg}} \times V^{*\mathrm{reg}})/\Delta Z(W)$. We denote by y_0 the point of the component that is the inverse image of $(v_{\mathbb{C}}, v_{\mathbb{C}}^*)\Delta Z(W)$. Let $x_0 = \rho_G(y_0)$. The choice of y_0 provides a bijection $H \setminus G \xrightarrow{\sim} \Upsilon^{-1}(x_0)$, $Hg \mapsto \rho_H(g \cdot y_0)$. The bijection $W \xrightarrow{\sim} H \setminus G \xrightarrow{\sim} \Upsilon^{-1}(x_0)$ is given by $w \mapsto (w(v_{\mathbb{C}}), v_{\mathbb{C}}^*)\Delta W$.

Let y_1 be a point of Y that lies in the irreducible component determined by $\mathfrak{r}$ and satisfies $\mathrm{Stab}_G(y_1) = G_{\mathfrak{r}}^I$.

We fix a path $\tilde{\gamma} : [0,1] \to Y$ such that $\tilde{\gamma}([0,1)) \subset Y^{\mathrm{nr}}$, $\tilde{\gamma}(0) = y_0$ and $\tilde{\gamma}(1) = y_1$. We denote by γ the image of $\tilde{\gamma}$ in X.

From §B.7 we deduce the following result.

Proposition 6.6.2. — *Two elements $w, w' \in W$ are in the same Calogero-Moser $\mathfrak{r}$-cell if and only if they are in the same Calogero-Moser γ-cell.*

PART III

CELLS AND FAMILIES

Notation. *We fix in this part a prime ideal* $\mathfrak{C}$ *of* $\mathbf{k}[\mathscr{C}]$*. Let* $\mathscr{C}(\mathfrak{C}) = \operatorname{Spec}\mathbf{k}[\mathscr{C}]/\mathfrak{C}$ *be the closed irreducible subscheme of* $\mathscr{C}$ *defined by* $\mathfrak{C}$*. We denote by* $\bar{\mathfrak{p}}_{\mathfrak{C}}$ *(resp.* $\mathfrak{p}_{\mathfrak{C}}^{\text{left}}$*, resp.* $\mathfrak{p}_{\mathfrak{C}}^{\text{right}}$*) the prime ideal of* P *corresponding to the closed irreducible subscheme* $\mathscr{C}(\mathfrak{C}) \times \{0\} \times \{0\}$ *(resp.* $\mathscr{C}(\mathfrak{C}) \times V/W \times \{0\}$*, resp.* $\mathscr{C}(\mathfrak{C}) \times \{0\} \times V^*/W$*). We set*

$$\begin{cases} \bar{P}_{\mathfrak{C}} = P/\bar{\mathfrak{p}}_{\mathfrak{C}} \simeq \mathbf{k}[\mathscr{C}]/\mathfrak{C}, \\ P_{\mathfrak{C}}^{\text{left}} = P/\mathfrak{p}_{\mathfrak{C}}^{\text{left}} \simeq \mathbf{k}[\mathscr{C}]/\mathfrak{C} \otimes \mathbf{k}[V]^W, \\ P_{\mathfrak{C}}^{\text{right}} = P/\mathfrak{p}^{\text{right}} \simeq \mathbf{k}[\mathscr{C}]/\mathfrak{C} \otimes \mathbf{k}[V^*]^W, \end{cases}$$

and we define

$$\begin{cases} \text{the } \bar{P}_{\mathfrak{C}}\text{-algebra } \bar{\mathbf{H}}_{\mathfrak{C}} = \mathbf{H}/\bar{\mathfrak{p}}_{\mathfrak{C}}\mathbf{H}, \\ \text{the } P_{\mathfrak{C}}^{\text{left}}\text{-algebra } \mathbf{H}_{\mathfrak{C}}^{\text{left}} = \mathbf{H}/\mathfrak{p}_{\mathfrak{C}}^{\text{left}}\mathbf{H}, \\ \text{the } P_{\mathfrak{C}}^{\text{right}}\text{-algebra } \mathbf{H}_{\mathfrak{C}}^{\text{right}} = \mathbf{H}/\mathfrak{p}_{\mathfrak{C}}^{\text{right}}\mathbf{H}. \end{cases}$$

We denote by $\bar{Z}_{\mathfrak{C}}$ *(resp.* $Z_{\mathfrak{C}}^{\text{left}}$*, resp.* $Z_{\mathfrak{C}}^{\text{right}}$*) the image of* Z *in* $\bar{\mathbf{H}}_{\mathfrak{C}}$ *(resp.* $\mathbf{H}_{\mathfrak{C}}^{\text{left}}$*, resp.* $\mathbf{H}_{\mathfrak{C}}^{\text{right}}$*). We also define*

$$\begin{cases} \bar{\mathbf{K}}_{\mathfrak{C}} = k_P(\bar{\mathfrak{p}}_{\mathfrak{C}}), \\ \mathbf{K}_{\mathfrak{C}}^{\text{left}} = k_P(\mathfrak{p}_{\mathfrak{C}}^{\text{left}}), \\ \mathbf{K}_{\mathfrak{C}}^{\text{right}} = k_P(\mathfrak{p}_{\mathfrak{C}}^{\text{right}}). \end{cases}$$

To simplify the notation, when $\mathfrak{C} = 0$*, the index* $\mathfrak{C}$ *will be omitted in all the previous notations (*$\bar{P}$*,* $\bar{\mathfrak{p}}$*,* $\mathfrak{p}^{\text{left}}$*,* $\mathbf{H}^{\text{right}}$*,* $\mathbf{K}^{\text{left}}$*,...). Given* $c \in \mathscr{C}$ *and* $\mathfrak{C} = \mathfrak{C}_c$*, the index* $\mathfrak{C}_c$ *will be replaced by* c *(*$\mathfrak{p}_c^{\text{right}}$*,* $\mathbf{H}_c^{\text{left}}$*,* $\mathbf{K}_c^{\text{right}}$*,* $\bar{\mathbf{K}}_c$*,...). Notice for instance that* $\bar{\mathfrak{p}}_{\mathfrak{C}} = \bar{\mathfrak{p}} + \mathfrak{C}\,P$ *(and similarly for* $\mathfrak{p}_{\mathfrak{C}}^{\text{left}}$ *and* $\mathfrak{p}_{\mathfrak{C}}^{\text{right}}$*) and that* $\bar{\mathbf{K}}_c \simeq \mathbf{k}$*.*

Definition. — *Fix a prime ideal* $\bar{\mathfrak{r}}_{\mathfrak{C}}$ *(resp.* $\mathfrak{r}_{\mathfrak{C}}^{\text{left}}$*, resp.* $\mathfrak{r}_{\mathfrak{C}}^{\text{right}}$*) of* R *lying over* $\bar{\mathfrak{p}}_{\mathfrak{C}}$ *(resp.* $\mathfrak{p}_{\mathfrak{C}}^{\text{left}}$*, resp.* $\mathfrak{p}_{\mathfrak{C}}^{\text{right}}$*). A* ***Calogero-Moser two-sided (resp. left, resp. right)*** $\mathfrak{C}$***-cell*** *is defined to be an* $\bar{\mathfrak{r}}_{\mathfrak{C}}$*-cell (resp.* $\mathfrak{r}_{\mathfrak{C}}^{\text{left}}$*-cell, resp.* $\mathfrak{r}_{\mathfrak{C}}^{\text{right}}$*-cell).*

When $\mathfrak{C} = 0$*, they will also be called* ***generic Calogero-Moser (two-sided, left or right) cells****. Given* $c \in \mathscr{C}$ *and* $\mathfrak{C} = \mathfrak{C}_c$*, they are called* ***Calogero-Moser (two-sided, left or right)*** c***-cells****.*

Remark. — Of course, the notion of Calogero-Moser (two-sided, left, or right) $\mathfrak{C}$-cell depends on the choice of the ideal $\bar{\mathfrak{r}}_{\mathfrak{C}}$, $\mathfrak{r}_{\mathfrak{C}}^{\text{left}}$ or $\mathfrak{r}_{\mathfrak{C}}^{\text{right}}$; however, as all the prime ideals of R lying over a prime ideal of P are G-conjugate, changing the ideal amounts to transforming the cells according to the action of G. ■

***Remark* (Semi-continuity).** — It is of course possible to choose the ideals $\bar{\mathfrak{r}}_{\mathfrak{C}}$, $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$ or $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{right}}$ so that $\bar{\mathfrak{r}}_{\mathfrak{C}}$ contains $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$ and $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{right}}$: in this case, by Remark 6.1.3, the Calogero-Moser two-sided $\mathfrak{C}$-cells are unions of Calogero-Moser left (resp. right) $\mathfrak{C}$-cells.

Similarly, if $\mathfrak{C}'$ is another prime ideal of $\mathbf{k}[\mathscr{C}]$ such that $\mathfrak{C} \subset \mathfrak{C}'$, then one can choose the ideals $\bar{\mathfrak{r}}_{\mathfrak{C}'}$, $\mathfrak{r}_{\mathfrak{C}'}^{\mathrm{left}}$ or $\mathfrak{r}_{\mathfrak{C}'}^{\mathrm{right}}$ in such a way that they contain respectively $\bar{\mathfrak{r}}_{\mathfrak{C}}$, $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$ or $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{right}}$. Then the Calogero-Moser two-sided (resp. left, resp. right) $\mathfrak{C}'$-cells are unions of Calogero-Moser two-sided (resp. left, resp. right) $\mathfrak{C}$-cells. ■

With the definition of Calogero-Moser two-sided, left or right cells given above, the first aim of this book is achieved. The aim of this part is now to study these particular cells, in relation with the representation theory of **H**: in each case, a family of *Verma modules* will help us in this study. More precisely:

- In Chapter 10, we will associate a *Calogero-Moser family* with each two-sided cell: the Calogero-Moser families define a partition of $\mathrm{Irr}(W)$.
- In Chapter 11, we will associate a *Calogero-Moser cellular character* with each left cell.

We conjecture that, whenever W is a Coxeter group, all these notions coincide with the analogous notions defined by Kazhdan-Lusztig in the framework of Coxeter groups. The conjectures will be stated precisely in § 15 and some evidence will be given (see § 15.3).

CHAPTER 7

REPRESENTATIONS

7.1. Highest weight categories

We define highest weight category structures on categories of (graded) representations of $\widetilde{\mathbf{H}}$, following Appendix F.2. The existence of such structures for the restricted Cherednik algebras (cf §9) is due to Bellamy and Thiel [**BelTh**].

We consider the $\mathbb{Z}$-grading on $A = \widetilde{\mathbf{H}}$ and take $k = \mathbf{k}[\widetilde{\mathscr{C}}]$. We have three graded k-subalgebras $B_- = \mathbf{k}[\widetilde{\mathscr{C}} \times V^*]$, $B_+ = \mathbf{k}[\widetilde{\mathscr{C}} \times V]$ and $H = \mathbf{k}[\widetilde{\mathscr{C}}]W$ of A. Theorem 3.1.5 and Example F.2.3 show that the conditions (i)-(ix) of Appendix F.2 are satisfied with $I = \{\mathbf{k}[\widetilde{\mathscr{C}}] \otimes E\}_{E \in \mathrm{Irr}(\mathbf{k}W)}$.

Let R be a noetherian commutative $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebra. Let $\tilde{\mathscr{O}}(R)$ be the category of finitely generated $\mathbb{Z}$-graded $(\widetilde{\mathbf{H}} \otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]} R)$-modules that are locally nilpotent for the action of V.

We put $\widetilde{\mathbf{H}}^- = \mathbf{k}[\widetilde{\mathscr{C}} \times V^*] \rtimes W$ and $\widetilde{\mathbf{H}}^+ = \mathbf{k}[\widetilde{\mathscr{C}} \times V] \rtimes W$. Given E a graded $(\mathbf{k}[V^*] \rtimes W)$-module, we define the Verma module

$$\tilde{\Delta}(E) = \mathrm{Ind}_{\widetilde{\mathbf{H}}^-}^{\widetilde{\mathbf{H}}}(\mathbf{k}[\widetilde{\mathscr{C}}] \otimes E) = \widetilde{\mathbf{H}} \otimes_{\widetilde{\mathbf{H}}^-} (\mathbf{k}[\widetilde{\mathscr{C}}] \otimes E).$$

The PBW decomposition gives a canonical isomorphism of $\widetilde{\mathbf{H}}^+$-modules

$$\mathbf{k}[\widetilde{\mathscr{C}} \times V] \otimes E \xrightarrow{\sim} \tilde{\Delta}(E) \tag{7.1.1}$$

where W acts diagonally and $\mathbf{k}[\widetilde{\mathscr{C}} \times V]$ acts by left multiplication on $\mathbf{k}[\widetilde{\mathscr{C}} \times V] \otimes E$. We will view $\mathbf{k}W$-modules as graded $(\mathbf{k}[V^*] \rtimes W)$-modules concentrated in degree 0 by letting V act by 0.

Theorem F.2.7 provides the following result.

Theorem 7.1.2. — *The category $\tilde{\mathscr{O}}(R)$ is a highest weight category over R, with set of standard modules $\{R\tilde{\Delta}(E)\langle i\rangle\}_{E \in \mathrm{Irr}(\mathbf{k}W), n \in \mathbb{Z}}$ and partial order $R\tilde{\Delta}(E)\langle i\rangle < \tilde{\Delta}(F)\langle j\rangle$ if $i < j$.*

We put $\tilde{\Delta}(\mathrm{co}) = \tilde{\Delta}(\mathbf{k}[V^*]^{\mathrm{co}(W)}) = \widetilde{\mathbf{H}}e \otimes_{\mathbf{k}[V^*]^W} \mathbf{k}$.

Lemma 7.1.3. — *We have $[\tilde{\Delta}(\mathrm{co})] = \sum_{E \in \mathrm{Irr}(W)} f_E(\mathbf{t}^{-1})[\tilde{\Delta}(E)]$ in $K_0(\tilde{\mathscr{O}}(R))$.*

Proof. — The $(\mathbf{k}[V^*] \rtimes W)$-module $M = \mathbf{k}[V^*]^{\mathrm{co}(W)}$ has a finite filtration given by $M^{\leqslant i} = (\mathbf{k}[V^*]^{\mathrm{co}(W)})^{\leqslant i}$ and the associated graded module is isomorphic to $\bigoplus_{E\in\mathrm{Irr}(W)} f_E(\mathbf{t}^{-1})E$. Consequently, we have a filtration of the $\tilde{\mathbf{H}}$-module $\tilde{\Delta}(\mathrm{co})$ given by $\tilde{\Delta}(\mathrm{co})^{\leqslant i} = \tilde{\Delta}((\mathbf{k}[V^*]^{\mathrm{co}(W)})^{\leqslant i})$ and the associated graded $\tilde{\mathbf{H}}$-module $\mathrm{gr}\,\tilde{\Delta}(\mathrm{co})$ is isomorphic to $\bigoplus_{E\in\mathrm{Irr}(W)} f_E(\mathbf{t}^{-1})\tilde{\Delta}(E)$. The result follows. □

Remark 7.1.4. — Note that, for the filtration introduced in the proof of Lemma 7.1.3, the module $\mathrm{gr}\,\tilde{\Delta}(\mathrm{co})$ is isomorphic to $\tilde{\Delta}(\mathbf{k}W)$, as an ungraded $\tilde{\mathbf{H}}$-module. ■

7.2. Euler action on Verma modules

The classical formula describing the action of the Euler element on Verma modules is given in the next proposition [**GGOR**, §3.1(4)].

Given $(\aleph, j) \in \boldsymbol{\aleph}^\circ$, $H \in \aleph$ and $E \in \mathrm{Irr}(W)$, we put

$$m^E_{\aleph,j} = \langle \mathrm{Res}^W_{W_H} E, \det{}^j \rangle_{W_H}.$$

and
$$C_E = \frac{1}{\dim_{\mathbf{k}} E} \sum_{s\in\mathrm{Ref}(W)} \varepsilon(s)\,\mathrm{Tr}(s,E)\; C_s.$$

There is a simple formula for C_E's [**BrMi**, 4.17].

Lemma 7.2.1. — *We have*

$$C_E = \sum_{(\aleph,j)\in\boldsymbol{\aleph}^\circ} \frac{m^E_{\aleph,j}|\aleph|e_\aleph}{\dim_{\mathbf{k}} E}\cdot K_{\aleph,j} \in \bigoplus_{(\aleph,j)\in\boldsymbol{\aleph}^\circ} \mathbb{Z}_{\geqslant 0} K_{\aleph,j}.$$

Proof. — We have

$$C_E = \sum_{(\aleph,j)\in\boldsymbol{\aleph}^\circ} \sum_{H\in\aleph} \frac{1}{\dim_{\mathbf{k}} E}\mathrm{Tr}(e_H\varepsilon_{H,j}, E) K_{\aleph,j}.$$

Now, given $(\aleph, j) \in \boldsymbol{\aleph}^\circ$, the central element $\sum_{H\in\aleph} e_H \varepsilon_{H,j}$ of $\mathbf{k}W$ acts on E by the scalar $\frac{m^E_{\aleph,j}|\aleph|e_\aleph}{\dim_{\mathbf{k}} E}$. The lemma follows. □

Proposition 7.2.2. — *Let $E \in \mathrm{Irr}(W)$. The element $\widetilde{\mathbf{eu}}$ acts on $\tilde{\Delta}(E)_i$ by multiplication by $Ti + C_E$.*

Proof. — Recall (§3.3) that if $(x_1, \dots, x_n)$ denotes a $\mathbf{k}$-basis of V^* and if $(y_1, \dots, y_n)$ denotes its dual basis, then

$$\widetilde{\mathbf{eu}} = \sum_{i=1}^n x_i y_i + \sum_{s\in\mathrm{Ref}(W)} \varepsilon(s) C_s s.$$

Let $h \in (\tilde{\mathbf{H}}^+)^i$, $v \in E$ and $m = h \otimes v \in \tilde{\Delta}(E)_i$. We have

$$\widetilde{\mathbf{eu}}\cdot m = h\widetilde{\mathbf{eu}} \otimes v + Tim = h \otimes (\sum_{s\in\mathrm{Ref}(W)} \varepsilon(s) C_s s\cdot v) + Tim.$$

Since $\sum_{s\in\mathrm{Ref}(W)}\varepsilon(s)C_s s$ acts on $\bar{\mathbf{K}}E$ by multiplication by C_E, the result follows. $\square$

7.3. Case $T=0$

Given E a $\mathbb{Z}$-graded $(\mathbf{k}[V^*]\rtimes W)$-module, we put $\Delta(E)=\tilde{\Delta}(E)\otimes_{\mathbf{k}[T]}\mathbf{k}[T]/(T)$: it is a graded $\mathbf{H}$-module. We define $\mathbf{H}^+=\tilde{\mathbf{H}}^+/T\tilde{\mathbf{H}}^+=\mathbf{k}[\mathscr{C}\times V]\rtimes W$ and $\mathbf{H}^-=\tilde{\mathbf{H}}^-/T\tilde{\mathbf{H}}^-=\mathbf{k}[\mathscr{C}\times V^*]\rtimes W$. Then

$$\Delta(E)=\mathrm{Ind}_{\mathbf{H}^-}^{\mathbf{H}}(\mathbf{k}[\mathscr{C}]\otimes E)=\mathbf{H}\otimes_{\mathbf{H}^-}(\mathbf{k}[\mathscr{C}]\otimes E).$$

If moreover $E\in\mathrm{Irr}(W)$, we denote by $\omega_E:\mathrm{Z}(\mathbf{k}W)\to\mathbf{k}$ its associated central character and we set

$$\Omega_E=(\mathrm{Id}_{\mathbf{k}[\mathscr{C}]}\otimes\omega_E)\circ\mathbf{\Omega}:Z\longrightarrow\mathbf{k}[\mathscr{C}],$$

where $\mathbf{\Omega}:Z\to\mathrm{Z}(\mathbf{k}[\mathscr{C}]W)$ is the morphism of algebras defined in §4.2.C. Note that Ω_E is a morphism of algebras.

Recall that Z^0 denotes the $\mathbb{Z}$-homogeneous component of Z of degree 0.

Proposition 7.3.1. — *Given $E\in\mathrm{Irr}(W)$, an element $b\in Z^0$ acts on $\Delta(E)$ by multiplication by $\Omega_E(b)$.*

Proof. — Using the PBW-decomposition, we can write

$$b=\sum_{i\in I}a_i f_i w_i g_i,$$

where $a_i\in\mathbf{k}[\mathscr{C}]$, $f_i\in\mathbf{k}[V]$, $w_i\in W$ and $g_i\in\mathbf{k}[V^*]$. Since b is homogeneous of degree 0, we can choose the f_i's and g_i's to be homogeneous elements such that $\deg_{\mathbb{Z}}(f_i)+\deg_{\mathbb{Z}}(g_i)=0$ for all $i\in I$.

Let $h\in\mathbf{H}$ and $v\in E$. We have

$$b\cdot(h\otimes_{\mathbf{H}^-}v)=bh\otimes_{\mathbf{H}^-}v=hb\otimes_{\mathbf{H}^-}v,$$

so

$$b\cdot(h\otimes_{\mathbf{H}^-}v)=\sum_{i\in I}a_i h f_i\otimes(w_i g_i\cdot v).$$

Let I_0 denote the set of $i\in I$ such that $\deg_{\mathbb{Z}}(f_i)=\deg_{\mathbb{Z}}(g_i)=0$. Then $g_i v=0$ if $i\notin I_0$, and $f_i,g_i\in\mathbf{k}$ if $i\in I_0$, so

$$b\cdot(h\otimes_{\mathbf{H}^-}v)=\sum_{i\in I_0}h\otimes(a_i f_i w_i g_i\cdot v).$$

But $\sum_{i\in I_0}a_i f_i g_i w_i=\mathbf{\Omega}(b)$ by definition, and the result follows. $\square$

Note that $\Omega_E(\mathbf{eu})=C_E$, so the next corollary follows also from Proposition 7.2.2.

Proposition 7.3.2. — *The element* $\mathbf{eu}$ *acts by $\Omega_E(\mathbf{eu})=C_E$ on $\Delta(E)$.*

Recall (Example 3.2.1) that the $(\mathbb{Z}/d\mathbb{Z})$-grading on $\mathbf{H}$ deduced from the $\mathbb{Z}$-grading is induced by conjugation by an element $w_z \in Z(W)$. As a consequence, the category of $\mathbb{Z}$-graded $\mathbf{H}$-modules decomposes as a direct sum, parametrized by $l \in \mathbb{Z}/z\mathbb{Z}$, of subcategories with objects the graded modules M such that w_z acts on M^i by ζ^{i+l}.

Given R a noetherian commutative $\mathbf{k}[\mathscr{C}]$-algebra, this induces a corresponding decomposition of the category $\mathscr{O}(R)$ of finitely generated graded $(\mathbf{H}\otimes_{\mathbf{k}[\mathscr{C}]} R)$-modules that are locally nilpotent for the action of V.

7.4. Automorphisms

The group $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ acts on $\tilde{\mathbf{H}}$, hence it acts on the category of $\tilde{\mathbf{H}}$-modules. The action is given as follows. Let M be an $\tilde{\mathbf{H}}$-module and let $\tau \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$. We denote by ${}^\tau M$ the $\tilde{\mathbf{H}}$-module whose underlying $\mathbf{k}$-module is M and where the action of $h \in \tilde{\mathbf{H}}$ on an element of ${}^\tau M$ is given by the action of ${}^{\tau^{-1}}h$ on the corresponding element of M.

This defines a functor

$$\tau : \tilde{\mathbf{H}}\text{-mod} \longrightarrow \tilde{\mathbf{H}}\text{-mod}$$

and this induces an action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ on the category $\tilde{\mathbf{H}}$-mod. Similarly, we can define a functor

$$\tau : \mathbf{A}^0\text{-mod} \longrightarrow \mathbf{A}^0\text{-mod}$$

and an action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ on the category $\mathbf{A}^0$-mod. There is a commutative diagram

(7.4.1)
$$\begin{tikzcd}
\tilde{\mathbf{H}}^-\text{-modgr} \arrow[r, "\mathrm{Ind}"] \arrow[d, "\tau"'] & \tilde{\mathbf{H}}\text{-modgr} \arrow[d, "\tau"] \\
\tilde{\mathbf{H}}^-\text{-modgr} \arrow[r, "\mathrm{Ind}"'] & \tilde{\mathbf{H}}\text{-modgr}
\end{tikzcd}$$

The next proposition is now clear.

Proposition 7.4.2. — *Given $E \in \mathrm{Irr}(\mathbf{k}W)$ and $\tau = (\xi, \xi', \gamma \rtimes g) \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$, we have*

$${}^\tau\tilde{\Delta}(E) \simeq \tilde{\Delta}({}^g E \otimes \gamma^{-1})$$

and

$$\Omega_E({}^\tau z) = {}^\tau\big(\Omega_{{}^g E \otimes \gamma^{-1}}(z)\big)$$

for all $z \in Z$.

Corollary 7.4.3. — *Given* $E \in \mathrm{Irr}(\mathbf{k}W)$, *then* $\Omega_E : Z \to \mathbf{k}[\mathscr{C}]$ *is a bigraded morphism. In particular,* $\mathrm{Ker}(\Omega_E)$ *is a bi-homogeneous ideal of* Z.

7.5. Case $T = 1$

It follows from Proposition 3.3.2 that the $\mathbb{Z}$-grading on $\dot{\mathbf{H}}$ is inner. We put $\dot{\Delta}(E) = \dot{\mathbf{H}} \otimes_{\tilde{\mathbf{H}}} \tilde{\Delta}(E) = \tilde{\Delta}(E) \otimes_{\mathbf{k}[T]} \big(\mathbf{k}[T]/(T-1)\big)$.

Let R be a commutative noetherian $\mathbf{k}[\mathscr{C}]$-algebra. We assume that given any family $E_1, E_2, \ldots, E_{n-1}, E_n = E_1 \in \mathrm{Irr}(\mathbf{k}W)$ and given any family $a_1, \ldots, a_n \in \mathbb{Z}$ with $a_i \neq a_{i+1}$ and $(C_{E_i} - C_{E_{i+1}} + a_i - a_{i+1})1_R$ non invertible in R for $1 \leqslant i \leqslant n-1$, then $a_1 = a_n$. Note that this assumption is automatically satisfied if R is a local ring.

Let $\dot{\mathscr{O}}(R)$ be the category of finitely generated $R\dot{\mathbf{H}}$-modules that are locally nilpotent for the action of V. Theorem F.2.10 shows that this is a highest weight category [**GGOR**, §3].

Theorem 7.5.1. — $\dot{\mathscr{O}}(R)$ *is a highest weight category over* R *with set of standard objects* $\{R\dot{\Delta}(E))\}_{E \in \mathrm{Irr}(\mathbf{k}W)}$. *The order is given by* $E > F$ *if* $(C_E - C_F)1_R \in \mathbb{Z}_{>0}$.

Let $\mathscr{F}_1$ be the set of height one prime ideals $\mathfrak{p}$ of $\mathbf{k}[\mathscr{C}]$ such that $\tilde{\mathscr{O}}(\mathbf{k}(\langle \mathfrak{p}, T-1\rangle))$ is not semisimple. Note that given $\mathfrak{m}$ a maximal ideal of $\mathbf{k}[\mathscr{C}]$, the category $\dot{\mathscr{O}}(\mathbf{k}(\mathfrak{m}))$ is not semisimple if and only if there is $\mathfrak{p} \in \mathscr{F}_1$ such that $\mathfrak{p} \subset \mathfrak{m}$.

We have the following classical semisimplicity result (an improvement of which will be given in Corollary 8.3.3 below, using the KZ functor).

Theorem 7.5.2. — *The ideals in* $\mathscr{F}_1$ *are of the form* $(C_E - C_F - r)$ *for some* $E, F \in \mathrm{Irr}(W)$ *such that* $C_E \neq C_F$ *and some* $r \in \mathbb{Z} \setminus \{0\}$. *They correspond to affine hyperplanes in* $\mathscr{K}(\mathbb{Q})$.

Assume R *is a field and* $(C_E - C_F)1_R \notin \mathbb{Z} - \{0\}$ *for all* $E, F \in \mathrm{Irr}(\mathbf{k}W)$. *Then the category* $\dot{\mathscr{O}}(R)$ *is semisimple.*

In particular, if R *is a field and* $\mathbb{Q} \cap \big(\sum_{(\aleph, j) \in \aleph^\circ} \mathbb{Z} K_{\aleph, j} 1_R\big) = \{0\}$, *then* $\dot{\mathscr{O}}(R)$ *is semisimple.*

Proof. — Under the assumption on C_E's, the order on $\mathrm{Irr}(W)$ is trivial, hence $\dot{\mathscr{O}}(R)$ is semisimple. The second assertion follows from Lemma 7.2.1. □

Consider now the case $R = \mathbb{C}$ and consider the principal value logarithm $\log : \mathbb{C} \to \mathbb{C}$. Proposition F.2.9 shows the following.

Proposition 7.5.3. — *Let* $c \in \mathscr{C}(\mathbb{C})$ *and let* $\tilde{c} = (T = 1, c) \in \tilde{\mathscr{C}}(\mathbb{C})$. *There is an equivalence of graded highest weight categories*

$$\dot{\mathscr{O}}(c)^{(\mathbb{Z})} \xrightarrow{\sim} \tilde{\mathscr{O}}(\tilde{c}), \; \dot{\Delta}(E) \mapsto \tilde{\Delta}(E)\langle \log(e^{C_E(c)}) - C_E(c)\rangle.$$

CHAPTER 8

HECKE ALGEBRAS

Notation. *From now on, and until the end of this book, we fix a number field F contained in $\mathbf{k}$, which is Galois over $\mathbb{Q}$ and contains all the traces of elements of W, and we denote by $\mathcal{O}$ the integral closure of $\mathbb{Z}$ in F. We also fix an embedding $F \hookrightarrow \mathbb{C}$. By Proposition 2.4.1, there exists a W-stable F-vector subspace V_F of V such that $V = \mathbf{k} \otimes_F V_F$. Let $a \mapsto \bar{a}$ denote the complex conjugation (it stabilizes F since F is Galois over $\mathbb{Q}$). Finally, we denote by $\boldsymbol{\mu}_W$ the group of roots of unity of the field generated by the traces of elements of W.*

The existence of such a field F is easy: we can take the field generated by the traces of elements of W (it is Galois over $\mathbb{Q}$ as it is contained in a cyclotomic number field). Note also that F contains all the roots of unity of the form ζ_{e_H}, where $H \in \mathcal{A}$.

8.1. Definitions

8.1.A. Braid groups. — Set $V_{\mathbb{C}} = \mathbb{C} \otimes_F V_F$. Given $H \in \mathcal{A}$, let $H_{\mathbb{C}} = \mathbb{C} \otimes_F (H \cap V_F)$. We define

$$V_{\mathbb{C}}^{\mathrm{reg}} = V_{\mathbb{C}} \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}}.$$

We have fixed a point $v_{\mathbb{C}} \in V_{\mathbb{C}}^{\mathrm{reg}}$ (cf §6.6). Given $v \in V_{\mathbb{C}}$, we denote by $\bar{v}$ its image in the quotient variety $V_{\mathbb{C}}/W$. The *braid group* associated with W, denoted B_W, is defined as

$$B_W = \pi_1(V_{\mathbb{C}}^{\mathrm{reg}}/W, \bar{v}_{\mathbb{C}}).$$

The *pure braid group* associated with W, denoted by P_W, is then defined as

$$P_W = \pi_1(V_{\mathbb{C}}^{\mathrm{reg}}, v_{\mathbb{C}}).$$

The covering $V_{\mathbb{C}}^{\mathrm{reg}} \to V_{\mathbb{C}}^{\mathrm{reg}}/W$ being unramified (Steinberg's Theorem 2.3.2), we obtain an exact sequence

$$1 \longrightarrow P_W \longrightarrow B_W \xrightarrow{p_W} W \longrightarrow 1. \tag{8.1.1}$$

Given $H \in \mathscr{A}$, we denote by $\boldsymbol{\sigma}_H$ a *generator of the monodromy* around the hyperplane H, as defined in [**BrMaRo**, §2.A], and such that $p_W(\boldsymbol{\sigma}_H) = s_H$. This is an element of B_W well-defined up to conjugacy by an element of P_W. Recall [**BrMaRo**, Theorem 2.17] that

$$B_W \textit{ is generated by } (g\boldsymbol{\sigma}_H g^{-1})_{H\in\mathscr{A}, g\in P_W}. \tag{8.1.2}$$

It can be proven [**Bes2, BrMaRo**] that B_W is already generated by $(\boldsymbol{\sigma}_H)_{H\in\mathscr{A}}$, for a suitable choice of the elements $\boldsymbol{\sigma}_H$.

We denote by $\boldsymbol{\pi}_z$ the image in B_W of the path in $V_{\mathbb{C}}^{\mathrm{reg}}$ defined by

$$\begin{array}{rccc} \boldsymbol{\pi}_z : & [0,1] & \longrightarrow & V_{\mathbb{C}}^{\mathrm{reg}} \\ & t & \longmapsto & e^{2i\pi t/z} v_{\mathbb{C}}. \end{array}$$

Note that [**BrMaRo**, Lemma 2.22]

$$\boldsymbol{\pi}_z \in \mathrm{Z}(B_W). \tag{8.1.3}$$

The image of $\boldsymbol{\pi}_z$ in W is the generator $w_z = e^{2i\pi/z}\,\mathrm{Id}_V$ of $W \cap Z(\mathrm{GL}_{\mathbf{k}}(V))$. We put $\boldsymbol{\pi} = (\boldsymbol{\pi}_z)^z \in P_W \cap \mathrm{Z}(B_W)$.

8.1.B. Generic Hecke algebra. — Recall that $\boldsymbol{\aleph}^\circ$ is the set of pairs $(\aleph, j)$ with $\aleph \in \mathscr{A}/W$ and $0 \leqslant j \leqslant e_\aleph - 1$ (see §2.3).

Consider the affine variety $\boldsymbol{\mathcal{Q}} = (\mathbb{G}_m)^{\boldsymbol{\aleph}^\circ}$ over $\mathscr{O}$ and its integral and integrally closed commutative ring of functions $\mathscr{O}[\boldsymbol{\mathcal{Q}}] = \mathscr{O}[(\mathbf{q}_{\aleph,j}^{\pm 1})_{(\aleph,j)\in\boldsymbol{\aleph}^\circ}]$. Its fraction field is $F(\boldsymbol{\mathcal{Q}})$. Given $\aleph \in \mathscr{A}/W$, $H \in \aleph$ and $0 \leqslant j \leqslant e_H - 1 = e_\aleph - 1$, we put $\mathbf{q}_{H,j} = \mathbf{q}_{\aleph,j}$.

The *generic Hecke algebra* associated with W, denoted by $\mathscr{H}$, is the quotient of the group algebra $\mathscr{O}[\boldsymbol{\mathcal{Q}}]B_W$ by the ideal generated by the elements

$$\prod_{j=0}^{e_H-1} (\boldsymbol{\sigma}_H - \zeta_{e_H}^j \mathbf{q}_{H,j}^{|\boldsymbol{\mu}_W|}), \tag{8.1.4}$$

where H runs over $\mathscr{A}$. Given $H \in \mathscr{A}$, let $\mathbf{T}_H$ denote the image of $\boldsymbol{\sigma}_H$ in $\mathscr{H}$. By (8.1.2),

$$\mathscr{H} \textit{ is generated by } (b\mathbf{T}_H b^{-1})_{H\in\mathscr{A}, b\in P_W}, \tag{8.1.5}$$

where we still denote by b the image in $\mathscr{H}$ of an element $b \in B_W$. If $H \in \mathscr{A}$, then

$$\prod_{j=0}^{e_H-1} (\mathbf{T}_H - \zeta_{e_H}^j \mathbf{q}_{H,j}^{|\boldsymbol{\mu}_W|}) = 0. \tag{8.1.6}$$

Note that

(8.1.7) $\mathbf{T}_H$ *is invertible in* $\mathcal{H}$.

The following lemma follows immediately from [**BrMaRo**, Proposition 2.18]:

Lemma 8.1.8. — *The specialization* $\mathbf{q}_{\aleph,j} \mapsto 1$ *gives an isomorphism of* $\mathcal{O}$*-algebras* $\mathcal{O} \otimes_{\mathcal{O}[\mathcal{Q}]} \mathcal{H} \xrightarrow{\sim} \mathcal{O}W$.

Let $a \mapsto \bar{a}$ denote the unique automorphism of the $\mathbb{Z}$-algebra $\mathcal{O}[\mathcal{Q}]$ which extends the complex conjugation on $\mathcal{O}$ and such that $\overline{\mathbf{q}}_{\aleph,j} = \mathbf{q}_{\aleph,j}^{-1}$.

Let us now state a basic conjecture (cf [**BrMaRo**, §4.C] for (1) and [**BrMaMi1**, §2.A] for (2)).

Conjecture 8.1.8. — (1) $\mathcal{H}$ *is a free* $\mathcal{O}[\mathcal{Q}]$*-module of rank* $|W|$.

(2) *There exists a symmetrizing form* $\tau_{\mathcal{H}} : \mathcal{H} \to \mathcal{O}[\mathcal{Q}]$ *such that:*

   (a) *After the specialization of Lemma 8.1.8 (i.e.* $\mathbf{q}_{\aleph,j} \mapsto 1$*),* $\tau_{\mathcal{H}}$ *specializes to the canonical symmetrizing form of* $\mathcal{O}W$ *(i.e.* $w \mapsto \delta_{w,1}$*).*

   (b) *If* $b \in B_W$*, then*

$$\tau_{\mathcal{H}}(\pi)\overline{\tau_{\mathcal{H}}(b^{-1})} = \tau_{\mathcal{H}}(b\pi).$$

Some remarks.

- There is at most one form $\tau_{\mathcal{H}}$ satisfying (1) and (2).

- Conjecture 8.1.8 is known to hold for all but finitely many irreducible W's (cf [**Marin**] for a report on the status on this conjecture). It holds in particular when W is a Coxeter group and when W has type $G(de, e, r)$.

- If Conjecture 8.1.8 holds, then $\tau_{\mathcal{H}}(\pi) \neq 0$ since, by the property (1) of the statement (b) and by (8.1.3), $\tau_{\mathcal{H}}(\pi)$ specializes to 1 through $\mathbf{q}_{H,j} \mapsto 1$.

- Assumption (1) is known to hold after tensoring by F [**ER, Lo, MP, Et**].

When considering Hecke algebras over a base ring that does not contain $\mathbb{Q}$, we will assume that Part (1) of Conjecture 8.1.8 holds.

8.1.C. Cyclotomic Hecke algebras. — We will not use here the classical definition of cyclotomic Hecke algebras [**BrMaMi1**, §6.A], [**Chl4**, Definition 4.3.1], since we will need to work over a sufficiently large ring allowing us to let the parameters vary as much as possible.

Notation. *Following* [**Bon1**], [**Bon2**] *and* [**Bon3**], *we will use an exponential notation for the group algebra* $\mathcal{O}[\mathbb{R}]$, *which will be denoted by* $\mathcal{O}[\mathbf{q}^{\mathbb{R}}]$: $\mathcal{O}[\mathbf{q}^{\mathbb{R}}] = \bigoplus_{r\in\mathbb{R}} \mathcal{O}\,\mathbf{q}^r$, *with* $\mathbf{q}^r\mathbf{q}^{r'} = \mathbf{q}^{r+r'}$. *Since* $\mathcal{O}$ *is integral and* $\mathbb{R}$ *is torsion-free,* $\mathcal{O}[\mathbf{q}^{\mathbb{R}}]$*is also integral and we denote by* $F(\mathbf{q}^{\mathbb{R}})$ *its fraction field. If* $a = \sum_{r\in\mathbb{R}} a_r\mathbf{q}^r$, *we denote by* $\deg(a)$ *(respectively* $\mathrm{val}(a)$*) its degree (respectively its valuation), that is, the element of* $\mathbb{R}\cup\{-\infty\}$ *(respectively* $\mathbb{R}\cup\{+\infty\}$*) defined by*

$$\deg(a) = \max\{r \in \mathbb{R} \mid a_r \neq 0\}$$

$$\textit{(respectively}\quad \mathrm{val}(a) = \min\{r \in \mathbb{R} \mid a_r \neq 0\}\textit{).}$$

We have $\deg(a) = -\infty$ (respectively $\mathrm{val}(a) = +\infty$) if and only if $a = 0$. The usual properties of degree and valuation (with respect to the sum and the product) are of course satisfied. Let us start with an easy remark:

Lemma 8.1.9. — *The ring* $\mathcal{O}[\mathbf{q}^{\mathbb{R}}]$ *is integrally closed.*

Proof. — This follows from the fact that $\mathcal{O}[\mathbf{q}^{\mathbb{R}}] = \bigcup_{\Lambda\subset\mathbb{R}} \mathcal{O}[\mathbf{q}^{\Lambda}]$, where Λ runs over the finitely generated subgroups of $\mathbb{R}$, and that, if Λ has $\mathbb{Z}$-rank e, then $\mathcal{O}[\mathbf{q}^{\Lambda}] \simeq \mathcal{O}[\mathbf{t}_1^{\pm 1}, \dots, \mathbf{t}_e^{\pm 1}]$ is integrally closed. ☐

Fix a family $k = (k_{\aleph,j})_{(\aleph,j)\in\boldsymbol{\aleph}^\circ}$ of real numbers (as usual, if $H \in \aleph$ and $0 \leqslant i \leqslant e_H - 1$, then we set $k_{H,j} = k_{\aleph,j}$). The *cyclotomic Hecke algebra* (*with parameter* k) is the $\mathcal{O}[\mathbf{q}^{\mathbb{R}}]$-algebra $\mathcal{H}^{\mathrm{cyc}}(k) = \mathcal{O}[\mathbf{q}^{\mathbb{R}}] \otimes_{\mathcal{O}[\mathcal{Q}]} \mathcal{H}$, where $\mathcal{O}[\mathbf{q}^{\mathbb{R}}]$ is viewed as an $\mathcal{O}[\mathcal{Q}]$-algebra through the morphism

$$\begin{array}{rccc} \Theta_k^{\mathrm{cyc}}: & \mathcal{O}[\mathcal{Q}] & \longrightarrow & \mathcal{O}[\mathbf{q}^{\mathbb{R}}] \\ & \mathbf{q}_{\aleph,j} & \longmapsto & \mathbf{q}^{k_{\aleph,j}}. \end{array}$$

Let T_H denote the image of $\mathbf{T}_H$ in $\mathcal{H}^{\mathrm{cyc}}(k)$; then

(8.1.10) $$\mathcal{H}^{\mathrm{cyc}}(k) \textit{ is generated by } (\bar{g}\,T_H\,\bar{g}^{-1})_{H\in\mathcal{A},g\in P_W}$$

and, if $H \in \mathcal{A}$, then

(8.1.11) $$\prod_{j=0}^{e_H-1} (T_H - \zeta_{e_H}^j \mathbf{q}^{|\mu_W| k_{H,j}}) = 0.$$

Remark 8.1.12. — It follows from Lemma 8.1.8 that, after the specialization $\mathcal{O}[\mathbf{q}^{\mathbb{R}}] \to \mathcal{O}$, $\mathbf{q}^r \mapsto 1$ (this is the augmentation morphism for the group $\mathbb{R}$), we obtain $\mathcal{O} \otimes_{\mathcal{O}[\mathbf{q}^{\mathbb{R}}]} \mathcal{H}^{\mathrm{cyc}}(k) \simeq \mathcal{O}W$.

Similarly, $\mathcal{H}^{\mathrm{cyc}}(0) \simeq \mathcal{O}[\mathbf{q}^{\mathbb{R}}]W$. ∎

Remark 8.1.13. — Let $(\lambda_{\aleph})_{\aleph \in \mathcal{A}/W}$ be a family of real numbers and, if $H \in \aleph$, set $\lambda_H = \lambda_{\aleph}$. Let $k'_{\aleph,j} = k_{\aleph,j} + \lambda_{\aleph}$ and let $k' = (k'_{\aleph,j})_{(\aleph,j)\in\aleph^{\circ}}$. The map $\bar{g}T_H\bar{g}^{-1} \mapsto \mathbf{q}^{-\lambda_H}\bar{g}T_H\bar{g}^{-1}$ extends to an isomorphism of $\mathcal{O}[\mathbf{q}^{\mathbb{R}}]$-algebras $\mathcal{H}^{\mathrm{cyc}}(k) \xrightarrow{\sim} \mathcal{H}^{\mathrm{cyc}}(k')$.

Hence, if we take $\lambda_{\aleph} = -(k_{\aleph,0} + k_{\aleph,1} + \cdots + k_{\aleph,e_{\aleph}-1})/e_{\aleph}$, then $\mathcal{H}^{\mathrm{cyc}}(k) \simeq \mathcal{H}^{\mathrm{cyc}}(k')$, with $k' \in \mathcal{K}(\mathbb{R})$.

This shows that, in the study of cyclotomic Hecke algebras, it is enough to consider the case of parameters in the subspace $\mathcal{K}$ of $\mathbb{R}^{\aleph^{\circ}}$. ∎

Remark 8.1.14. — The group algebra $\mathcal{O}[\mathbf{q}^K]$ of any characteristic 0 field K is integrally closed. ∎

8.2. Coxeter groups

We assume in §8.2 that W is a Coxeter group (cf §2.6 for the notations). We assume $F \subset \mathbb{R}$ and $v_{\mathbb{C}} \in C_{\mathbb{R}}$.

8.2.A. Braid groups. — For $s, t \in S$, let m_{st} denote the order of st in W. For $s \in S$ and $H = \mathrm{Ker}(s - \mathrm{Id}_V)$, let $\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_H$ be the loop in $V_{\mathbb{C}}^{\mathrm{reg}}/W$ that is the image of the path

$$\begin{array}{rcc} [0,1] & \longrightarrow & V_{\mathbb{C}}^{\mathrm{reg}} \\ t & \longmapsto & e^{i\pi t}\Big(\dfrac{v_{\mathbb{R}} - s(v_{\mathbb{R}})}{2} + \dfrac{v_{\mathbb{R}} + s(v_{\mathbb{R}})}{2}\Big) \end{array}$$

from $v_{\mathbb{R}}$ to $s(v_{\mathbb{R}})$. With this notation, B_W admits the following presentation **[Bri]**:

$$B_W \quad : \quad \begin{cases} \text{Generators:} & (\boldsymbol{\sigma}_s)_{s\in S}, \\ \text{Relations:} & \forall\ s,t \in S,\ \underbrace{\boldsymbol{\sigma}_s\boldsymbol{\sigma}_t\boldsymbol{\sigma}_s\cdots}_{m_{st}\text{ times}} = \underbrace{\boldsymbol{\sigma}_t\boldsymbol{\sigma}_s\boldsymbol{\sigma}_t\cdots}_{m_{st}\text{ times}}. \end{cases} \tag{8.2.1}$$

Given $w = s_1 s_2 \cdots s_l$ a reduced decomposition of w, we set $\boldsymbol{\sigma}_w = \boldsymbol{\sigma}_{s_1}\boldsymbol{\sigma}_{s_2}\cdots\boldsymbol{\sigma}_{s_l}$: it is a classical fact that $\boldsymbol{\sigma}_w$ does not depend on the choice of the reduced decomposition. Moreover,

$$\pi = \boldsymbol{\sigma}_{w_0}^2. \tag{8.2.2}$$

8.2.B. Hecke algebras. —

Generic case. — Given $s \in S$, we put $\mathbf{q}_{s,j} = \mathbf{q}_{\mathrm{Ker}(s-\mathrm{Id}_V),j}$. It follows from (8.2.1) that the generic Hecke algebra $\mathscr{H}$ admits the following presentation, where $\mathbf{T}_s$ denotes the image of $\boldsymbol{\sigma}_s$ in $\mathscr{H}$:

(8.2.3)
$$\mathscr{H} \quad : \quad \begin{cases} \text{Generators:} & (\mathbf{T}_s)_{s\in S}, \\ \text{Relations:} & \forall\ s \in S,\ (\mathbf{T}_s - \mathbf{q}_{s,0}^2)(\mathbf{T}_s + \mathbf{q}_{s,1}^2) = 0, \\ & \forall\ s,t \in S,\ \underbrace{\mathbf{T}_s\mathbf{T}_t\mathbf{T}_s\cdots}_{m_{st}\text{ times}} = \underbrace{\mathbf{T}_t\mathbf{T}_s\mathbf{T}_t\cdots}_{m_{st}\text{ times}}. \end{cases}$$

Given $w = s_1 s_2 \cdots s_l$ a *reduced decomposition* of w, we set $\mathbf{T}_w = \mathbf{T}_{s_1}\mathbf{T}_{s_2}\cdots\mathbf{T}_{s_l}$. This is the image of $\boldsymbol{\sigma}_w$ in $\mathscr{H}$, hence $\mathbf{T}_w$ does not depend on the choice of the reduced decomposition. Moreover,

(8.2.4)
$$\mathscr{H} = \bigoplus_{w\in W} \mathscr{O}[\boldsymbol{\mathscr{Q}}]\mathbf{T}_w.$$

Note that $\mathbf{T}_w\mathbf{T}_{w'} = \mathbf{T}_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$. Note also that the basis $(\mathbf{T}_w)_{w\in W}$ of $\mathscr{H}$ depends on the choice of S, that is, of $v_{\mathbb{R}}$.

Cyclotomic case. — We take $k = (k_{\aleph,j})_{\aleph\in\mathscr{A}/W, j\in\{0,1\}} \in \mathscr{K}(\mathbb{R})$. Remark 8.1.13 shows that assuming $k_{\aleph,0} + k_{\aleph,1} = 0$ does not restrict the class of algebras we are interested in. Recall that for $H \in \mathscr{A}$, we set $c_{s_H} = k_{H,0} - k_{H,1} = 2k_{H,0} = -2k_{H,1}$. The cyclotomic Hecke algebra $\mathscr{H}^{\mathrm{cyc}}(k)$ is the $\mathscr{O}[\mathbf{q}^{\mathbb{R}}]$-algebra with the following presentation:

(8.2.5)
$$\mathscr{H}^{\mathrm{cyc}}(k) \quad : \quad \begin{cases} \text{Generators:} & (T_s)_{s\in S}, \\ \text{Relations:} & \forall\ s \in S,\ (T_s - \mathbf{q}^{c_s})(T_s + \mathbf{q}^{-c_s}) = 0, \\ & \forall\ s,t \in S,\ \underbrace{T_sT_tT_s\cdots}_{m_{st}\text{ times}} = \underbrace{T_tT_sT_t\cdots}_{m_{st}\text{ times}}. \end{cases}$$

8.3. KZ functor

In this section §8.3, we assume that $\mathbf{k} = \mathbb{C}$ and we consider $\mathscr{C}$ and $\boldsymbol{\mathscr{Q}}$ as complex analytic manifolds.

8.3.A. Analytic manifolds. — We denote by $\mathbb{C}\{X\}$ the ring of analytic functions on a complex manifold X.

Let $\exp : \mathscr{C} \to \boldsymbol{\mathscr{Q}}$ be the analytic map given by $\mathbf{q}_{H,j} = e^{2i\pi K_{H,-j}/|\mu_W|}$.

Let U be a submanifold of $\mathscr{C}$ such that

- exp restricts to an isomorphism $U \xrightarrow{\sim} \exp(U)$
- given any $E_1, E_2, \ldots, E_{n-1}, E_n = E_1 \in \mathrm{Irr}(\mathbf{k}W)$ and $u_1, \ldots, u_{n-1} \in U$ such that $C_{E_i} - C_{E_{i+1}}$ takes a non-zero integral value at u_i for $1 \leqslant i \leqslant n-1$, then $C_{E_1}(u_1) = C_{E_1}(u_{n-1})$.

Our assumptions on U ensure that $\dot{\mathscr{O}}(\mathbb{C}\{U\})$ is a highest weight category over $\mathbb{C}\{U\}$ (Theorem 7.5.1).

8.3.B. KZ functor and properties. — Let us recall the construction of the KZ functor $\mathrm{KZ} : \dot{\mathcal{O}}(\mathbb{C}\{U\}) \to (\mathbb{C}\{U\}\mathscr{H})\text{-mod}$ and its main property as in [**GGOR**, §5.3]. Our change of Dunkl operators corresponds to a twist of the monodromy representation of [**GGOR**] by the one-dimensional representation of B_W given by $\boldsymbol{\sigma}_H \mapsto \mathbf{q}_{H,0}$.

Let $M \in \dot{\mathcal{O}}(\mathbb{C}\{U\})$ and let $M^{\mathrm{reg}} = \mathbb{C}\{U\}\dot{\mathbf{H}}^{\mathrm{reg}} \otimes_{\tilde{\mathbf{H}}} M$. The isomorphism Θ^{reg} (Theorem 3.1.11(d)) makes M^{reg} into a $(\mathbb{C}\{U\}\otimes\mathscr{D}(V^{\mathrm{reg}})\rtimes W)$-module and the Morita equivalence of Lemma 3.1.8(b) produces a $(\mathbb{C}\{U\}\otimes\mathscr{D}(V^{\mathrm{reg}}/W))$-module $\bar{M}^{\mathrm{reg}}$. It has regular singularities and taking horizontal sections, we obtain a $\mathbb{C}\{U\}B_W$-module $\mathrm{dR}(\bar{M}^{\mathrm{reg}})_{\bar{v}_{\mathbb{C}}}$, finitely generated as a $\mathbb{C}\{U\}$-module. The action of $\mathbb{C}\{U\}B_W$ on $\mathrm{dR}(\bar{M}^{\mathrm{reg}})_{\bar{v}_{\mathbb{C}}}$ factors through an action of $\mathbb{C}\{U\}\mathscr{H}$: the resulting $(\mathbb{C}\{U\}\mathscr{H})$-module is $\mathrm{KZ}(M)$.

The KZ functor satisfies a "double endomorphism Theorem" property [**GGOR**, Theorems 5.14 and 5.16].

Theorem 8.3.1. — *The functor* $\mathrm{KZ} : \dot{\mathcal{O}}(\mathbb{C}\{U\}) \to (\mathbb{C}\{U\}\mathscr{H})\text{-mod}$ *is exact and its restriction to* $\mathrm{Proj}(\dot{\mathcal{O}}(\mathbb{C}\{U\}))$ *is fully faithful. It induces an isomorphism*

$$Z(\dot{\mathcal{O}}(\mathbb{C}\{U\})) \xrightarrow{\sim} Z(\mathbb{C}\{U\}\mathscr{H})$$

and an equivalence

$$\dot{\mathcal{O}}(\mathbb{C}\{U\})/\{M | M^{\mathrm{reg}} = 0\} \xrightarrow{\sim} (\mathbb{C}\{U\}\mathscr{H})\text{-mod}.$$

8.3.C. Semi-simplicity. — Let $c \in \mathscr{C}(\mathbb{C})$ and $q = \exp(c)$. Theorem 8.3.1 shows that the semisimplicity of $\dot{\mathcal{O}}(\mathfrak{C}_c)$ is equivalent to that of $\mathbb{C}[q]\mathscr{H}$. From Theorem 7.5.2, we deduce the following [**Rou**, Proposition 5.4]. Note that this result is equivalent to a result of Chlouveraki on Schur elements obtained independently [**Chl4**, Theorem 4.2.5].

Corollary 8.3.2. — *If the subgroup of* $\mathbb{C}^\times$ *generated by* $\{q_{\aleph,j}^{|\mu_W|}\}_{(\aleph,j)\in\boldsymbol{\aleph}^\circ}$ *is torsion-free, then* $\mathbb{C}[q]\mathscr{H}$ *is semisimple.*

Proof. — Let $k = \kappa(c)$ and let Γ_0 be the subgroup of $\mathbb{C}$ generated by the $k_{\aleph,j}$'s for $(\aleph, j) \in \boldsymbol{\aleph}^\circ$. By assumption, $\Gamma_0/(\mathbb{Z} \cap \Gamma_0)$ is torsion free. As a consequence, there is a subgroup Γ' of Γ_0 such that $\Gamma_0 = \Gamma' \times (\mathbb{Z} \cap \Gamma_0)$. Let $p : \Gamma_0 \to \Gamma'$ be the projection, let $k' = p(k)$ and let $c' = \kappa^{-1}(k')$. Theorem 7.5.2 shows that $\dot{\mathcal{O}}(\mathfrak{C}_{c'})$ is semisimple, hence $\mathbb{C}[q]\mathscr{H}$ is semisimple, since $\mathbb{C}[\exp(c')]\mathscr{H} \simeq \mathbb{C}[\exp(c)]\mathscr{H}$. □

Corollary 8.3.2 provides an improvement of Theorem 7.5.2 using now that the semisimplicity of $\mathbb{C}[q]\mathscr{H}$ implies that of $\dot{\mathcal{O}}(\mathfrak{C}_c)$ [**Rou**, Proposition 5.4].

Corollary 8.3.3. — *The prime ideals in* $\mathscr{F}_1$ *correspond to affine hyperplanes of* $\mathscr{K}$ *of the form* $\sum_{(\aleph,j)\in\boldsymbol{\aleph}^\circ} a_{\aleph,j} K_{\aleph,j} = \frac{b}{r}$ *for some* $a \in \mathbb{Z}^{\boldsymbol{\aleph}^\circ}$ *with* $\gcd(\{a_{\aleph,j}\}) = 1$ *and* $r, b \in \mathbb{Z}$, $r \geqslant 2$, $b \geqslant 1$ *and* $\gcd(r, b) = 1$.

If $(\mathbb{Q} - \mathbb{Z}) \cap \big(\sum_{(\aleph,j)\in\boldsymbol{\aleph}^\circ} \mathbb{Z} k_{\aleph,j}\big) = \emptyset$, *then* $\dot{\mathcal{O}}(\mathfrak{C}_c)$ *is semisimple.*

Note that the affine hyperplane of $\mathscr{K}$ coming from a prime ideal $\mathfrak{p}$ in $\mathscr{F}_1$ has a unique equation of the form $\sum_{(\aleph,j)\in\aleph^\circ} a_{\mathfrak{p},\aleph,j} K_{\aleph,j} = \frac{b_\mathfrak{p}}{r_\mathfrak{p}}$ as in the corollary. We define a map $m' : \mathscr{F}_1 \to \mathscr{K}(\mathbb{Z})$ by $m'(\mathfrak{p}) = \sum_{(\aleph,j)\in\aleph^\circ} a_{\mathfrak{p},\aleph,j} K_{\aleph,j}$.

Given $\mathfrak{p} \in \mathscr{F}_1$, let $r'_\mathfrak{p}$ be the minimal positive integer such that $|\boldsymbol{\mu}_W| r'_\mathfrak{p} \mathrm{sec}^*(m'(\mathfrak{p})) \in \mathbb{Z}^{\aleph^\circ}$. We define a map $\mathscr{F}_1 \to \mathbb{Z}^{\aleph^\circ}$ by setting $m(\mathfrak{p})_{\aleph,j} = |\boldsymbol{\mu}_W| r'_\mathfrak{p} (\mathrm{sec}^*(m'(\mathfrak{p})))_{\aleph,-j}$. The elements of $\mathbb{Z}^{\aleph^\circ}$ can be viewed as functions on $(\mathbb{G}_m)^{\aleph^\circ}$. The previous results have the following corollary.

Corollary 8.3.4. — *The algebra $\mathbb{C}[q]\mathscr{H}$ is semisimple if and only if $m(\mathfrak{p})(q) \neq e^{2i\pi|\boldsymbol{\mu}_W|b_\mathfrak{p}/r_\mathfrak{p}}$ for all $\mathfrak{p} \in \mathscr{F}_1$.*

Note that the set $\{(m(\mathfrak{p}), |\boldsymbol{\mu}_W| b_\mathfrak{p}/r_\mathfrak{p} \pmod{\mathbb{Z}})\}_{\mathfrak{p}\in\mathscr{F}_1}$ is finite, so that Corollary 8.3.4 provides a finite set of conditions. Given $\mathfrak{p} \in \mathscr{F}_1$, let $\Psi_\mathfrak{p} \in F[t]$ be the minimal polynomial of $e^{2i\pi|\boldsymbol{\mu}_W|b_\mathfrak{p}/r_\mathfrak{p}}$.

8.4. Representations

The following result is due to Malle [**Mal3**, Theorem 5.2], the difficulty being the statement on splitting.

***Theorem 8.4.1* (Malle)**. — *The $F(\mathscr{Q})$-algebra $F(\mathscr{Q})\mathscr{H}$ is split semisimple.*

Since the algebra FW is also split semisimple (by Benard-Bessis Theorem 2.4.2), it follows from Tits Deformation Theorem [**GePf**, Theorem 7.4.6] that we have a bijective map

$$\begin{array}{ccc} \mathrm{Irr}(W) & \xrightarrow{\sim} & \mathrm{Irr}(F(\mathscr{Q})\mathscr{H}) \\ E & \longmapsto & E^{\mathrm{gen}} \end{array}$$

defined by the following property: the character of E is the specialization of the character of E^{gen} through $\mathbf{q}_{\aleph,j} \mapsto 1$.

Let $\boldsymbol{\omega}_E^{\mathrm{gen}} : \mathrm{Z}(F(\mathscr{Q})\mathscr{H}) \longrightarrow F(\mathscr{Q})$ denote the central character associated with the representation E: given $a \in \mathrm{Z}(F(\mathscr{Q})\mathscr{H})$, we define $\boldsymbol{\omega}_E^{\mathrm{gen}}(a)$ as the element of $F(\mathscr{Q})$ by which a acts on E^{gen}. This is a morphism of $F(\mathscr{Q})$-algebras. Since $\mathscr{O}[\mathscr{Q}]$ is integrally closed, $\boldsymbol{\omega}_E^{\mathrm{gen}}$ restricts to a morphism of $\mathscr{O}[\mathscr{Q}]$-algebras $\boldsymbol{\omega}_E^{\mathrm{gen}} : \mathrm{Z}(\mathscr{H}) \longrightarrow \mathscr{O}[\mathscr{Q}]$. We denote by $\omega_E : \mathrm{Z}(\mathscr{O}W) \to \mathscr{O}$ the usual central character (specialization of $\boldsymbol{\omega}_E^{\mathrm{gen}}$ at $\mathbf{q} = 1$).

The image of $\boldsymbol{\pi}_z \in \mathrm{Z}(B_W)$ in $\mathscr{H}$ belongs to the center of this algebra. Hence, one can evaluate $\boldsymbol{\omega}_E^{\mathrm{gen}}$ at $\boldsymbol{\pi}_z$ and we recover the formula of [**BrMi**, Proposition 4.16].

Proposition 8.4.2. — *Given* $E \in \mathrm{Irr}(W)$, *we have*

$$\boldsymbol{\omega}_E^{\mathrm{gen}}(\pi_z) = \omega_E(w_z) \prod_{(\aleph,j)\in\boldsymbol{\aleph}^\circ} \mathbf{q}_{\aleph,j}^{\frac{|\mu_W|}{z}\frac{m_{\aleph,j}^E|\aleph|e_\aleph}{\dim E}}.$$

Proof. — Let U be an open submanifold of $\mathscr{C}$ satisfying the assumptions of §8.3.A. The element $w_z e^{2i\pi\mathbf{eu}/z}$ acts on objects M of $\mathscr{O}_U$ and defines an element of $\mathrm{Z}(\mathscr{O}_U)^\times$. Its action on $\mathrm{KZ}(M)$ is given by π_z. We deduce from Proposition 7.2.2 that π_z acts on $\mathrm{KZ}(\Delta(E))$ by $\omega_E(w_z)e^{2iC_E/z}$ and the proposition follows from Lemma 7.2.1. □

The following result follows from Corollary 8.3.2.

Corollary 8.4.3. — *The* $F(\mathbf{q}^{\mathbb{R}})$*-algebra* $F(\mathbf{q}^{\mathbb{R}})\mathscr{H}^{\mathrm{cyc}}(k)$ *is split semisimple.*

By Tits Deformation Theorem, we get a sequence of bijective maps

$$\begin{array}{ccccc} \mathrm{Irr}(W) & \xrightarrow{\sim} & \mathrm{Irr}(F(\mathbf{q}^{\mathbb{R}})\mathscr{H}^{\mathrm{cyc}}(k)) & \xrightarrow{\sim} & \mathrm{Irr}(F(\boldsymbol{\mathcal{Q}})\mathscr{H}) \\ \chi & \longmapsto & \chi_k^{\mathrm{cyc}} & \longmapsto & \chi^{\mathrm{gen}} \end{array}$$

such that $\chi_k^{\mathrm{cyc}} = \Theta_k^{\mathrm{cyc}} \circ \chi^{\mathrm{gen}}$.

Finally, let $\omega_{\chi,k}^{\mathrm{cyc}} : \mathrm{Z}(\mathscr{H}^{\mathrm{cyc}}(k)) \longrightarrow \mathscr{O}[\mathbf{q}^{\mathbb{R}}]$ denote the central character associated with χ_k^{cyc}. It follows from Proposition 8.4.2 that

(8.4.4) $$\omega_{\chi,k}^{\mathrm{cyc}}(\pi) = \mathbf{q}^{|\mu_W|C_\chi(k)}.$$

8.5. Hecke families

We assume in §8.5 that Part (1) of Conjecture 8.1.8 holds.

8.5.A. Definition. — We will call *Hecke ring*, and we denote by $\mathscr{O}^{\mathrm{cyc}}[\mathbf{q}^{\mathbb{R}}]$, the ring

$$\mathscr{O}^{\mathrm{cyc}}[\mathbf{q}^{\mathbb{R}}] = \mathscr{O}[\mathbf{q}^{\mathbb{R}}]\big[\big((1-\mathbf{q}^r)^{-1}\big)_{r\in\mathbb{R}\setminus\{0\}}\big].$$

Given b a central idempotent (not necessarily primitive) of $\mathscr{O}^{\mathrm{cyc}}[\mathbf{q}^{\mathbb{R}}]\mathscr{H}^{\mathrm{cyc}}(k)$, we denote by $\mathrm{Irr}_{\mathscr{H}}(W,b)$ the set of irreducible representations E of W such that $E_k^{\mathrm{cyc}} \in \mathrm{Irr}(F(\mathbf{q}^{\mathbb{R}})\mathscr{H}^{\mathrm{cyc}}(k)b)$.

Definition 8.5.1. — *A* ***Hecke*** $\boldsymbol{k}$***-family*** *is a subset of* $\mathrm{Irr}(W)$ *of the form* $\mathrm{Irr}_{\mathscr{H}}(W,b)$, *where* b *is a primitive central idempotent of* $\mathscr{O}^{\mathrm{cyc}}[\mathbf{q}^{\mathbb{R}}]\mathscr{H}^{\mathrm{cyc}}(k)$.

The Hecke k-families form a partition of $\mathrm{Irr}(W)$.

***Lemma 8.5.2* (Broué-Kim)**. — *If* E *and* E' *are in the same Hecke* k*-family, then* $C_E(k) = C_{E'}(k)$.

Proof. — We could apply the argument contained in [**BrKi**, Proposition 2.9(2)]. However, our framework is slightly different and we propose a different proof, based on the particular form of $\omega_{E,k}^{\mathrm{cyc}}(\pi)$ (see 8.4.4).

Let $\mathscr{E} = \{r_1, r_2, \dots, r_m\}$, with $r_i \neq r_j$ if $i \neq j$, denote the image of the map $\mathrm{Irr}(W) \to \mathbb{R}$, $E \mapsto |\boldsymbol{\mu}_W| C_E(k)$. If $1 \leqslant j \leqslant m$, we set

$$\mathscr{F}_j = \{E \in \mathrm{Irr}(W) \mid |\boldsymbol{\mu}_W| C_E(k) = r_j\}.$$

Given $E \in \mathrm{Irr}(W)$, let $e_{E,k}$ denote the associated primitive central idempotent of $F(\mathbf{q}^{\mathbb{R}})\mathscr{H}^{\mathrm{cyc}}(k)$. We set

$$b_j = \sum_{E \in \mathscr{F}_j} e_{E,k}.$$

To show the lemma, it is sufficient to check that $b_j \in \mathscr{O}^{\mathrm{cyc}}[\mathbf{q}^{\mathbb{R}}]\mathscr{H}^{\mathrm{cyc}}(k)$. In $\mathscr{O}^{\mathrm{cyc}}[\mathbf{q}^{\mathbb{R}}]\mathscr{H}^{\mathrm{cyc}}(k)$, we have

$$\pi = \mathbf{q}^{r_1} b_1 + \mathbf{q}^{r_2} b_2 + \cdots + \mathbf{q}^{r_m} b_m.$$

Hence,

$$\begin{cases} b_1 + b_2 + \cdots + b_m = 1 \\ \mathbf{q}^{r_1} b_1 + \mathbf{q}^{r_2} b_2 + \cdots + \mathbf{q}^{r_m} b_m = \pi \\ \qquad \cdots \\ \mathbf{q}^{(m-1)r_1} b_1 + \mathbf{q}^{(m-1)r_2} b_2 + \cdots + \mathbf{q}^{(m-1)r_m} b_m = \pi^{m-1}. \end{cases}$$

The determinant of this system is a Vandermonde determinant, equal to

$$\prod_{1 \leqslant i < j \leqslant m} (\mathbf{q}^{r_i} - \mathbf{q}^{r_j}),$$

which is invertible in the Hecke ring $\mathscr{O}^{\mathrm{cyc}}[\mathbf{q}^{\mathbb{R}}]$ by construction. Since $1, \pi, \dots, \pi^{m-1} \in \mathscr{H}^{\mathrm{cyc}}(k)$, the result follows. □

8.5.B. Reflections of order 2. — In this section §8.5.B, we assume that all the reflections of W have order 2. We thank Maria Chlouveraki for explaining us the main result of this section.

Let $\mathscr{O}[\mathscr{Q}] \to \mathscr{O}[\mathscr{Q}]$, $f \mapsto f^\dagger$ denote the unique involutive automorphism of $\mathscr{O}$-algebra exchanging $\mathbf{q}_{\aleph,0}$ and $\mathbf{q}_{\aleph,1}$ for all $\aleph \in \mathscr{A}/W$. Let $\mathscr{O}[\mathscr{Q}]B_W \to \mathscr{O}[\mathscr{Q}]B_W$, $a \mapsto a^\dagger$ denote also the unique semilinear (for the involution $f \mapsto f^\dagger$ of $\mathscr{O}[\mathscr{Q}]$) automorphism such that $\beta^\dagger = \varepsilon(p_W(\beta))\beta$ for all $\beta \in B_W$. The relations (8.1.4) are stable under this automorphism. So it induces a semilinear automorphism $\mathscr{H} \to \mathscr{H}$, $h \mapsto h^\dagger$ of the generic Hecke algebra.

This automorphism, after the specialization $\mathbf{q}_{\aleph,j} \mapsto 1$, becomes the unique $\mathscr{O}$-linear automorphism of $\mathscr{O}W$ which sends $w \in W$ to $\varepsilon(w)w$. In other words, it is the automorphism induced by the linear character ε.

Similarly, since $k_{\aleph,0}+k_{\aleph,1}=0$, if we still denote by $\mathcal{O}[\mathbf{q}^{\mathbb{R}}]\to\mathcal{O}[\mathbf{q}^{\mathbb{R}}]$, $f\mapsto f^{\dagger}$ the unique automorphism of $\mathcal{O}$-algebra such that $(\mathbf{q}^r)^{\dagger}=\mathbf{q}^{-r}$, then the specialization $\mathbf{q}_{\aleph,j}\mapsto\mathbf{q}^{k_{\aleph,j}}$ induces an $\mathcal{O}[\mathbf{q}^{\mathbb{R}}]$-semilinear automorphism of the algebra $\mathcal{H}^{\mathrm{cyc}}$, still denoted by $h\mapsto h^{\dagger}$. If $\chi\in\mathrm{Irr}(W)$, let $(\chi^{\mathrm{gen}})^{\dagger}$ (respectively $(\chi_k^{\mathrm{cyc}})^{\dagger}$) denote the composition of χ^{gen} (respectively χ_k^{cyc}) with the automorphism $\dagger$: it is a new irreducible character of $F(\mathcal{Q})\mathcal{H}$ (respectively $F(\mathbf{q}^{\mathbb{R}})\mathcal{H}^{\mathrm{cyc}}(k)$). Since it is determined by its specialization through $\mathbf{q}_{\aleph,j}\mapsto 1$ (respectively $\mathbf{q}^r\mapsto 1$), we have

$$(\chi^{\mathrm{gen}})^{\dagger}=(\chi\varepsilon)^{\mathrm{gen}}\qquad\text{and}\qquad(\chi_k^{\mathrm{cyc}})^{\dagger}=(\chi\varepsilon)_k^{\mathrm{cyc}}.\tag{8.5.3}$$

As the automorphism $f\mapsto f^{\dagger}$ of $\mathcal{O}[\mathbf{q}^{\mathbb{R}}]$ extends to the ring $\mathcal{O}^{\mathrm{cyc}}[\mathbf{q}^{\mathbb{R}}]$, the next lemma follows immediately:

Lemma 8.5.4. — *Assume that all the reflections of W have order 2. If $\mathcal{F}$ is a Hecke k-family, then $\mathcal{F}\varepsilon$ is a Hecke k-family.*

8.5.C. About the coefficient ring. — It might seem strange to work with such a large coefficient ring (far from being Noetherian for instance). A first argument for this choice is that this ring is still integral and integrally closed.

Moreover, this choice allows to work with a fixed ring, whatever the value of the parameter k is: as we let k vary in a *real* vector space of parameters, this choice becomes more natural. Also, as it has been seen in Corollary 8.4.3, the fact that it is possible to extract n-th roots of all "powers" of $\mathbf{q}$ implies immediately the splitness of all the cyclotomic Hecke algebras over the same fixed ring.

This ring is of the form $\mathcal{O}[\Gamma]$, where Γ is a totally ordered abelian group: this allows to define for instance, thanks to the notion of degree and valuation, the $\mathbf{a}$ and $\mathbf{A}$-invariants associated with irreducible characters of W (even though we will not used them in this book). Finally, as we will see in §8.6, it is also the general framework for Kazhdan-Lusztig theory, which we aim to generalize to complex reflection groups.

It is nevertheless necessary to compare the notion of Hecke families we have introduced in §8.5 with the classical definitions. In order to do so, let B be a commutative integral $\mathcal{O}[\mathcal{Q}]$-algebra, with fraction field F_B. Let L_B denote the subgroup of $B^{\times}$ generated by $\{1_B\mathbf{q}_{\aleph,j}\}_{(\aleph,j)\in\aleph^{\circ}}$, a quotient of $\mathbb{Z}^{\aleph^{\circ}}$. We assume that L_B has no torsion, that $\mathcal{O}[L_B]$ embeds in B, and that $F[L_B]\cap B=\mathcal{O}[L_B]$. As in Corollary 8.4.3, the F_B-algebra $F_B\mathcal{H}$ is split semisimple and we get a bijective map $\mathrm{Irr}(W)\xrightarrow{\sim}\mathrm{Irr}(F_B\mathcal{H})$.

Example 8.5.5. — Let Λ be a torsion-free abelian group. As in Lemma 8.1.9, note that $\mathcal{O}[\Lambda]$ is integrally closed. Let $q : \aleph^\circ \to \Lambda$ be a map. It extends to a morphism of groups $\mathbb{Z}^{\aleph^\circ} \to \Lambda$ and to a morphism between group algebras $\mathcal{O}[\mathcal{Q}] \to \mathcal{O}[\Lambda]$. The algebra $B = \mathcal{O}[\Lambda]$ satisfies the previous assumption. ■

Let $B^{\mathrm{cyc}} = B[(1-v)^{-1}_{v\in L_B-\{0\}}]$ (we note additively the abelian group L_B). As in §8.5.A, we have a notion of *Hecke B-family*.

Proposition 8.5.6. — *The Hecke B-families coincide with the Hecke $\mathcal{O}[L_B]$-families.*

Proof. — Let $\Lambda = L_B$. We have $B^{\mathrm{cyc}} \cap F_{\mathcal{O}[\Lambda]} = \mathcal{O}[\Lambda]^{\mathrm{cyc}}$. We deduce that, if b is a primitive central idempotent of $F_{\mathcal{O}[\Lambda]}\mathcal{H}$ such that $b \in B^{\mathrm{cyc}}\mathcal{H}$, then $b \in \mathcal{O}[\Lambda]^{\mathrm{cyc}}\mathcal{H}$. □

The previous proposition reduces the study of Hecke families to the case where $B = \mathcal{O}[\Lambda]$ and Λ is a torsion-free quotient of $\mathbb{Z}^{\aleph^\circ}$.

Let $\mathcal{M} = m(\mathcal{F}_1)$, a finite subset $\mathbb{Z}^{\aleph^\circ}$ (cf end of §8.3.C). Consider now a torsion-free abelian group Λ and a map $q : \aleph^\circ \to \Lambda$ as in Example 8.5.5. Let Λ' be a torsion-free abelian group and let $f : \Lambda \to \Lambda'$ be a surjective morphism of groups.

Proposition 8.5.7. — *If $q(\mathcal{M}) \cap \mathrm{Ker} f = \{0\}$, then the Hecke $\mathcal{O}[\Lambda]$-families coincide with the Hecke $\mathcal{O}[\Lambda']$-families.*

Proof. — The morphism f induces a surjective morphism between group algebras $\mathcal{O}[\Lambda] \to \mathcal{O}[\Lambda']$ which extends to a surjective morphism of algebras between localizations $f : \mathcal{O}[\Lambda][(1-v)^{-1}_{v\in\Lambda-\mathrm{Ker} f}] \to \mathcal{O}[\Lambda']^{\mathrm{cyc}}$. Let $h \in F[\Lambda][\{\Psi_{\mathfrak{p}}(m(\mathfrak{p}))^{-1}\}_{\mathfrak{p}\in\mathcal{F}_1, m(\mathfrak{p})\notin \mathrm{Ker} q}]$ (cf end of §8.3.C). If $h \in \mathcal{O}[\Lambda]^{\mathrm{cyc}}$, then $h \in f^{-1}(\mathcal{O}[\Lambda']^{\mathrm{cyc}})$.

It follows from Corollary 8.3.4 that the idempotents of $Z(F(\Lambda)\mathcal{H})$ are in the algebra $F[\Lambda][\{\Psi_{\mathfrak{p}}(m(\mathfrak{p}))^{-1}\}_{\mathfrak{p}\in\mathcal{F}_1, m(\mathfrak{p})\notin \mathrm{Ker} q}]\mathcal{H}$. Consequently, any idempotent of $Z(\mathcal{O}[\Lambda]^{\mathrm{cyc}}\mathcal{H})$ is contained in $\mathcal{O}[\Lambda][(1-v)^{-1}_{v\in\Lambda-\mathrm{Ker} f}]\mathcal{H}$. Proposition D.1.2 shows that the central idempotents of $\mathcal{O}[\Lambda][(1-v)^{-1}_{v\in\Lambda-\mathrm{Ker} f}]\mathcal{H}$ are in bijection with those of $\mathcal{O}[\Lambda']^{\mathrm{cyc}}\mathcal{H}$ and the result follows. □

Given Λ and q as above, there exists a morphism of groups $f : \Lambda \to \mathbb{Z}$ such that $\mathrm{Ker} f \cap q(\mathcal{M}) = \{0\}$. So Proposition 8.5.7 reduces the study of Hecke $\mathcal{O}[\Lambda]$-families (and so of Hecke B-families, by the above arguments) to the case of Hecke $\mathcal{O}[t^{\pm 1}]$-families, for a choice of integers $m_{\aleph,j} \in \mathbb{Z}$ defining a morphism of groups $\mathbb{Z}^{\aleph^\circ} \to t^{\mathbb{Z}}$, $q_{\aleph,j} \mapsto t^{m_{\aleph,j}}$. This is the usual framework for Hecke families.

8.6. Kazhdan-Lusztig cells

Assumption. *From now on, and until the end of §8.6, we assume that W is a Coxeter group and we fix a family $k = (k_{\aleph,j})_{\aleph\in\mathscr{A}/W, j\in\{0,1\}} \in \mathscr{C}_{\mathbb{R}}$. We denote by $c : \mathrm{Ref}(W) \to \mathbb{R}$, the map defined by $c_{s_H} = 2k_{H,0}$ for all $H \in \mathscr{A}$. It is constant on conjugacy classes.*

Giving the map $c : \mathrm{Ref}(W) \to \mathbb{R}$ constant on conjugacy classes of reflections is equivalent to giving the family $k \in \mathscr{C}_{\mathbb{R}}$.

8.6.A. Kazhdan-Lusztig basis. — The involution $a \mapsto \bar{a}$ of $\mathscr{O}[\mathbf{q}^{\mathbb{R}}]$ extends to an $\mathscr{O}[\mathbf{q}^{\mathbb{R}}]$-semilinear involution of the algebra $\mathscr{H}^{\mathrm{cyc}}(k)$ by setting

$$\overline{T}_w = T_{w^{-1}}^{-1}.$$

If $\mathbb{X}$ is a subset of $\mathbb{R}$, we set $\mathscr{O}[\mathbf{q}^{\mathbb{X}}] = \bigoplus_{r\in\mathbb{X}} \mathscr{O}\, \mathbf{q}^r$. We also set

$$\mathscr{H}^{\mathrm{cyc}}(k)_{>0} = \bigoplus_{w\in W} \mathscr{O}[\mathbf{q}^{\mathbb{R}_{>0}}]\ T_w.$$

The following theorem is proven in **[KaLu]** (equal parameter case) and **[Lus1]** (general case).

***Theorem 8.6.1* (Kazhdan-Lusztig)**. — *For $w \in W$, there exists a unique $C_w \in \mathscr{H}^{\mathrm{cyc}}(k)$ such that*

$$\begin{cases} \overline{C}_w = C_w, \\ C_w \equiv T_w \mod \mathscr{H}^{\mathrm{cyc}}(k)_{>0}. \end{cases}$$

The family $(C_w)_{w\in W}$ is an $\mathscr{O}[\mathbf{q}^{\mathbb{R}}]$-basis of $\mathscr{H}^{\mathrm{cyc}}(k)$.

Note that C_w depends only on k (i.e., on c). For example, if $s \in S$, then

$$C_s = \begin{cases} T_s - q^{c_s} & \text{if } c_s > 0, \\ T_s & \text{si } c_s = 0, \\ T_s + q^{-c_s} & \text{if } c_s < 0. \end{cases}$$

Similarly, as well as T_w, C_w depends on the choice of S. The basis $(C_w)_{w\in W}$ is called the *Kazhdan-Lusztig basis* of $\mathscr{H}^{\mathrm{cyc}}(k)$.

8.6.B. Definition. — For $x, y \in W$, we will write $x \overset{L,c}{\longleftarrow} y$ if there exists $h \in \mathcal{H}^{\mathrm{cyc}}(k)$ such that C_x appears with a non-zero coefficient in the decomposition of hC_y in the Kazhdan-Lusztig basis. Let $\leqslant_L^c$ denote the transitive closure of this relation; it is a pre-order and we denote by $\sim_L^{\mathrm{KL},c}$ the associated equivalence relation.

We define similarly $\overset{R,c}{\longleftarrow}$ by multiplying on the right by h as well as $\leqslant_R^c$ and $\sim_R^{\mathrm{KL},c}$. Let $\leqslant_{LR}^c$ be the transitive relation generated by $\leqslant_L^c$ and $\leqslant_R^c$, and let $\sim_{LR}^{\mathrm{KL},c}$ denote the associated equivalence relation.

Definition 8.6.2. — *We call **Kazhdan-Lusztig left c-cells** of W the equivalence classes for the relation $\sim_L^{\mathrm{KL},c}$. We define similarly **Kazhdan-Lusztig right c-cells** and **Kazhdan-Lusztig two-sided c-cells** using $\sim_R^{\mathrm{KL},c}$ and $\sim_{LR}^{\mathrm{KL},c}$ respectively. If $? \in \{L,R,LR\}$, we denote by ${}^{\mathrm{KL}}\mathrm{Cell}_?^c(W)$ the corresponding set of Kazhdan-Lusztig c-cells in W.*

If $? \in \{L,R,LR\}$ and if Γ is an equivalence class for the relation $\sim_?^{\mathrm{KL},c}$ (that is, a Kazhdan-Lusztig c-cell of the type associated with ?), we set

$$\mathcal{H}^{\mathrm{cyc}}(k)_{\leqslant_?^{\mathrm{KL},c}\Gamma} = \bigoplus_{w \leqslant_?^{\mathrm{KL},c}\Gamma} \mathcal{O}[\mathbf{q}^{\mathbb{R}}]\, C_w \qquad \text{and} \qquad \mathcal{H}^{\mathrm{cyc}}(k)_{<_?^{\mathrm{KL},c}\Gamma} = \bigoplus_{w <_?^{\mathrm{KL},c}\Gamma} \mathcal{O}[\mathbf{q}^{\mathbb{R}}]\, C_w,$$

as well as

$$\mathcal{M}_\Gamma^? = \mathcal{H}^{\mathrm{cyc}}(k)_{\leqslant_?^{\mathrm{KL},c}\Gamma} / \mathcal{H}^{\mathrm{cyc}}(k)_{<_?^{\mathrm{KL},c}\Gamma}.$$

By construction, $\mathcal{H}^{\mathrm{cyc}}(k)_{\leqslant_?^{\mathrm{KL},c}\Gamma}$ and $\mathcal{H}^{\mathrm{cyc}}(k)_{<_?^{\mathrm{KL},c}\Gamma}$ are ideals (left ideals if $? = L$, right ideals if $? = R$ or two-sided ideals if $? = LR$) and $\mathcal{M}_\Gamma^?$ is a left (respectively right) $\mathcal{H}^{\mathrm{cyc}}(k)$-module if $? = L$ (respectively $? = R$), or an $(\mathcal{H}^{\mathrm{cyc}}(k), \mathcal{H}^{\mathrm{cyc}}(k))$-bimodule if $? = LR$. Note that

(8.6.3) *$\mathcal{M}_\Gamma^?$ is a free $\mathcal{O}[\mathbf{q}^{\mathbb{R}}]$-module with basis the image of $(C_w)_{w\in\Gamma}$.*

Definition 8.6.4. — *If C is a Kazhdan-Lusztig left c-cell of W, we denote by $[\,C\,]_c^{\mathrm{KL}}$ the class of $\mathbf{k} \otimes_{\mathcal{O}[\mathbf{q}^{\mathbb{R}}]} \mathcal{M}_C^L$ in the Grothendieck group $K_0(\mathbf{k}W) = \mathbb{Z}\mathrm{Irr}(W)$ (here, the tensor product $\mathbf{k} \otimes_{\mathcal{O}[\mathbf{q}^{\mathbb{R}}]} -$ is viewed through the specialization $\mathbf{q}^r \mapsto 1$). We will call **c-cellular** KL-**character** of W every character of the form $[\,C\,]_c^{\mathrm{KL}}$, where C is a left Kazhdan-Lusztig c-cell.*

*If Γ is a Kazhdan-Lusztig two-sided c-cell of W, we denote by $\mathrm{Irr}_\Gamma^{\mathrm{KL}}(W)$ the set of irreducible characters of W appearing in $\mathbf{k} \otimes_{\mathcal{O}[\mathbf{q}^{\mathbb{R}}]} \mathcal{M}_\Gamma^{LR}$, viewed as a left $\mathbf{k}W$-module. We will call **Kazhdan-Lusztig c-family** every subset of $\mathrm{Irr}(W)$ of the form $\mathrm{Irr}_\Gamma^{\mathrm{KL}}(W)$ where Γ is a Kazhdan-Lusztig two-sided c-cell. We will say that $\mathrm{Irr}_\Gamma^{\mathrm{KL}}(W)$ is the Kazhdan-Lusztig c-family **associated** with Γ, or that Γ is the Kazhdan-Lusztig two-sided c-cell **covering** $\mathrm{Irr}_\Gamma^{\mathrm{KL}}(W)$.*

Since $\mathbf{k}W$ is semisimple and since $\mathbf{k}\otimes_{\mathscr{O}[\mathbf{q}^{\mathbb{R}}]}\mathscr{M}_{\Gamma}^{LR}$ is a quotient of two-sided ideals of $\mathbf{k}W$, the Kazhdan-Lusztig c-families form a partition of $\mathrm{Irr}(W)$

(8.6.5)
$$\mathrm{Irr}(W)=\coprod_{\Gamma\in{}^{\mathrm{KL}}\mathrm{Cell}_{LR}^{c}(W)}\mathrm{Irr}_{\Gamma}^{\mathrm{KL}}(W)$$

and, since $\mathbf{k}W$ is split,

(8.6.6)
$$|\Gamma|=\sum_{\chi\in\mathrm{Irr}_{\Gamma}^{\mathrm{KL}}(W)}\chi(1)^2.$$

Moreover, if C is a Kazhdan-Lusztig *left* c-cell of W, we set

$$[\,C\,]_c^{\mathrm{KL}}=\sum_{\chi\in\mathrm{Irr}(W)}\mathrm{mult}_{C,\chi}^{\mathrm{KL}}\,\chi,$$

where $\mathrm{mult}_{C,\chi}^{\mathrm{KL}}\in\mathbb{N}$. Then:

Lemma 8.6.7. — *With the previous notation, we have:*

(a) *If* $C\in{}^{\mathrm{KL}}\mathrm{Cell}_L^c(W)$, *then* $\displaystyle\sum_{\chi\in\mathrm{Irr}(W)}\mathrm{mult}_{C,\chi}^{\mathrm{KL}}\,\chi(1)=|C|$.

(b) *If* $\chi\in\mathrm{Irr}(W)$, *then* $\displaystyle\sum_{C\in{}^{\mathrm{KL}}\mathrm{Cell}_L^c(W)}\mathrm{mult}_{C,\chi}^{\mathrm{KL}}=\chi(1)$.

Proof. — The equality (a) simply says that the dimension of $[\,C\,]_c^{\mathrm{KL}}$ is equal to $|C|$ by (8.6.3). The equality (b) translates the fact that, since W is a disjoint union of Kazhdan-Lusztig left c-cells, we have $[\,\mathbf{k}W\,]_{\mathbf{k}W}=\sum_{C\in{}^{\mathrm{KL}}\mathrm{Cell}_L^c(W)}[\,C\,]_c^{\mathrm{KL}}$. □

8.6.C. Properties of cells. — The algebra $\mathscr{H}^{\mathrm{cyc}}(k)$ is endowed with an $\mathscr{O}[\mathbf{q}^{\mathbb{R}}]$-linear involutive anti-automorphism which sends T_w on $T_{w^{-1}}$: it will be denoted by $h\mapsto h^*$. It is immediate that

(8.6.8)
$$C_w^*=C_{w^{-1}},$$

which implies that, if x and y are two elements of W, then

(8.6.9)
$$x\leqslant_L^c y \textit{ if and only if } x^{-1}\leqslant_R^c y^{-1}$$

and so

(8.6.10)
$$x\sim_L^{\mathrm{KL},c} y \textit{ if and only if } x^{-1}\sim_R^{\mathrm{KL},c} y^{-1}.$$

In other words, the map ${}^{\mathrm{KL}}\mathrm{Cell}_L^c(W)\longrightarrow{}^{\mathrm{KL}}\mathrm{Cell}_R^c(W)$, $\Gamma\mapsto\Gamma^{-1}$ is well-defined and bijective.

The next property is less obvious [**Lus4**, Corollary 11.7]

(8.6.11)
$$x\leqslant_?^c y \textit{ if and only if } w_0 y\leqslant_?^c w_0 x \textit{ if and only if } y w_0\leqslant_?^c x w_0.$$

It follows that

(8.6.12)
$$x\sim_?^c y \textit{ if and only if } w_0 x\sim_?^c w_0 y \textit{ if and only if } x w_0\sim_?^c y w_0.$$

Moreover, if $C \in {}^{\mathrm{KL}}\mathrm{Cell}_L^c(W)$, then [**Lus4**, Proposition 21.5]

$$[\, w_0 C\,]_c^{\mathrm{KL}} = [\, C w_0\,]_c^{\mathrm{KL}} = [\, C\,]_c^{\mathrm{KL}} \cdot \varepsilon. \tag{8.6.13}$$

Similarly, if $\Gamma \in {}^{\mathrm{KL}}\mathrm{Cell}_{LR}^c(W)$, then [**Lus4**, proposition 21.5]

$$\mathrm{Irr}_{w_0\Gamma}^{\mathrm{KL}}(W) = \mathrm{Irr}_{\Gamma w_0}^{\mathrm{KL}}(W) = \mathrm{Irr}_{\Gamma}^{\mathrm{KL}}(W) \cdot \varepsilon. \tag{8.6.14}$$

This shows in particular that

$$w_0 \Gamma w_0 = \Gamma. \tag{8.6.15}$$

So tensoring by ε induces a permutation of Kazhdan-Lusztig c-families and of c-cellular KL-characters.

If $\gamma : W \to \mathbf{k}^\times$ is a linear character (note that γ has values in $\{1,-1\}$), we set $\gamma \cdot c : \mathrm{Ref}(W) \to \mathbb{R}$, $s \mapsto \gamma(s)c_s$. The following Lemma is proven in [**Bon2**, Corollary 2.5 and 2.6]:

Lemma 8.6.16. — *If $\gamma \in W^\wedge$ and let $? \in \{L, R, LR\}$. Then:*

(a) *The relations $\leqslant_?^c$ and $\leqslant_?^{\gamma\cdot c}$ coincide.*
(b) *The relations $\sim_?^{\mathrm{KL},c}$ and $\sim_?^{\mathrm{KL},\gamma\cdot c}$ coincide.*
(c) *If $C \in {}^{\mathrm{KL}}\mathrm{Cell}_L^c(W) = {}^{\mathrm{KL}}\mathrm{Cell}_L^{\gamma\cdot c}(W)$, then $[\, C\,]_{\gamma\cdot c}^{\mathrm{KL}} = \gamma \cdot [\, C\,]_c^{\mathrm{KL}}$.*
(d) *If $\Gamma \in {}^{\mathrm{KL}}\mathrm{Cell}_{LR}^c(W) = {}^{\mathrm{KL}}\mathrm{Cell}_{LR}^{\gamma\cdot c}(W)$, then $\mathrm{Irr}_\Gamma^{\mathrm{KL},\gamma\cdot c}(W) = \gamma \cdot \mathrm{Irr}_\Gamma^{\mathrm{KL},c}(W)$.*

The next result is easy [**Lus4**, Lemma 8.6]:

Lemma 8.6.17. — *Assume that $c_s \neq 0$ for all $s \in \mathrm{Ref}(W)$. Then:*

(a) *$\{1\}$ and $\{w_0\}$ are Kazhdan-Lusztig left, right or two-sided c-cells.*
(b) *Let $\gamma : W \to \mathbf{k}^\times$ be the unique linear character such that $\gamma(s) = 1$ if $c_s > 0$ and $\gamma(s) = -1$ if $c_s < 0$. Then $[\, 1\,]_c^{\mathrm{KL}} = \gamma$ and $[\, w_0\,]_c^{\mathrm{KL}} = \gamma\varepsilon$.*

Remark 8.6.18. — In fact, [**Lus4**, Lemma 8.6] is proven whenever $c_s > 0$ for all s. To obtain the general statement of Lemma 8.6.17, it is sufficient to apply Lemma 8.6.16. ■

***Example 8.6.19* (Vanishing parameters)**. — If $c = 0$ (i.e. if $c_s = 0$ for all s), then $C_w = T_w$, $\mathscr{H}^{\mathrm{cyc}}(0) = \mathscr{O}[\mathbf{q}^{\mathbb{R}}][W]$ there is only one Kazhdan-Lusztig left, right or two-sided 0-cell, namely W. We then have $\mathrm{Irr}_W^{\mathrm{KL},0}(W) = \mathrm{Irr}(W)$ and $[\, W\,]_0^{\mathrm{KL}} = \sum_{\chi \in \mathrm{Irr}(W)} \chi(1)\chi$. ■

CHAPTER 9

RESTRICTED CHEREDNIK ALGEBRA AND CALOGERO-MOSER FAMILIES

In this chapter, we start by recalling in §9.1 and §9.2 some results of Gordon **[Gor1]** on the representations of the restricted Cherednik algebras. We do not need to extend the defining field for representations, as the algebras are split. This is useful to derive consequences about the partition into Calogero-Moser two-sided cells (§ 10).

9.1. Representations of restricted Cherednik algebras

The *restricted Cherednik algebra* is the $\mathbf{k}[\mathscr{C}]$-algebra $\bar{\mathbf{H}}$ defined by

$$\bar{\mathbf{H}} = \mathbf{H}/\bar{\mathfrak{p}}\mathbf{H} = \mathbf{k}[\mathscr{C}] \otimes_P \mathbf{H}.$$

Theorem 4.1.2 gives a PBW decomposition for that algebra.

Proposition 9.1.1. — *The map* $\mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V]^{\mathrm{co}(W)} \otimes \mathbf{k}W \otimes \mathbf{k}[V^*]^{\mathrm{co}(W)} \to \bar{\mathbf{H}}$ *induced by the product is an isomorphism of* $\mathbf{k}[\mathscr{C}]$*-modules. In particular,* $\bar{\mathbf{H}}$ *is a free* $\mathbf{k}[\mathscr{C}]$*-module of rank* $|W|^3$.

The algebra $\bar{\mathbf{H}}$ inherits a $(\mathbb{N}\times\mathbb{N})$-grading, a $\mathbb{Z}$-grading and a $\mathbb{N}$-grading from $\widetilde{\mathbf{H}}$ (cf §3.2).

Given $E \in \mathrm{Irr}(\mathbf{k}W)$, we put $\bar{\Delta}(E) = \Delta(E) \otimes_P \mathbf{k}[\mathscr{C}]$, a $\mathbb{Z}$-graded $\bar{\mathbf{H}}$-module. Note that $\bar{\Delta}(E)$ is isomorphic to $\mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V]^{\mathrm{co}(W)} \otimes E$ as a graded $(\mathbf{k}[\mathscr{C}]W)$-module.

We put $\bar{\Delta}(\mathrm{co}) = \Delta(\mathrm{co}) \otimes_P \mathbf{k}[\mathscr{C}] = \Delta(\mathrm{co}) \otimes_{\mathbf{k}[V]^W} \mathbf{k} = \bar{\mathbf{H}}e$.

Let $\mathfrak{C}$ be a prime ideal of $\mathscr{C}$. We put $\bar{\mathbf{H}}_{\mathfrak{C}} = \bar{\mathbf{H}} \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{k}(\mathfrak{C})$. Theorem F.2.7 has the following consequence.

Theorem 9.1.2 **([BelTh])**. — $\bar{\mathbf{H}}_{\mathfrak{C}}$-modgr *is a split highest weight category over* $\mathbf{k}(\mathfrak{C})$ *with standard objects the baby Verma modules* $\bar{\Delta}_{\mathfrak{C}}(E) = \bar{\Delta}(E) \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{k}(\mathfrak{C})$, $E \in \mathrm{Irr}(\mathbf{k}W)$, *and their shifts.*

Proposition 9.1.3. — *Given* $E \in \mathrm{Irr}(\mathbf{k}W)$*, the* $\bar{\mathbf{H}}_{\mathfrak{C}}$*-module* $\bar{\Delta}_{\mathfrak{C}}(E)$ *has a unique simple quotient* $L_{\mathfrak{C}}(E)$*. The map* $\mathrm{Irr}(\mathbf{k}W) \longrightarrow \mathrm{Irr}(\bar{\mathbf{H}}_{\mathfrak{C}})$*,* $E \mapsto L_{\mathfrak{C}}(E)$ *is bijective and the algebra* $\bar{\mathbf{H}}_{\mathfrak{C}}$ *is split.*

9.2. Calogero-Moser families

Let $\theta_{\mathfrak{C}} : \mathbf{k}[\mathscr{C}] \to \mathbf{k}(\mathfrak{C})$ be the quotient map and let $\Omega^{\mathfrak{C}}_E = \theta_{\mathfrak{C}} \circ \Omega_E = \theta_{\mathfrak{C}} \circ \omega_E \circ \mathbf{\Omega} : Z \to \mathbf{k}(\mathfrak{C})$ for $E \in \mathrm{Irr}(W)$.

Lemma 9.2.1. — *If* $z \in Z$*, then* z *acts by multiplication by* $\Omega^{\mathfrak{C}}_E(z)$ *on* $L_{\mathfrak{C}}(E)$*.*

Proof. — We may assume that z is $\mathbb{Z}$-homogeneous of degree i. If $i = 0$, then the lemma follows from Proposition 7.3.1. If $i \neq 0$, then, as $L_{\mathfrak{C}}(E)$ is $\mathbb{Z}$-graded and finite-dimensional, z acts nilpotently on $L_{\mathfrak{C}}(E)$ and, as it also acts by a scalar, this scalar must be equal to 0. But $\mathbf{\Omega}(z) = 0$ by (4.2.10), and the result follows. $\square$

Given $b \in \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(\bar{\mathbf{H}}_{\mathfrak{C}}))$, let $\mathrm{Irr}_{\mathbf{H}}(W, b)$ denote the set of irreducible representations E of W such that $\bar{\Delta}_{\mathfrak{C}}(E)$ is in the block $\bar{\mathbf{H}}_{\mathfrak{C}} b$, i.e., $b\bar{\Delta}_{\mathfrak{C}}(E) \neq 0$. Note that $E \in \mathrm{Irr}_{\mathbf{H}}(W, b)$ if and only if $L_{\mathfrak{C}}(E)$ belongs to $\mathrm{Irr}(\bar{\mathbf{H}}_{\mathfrak{C}} b)$. It follows from Proposition D.1.2 that

$$\mathrm{Idem}_{\mathrm{pr}}(\bar{Z}_{\mathfrak{C}}) = \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(\bar{\mathbf{H}}_{\mathfrak{C}})).$$

So,

$$\mathrm{Irr}(W) = \coprod_{b \in \mathrm{Idem}_{\mathrm{pr}}(\bar{Z}_{\mathfrak{C}})} \mathrm{Irr}_{\mathbf{H}}(W, b). \tag{9.2.2}$$

A *Calogero-Moser* $\mathfrak{C}$*-family* is a subset of $\mathrm{Irr}\, W$ of the form $\mathrm{Irr}_{\mathbf{H}}(W, b)$, where $b \in \mathrm{Idem}_{\mathrm{pr}}(\bar{Z}_{\mathfrak{C}})$.

The next lemma follows from Corollary D.2.4, Proposition 9.1.3 and Lemma 9.2.1.

Lemma 9.2.3. — *Let* E*,* $E' \in \mathrm{Irr}\, W$*. Then* E *and* E' *are in the same Calogero-Moser* $\mathfrak{C}$*-family if and only if* $\Omega^{\mathfrak{C}}_E = \Omega^{\mathfrak{C}}_{E'}$*. Moreover, the map*

$$\begin{array}{rccc} \Theta_{\mathfrak{C}}: & \mathrm{Irr}\, W & \longrightarrow & \Upsilon^{-1}(\bar{\mathfrak{p}}_{\mathfrak{C}}) \\ & E & \longmapsto & \mathrm{Ker}\ \Omega^{\mathfrak{C}}_E \end{array} \tag{9.2.4}$$

is surjective and its fibers are the Calogero-Moser $\mathfrak{C}$*-families.*

Let $b \in \mathrm{Idem}_{\mathrm{pr}}(\bar{Z}_{\mathfrak{C}})$. We denote by $\Omega^{\mathfrak{C}}_b$ the common value of the $\Omega^{\mathfrak{C}}_E$ for $E \in \mathrm{Irr}_{\mathbf{H}}(W, b)$. When $\mathfrak{C} = 0$, we put $\Omega_b = \Omega^{\mathfrak{C}}_b$. When $\mathfrak{C} = \mathfrak{C}_c$ for some $c \in \mathscr{C}$, we put $\Omega^c_b = \Omega^{\mathfrak{C}_c}_b$.

Corollary 9.2.5. — *If $\mathfrak{z}$ is a prime ideal of Z lying over $\bar{\mathfrak{p}}_{\mathfrak{C}}$, then the inclusion $P \hookrightarrow Z$ induces an isomorphism $P/\bar{\mathfrak{p}}_{\mathfrak{C}} \xrightarrow{\sim} Z/\mathfrak{z}$.*

Proof. — By Lemma 9.2.3, there exists $E \in \operatorname{Irr}(W)$ such that $\mathfrak{z} = \operatorname{Ker}(\Omega_E^{\mathfrak{C}})$. Since $\Omega_E^{\mathfrak{C}} : Z \to \mathbf{k}(\mathfrak{C})$ factors through a surjective morphism $Z \to P/\bar{\mathfrak{p}}_{\mathfrak{C}}$, the corollary follows. □

Example 9.2.6. — We will call *generic Calogero-Moser family* every Calogero-Moser $\mathfrak{C}'$-family, where $\mathfrak{C}' = 0$. In this case, the map $\Theta_{\mathfrak{C}}$ will be simply denoted by Θ. Every Calogero-Moser $\mathfrak{C}$-family is a union of generic Calogero-Moser families. ■

Example 9.2.7. — Given $c \in \mathscr{C}$, we define a *Calogero-Moser c-family* to be a Calogero-Moser $\mathfrak{C}_c$-family. In this case, $\Omega_\chi^{\mathfrak{C}_c}$ will be denoted by Ω_χ^c and $\Theta_{\mathfrak{C}_c}$ will be denoted by Θ_c. ■

Example 9.2.8. — We have $|\Upsilon_0^{-1}(0)| = 1$, hence there is a unique Calogero-Moser 0-family. ■

9.3. Linear characters and Calogero-Moser families

From Proposition 7.4.2 we deduce the following result.

Proposition 9.3.1. — *Given $E \in \operatorname{Irr}(W)$ and $\tau = (\xi, \xi', \gamma \rtimes g) \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ stabilizing $\mathfrak{C}$, we have*

$$ {}^\tau L_{\mathfrak{C}}(E) \simeq L_{\mathfrak{C}}({}^g E \otimes \gamma^{-1}). $$

The following result follows from Proposition 7.4.2 with $\xi = \xi' = 1$.

Corollary 9.3.2. — *Let $c \in \mathscr{C}$, let γ be a linear character of W and let $\mathscr{F}$ be a Calogero-Moser c-family. Then $\mathscr{F}\gamma$ is a Calogero-Moser $\gamma \cdot c$-family.*

Using Proposition 7.4.2 again, we obtain the following result.

Corollary 9.3.3. — *Let $\tau = (\xi, \xi', \gamma \rtimes g) \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ and let $\mathscr{F}$ be a Calogero-Moser $\mathfrak{C}$-family. If τ stabilizes $\mathfrak{C}$, then $\mathscr{F}\gamma$ is a Calogero-Moser $\mathfrak{C}$-family.*

Corollary 9.3.4. — *Let γ be a linear character of W and let $\mathscr{F}$ be a* **generic** *Calogero-Moser family. Then $\mathscr{F}\gamma$ is a generic Calogero-Moser family.*

Corollary 9.3.5. — *Assume that all the reflections of W have order 2. Let $\mathscr{F}$ be a Calogero-Moser $\mathfrak{C}$-family. Then $\mathscr{F}\varepsilon$ is a Calogero-Moser $\mathfrak{C}$-family (recall that ε is the determinant).*

Proof. — The element $\tau = (-1, 1, \varepsilon \rtimes 1)$ of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathcal{N})$ acts trivially on $\mathbf{k}[\mathscr{C}]$. The result follows now from Corollary 9.3.3. $\square$

***Example 9.3.6* (Generic families and linear characters).** — Let $\gamma \in W^\wedge$ and $\chi \in \mathrm{Irr}(W)$ be in the same *generic* Calogero-Moser family. Then $\Omega_\chi(\mathbf{eu}) = \Omega_\gamma(\mathbf{eu})$, hence $\chi(s) = \gamma(s)\chi(1)$ for all $s \in \mathrm{Ref}(W)$. In other words, all the reflections of W are in the center of χ (that is, the normal subgroup of W consisting of elements which acts on E_χ by scalar multiplication). It follows that the center of χ is W itself, hence $\chi = \gamma$.

Consequently, a linear character is alone in its generic Calogero-Moser family. This result applies in particular to $\mathbf{1}_W$ and ε, and is compatible with Corollary 9.3.4. ■

9.4. Graded dimension, b-invariant

By Proposition C.1.7, the elements of $\mathrm{Idem}_{\mathrm{pr}}(\bar{Z}_{\mathfrak{C}})$ have $\mathbb{Z}$-degree 0. In particular, given $b \in \mathrm{Idem}_{\mathrm{pr}}(\bar{Z}_{\mathfrak{C}})$, then $b\bar{Z}_{\mathfrak{C}}$ is a finite-dimensional graded $\mathbf{k}(\mathfrak{C})$-algebra. The aim of this section is to study this grading. We put $\bar{\Delta}_{\mathfrak{C}}(\mathrm{co}) = \bar{\Delta}(\mathrm{co}) \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{k}(\mathfrak{C})$.

Theorem 9.4.1. — *Let $b \in \mathrm{Idem}_{\mathrm{pr}}(\bar{Z}_{\mathfrak{C}})$ and let $\mathscr{F} = \mathrm{Irr}_{\mathbf{H}}(W, b)$. Then:*

(a) $\dim^{\mathrm{gr}}_{\mathbf{k}(\mathfrak{C})} b\bar{Z}_{\mathfrak{C}} = \sum_{\chi \in \mathscr{F}} f_\chi(\mathbf{t}^{-1})\, f_\chi(\mathbf{t})$.

(b) *There exists a unique $\chi \in \mathscr{F}$ with minimal* $\mathbf{b}$*-invariant, which we denote by $\chi_{\mathscr{F}}$.*

(c) *The coefficient of $\mathbf{t}^{\mathbf{b}_{\chi_{\mathscr{F}}}}$ in $f_{\chi_{\mathscr{F}}}(\mathbf{t})$ is equal to* 1.

(d) *$b\bar{\Delta}_{\mathfrak{C}}(\mathrm{co}) = b\bar{\mathbf{H}}_{\mathfrak{C}}e$ is a projective cover of $L_{\mathfrak{C}}(\chi_{\mathscr{F}})$.*

(e) *The algebra $\mathrm{End}_{\bar{\mathbf{H}}_{\mathfrak{C}}}(\bar{\Delta}_{\mathfrak{C}}(\chi_{\mathscr{F}}))$ is a quotient of $b\bar{Z}_{\mathfrak{C}}$. In particular, it is commutative.*

By (2.5.6), we obtain the following immediate consequence:

Corollary 9.4.2. — *Given $b \in \mathrm{Idem}_{\mathrm{pr}}(\bar{Z}_{\mathfrak{C}})$, we have*

$$\dim_{\mathbf{k}(\mathfrak{C})} b\bar{Z}_{\mathfrak{C}} = \sum_{\chi \in \mathrm{Irr}_{\mathbf{H}}(W,b)} \chi(1)^2.$$

Remark 9.4.3. — Theorem 9.4.1 generalizes [**Gor1**, Theorem 5.6] and Corollary 9.4.2 generalizes [**Gor1**, Corollary 5.8] (case of families with only one element). As pointed out to us by Gordon, in [**Gor1**, Theorem 5.6], $p_\chi(\mathbf{t}) = \mathbf{t}^{\mathbf{b}_{\chi^*}-\mathbf{b}_\chi} f_\chi(\mathbf{t}) f_{\chi^*}(\mathbf{t}^{-1})$ should be replaced by $p_\chi(\mathbf{t}) = f_{\chi^*}(\mathbf{t}) f_{\chi^*}(\mathbf{t}^{-1})$ with the notation of [**Gor1**]. Note that the difference with our result comes from the fact that we have used a $\mathbb{Z}$-grading opposed to the one of [**Gor1**, §4.1], which amounts to swapping V and V^* and so to swap, in this formula, χ and χ^*. ■

Proof of Theorem 9.4.1. — Lemma 7.1.3 shows that

$$[b\bar{\Delta}_{\mathfrak{C}}(\mathrm{co})]^{\mathrm{gr}}_{\bar{\mathbf{H}}_{\mathfrak{C}}} = \sum_{\chi\in\mathscr{F}} f_\chi(\mathbf{t}^{-1})[\bar{\Delta}_{\mathfrak{C}}(\chi)]^{\mathrm{gr}}_{\bar{\mathbf{H}}_{\mathfrak{C}}}.$$

The right action of $b\bar{Z}_{\mathfrak{C}}$ on $b\bar{\mathbf{H}}_{\mathfrak{C}}e$ induces an isomorphism of graded algebras $b\bar{Z}_{\mathfrak{C}} \xrightarrow{\sim} e\,b\bar{\mathbf{H}}_{\mathfrak{C}}e$ (Corollary 4.2.7). We deduce now assertion (a):

$$\dim^{\mathrm{gr}}_{\mathbf{k}(\mathfrak{C})}(b\bar{Z}_{\mathfrak{C}}) = \dim^{\mathrm{gr}}_{\mathbf{k}(\mathfrak{C})}(e\,b\bar{\Delta}_{\mathfrak{C}}(\mathrm{co})) = \sum_{\chi\in\mathscr{F}} f_\chi(\mathbf{t}^{-1})\dim^{\mathrm{gr}}_{\mathbf{k}(\mathfrak{C})}\big((\mathbf{k}(\mathfrak{C})[V]^{\mathrm{co}(W)}\otimes E_\chi)^W\big) = \sum_{\chi\in\mathscr{F}} f_\chi(\mathbf{t}^{-1})f_\chi(\mathbf{t}).$$

Since $\mathrm{End}_{\bar{\mathbf{H}}_{\mathfrak{C}}}(b\bar{\mathbf{H}}_{\mathfrak{C}}e)$ is isomorphic to the local commutative algebra $b\bar{Z}_{\mathfrak{C}}$, the $\bar{\mathbf{H}}_{\mathfrak{C}}$-module $b\bar{\mathbf{H}}_{\mathfrak{C}}e$ is indecomposable (and of course projective), so it admits a unique simple quotient $L_{\mathfrak{C}}(\chi_{\mathscr{F}})$, for some $\chi_{\mathscr{F}} \in \mathscr{F}$. The highest weight category structure of $\bar{\mathbf{H}}_{\mathfrak{C}}$-modgr shows that

$$[b\bar{\mathbf{H}}_{\mathfrak{C}}e] - \mathbf{t}^{b_{\chi_{\mathscr{F}}}}[\bar{\Delta}_{\mathfrak{C}}(\chi_{\mathscr{F}})] \in \bigoplus_{\chi\in\mathscr{F}} \mathbf{t}^{b_{\chi_{\mathscr{F}}}+1}\mathbb{Z}[\mathbf{t}][\bar{\Delta}_{\mathfrak{C}}(\chi)].$$

The assertions (b), (c) and (d) follow.

Let M be the kernel of a surjection $b\bar{\mathbf{H}}_{\mathfrak{C}}e \to \bar{\Delta}_{\mathfrak{C}}(\chi_{\mathscr{F}})$. Since $\mathrm{End}_{\bar{\mathbf{H}}_{\mathfrak{C}}}(b\bar{\mathbf{H}}_{\mathfrak{C}}e)$ is generated by $\bar{Z}_{\mathfrak{C}}$, it follows that the $\bar{\mathbf{H}}_{\mathfrak{C}}$-endomorphisms of $b\bar{\mathbf{H}}_{\mathfrak{C}}e$ stabilize M. We obtain by restriction a morphism of $\mathbf{k}(\mathfrak{C})$-algebras $b\bar{Z}_{\mathfrak{C}} \to \mathrm{End}_{\bar{\mathbf{H}}_{\mathfrak{C}}}(\bar{\Delta}_{\mathfrak{C}}(\chi_{\mathscr{F}}))$ which is surjective since $b\bar{\mathbf{H}}_{\mathfrak{C}}e$ is projective. □

Corollary 9.4.4. — *Let $z \in Z$ and let $\mathrm{car}_z(\mathbf{t}) \in P[\mathbf{t}]$ denote the characteristic polynomial of the multiplication by z in the P-module Z. Then*

$$\mathrm{car}_z(\mathbf{t}) \equiv \prod_{\chi\in\mathrm{Irr}(W)} (\mathbf{t}-\Omega_\chi(z))^{\chi(1)^2} \mod \bar{\mathfrak{p}}.$$

Proof. — Let $b \in \mathrm{Idem}_{\mathrm{pr}}(\mathbf{k}(\mathscr{C})\bar{Z})$. Since $z - \Omega_b(z)$ is a nilpotent endomorphism of $b\mathbf{k}(\mathscr{C})\bar{Z}$, the characteristic polynomial of z on $b\mathbf{k}(\mathscr{C})\bar{Z}$ is $(\mathbf{t}-\Omega_b(z))^{\dim_{\mathbf{k}(\mathscr{C})} b\mathbf{k}(\mathscr{C})\bar{Z}}$. Consequently,

$$\mathrm{car}_z(\mathbf{t}) \equiv \prod_{b\in\mathrm{Idem}_{\mathrm{pr}}(\mathbf{k}(\mathscr{C})\bar{Z})} (\mathbf{t}-\Omega_b(z))^{\dim_{\mathbf{k}(\mathscr{C})} b\mathbf{k}(\mathscr{C})\bar{Z}} \mod \bar{\mathfrak{p}}.$$

Since $\Omega_b(z) = \Omega_\chi(z)$ for all $\chi \in \mathrm{Irr}_{\mathbf{H}}(W, b)$, the result follows from (9.2.2) and from Corollary 9.4.2. □

Corollary 9.4.5. — *Let $\gamma : W \longrightarrow \mathbf{k}^\times$ be a linear character. Then Z is unramified over P at* $\mathrm{Ker}(\Omega_\gamma)$.

Proof. — Indeed, if b_γ denotes the primitive idempotent of $\mathbf{k}(\mathscr{C})\bar{Z}$ associated with γ, then $\mathrm{Irr}_{\mathbf{H}}(W, b_\gamma) = \{\gamma\}$ by Example 9.3.6. This implies, by Corollary 9.4.2, that $\dim_{\mathbf{k}(\mathscr{C})}(b_\gamma \mathbf{k}(\mathscr{C})\bar{Z}) = 1$.

Set $\mathfrak{z}_\gamma = \mathrm{Ker}(\Omega_\gamma)$ (we have $\mathfrak{z}_\gamma \cap P = \bar{\mathfrak{p}}$). Then $Z/\mathfrak{z}_\gamma \simeq \mathbf{k}[\mathscr{C}]$, hence $Z_{\mathfrak{z}_\gamma}/\mathfrak{z}_\gamma Z_{\mathfrak{z}_\gamma} \simeq \mathbf{k}(\mathscr{C})$. But, on the other hand, $Z_{\mathfrak{z}_\gamma}/\bar{\mathfrak{p}} Z_{\mathfrak{z}_\gamma} = b_\gamma \mathbf{k}(\mathscr{C})\bar{Z}$. So $\dim_{\mathbf{k}(\mathscr{C})}(Z_{\mathfrak{z}_\gamma}/\bar{\mathfrak{p}} Z_{\mathfrak{z}_\gamma}) = 1$, which implies that $\bar{\mathfrak{p}} Z_{\mathfrak{z}_\gamma} = \mathfrak{z}_\gamma Z_{\mathfrak{z}_\gamma}$, as desired. □

9.5. Exchanging V and V^*

If E is a graded $(\mathbf{k}[V] \rtimes W)$-module, one can define

$$\Delta^*(E) = \mathrm{Ind}_{\mathbf{H}^+}^{\mathbf{H}}(\mathbf{k}[\mathscr{C}] \otimes E),$$

which is a graded $\mathbf{H}$-module, as well as its reduction modulo $\bar{\mathfrak{p}}$, denoted by $\bar{\Delta}^*(E)$, which is a graded $\bar{\mathbf{H}}$-module. If $\mathfrak{C}$ is a prime ideal, we also set $\bar{\Delta}^*_{\mathfrak{C}}(E) = \mathbf{k}(\mathfrak{C}) \otimes_{\mathbf{k}[\mathscr{C}]} \bar{\Delta}^*(E)$, which is a graded $\bar{\mathbf{H}}_{\mathfrak{C}}$-module.

Assume now that $E \in \mathrm{Irr}(W)$. Then, as in Proposition 9.1.3, the $\bar{\mathbf{H}}_{\mathfrak{C}}$-module $\bar{\Delta}^*_{\mathfrak{C}}(E)$ is indecomposable and admits a unique simple quotient, which will be denoted by $L^*_{\mathfrak{C}}(E)$. Moreover, the map

$$\begin{array}{ccc} \mathrm{Irr}(W) & \longrightarrow & \mathrm{Irr}(\bar{\mathbf{H}}_{\mathfrak{C}}) \\ E & \longmapsto & L^*_{\mathfrak{C}}(E) \end{array} \tag{9.5.1}$$

is bijective.

Remark 9.5.2. — From Proposition 9.1.3 and (9.5.1), it follows that there exists a unique permutation $\bigstar_{\mathfrak{C}}$ of $\mathrm{Irr}(W)$ such that

$$L^*_{\mathfrak{C}}(E) = L_{\mathfrak{C}}(\bigstar_{\mathfrak{C}}(E))$$

for all $E \in \mathrm{Irr}(W)$. It turns out that the permutation $\bigstar_{\mathfrak{C}}$ is in general difficult to compute, and that it depends heavily on the prime ideal $\mathfrak{C}$, as the reader can already check when $\dim_{\mathbf{k}} V = 1$. ■

We say that E and F belong to the same $*$-Calogero-Moser $\mathfrak{C}$-family if $L^*_{\mathfrak{C}}(E)$ and $L^*_{\mathfrak{C}}(F)$ are two simple modules belonging to the same block of $\bar{\mathbf{H}}_{\mathfrak{C}}$. It follows from Remark 9.5.2 that E and F belong to the same $*$-Calogero-Moser $\mathfrak{C}$-family if and only if $\bigstar_{\mathfrak{C}}(E)$ and $\bigstar_{\mathfrak{C}}(F)$ belong to the same Calogero-Moser $\mathfrak{C}$-family.

However, there is another easier description of $*$-families which has been explained to us by Gwyn Bellamy. It follows from (7.1.1) that

$$\operatorname{Res}^{\bar{\mathbf{H}}_{\mathfrak{C}}}_{\bar{\mathbf{H}}^+_{\mathfrak{C}}} \bar{\Delta}_{\mathfrak{C}}(E) \simeq \mathbf{k}(\mathfrak{C}) \otimes \mathbf{k}[V]^{\operatorname{co}(W)} \otimes E.$$

Recall that $N = |\operatorname{Ref}(W)|$ and that $\mathbf{k}[V]^{\operatorname{co}(W)}_N$ is one dimensional, affording the character ε (as a $\mathbf{k}W$-module) and that $\mathbf{k}[V]^{\operatorname{co}(W)}_{>N} = 0$. So one gets an injective morphism of $\bar{\mathbf{H}}^+_{\mathfrak{C}}$-modules $E \otimes \varepsilon \hookrightarrow \operatorname{Res}^{\bar{\mathbf{H}}_{\mathfrak{C}}}_{\bar{\mathbf{H}}^+_{\mathfrak{C}}} \bar{\Delta}_{\mathfrak{C}}(E)$. By adjunction, one gets a non-zero morphism

$$\bar{\Delta}^*_{\mathfrak{C}}(E \otimes \varepsilon) \longrightarrow \bar{\Delta}_{\mathfrak{C}}(E). \tag{9.5.3}$$

In particular,

(9.5.4) *the simple module $L^*_{\mathfrak{C}}(E \otimes \varepsilon)$ is a composition factor of $\bar{\Delta}_{\mathfrak{C}}(E)$.*

Since $\bar{\Delta}^*_{\mathfrak{C}}(E)$ is indecomposable with unique simple quotient $L^*_{\mathfrak{C}}(E)$, this implies that

(9.5.5) *the simple modules $L^*_{\mathfrak{C}}(E \otimes \varepsilon)$ and $L_{\mathfrak{C}}(E)$ belong to the same block of $\bar{\mathbf{H}}_{\mathfrak{C}}$.*

Proposition 9.5.6. — *The simple $\mathbf{k}W$-modules E and F belong to the same Calogero-Moser $\mathfrak{C}$-family if and only if $E \otimes \varepsilon$ and $F \otimes \varepsilon$ belong to the same $*$-Calogero-Moser $\mathfrak{C}$-family.*

The following consequence was announced in Remark 4.2.12.

Corollary 9.5.7. — *If $z \in Z$, then $\mathbf{\Omega}(z) = {}^{\varepsilon}\mathbf{\Omega}^*(z)$.*

Proof. — It is sufficient to prove that, for all $E \in \operatorname{Irr}(W)$, we have $\omega_E(\mathbf{\Omega}(z)) = \omega_E({}^{\varepsilon}\mathbf{\Omega}^*(z))$. Since $\omega_E({}^{\varepsilon}\mathbf{\Omega}^*(z)) = \omega_{E\otimes\varepsilon}(\mathbf{\Omega}^*(z))$, it is sufficient to prove that

$$\omega_E(\mathbf{\Omega}(z)) = \omega_{E\otimes\varepsilon}(\mathbf{\Omega}^*(z)).$$

Take $\mathfrak{C}$ to be the zero ideal of $\mathbf{k}[\mathscr{C}]$, so that $\mathbf{k}(\mathfrak{C}) = \mathbf{k}(\mathscr{C})$. By Lemma 9.2.1 (and its version obtained by swapping V and V^*), we get that $\omega_E(\mathbf{\Omega}(z))$ is the scalar by which z acts on the simple module $L_{\mathfrak{C}}(E)$ while $\omega_{E\otimes\varepsilon}(\mathbf{\Omega}^*(z))$ is the scalar by which z acts on the simple module $L^*_{\mathfrak{C}}(E \otimes \varepsilon)$. So the result follows from (9.5.5). □

9.6. Geometry

The composition

(9.6.1)
$$\mathbf{k}[\mathscr{C}] \hookrightarrow Z \xrightarrow{\Omega_b} \mathbf{k}[\mathscr{C}]$$

is the identity, which means that the morphism of $\mathbf{k}$-varieties $\Omega_b^\sharp : \mathscr{C} \longrightarrow \mathscr{Z}$ induced by Ω_b is a section of the morphism $\pi \circ \Upsilon : \mathscr{Z} \longrightarrow \mathscr{C}$ (see the diagram 5.7.2). Lemma 9.2.3 says that the map

$$\begin{array}{ccc} \mathrm{Idem}_{\mathrm{pr}}(\mathbf{k}(\mathscr{C})\bar{Z}) & \longrightarrow & \Upsilon^{-1}(\bar{\mathfrak{p}}) \\ b & \longmapsto & \mathrm{Ker}(\Omega_b) \end{array}$$

is bijective or, in geometric terms, that the irreducible components of $\Upsilon^{-1}(\mathscr{C} \times 0)$ are in bijection with $\mathrm{Idem}_{\mathrm{pr}}(\mathbf{k}(\mathscr{C})\bar{Z})$, through the map $b \mapsto \Omega_b^\sharp(\mathscr{C})$. We deduce the following proposition.

Proposition 9.6.2. — *Let $c \in \mathscr{C}$. Then the following are equivalent:*

(1) $|\mathrm{Idem}_{\mathrm{pr}}(\mathbf{k}(\mathscr{C})\bar{Z})| = |\mathrm{Idem}_{\mathrm{pr}}(\bar{\mathbf{K}}_c\bar{Z})|$.

(2) $|\Upsilon_c^{-1}(0)|$ *is equal to the number of irreducible components of* $\Upsilon^{-1}(\mathscr{C} \times 0)$.

(3) *Every element of* $\Upsilon_c^{-1}(0)$ *belongs to a unique irreducible component of* $\Upsilon^{-1}(\mathscr{C} \times 0)$.

(4) *If b and b' are two distinct elements of* $\mathrm{Idem}_{\mathrm{pr}}(\mathbf{k}(\mathscr{C})\bar{Z})$, *then* $\theta_c \circ \Omega_b \neq \theta_c \circ \Omega_{b'}$.

We say that $c \in \mathscr{C}$ is *generic* if it satisfies one of the equivalent conditions of Proposition 9.6.2. It will be called *particular* otherwise. We will denote by $\mathscr{C}_{\mathrm{gen}}$ (respectively $\mathscr{C}_{\mathrm{par}}$) the set of generic (respectively particular) elements of $\mathscr{C}$.

Corollary 9.6.3. — *$\mathscr{C}_{\mathrm{gen}}$ is a Zariski dense and open subset of $\mathscr{C}$ and $\mathscr{C}_{\mathrm{par}}$ is Zariski closed in $\mathscr{C}$. If $W \neq 1$, then $\mathscr{C}_{\mathrm{par}}$ is of pure codimension* 1 *and contains* 0.

Moreover, $\mathscr{C}_{\mathrm{gen}}$ and $\mathscr{C}_{\mathrm{par}}$ are stable under the action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$.

Proof. — The stability under the action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ is obvious. The fact that $\mathscr{C}_{\mathrm{gen}}$ (respectively $\mathscr{C}_{\mathrm{par}}$) is open (respectively closed) follows from Proposition D.2.11(2). Whenever $W \neq 1$, the trivial character is alone in its generic Calogero-Moser family (see Example 9.3.6) while its Calogero-Moser 0-family is $\mathrm{Irr}(W)$. This shows that $0 \in \mathscr{C}_{\mathrm{par}}$ and, by Proposition D.2.11(1), $\mathscr{C}_{\mathrm{par}}$ has pure codimension 1. ☐

We deduce the following from Example 9.3.6.

Corollary 9.6.4. — *If $c \in \mathscr{C}$ is generic, then any linear character of W is alone in its Calogero-Moser c-family.*

Corollary 9.6.5. — *Let $\gamma : W \longrightarrow \mathbf{k}^\times$ be a linear character and assume that c is generic. Then Z is unramified over P at* $\mathrm{Ker}(\Omega^c_\gamma)$.

Let us conclude with a short study of the smoothness of $\mathscr{Z}$. Let $b \in \mathrm{Idem}_{\mathrm{pr}}(\bar{Z}_{\mathfrak{C}})$ and let $\bar{\mathfrak{z}}_b$ denote the prime ideal of Z equal to the kernel of $\Omega^{\mathfrak{C}}_b : Z \to \mathbf{k}(\mathfrak{C})$. By **[Gor1**, §5], we have the following characterization of smoothness.

Proposition 9.6.6. — *The ring Z is regular at $\bar{\mathfrak{z}}_b$ if and only if $|\mathrm{Irr}_{\mathbf{H}}(W,b)| = 1$. Moreover,*

$$\bar{\mathbf{H}}_{\mathfrak{C}} b \simeq \mathrm{Mat}_{|W|}(b\bar{Z}_{\mathfrak{C}})$$

and $b\bar{Z}_{\mathfrak{C}}$ is a local finite dimensional $\mathbf{k}(\mathfrak{C})$-algebra with residue field $\mathbf{k}(\mathfrak{C})$.

Now let $c \in \mathscr{C}$ and assume that $\mathfrak{C} = \mathfrak{C}_c$. Let z_b denote the point of $\Upsilon_c^{-1}(0) \subset \mathscr{Z}_c \subset \mathscr{Z}$ corresponding to b.

Proposition 9.6.7. — *With the above notation, the following are equivalent:*

(1) *$\mathscr{Z}$ is smooth at z_b.*
(2) *$\mathscr{Z}_c$ is smooth at z_b.*

Proof. — Let us first recall the following Lemma:

> ***Lemma 9.6.8.*** — *Let $\varphi : \mathscr{Y} \to \mathscr{X}$ be a morphism of $\mathbf{k}$-varieties (not necessarily reduced), let $y \in \mathscr{Y}$ and let $x = \varphi(y)$. We assume that there exists a morphism of $\mathbf{k}$-varieties $\sigma : \mathscr{X} \to \mathscr{Y}$ such that $y = \sigma(x)$ and $\varphi \circ \sigma = \mathrm{Id}_{\mathscr{X}}$. Then*
>
> $$\mathscr{T}_y(\mathscr{Y}) = \mathscr{T}_y(\varphi^*(x)) \oplus \mathscr{T}_y(\sigma(\mathscr{X})).$$
>
> *Here, $\mathscr{T}_y(\mathscr{Y})$ denotes the tangent space to the $\mathbf{k}$-variety $\mathscr{Y}$ and $\varphi^*(x)$ denotes the (scheme-theoretic) fiber of φ at x, viewed as a closed $\mathbf{k}$-subvariety of $\mathscr{Y}$, not necessarily reduced.*

Let $\chi \in \mathrm{Irr}_{\mathbf{H}}(W,b)$. The morphism of varieties $\Omega^\sharp_\chi : \mathscr{C} \to \mathscr{Z}$ is a section of the morphism $\pi \circ \Upsilon : \mathscr{Z} \to \mathscr{C}$. Moreover, by assumption, $z_b = \Omega^\sharp_\chi(c)$. By Lemma 9.6.8 above, we have

$$\mathscr{T}_{z_b}(\mathscr{Z}) = \mathscr{T}_{z_b}(\mathscr{Z}_c) \oplus \mathscr{T}_{z_b}(\Omega^\sharp_\chi(\mathscr{C})).$$

Since $\mathscr{T}_{z_b}(\Omega^\sharp_\chi(\mathscr{C})) \simeq \mathscr{T}_c(\mathscr{C})$, the Proposition follows from the smoothness of $\mathscr{C}$ and from the fact that $\dim(\mathscr{Z}) = \dim(\mathscr{Z}_c) + \dim(\mathscr{C})$. $\square$

After the work of Etingof-Ginzburg [**EtGi**], Ginzburg-Kaledin [**GiKa**], Gordon [**Gor1**] and Bellamy [**Bel2**], a complete classification of complex reflection groups W such that there exists $c \in \mathscr{C}$ such that $\mathscr{Z}_c$ is smooth has been obtained. Note that the statements *"There exists $c \in \mathscr{C}$ such that $\mathscr{Z}_c$ is smooth"* and *"The ring $\mathbf{k}(\mathscr{C}) \otimes_{\mathbf{k}[\mathscr{C}]} Z = \mathbf{k}(\mathscr{C})Z$ is regular"* are equivalent. We recall now the result.

***Theorem 9.6.9* (Etingof-Ginzburg, Ginzburg-Kaledin, Gordon, Bellamy)**
Assume that W is irreducible. Then the ring $\mathbf{k}(\mathscr{C})Z$ is regular if and only if we are in one of the following two cases:

(1) *W has type $G(d,1,n)$, with $d,\ n \geqslant 1$.*
(2) *W is the group denoted G_4 in the Shephard-Todd classification.*

The following proposition is a consequence of the work of Etingof-Ginzburg [**EtGi**], Gordon [**Gor1**] and Bellamy [**Bel2**].

Proposition 9.6.10. — *The following are equivalent:*

(1) *There exists $c \in \mathscr{C}$ such that $\mathscr{Z}_c$ is smooth.*
(2) *There exists $c \in \mathscr{C}$ such that the points of $\Upsilon_c^{-1}(0)$ are smooth in $\mathscr{Z}_c$.*

Note that the proof of this fact relies on the Shephard-Todd classification of complex reflection groups.

9.7. Blocks and Calogero-Moser families

Calogero-Moser families and blocks of the category $\mathcal{O}$ are closely related, as the following lemma shows. Given $E \in \mathrm{Irr}(W)$, we put $\Delta_{\mathfrak{C}}(E) = \Delta(E) \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{k}(\mathfrak{C})$. As in Example 3.2.1, we consider $w_z = \zeta^{-1}\,\mathrm{Id}_V$ a generator of $W \cap Z(\mathrm{GL}_{\mathbf{k}}(V))$.

Proposition 9.7.1. — *Let $E, F \in \mathrm{Irr}(W)$ and let $i \in \mathbb{Z}$. The standard objects $\Delta_{\mathfrak{C}}(E)$ and $\Delta_{\mathfrak{C}}(F)\langle i\rangle$ are in the same block of $\mathcal{O}(\mathbf{k}(\mathfrak{C}))$ if and only if E and F are in the same Calogero-Moser $\mathfrak{C}$-family and $\omega_E(w_z) = \zeta^i\omega_F(w_z)$.*

Proof. — We use the notations of Example 3.2.1. The element $w_z \in Z(W)$ acts on the degree r part of $\Delta_{\mathfrak{C}}(F)\langle i\rangle$ by $\omega_F(w_z)\zeta^{r+i}$. It follows from §7.3 that if $\Delta_{\mathfrak{C}}(E) \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{k}(\mathfrak{C})$ and $\Delta_{\mathfrak{C}}(F) \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{k}(\mathfrak{C})\langle i\rangle$ are in the same block of $\mathcal{O}(\mathbf{k}(\mathfrak{C}))$, then $\omega_E(w_z) = \omega_F(w_z)\zeta^i$.

Note that $\Delta_{\mathfrak{C}}(E)$ has a filtration whose successive quotients are isomorphic to $\bar{\Delta}_{\mathfrak{C}}(E) \otimes_{\mathbf{k}} (S(V^*)^G)^i$. As a consequence, $\bar{\Delta}_{\mathfrak{C}}(E)$ and $\bar{\Delta}_{\mathfrak{C}}(E)\langle -i\rangle$ are in the same block, whenever $(S(V^*)^G)^i \neq 0$. Since $z = \gcd(d_1, \ldots, d_n)$, it follows that $\bar{\Delta}_{\mathfrak{C}}(E)$ and $\bar{\Delta}_{\mathfrak{C}}(E)\langle z\rangle$ are in the same block.

Assume now E and F are in the same Calogero-Moser $\mathfrak{C}$-family and $\omega_E(w_z) = \zeta^i \omega_F(w_z)$. There is an integer j such that $\bar{\Delta}_{\mathfrak{C}}(E)$ and $\bar{\Delta}_{\mathfrak{C}}(F)\langle j\rangle$ are in the same block of $\mathscr{O}(\mathbf{k}(\mathfrak{C}))$. It follows that $\omega_E(w_z)=\omega_F(w_z)\zeta^j$, hence $d|(i-j)$. So, $\bar{\Delta}_{\mathfrak{C}}(E)$ and $\bar{\Delta}_{\mathfrak{C}}(F)\langle i\rangle$ are in the same block of $\mathscr{O}(\mathbf{k}(\mathfrak{C}))$. □

Let $\tilde{\mathscr{F}}$ be the set of height one prime ideals $\mathfrak{p}$ of $\mathbf{k}[\widetilde{\mathscr{C}}]$ such that the blocks of $\tilde{\mathscr{O}}(\mathbf{k}(\mathfrak{p}))$ are not trivial, ie, there exists $E,F\in\mathrm{Irr}(W)$ and $r\in\mathbb{Z}$ such that $\tilde{\Delta}_{\mathfrak{p}}(E)$ and $\tilde{\Delta}_{\mathfrak{p}}(F)\langle r\rangle$ are in the same block and $E\not\simeq F\langle r\rangle$. Note that the ideals of $\tilde{\mathscr{F}}$ are homogeneous for the $\mathbb{Z}$-grading on $\mathbf{k}[\widetilde{\mathscr{C}}]$.

We assume for the remainder of §9.7 that $V\neq 0$.

Proposition 9.7.2. — *The ideals in $\tilde{\mathscr{F}}$ are $(T=0)$ and the ideals (C_E-C_F-rT) such that $(C_E-C_F-r)\in\mathscr{F}_1$, where $E,F\in\mathrm{Irr}(W)$ and $r\in\mathbb{Z}\setminus\{0\}$.*

Proof. — Proposition 9.7.1 shows that $\mathscr{O}(\mathbf{k}(\mathfrak{C}))$ has non-trivial blocks for all prime ideals $\mathfrak{C}$ of $\mathbf{k}[\mathscr{C}]$. It follows that $(T=0)\in\tilde{\mathscr{F}}$.

Let $\mathfrak{p}$ be an ideal of $\tilde{\mathscr{F}}$ distinct from $(T=0)$. Since $\mathfrak{p}$ is homogeneous, it follows that it is generated by some irreducible polynomial $P(T)=\sum_{i=0}^r a_iT^i$, where $r\geqslant 0$ and a_i is a homogeneous polynomial of degree $d-i$ in the indeterminates C_s, for some $d\in\mathbb{Z}$. Consider the proper ideal $\mathfrak{q}=(T-1,P(T))$ of $\mathbf{k}[\widetilde{\mathscr{C}}]$. Since $\mathfrak{p}\in\tilde{\mathscr{F}}$ and $\mathfrak{p}\subset\mathfrak{q}$, it follows that $P(1)\mathbf{k}[\mathscr{C}]$ contains an ideal in $\mathscr{F}_1$, hence $P(1)$ is divisible by C_E-C_F-r for some $E,F\in\mathrm{Irr}(W)$ with $C_E\neq C_F$ and some $r\in\mathbb{Z}\setminus\{0\}$. Since $P(T)$ is homogeneous, it follows that it is divisible by C_E-C_F-rT. We deduce that $(P(T))=(C_E-C_F-rT)$. □

Define $\mathscr{F}_0$ to be the set of height one prime ideals $\mathfrak{p}$ of $\mathbf{k}[\mathscr{C}]$ such that the Calogero-Moser $\mathfrak{p}$-families are different from the generic Calogero-Moser families. Propositions 9.7.1 and 9.7.2 have the following consequence. The fact that the ideals in $\mathscr{F}_0$ define hyperplanes of $\mathscr{C}$ (and not merely hypersurfaces) is due to Bellamy, Schedler and Thiel [**BelSchTh**, Theorem 5.1].

Corollary 9.7.3. — *The ideals in $\mathscr{F}_0$ are those ideals of the form (C_E-C_F) for some $E,F\in$ $\mathrm{Irr}(W)$ such that there exists $r\in\mathbb{Z}\setminus\{0\}$ with $(C_E-C_F-r)\in\mathscr{F}_1$.*

Theorem 9.7.4. — *Let $c\in\mathscr{C}$. The Calogero-Moser c-families are the smallest subsets of* $\mathrm{Irr}(W)$ *that are unions of generic Calogero-Moser families and unions of blocks of $\dot{\mathscr{O}}(\mathbf{k}(\hbar))$ for all morphisms of $\mathbf{k}$-algebras $\mathbf{k}[\mathscr{C}]\to\mathbf{k}(\hbar)$ of the form $C\mapsto\hbar c+c'$ with $\kappa(c')\in\mathscr{H}(\mathbb{Q})$.*

Proof. — Let I be the set of prime ideals $\mathfrak{p}=(C_E-C_F-r)\in\mathscr{F}_1$ such that $C_E(c)=C_F(c)$.

By Propositions 9.7.1 and 9.7.2, the Calogero-Moser c-families are the smallest subsets of $\mathrm{Irr}(W)$ that are unions of generic Calogero-Moser families and unions of blocks of $\dot{\mathscr{O}}(\mathfrak{p})$ for all $\mathfrak{p}\in I$.

Consider a morphism of $\mathbf{k}$-algebras $\mathbf{k}[\mathscr{C}]\to\mathbf{k}(\hbar)$ of the form $C\mapsto\hbar c+c'$ with $\kappa(c')\in\mathscr{H}(\mathbb{Q})$. Let $I(c')$ be the set of $\mathfrak{p}=(C_E-C_F-r)\in\mathscr{F}_1$ such that $(C_E-C_F-$

$r)(\hbar c + c') = 0$, i.e., $C_E(c) = C_F(c)$ and $(C_E - C_F)(c') = r$. The blocks of $\dot{\mathscr{O}}(\mathbf{k}(\hbar))$ are the smallest subsets of $\mathrm{Irr}(W)$ that are unions of blocks of $\dot{\mathscr{O}}(\mathfrak{p})$ for all $\mathfrak{p} \in I(c')$. Since $I = \bigcup_{c' \in \mathscr{C}(\mathbb{Q})} I(c')$, the theorem follows. □

Since blocks of $\mathscr{O}$ correspond to blocks of the Hecke algebra (cf §8.3.C), we can reformulate the previous result.

Theorem 9.7.5. — *Let $c \in \boldsymbol{\mathscr{C}}$ and $k\kappa(c)$. The Calogero-Moser c-families are the smallest subsets of* $\mathrm{Irr}(W)$ *that are unions of generic Calogero-Moser families and unions of blocks of $\mathbb{C}(\mathbf{q}^{\mathbf{k}})\mathscr{H}$ for all morphisms of $\mathbb{C}$-algebras $\mathbb{C}[\boldsymbol{\mathscr{Q}}] \to \mathbb{C}(\mathbf{q}^{\mathbf{k}})$ of the form $\mathbf{q} \mapsto \zeta \mathbf{q}^k$ where $\zeta = (\zeta_{\aleph,j})_{(\aleph,j)\in \boldsymbol{\aleph}^\circ}$ is a family of roots of unity.*

Using Proposition 9.6.6, we deduce a description of Calogero-Moser families from blocks of the Hecke algebra, when the Calogero-Moser space is smooth for generic values of the parameter.

Corollary 9.7.6. — *Assume $\mathscr{Z}_\eta$ is smooth, for η the generic point of $\boldsymbol{\mathscr{C}}$. Let $c \in \mathscr{C}(\mathbf{k})$. Let I be a subset of* $\mathrm{Irr}(W)$. *The following are equivalent:*

- *I is a union of Calogero-Moser c-families.*
- *I is a union blocks of $\dot{\mathscr{O}}(\mathbf{k}(\hbar))$ for all morphisms of $\mathbf{k}$-algebras $\mathbf{k}[\boldsymbol{\mathscr{C}}] \to \mathbf{k}(\hbar)$ of the form $C \mapsto \hbar c + c'$ with $\kappa(c') \in \mathscr{K}(\mathbb{Q})$.*
- *I is a union blocks of $\mathbb{C}(\mathbf{q}^{\mathbf{k}})\mathscr{H}$ for all morphisms of $\mathbb{C}$-algebras $\mathbb{C}[\boldsymbol{\mathscr{Q}}] \to \mathbb{C}[\mathbf{q}^{\mathbf{k}}]$ of the form $\mathbf{q} \mapsto \zeta \mathbf{q}^k$ where $\zeta = (\zeta_{\aleph,j})_{(\aleph,j)\in \boldsymbol{\aleph}^\circ}$ is a family of roots of unity.*

Remark 9.7.7. — When W has a unique conjugacy class of reflections, the previous results are trivial: when $c \neq 0$, the algebras $\mathbb{C}(\mathbf{q}^{\mathbf{k}})\mathscr{H}$ in Theorem 9.7.5 and Corollary 9.7.6 are semisimple. ■

CHAPTER 10

CALOGERO-MOSER TWO SIDED CELLS

In this chapter §10, we fix a prime ideal $\mathfrak{C}$ of $\mathbf{k}[\mathscr{C}]$. We study the relation between Calogero-Moser two-sided $\mathfrak{C}$-cells and families.

10.1. Choices

In order to relate the Calogero-Moser two-sided cells with the Kazhdan-Lusztig ones, we need to make an appropriate choice of a prime ideal $\bar{\mathfrak{r}}_{\mathfrak{C}}$ of R lying over $\bar{\mathfrak{p}}_{\mathfrak{C}}$. We do not have a procedure to make this choice, but we will give at least some hints for this.

Recall that $\bar{\mathfrak{p}}$ denotes the prime ideal of P corresponding to the closed irreducible subvariety $\mathscr{C} \times \{0\} \times \{0\}$ of $\mathscr{P}$ (c.f. p.88). There are several prime ideals of Z lying over $\bar{\mathfrak{p}}$. They are described in Lemma 9.2.3, which says that they are in bijection with the set of generic Calogero-Moser families of W. Recall also that (Example 9.3.6) the trivial character $\mathbf{1}_W$ of W is itself a generic Calogero-Moser family. We denote by $\bar{\mathfrak{z}}$ the associated prime ideal of Z:

$$\bar{\mathfrak{z}} = \operatorname{Ker}(\Omega_{\mathbf{1}_W}).$$

We put $\bar{\mathfrak{q}} = \operatorname{cop}(\bar{\mathfrak{z}})$.

Lemma 10.1.1. — *The ideal $\bar{\mathfrak{q}}$ of Q is the unique prime ideal lying over $\bar{\mathfrak{p}}$ and containing* $\mathrm{eu} - \sum_{H\in\mathscr{A}} e_H K_{H,0}$. *The algebra Q is étale over P at $\bar{\mathfrak{q}}$.*

Proof. — It follows from Corollary 9.4.5 that Q is unramified over P at $\bar{\mathfrak{q}}$: as Q is a free (hence flat) P-module and since, in characteristic zero, all the field extensions are separable, we deduce that Q is étale over P at $\bar{\mathfrak{q}}$. The fact that $\mathbf{eu} - \sum_{H\in\mathscr{A}} e_H K_{H,0} \in \bar{\mathfrak{z}}$ follows from Corollary 7.3.2. Now, if $\bar{\mathfrak{z}}'$ is a prime ideal of Z lying over $\bar{\mathfrak{p}}$ and containing $\mathbf{eu} - \sum_{H\in\mathscr{A}} e_H K_{H,0}$, then there exists $\chi \in \operatorname{Irr}(W)$ such that $\bar{\mathfrak{z}}' = \operatorname{Ker}(\Omega_\chi)$. In particular, $\Omega_\chi(\mathbf{eu}) = \sum_{H\in\mathscr{A}} e_H K_{H,0}$ and so $\Omega_\chi(\mathbf{eu}) = \Omega_{\mathbf{1}_W}(\mathbf{eu})$. It has been shown in Example 9.3.6 that this implies $\chi = \mathbf{1}_W$. □

Let us use now the notation of §9. Lemma 9.2.3 says that the set of prime ideals of Q lying over $\bar{\mathfrak{p}}_\mathfrak{C}$ is in bijection with the set of Calogero-Moser $\mathfrak{C}$-families. Let $\bar{\mathfrak{z}}_\mathfrak{C}$ denote the prime ideal corresponding to the $\mathfrak{C}$-family containing the trivial character $\mathbf{1}_W$ of W:

$$\bar{\mathfrak{z}}_\mathfrak{C} = \operatorname{Ker}(\Omega^\mathfrak{C}_{\mathbf{1}_W}).$$

We set $\bar{\mathfrak{q}}_\mathfrak{C} = \operatorname{cop}(\bar{\mathfrak{z}}_\mathfrak{C})$.

Lemma 10.1.2. — *We have* $\bar{\mathfrak{q}}_\mathfrak{C} = \bar{\mathfrak{q}} + \mathfrak{C}Q$.

Proof. — The morphism $\Omega_{\mathbf{1}_W} : Z \to \mathbf{k}[\mathscr{C}]$ induces an isomorphism $Z/\bar{\mathfrak{z}} \xrightarrow{\sim} \mathbf{k}[\mathscr{C}]$. Since $\bar{\mathfrak{z}}_\mathfrak{C}$ contains $\mathfrak{C}$, it follows that $\bar{\mathfrak{z}} + \mathfrak{C}Z$ is a prime ideal of Z, corresponding to the closed irreducible subscheme $\mathscr{C}_\mathfrak{C}$ of $\mathscr{C}$. □

Corollary 10.1.3. — *The ideal* $\bar{\mathfrak{q}}_\mathfrak{C}$ *of* Q *is the unique prime ideal lying over* $\bar{\mathfrak{p}}_\mathfrak{C}$ *and containing* $\mathrm{eu} - \sum_{H\in\mathscr{A}} e_H K_{H,0}$.

10.2. Two-sided cells

> **Assumption.** *From now on, and until the end of §10.2, we fix a prime ideal* $\bar{\mathfrak{r}}_\mathfrak{C}$ *of* R *lying over* $\bar{\mathfrak{q}}_\mathfrak{C}$. *Recall that* $\bar{\mathbf{K}}_\mathfrak{C} = k_P(\bar{\mathfrak{p}}_\mathfrak{C}) = k_{\mathbf{k}[\mathscr{C}]}(\mathfrak{C})$. *We put* $\bar{\mathbf{L}}_\mathfrak{C} = k_Q(\bar{\mathfrak{q}}_\mathfrak{C})$ *and* $\bar{\mathbf{M}}_\mathfrak{C} = k_R(\bar{\mathfrak{r}}_\mathfrak{C})$. *We denote by* $\bar{\mathbf{K}}$, $\bar{\mathbf{L}}$ *and* $\bar{\mathbf{M}}$ *(respectively* $\bar{\mathbf{K}}_c$, $\bar{\mathbf{L}}_c$ *and* $\bar{\mathbf{M}}_c$*) the fields* $\bar{\mathbf{K}}_\mathfrak{C}$, $\bar{\mathbf{L}}_\mathfrak{C}$ *and* $\bar{\mathbf{M}}_\mathfrak{C}$ *whenever* $\mathfrak{C} = 0$ *(respectively* $\mathfrak{C} = \mathfrak{C}_c$ *for some* $c \in \mathscr{C}$*).*
>
> *The decomposition (respectively inertia) group of* $\bar{\mathfrak{r}}_\mathfrak{C}$ *will be denoted by* $\bar{D}_\mathfrak{C}$ *(respectively* $\bar{I}_\mathfrak{C}$*). We define similarly* $\bar{D}$, $\bar{I}$, $\bar{D}_c$ *and* $\bar{I}_c$.

10.2.A. Galois theory. — Recall that, by Corollary 9.2.5, the canonical embedding $P/\bar{\mathfrak{p}}_\mathfrak{C} \hookrightarrow Q/\bar{\mathfrak{q}}_\mathfrak{C}$ is an isomorphism, hence $\bar{\mathbf{K}}_\mathfrak{C} = \bar{\mathbf{L}}_\mathfrak{C}$. Since $\mathrm{Gal}(\bar{\mathbf{M}}_\mathfrak{C}/\bar{\mathbf{L}}_\mathfrak{C}) = \bar{D}_\mathfrak{C}/\bar{I}_\mathfrak{C}$ (Theorem B.2.4), we deduce that

$$(\bar{D}_\mathfrak{C} \cap H)/(\bar{I}_\mathfrak{C} \cap H) \simeq \bar{D}_\mathfrak{C}/\bar{I}_\mathfrak{C}. \tag{10.2.1}$$

In particular,

$$(\bar{D}_\mathfrak{C} \cap H)\bar{I}_\mathfrak{C} = \bar{D}_\mathfrak{C}. \tag{10.2.2}$$

Moreover, since the algebra Q is unramified over P at $\bar{\mathfrak{q}}$ (Lemma 10.1.1), it follows from Theorem B.2.6 that $\bar{I} \subset H$. Combined with (10.2.2), we obtain

(10.2.3) $$\bar{I} \subset \bar{D} \subset H.$$

Note that this last result is not true in general for $\bar{I}_{\mathfrak{C}}$, as shown by (17.1.2),

To conclude with basic Galois properties, note that, by Corollaries 9.2.5 and B.3.8, we have

(10.2.4) $$\bar{I}_{\mathfrak{C}} \backslash G/H = \bar{D}_{\mathfrak{C}} \backslash G/H.$$

10.2.B. Two-sided cells and grading. — Let $\widetilde{\mathfrak{C}} = \bigoplus_{i \geqslant 0} \mathfrak{C} \cap \mathbf{k}[\mathscr{C}]^{\mathbb{N}}[i]$ be the maximal homogeneous ideal of $\mathbf{k}[\mathscr{C}]$ contained in $\mathfrak{C}$. Then $\widetilde{\mathfrak{C}}$ is a prime ideal of $\mathbf{k}[\mathscr{C}]$ (see Lemma C.2.9). Let $\bar{\mathfrak{r}}_{\widetilde{\mathfrak{C}}}$ denote the maximal homogeneous ideal of R contained in $\bar{\mathfrak{r}}_{\mathfrak{C}}$: it is a prime ideal of R lying over $\bar{\mathfrak{q}}_{\widetilde{\mathfrak{C}}}$ (see Corollary C.2.12). The following is a consequence of Proposition 6.1.5.

Lemma 10.2.5. — *We have $\bar{I}_{\mathfrak{C}} = \bar{I}_{\widetilde{\mathfrak{C}}}$. The Calogero-Moser two-sided $\mathfrak{C}$-cells and the Calogero-Moser two-sided $\widetilde{\mathfrak{C}}$-cells coincide.*

10.2.C. Two-sided cells and families. — The $\mathbf{k}[\mathscr{C}]$-algebra $\bar{\mathbf{M}}_{\mathfrak{C}}$ is a finite field extension of $\bar{\mathbf{K}}_{\mathfrak{C}} = k_P(\bar{\mathfrak{p}}_{\mathfrak{C}}) = \mathrm{Frac}(\mathbf{k}[\mathscr{C}]/\mathfrak{C})$ and $\bar{\mathbf{M}}_{\mathfrak{C}}\mathbf{H} = \bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}}$. Theorem 6.2.2 says that there is a bijection between the Calogero-Moser two-sided $\mathfrak{C}$-cells and the Calogero-Moser $\mathfrak{C}$-families: given $b \in \mathrm{Idem}_{\mathrm{pr}}(R_{\bar{\mathfrak{r}}_{\mathfrak{C}}} Z)$, this bijection sends $\mathrm{CM}_{\bar{\mathfrak{r}}_{\mathfrak{C}}}(b)$ to $\mathrm{Irr}_{\mathbf{H}}(W, \bar{b})$, where $\bar{b}$ denotes the image of b in $\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}}$, and $\bar{b} \in \bar{\mathbf{K}}_{\mathfrak{C}}\bar{\mathbf{H}}$ since $\bar{\mathbf{K}}_{\mathfrak{C}}\bar{\mathbf{H}}$ is split.

> **Terminology, notation.** *Given $b \in \mathrm{Idem}_{\mathrm{pr}}(\bar{\mathbf{M}}_{\mathfrak{C}} Z)$, we say that the Calogero-Moser two-sided $\mathfrak{C}$-cell $\mathrm{CM}_{\bar{\mathfrak{r}}_{\mathfrak{C}}}(b)$ **covers** the Calogero-Moser $\mathfrak{C}$-family $\mathrm{Irr}_{\mathbf{H}}(W, b)$. Given Γ a Calogero-Moser two-sided $\mathfrak{C}$-cell, we denote by $\mathrm{Irr}_{\Gamma}^{\mathrm{CM}}(W)$ the Calogero-Moser $\mathfrak{C}$-family covered by Γ. The set of Calogero-Moser two-sided $\mathfrak{C}$-cells will be denoted by ${}^{\mathrm{CM}}\mathrm{Cell}_{LR}^{\mathfrak{C}}(W)$.*

Remark 10.2.6. — The definition of Calogero-Moser two-sided cells depends on the choice of the prime ideal $\bar{\mathfrak{r}}_{\mathfrak{C}}$ lying over $\bar{\mathfrak{q}}_{\mathfrak{C}}$. Given $\bar{\mathfrak{r}}'_{\mathfrak{C}}$ another prime ideal of R lying over $\bar{\mathfrak{q}}_{\mathfrak{C}}$, there exists $h \in H$ such that $\bar{\mathfrak{r}}'_{\mathfrak{C}} = h(\bar{\mathfrak{r}}_{\mathfrak{C}})$ and the Calogero-Moser $\bar{\mathfrak{r}}'_{\mathfrak{C}}$-cells are obtained from the Calogero-Moser $\bar{\mathfrak{r}}_{\mathfrak{C}}$-cells via the action of h on $W \stackrel{\sim}{\longleftrightarrow} G/H$. ■

The link between Calogero-Moser two-sided $\mathfrak{C}$-cells and Calogero-Moser $\mathfrak{C}$-families is strengthened by the following theorem.

Theorem 10.2.7. — (a) *The decomposition group $\bar{D}_{\mathfrak{C}}$ acts trivially on ${}^{\mathrm{CM}}\mathrm{Cell}_{LR}^{\mathfrak{C}}(W)$.*

(b) *Given $w \in W$, $b \in \mathrm{Idem}_{\mathrm{pr}}(R_{\bar{\mathfrak{r}}_{\mathfrak{C}}}Q)$ and $\chi \in \mathrm{Irr}(W)$, the following are equivalent*

- *the Calogero-Moser two-sided $\mathfrak{C}$-cell of w is associated with the Calogero-Moser $\mathfrak{C}$-family of χ*
- $w^{-1}(\bar{\mathfrak{r}}_{\mathfrak{C}}) \cap Q = \mathrm{cop}(\mathrm{Ker}(\Omega_\chi^{\mathfrak{C}}))$
- $(w(q) \mod \bar{\mathfrak{r}}_{\mathfrak{C}}) = \Omega_\chi^{\mathfrak{C}}(\mathrm{cop}^{-1}(q)) \in \bar{\mathbf{M}}_{\mathfrak{C}} = k_R(\bar{\mathfrak{r}}_{\mathfrak{C}})$ *for all* $q \in Q$.

(c) *Given Γ is a Calogero-Moser two-sided $\mathfrak{C}$-cell, we have* $|\Gamma| = \sum_{\chi \in \mathrm{Irr}_\Gamma^{\mathrm{CM}}(W)} \chi(1)^2$.

Proof. — (a) follows from 10.2.4.

(b) Let $\bar{\omega}_w : Q \to R/\bar{\mathfrak{r}}_{\mathfrak{C}}$ denote the morphism of P-algebras which sends $q \in Q$ to the image of $\omega_w(q) = w(q) \in R$ in $R/\bar{\mathfrak{r}}_{\mathfrak{C}}$. Then w belongs to the Calogero-Moser two-sided $\mathfrak{C}$-cell associated with the Calogero-Moser $\mathfrak{C}$-family of χ if and only if $\bar{\omega}_w = \Omega_\chi$. But, by Lemma 9.2.3, this is equivalent to say that $\mathrm{Ker}(\bar{\omega}_w) = \mathrm{cop}(\mathrm{Ker}(\Omega_\chi^{\mathfrak{C}}))$. Since $\mathrm{Ker}(\bar{\omega}_w) = w^{-1}(\bar{\mathfrak{r}}_{\mathfrak{C}}) \cap Q$, the first equivalence follows. Since $Q = (w^{-1}(\mathfrak{r}) \cap Q) + \mathbf{k}[\mathscr{C}]$ (Corollary 9.2.5), the second equivalence follows.

The assertion (c) follows from Corollaries 9.4.2 and 6.2.5. ☐

Corollary 10.2.8. — *Let $\mathfrak{C}'$ be a prime ideal of $\mathbf{k}[\mathscr{C}]$ contained in $\mathfrak{C}$ and let $\bar{\mathfrak{r}}_{\mathfrak{C}'}$ be a prime ideal of R lying above $\bar{\mathfrak{p}}_{\mathfrak{C}'}$ and contained in $\bar{\mathfrak{r}}_{\mathfrak{C}}$. Then the Calogero-Moser two-sided $\mathfrak{C}$-cells are unions of Calogero-Moser two-sided $\mathfrak{C}'$-cells. Moreover, if Γ is a Calogero-Moser two-sided $\mathfrak{C}$-cell and if $\Gamma = \Gamma_1 \coprod \cdots \coprod \Gamma_r$ where the Γ_i's are Calogero-Moser two-sided $\mathfrak{C}'$-cells, then*

$$\mathrm{Irr}_\Gamma^{\mathrm{CM},\mathfrak{C}}(W) = \mathrm{Irr}_{\Gamma_1}^{\mathrm{CM},\mathfrak{C}'}(W) \coprod \cdots \coprod \mathrm{Irr}_{\Gamma_r}^{\mathrm{CM},\mathfrak{C}'}(W).$$

Corollary 10.2.9. — *Assume that all the reflections of W have order 2 and that $w_0 = -\mathrm{Id}_V \in W$. If Γ is a Calogero-Moser two-sided $\mathfrak{C}$-cell covering the Calogero-Moser $\mathfrak{C}$-family $\mathscr{F}$, then $w_0\Gamma = \Gamma w_0$ is the Calogero-Moser two-sided $\mathfrak{C}$-cell covering the Calogero-Moser $\mathfrak{C}$-family $\varepsilon\mathscr{F}$.*

Proof. — First of all, $w_0\Gamma = \Gamma w_0$ is a Calogero-Moser two-sided $\mathfrak{C}$-cell by Example 6.1.4 whereas $\varepsilon\mathscr{F}$ is a Calogero-Moser $\mathfrak{C}$-family by Corollary 9.3.5.

Let $w \in \Gamma$, $\chi \in \mathscr{F}$ and $q \in Q$. By Theorem 10.2.7(b), we only need to show that $ww_0(q) \equiv \Omega_{\chi\varepsilon}(q) \mod \bar{\mathfrak{r}}_{\mathfrak{C}}$. Let $\tau_0 = (-1, 1, \varepsilon) \in \mathbf{k}^\times \times \mathbf{k}^\times \times W^\wedge$. By Proposition 5.5.2, we have $w_0(q) = {}^{\tau_0}q$ for all $q \in Q$. Moreover, by Corollary 7.4.2, we have $\Omega_{\chi\varepsilon}(q) = {}^{\tau_0}\Omega_\chi({}^{\tau_0}q)$ (because τ_0 has order 2). Since τ_0 acts trivially on $\mathbf{k}[\mathscr{C}]$, we have $\Omega_{\chi\varepsilon}(q) =$

$\Omega_\chi({}^{\tau_0}q)$. It is now sufficient to show that $w({}^{\tau_0}q) \equiv \Omega_\chi({}^{\tau_0}q) \mod \bar{\mathfrak{r}}_{\mathfrak{C}})$. This follows from Theorem 10.2.7(b). □

***Example 10.2.10* (Smoothness).** — If the ring Q is regular at $\bar{\mathfrak{q}}_b$ and if χ denotes the unique element of $\mathrm{Irr}_{\mathbf{H}}(W,b)$ (see Proposition 9.6.6), then $|\mathrm{CM}_{\mathfrak{r}}(b)| = \chi(1)^2$ by Theorem 10.2.7(c). ■

Remark 10.2.11. — If $\bar{\mathfrak{r}}_{\mathfrak{C}}$ and $\bar{\mathfrak{r}}$ are chosen so that $\bar{\mathfrak{r}} \subset \bar{\mathfrak{r}}_{\mathfrak{C}}$, then $\bar{I} \subset \bar{I}_{\mathfrak{C}}$ and so every Calogero-Moser $\mathfrak{C}$-cell is a union of generic Calogero-Moser two-sided cells. It is the "cell version" of the corresponding result on families. ■

If γ is a linear character, then it is alone in its generic Calogero-Moser family (Example 9.3.6) and its covering generic Calogero-Moser two-sided cell contains only one element (Theorem 10.2.7(c)). Let w_γ denote this element. By Theorem 10.2.7(b), we have

$$w_\gamma^{-1}(\bar{\mathfrak{r}}) \cap Q = \mathrm{Ker}(\Omega_\gamma). \tag{10.2.12}$$

Corollary 10.2.13. — *We have $w_{\mathbf{1}_W} = 1$. In other words 1 is alone in its generic Calogero-Moser two-sided cell and covers the generic Calogero-Moser family of the trivial character $\mathbf{1}_W$ (which is a singleton).*

Proof. — By Theorem 10.2.7 and (10.2.12), $w_{\mathbf{1}_W}$ is the unique element $w \in W$ such that $w^{-1}(\bar{\mathfrak{r}}) \cap Q = \mathrm{Ker}(\Omega_{\mathbf{1}_W}) = \bar{\mathfrak{q}}$. Since $\bar{\mathfrak{r}} \cap Q = \bar{\mathfrak{q}}$, we have $w_{\mathbf{1}_W} = 1$. □

Proposition 10.2.14. — *Let $\gamma \in W^\wedge$. Then $\bar{I} \subset w_\gamma H w_\gamma^{-1}$.*

Proof. — We shall give two proofs of this fact. First, w_γ is alone in its generic Calogero-Moser two-sided cell, so $\bar{I}\, w_\gamma H = w_\gamma H$, whence the result.

Let us now give a second proof. By Corollary 9.4.5, Q is unramified over P at $\mathrm{Ker}(\Omega_\gamma) = w_\gamma^{-1}(\bar{\mathfrak{r}}) \cap Q$. So, by Theorem B.2.6, $I_{w_\gamma^{-1}(\bar{\mathfrak{r}})} \subset H$, which is exactly the desired statement, since $I_{w_\gamma^{-1}(\bar{\mathfrak{r}})} = w_\gamma^{-1} \bar{I}\, w_\gamma$. □

Remark 10.2.15. — The action of H on $W \stackrel{\sim}{\longleftrightarrow} G/H$ stabilizes the identity element (that is, H stabilizes $\mathbf{eu}$). This shows that the statement of Corollary 10.2.13 does not depend on the choice of $\bar{\mathfrak{r}}$. ■

CHAPTER 11

CALOGERO-MOSER LEFT AND RIGHT CELLS

In §11, $\mathfrak{C}$ denotes a prime ideal of $\mathbf{k}[\mathscr{C}]$. We use Verma modules for $\mathbf{H}^{\mathrm{left}}$ to define the notion of *Calogero-Moser $\mathfrak{C}$-cellular characters*. We expect that they coincide with the Kazhdan-Lusztig cellular characters whenever W is a Coxeter group. We relate Calogero-Moser cellular characters to Calogero-Moser left cells.

This chapter will mainly consider *left* Calogero-Moser $\mathfrak{C}$-cells and (left) Verma modules: definitions and results can be immediately transposed to the *right* setting.

11.1. Verma modules and cellular characters

11.1.A. Morita equivalence. — Let $P^{\mathrm{reg,left}} = \mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}/W \times \{0\}] = P^{\mathrm{reg}} \otimes_P P^{\mathrm{left}}$, $Z^{\mathrm{reg,left}} = P^{\mathrm{reg,left}} Z$, and $\mathbf{H}^{\mathrm{reg,left}} = P^{\mathrm{reg,left}}\mathbf{H}$. Note that, by Example 6.4.7, $\mathfrak{z}_{\mathrm{sing}} \cap P \not\subset \mathfrak{p}^{\mathrm{left}}$. Hence, Theorem 6.4.6 can be applied. Thanks to Corollary 4.3.5, we obtain the following result.

Theorem 11.1.1. — *The $(\mathbf{H}^{\mathrm{reg,left}}, Z^{\mathrm{reg,left}})$-bimodule $\mathbf{H}^{\mathrm{reg,left}}e$ induces a Morita equivalence between the algebras $\mathbf{H}^{\mathrm{reg,left}}$ and $Z^{\mathrm{reg,left}}$. Consequently, the $(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}, \mathbf{K}_{\mathfrak{C}}^{\mathrm{left}} Z^{\mathrm{left}})$-bimodule $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}e$ induces a Morita equivalence between the algebras $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$ and $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}} Z^{\mathrm{left}}$.*

If $\mathfrak{r}$ is a prime ideal of R lying over $\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}$, then the simple $k_R(\mathfrak{r})\mathbf{H}^{\mathrm{left}}$-modules are absolutely simple and have dimension $|W|$.

The Morita equivalence of Theorem 11.1.1 induces a bijection

$$\begin{array}{ccc} \mathrm{Irr}(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}) & \stackrel{\sim}{\longrightarrow} & \mathrm{Irr}(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}} Z^{\mathrm{left}}) \\ L & \longmapsto & eL. \end{array} \tag{11.1.2}$$

On the other hand, the (isomorphism classes of) simple $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}} Z^{\mathrm{left}}$-modules are in bijection with the maximal ideals of $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}} Z^{\mathrm{left}}$, that is, with the minimal prime ideals of $Z_{\mathfrak{C}}^{\mathrm{left}}$ or, in other words, with $\Upsilon^{-1}(\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}})$. Using (11.1.2), we obtain a bijection

$$\begin{array}{ccc} \Upsilon^{-1}(\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}) & \stackrel{\sim}{\longrightarrow} & \mathrm{Irr}(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}) \\ \mathfrak{z} & \longmapsto & L_{\mathfrak{C}}^{\mathrm{left}}(\mathfrak{z}). \end{array} \tag{11.1.3}$$

11.1.B. Cellular characters. — Let $\mathfrak{z} \in \Upsilon^{-1}(\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}})$ and let $e_{\mathfrak{z}}$ be the corresponding primitive idempotent of $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}} Z^{\mathrm{left}}$.

We identify $G_0((Z_{\mathfrak{C}}^{\mathrm{left}})_{\mathfrak{z}}[W])$ with $G_0(\mathbf{k}W)$ by $[(Z/\mathfrak{z})\otimes E]\mapsto[E]$ for $E\in\mathrm{Irr}(W)$.

The right action by multiplication of $\mathbf{k}W$ on $\mathbf{k}W$ induces a right action of $\mathbf{k}W$ on $\Delta(\mathbf{k}W)$.

Definition 11.1.4. — *We define the **Calogero-Moser $\mathfrak{C}$-cellular character** associated with $\mathfrak{z}$ as the character of W given by*

$$\gamma_{\mathfrak{z}}^{\mathrm{CM}} = ([Z_{\mathfrak{z}} e \mathbf{K}_{\mathfrak{C}}^{\mathrm{left}} \Delta(\mathbf{k}W)]_{(Z_{\mathfrak{C}}^{\mathrm{left}})_{\mathfrak{z}} W})^*.$$

When $\mathfrak{C} = 0$ (respectively $\mathfrak{C} = \mathfrak{C}_c$ for some $c \in \mathscr{C}$), they will be called ***generic Calogero-Moser cellular characters*** (respectively ***Calogero-Moser c-cellular characters***).

Given $\mathfrak{z} \in \Upsilon^{-1}(\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}})$ and $\chi \in \mathrm{Irr}(W)$, we denote by $\mathrm{mult}_{\mathfrak{z},\chi}^{\mathrm{CM}}$ the multiplicity of the simple module $L_{\mathfrak{C}}^{\mathrm{left}}(\mathfrak{z})$ in a composition series of the Verma module $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\Delta(\chi)$.

The above definition can be expressed in terms of length of $Z_{\mathfrak{z}}$-modules, through the Morita equivalence. We first need the following lemma. Given M a finitely generated $Z_{\mathfrak{C}}^{\mathrm{left}}$-module, the $Z_{\mathfrak{z}}$-module $M_{\mathfrak{z}}$ has finite length and we denote this length by $\mathrm{Length}_{Z_{\mathfrak{z}}}(M_{\mathfrak{z}})$.

Lemma 11.1.5. — *Let M be a finitely generated $\mathbf{H}_{\mathfrak{C}}^{\mathrm{left}}$-module. Then $eM_{\mathfrak{z}}$ is a $Z_{\mathfrak{z}}$-module of finite length and* $\mathrm{Length}_{Z_{\mathfrak{z}}}(eM_{\mathfrak{z}})$ *is equal to the multiplicity of $L_{\mathfrak{C}}^{\mathrm{left}}(\mathfrak{z})$ in the $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$-module $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}M$.*

Proof. — By construction, $\mathrm{Length}_{Z_{\mathfrak{z}}}(eM_{\mathfrak{z}})$ is equal to the multiplicity of $eL_{\mathfrak{C}}^{\mathrm{left}}(\mathfrak{z})$ in the $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}} Z^{\mathrm{left}}$-module $e\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}M$. The result follows now from the Morita equivalence of Theorem 11.1.1. ☐

Proposition 11.1.6. — *Let $\mathfrak{z}\in\Upsilon^{-1}(\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}})$. We have*

$$\gamma_{\mathfrak{z}}^{\mathrm{CM}} = \sum_{\chi\in\mathrm{Irr}(W)} \mathrm{Length}_{Z_{\mathfrak{z}}}\big(e\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\Delta(\chi)\big)_{\mathfrak{z}}\cdot\chi = \sum_{\chi\in\mathrm{Irr}(W)} \mathrm{mult}_{\mathfrak{z},\chi}^{\mathrm{CM}}\cdot\chi.$$

Proof. — We have $\Delta(\mathbf{k}W) = \bigoplus_{E\in\mathrm{Irr}(W)} \Delta(E)\otimes E^*$, hence $Z_{\mathfrak{z}}e\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\Delta(\mathbf{k}W) = \bigoplus_E Z_{\mathfrak{z}}e\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\Delta(E)\otimes E^*$. Since $[Z_{\mathfrak{z}}e\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\Delta(E)]_{(Z_{\mathfrak{C}}^{\mathrm{left}})_{\mathfrak{z}}} = \mathrm{Length}_{Z_{\mathfrak{z}}}\big(e\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\Delta(E)\big)_{\mathfrak{z}}[Z/\mathfrak{z}]$, we deduce the first equality. The second equality follows from Lemma 11.1.5. □

11.2. Choices

As in the case of two-sided cells, the notion of Calogero-Moser left $\mathfrak{C}$-cell depends on the choice of a prime ideal of R lying over $\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}$. We will use Verma modules to restrict choices.

Lemma 11.2.1. — *Given γ is a **linear** character of W, the $\mathbf{K}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$-module $\mathbf{K}^{\mathrm{left}}\Delta(\gamma)$ is absolutely simple.*

Proof. — This follows from Theorem 11.1.1 and (7.1.1). □

Fix now a linear character γ of W. By Lemma 11.2.1, the endomorphism algebra of $\mathbf{K}^{\mathrm{left}}\Delta(\gamma)$ is equal to $\mathbf{K}^{\mathrm{left}}$. This induces a morphism of P-algebras $\Omega_\gamma^{\mathrm{left}} : Z \to \mathbf{K}^{\mathrm{left}}$ whose restriction to P is the canonical morphism $P \to P^{\mathrm{left}}$. Since Z is integral over P, the image of $\Omega_\gamma^{\mathrm{left}}$ is integral over P^{left} and contained in $\mathbf{K}^{\mathrm{left}} = \mathrm{Frac}(P^{\mathrm{left}})$. Since $P^{\mathrm{left}} \simeq \mathbf{k}[\mathscr{C} \times V/W]$ is integrally closed, this forces $\Omega_\gamma^{\mathrm{left}}$ to factor through P^{left}. We set

$$\mathfrak{z}^{\mathrm{left}} = \mathrm{Ker}(\Omega_1^{\mathrm{left}}) \quad \text{and} \quad \mathfrak{q}^{\mathrm{left}} = \mathrm{cop}(\mathfrak{z}^{\mathrm{left}}).$$

Proposition 11.2.2. — *The ideal $\mathfrak{q}^{\mathrm{left}}$ of Q satisfies the following properties:*

(a) *$\mathfrak{q}^{\mathrm{left}}$ is a prime ideal of Q lying over $\mathfrak{p}^{\mathrm{left}}$.*

(b) *$\mathfrak{q}^{\mathrm{left}} \subset \bar{\mathfrak{q}}$.*

(c) *$P^{\mathrm{left}} = P/\mathfrak{p}^{\mathrm{left}} \simeq Q/\mathfrak{q}^{\mathrm{left}}$.*

Proof. — Since $\mathbf{K}^{\mathrm{left}}$ is a field, $\mathfrak{q}^{\mathrm{left}}$ is prime. Since the restriction of Ω_1^{left} to P is the canonical morphism $P \to P^{\mathrm{left}}$, $\mathfrak{q}^{\mathrm{left}} \cap P = \mathfrak{p}^{\mathrm{left}}$. This shows (a).

By construction, $\Delta(\gamma)/\bar{\mathfrak{p}}\Delta(\gamma) = \bar{\Delta}(\gamma)$ and so the morphism $\Omega_\gamma : Z \to \bar{P} = P/\bar{\mathfrak{p}}$ factors through the morphisms $\Omega_\gamma^{\mathrm{left}} : Z \to P^{\mathrm{left}}$ and $P^{\mathrm{left}} \to \bar{P}$, whence (b).

Finally, the isomorphism (c) follows from the fact that the image of Ω_γ is P^{left}. □

Proposition 11.2.2 allows us to choose a prime ideal of Q lying over $\mathfrak{p}^{\mathrm{left}}$ and compatible with our choice of $\bar{\mathfrak{q}}$. The next lemma shows that this choice is unique:

Lemma 11.2.3. — *We have $\mathfrak{p}^{\mathrm{left}}Q_{\bar{\mathfrak{q}}} = \mathfrak{q}^{\mathrm{left}}Q_{\bar{\mathfrak{q}}}$.*

Proof. — It is sufficient to prove that $\mathfrak{p}^{\mathrm{left}} Q_{\bar{\mathfrak{q}}}$ is a prime ideal of $Q_{\bar{\mathfrak{q}}}$. By Lemma 10.1.1, the local morphism of local rings $P_{\bar{\mathfrak{p}}} \to Q_{\bar{\mathfrak{q}}}$ is étale. Moreover, $P/\mathfrak{p}^{\mathrm{left}} \simeq \mathbf{k}[\mathscr{C} \times V^*/W]$ is integrally closed (it is a polynomial algebra) and so $P_{\bar{\mathfrak{p}}}/\mathfrak{p}^{\mathrm{left}} P_{\bar{\mathfrak{p}}}$ is also integrally closed. By base change, the ring morphism $P_{\bar{\mathfrak{p}}}/\mathfrak{p}^{\mathrm{left}} P_{\bar{\mathfrak{p}}} \hookrightarrow Q_{\bar{\mathfrak{q}}}/\mathfrak{p}^{\mathrm{left}} Q_{\bar{\mathfrak{q}}}$ is étale, which implies that $Q_{\bar{\mathfrak{q}}}/\mathfrak{p}^{\mathrm{left}} Q_{\bar{\mathfrak{q}}}$, which is a local ring (hence is connected), is also integrally closed (by [**SGA1**, Talk I, Corollary 9.11]) and so is a domain (because it is connected). This shows that $\mathfrak{p}^{\mathrm{left}} Q_{\bar{\mathfrak{q}}}$ is a prime ideal of $Q_{\bar{\mathfrak{q}}}$, as desired. □

Corollary 11.2.4. — *The ideal $\mathfrak{q}^{\mathrm{left}}$ is the unique prime ideal of Q lying over $\mathfrak{p}^{\mathrm{left}}$ and contained in $\bar{\mathfrak{q}}$. Moreover, Q is étale over P at $\mathfrak{q}^{\mathrm{left}}$.*

Since $Q/\mathfrak{q}^{\mathrm{left}} \simeq P/\mathfrak{p}^{\mathrm{left}} = \mathbf{k}[\mathscr{C} \times V/W]$, we get that $Q/(\mathfrak{q}^{\mathrm{left}} + \mathfrak{C} Q) \simeq \mathbf{k}[\mathscr{C}]/\mathfrak{C} \otimes \mathbf{k}[V/W]$ and so $\mathfrak{q}^{\mathrm{left}} + \mathfrak{C} Q$ is a prime ideal of Q. We will denote it by $\mathfrak{q}_{\mathfrak{C}}^{\mathrm{left}}$.

Corollary 11.2.5. — *We have $Q/\mathfrak{q}_{\mathfrak{C}}^{\mathrm{left}} \simeq P/\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}$. Moreover, $\mathfrak{q}_{\mathfrak{C}}^{\mathrm{left}}$ is the unique prime ideal of Q lying over $\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}$ and contained in $\bar{\mathfrak{q}}$.*

Proof. — The first statement is immediate and the second one follows from the first one. □

Choices, notation. *From now on, and until the end of this Part, we fix a prime ideal $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$ of R lying over $\mathfrak{q}_{\mathfrak{C}}^{\mathrm{left}}$ and contained in $\bar{\mathfrak{r}}_{\mathfrak{C}}$. We put*

$$\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}} = k_R(\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}).$$

The decomposition (respectively inertia) group of $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$ is denoted by $D_{\mathfrak{C}}^{\mathrm{left}}$ (respectively $I_{\mathfrak{C}}^{\mathrm{left}}$).

Whenever $\mathfrak{C} = 0$ (respectively $\mathfrak{C} = \mathfrak{C}_c$ with $c \in \mathscr{C}$), the objects $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$, $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}$, $D_{\mathfrak{C}}^{\mathrm{left}}$ and $I_{\mathfrak{C}}^{\mathrm{left}}$ are denoted respectively by $\mathfrak{r}^{\mathrm{left}}$, $\mathbf{K}^{\mathrm{left}}$, D^{left} and I^{left} (respectively $\mathfrak{r}_c^{\mathrm{left}}$, $\mathbf{K}_c^{\mathrm{left}}$, D_c^{left} and I_c^{left}).

Remark 11.2.6. — It has been shown in Corollary 9.2.5 that, if $\bar{\mathfrak{q}}_\star$ is a prime ideal of Q lying over $\bar{\mathfrak{p}}$, then $Q/\bar{\mathfrak{q}}_\star \simeq P/\bar{\mathfrak{p}}$. Even though $Q/\mathfrak{q}^{\mathrm{left}} \simeq P/\mathfrak{p}^{\mathrm{left}}$, we will see in Chapter 19 that this cannot be extended in general to other prime ideals of Q lying over $\mathfrak{p}^{\mathrm{left}}$: indeed, if W has type B_2, then there exists a prime ideal $\mathfrak{q}_\star^{\mathrm{left}}$ of Q lying over $\mathfrak{p}^{\mathrm{left}}$ such that $P/\mathfrak{p}^{\mathrm{left}}$ is a proper subring of $Q/\mathfrak{q}_\star^{\mathrm{left}}$ (see Lemma 19.7.12(c)). So, in general, $\mathbf{K}^{\mathrm{left}} \subsetneq \mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}$, where $\mathfrak{C} = 0$. ■

Corollary 11.2.7. — *We have* $I_{\mathfrak{C}}^{\text{left}} \subset D_{\mathfrak{C}}^{\text{left}} \subset H$. *If moreover* $\mathfrak{r}_{\mathfrak{C}}^{\text{left}}$ *contains* $\mathfrak{r}^{\text{left}}$, *then* $I^{\text{left}} \subset I_{\mathfrak{C}}^{\text{left}}$.

Proof. — By Corollary 11.2.5 and Theorem B.2.6, $\bar{I}_{\mathfrak{C}}^{\text{left}} \subset H$ and $k_P(\mathfrak{p}_{\mathfrak{C}}^{\text{left}}) = k_Q(\mathfrak{q}_{\mathfrak{C}}^{\text{left}})$. So, $(D_{\mathfrak{C}}^{\text{left}} \cap H)/(I_{\mathfrak{C}}^{\text{left}} \cap H) \simeq D_{\mathfrak{C}}^{\text{left}}/I_{\mathfrak{C}}^{\text{left}}$, and the first sequence of inclusions follows.

The last inclusion is obvious. □

We will prove that $\mathfrak{r}_{\mathfrak{C}}^{\text{left}}$ determines $\bar{\mathfrak{r}}_{\mathfrak{C}}$: for this, we will use the $\mathbb{Z}$-grading on R defined in §5.3. Set

$$R_{<0} = \bigoplus_{i<0} R^{\mathbb{Z}}[i] \quad \text{and} \quad R_{>0} = \bigoplus_{i>0} R^{\mathbb{Z}}[i].$$

Then:

Proposition 11.2.8. — $\bar{\mathfrak{r}}_{\mathfrak{C}} = \mathfrak{r}_{\mathfrak{C}}^{\text{left}} + \langle R_{>0}, R_{<0}\rangle$.

Proof. — Let I denote the ideal of R generated by $R_{>0}$ and $R_{<0}$. The ideal $\bar{\mathfrak{p}}_{\mathfrak{C}}$ of P is $\mathbb{Z}$-homogeneous (it is not necessarily $(\mathbb{N}\times\mathbb{N})$-homogeneous) so the ideal $\bar{\mathfrak{r}}_{\mathfrak{C}}$ of R is also $\mathbb{Z}$-homogeneous (see Corollary C.2.10). The extension $R/\bar{\mathfrak{r}}_{\mathfrak{C}}$ of $P/\bar{\mathfrak{p}}_{\mathfrak{C}}$ is integral and, since $P/\bar{\mathfrak{p}}_{\mathfrak{C}}$ has its $\mathbb{Z}$-grading concentrated in degree 0, it follows that the $\mathbb{Z}$-grading of $R/\bar{\mathfrak{r}}_{\mathfrak{C}}$ is concentrated in degree 0 (see Proposition C.2.4). In particular, $I \subset \bar{\mathfrak{r}}_{\mathfrak{C}}$ and so

$$\mathfrak{r}_{\mathfrak{C}}^{\text{left}} + I \subset \bar{\mathfrak{r}}_{\mathfrak{C}}.$$

Moreover, $(\mathfrak{r}_{\mathfrak{C}}^{\text{left}} + I) \cap P$ contains $\mathfrak{C}$ and $\bar{\mathfrak{p}} = \langle P_{<0}, P_{>0}\rangle$, so

$$\bar{\mathfrak{p}}_{\mathfrak{C}} \subset (\mathfrak{r}_{\mathfrak{C}}^{\text{left}} + I) \cap P.$$

It is now sufficient to show that $\mathfrak{r}_{\mathfrak{C}}^{\text{left}} + I$ is a prime ideal.

Set $I_0 = I \cap R_0$. Then the natural map $R_0 \hookrightarrow R \twoheadrightarrow R/I$ induces an isomorphism $R_0/I_0 \xrightarrow{\sim} R/I$. Consequently, $R/(\mathfrak{r}_{\mathfrak{C}}^{\text{left}} + I)$ is isomorphic to $R_0/((\mathfrak{r}_{\mathfrak{C}}^{\text{left}} + I) \cap R_0)$. So we only need to prove that $(\mathfrak{r}_{\mathfrak{C}}^{\text{left}} + I) \cap R_0$ is a prime ideal of R_0. In fact, we will prove that $(\mathfrak{r}_{\mathfrak{C}}^{\text{left}} + I) \cap R_0 = \mathfrak{r}_{\mathfrak{C}}^{\text{left}} \cap R_0$, and this will conclude the proof.

First of all, note that, since $\mathfrak{r}_{\mathfrak{C}}^{\text{left}}$ and I are $\mathbb{Z}$-homogeneous, we have $(\mathfrak{r}_{\mathfrak{C}}^{\text{left}} + I) \cap R_0 = (\mathfrak{r}_{\mathfrak{C}}^{\text{left}} \cap R_0) + I_0$. So it is sufficient to prove that

$$(*) \qquad I_0 \subset \mathfrak{r}_{\mathfrak{C}}^{\text{left}}.$$

Since $R/\mathfrak{r}_{\mathfrak{C}}^{\text{left}}$ is an integral extension of $P/\mathfrak{p}_{\mathfrak{C}}^{\text{left}}$ and $P/\mathfrak{p}_{\mathfrak{C}}^{\text{left}}$ is $\mathbb{N}$-graded, we deduce that $R/\mathfrak{r}_{\mathfrak{C}}^{\text{left}}$ is $\mathbb{N}$-graded (see Proposition C.2.4). So $R_{<0} \subset \mathfrak{r}_{\mathfrak{C}}^{\text{left}}$ and $I_0 = R_0 \cap (R \cdot R_{>0}) = R_0 \cap (R \cdot R_{<0}) = R_0 \cap (R_{<0} \cdot R_{>0}) \subset \mathfrak{r}_{\mathfrak{C}}^{\text{left}}$. □

Corollary 11.2.9. — $D_{\mathfrak{C}}^{\text{left}} \subset \bar{D}_{\mathfrak{C}}$.

Proof. — This follows immediately from Proposition 11.2.8 and from the fact that $R_{<0}$ and $R_{>0}$ are G-stable (see Proposition 5.3.1). $\square$

Remark 11.2.10. — The algebraic proof of Proposition 11.2.8 given here is in fact the translation of a geometric fact, as will be explained in Chapter 14. ■

Proposition 11.2.11. — *Let $\mathfrak{z}_*^{\mathrm{left}}$ be a prime ideal of Z lying over $\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}$. Then there exists a unique prime ideal of Z lying over $\bar{\mathfrak{p}}_{\mathfrak{C}}$ and containing $\mathfrak{z}_*^{\mathrm{left}}$: it is equal to $\mathfrak{z}_*^{\mathrm{left}} + \langle Z_{<0}, Z_{>0} \rangle$.*

Proof. — Since Z is integral over P, the proof of Proposition 11.2.8 can be applied word by word in this situation, and provides the same conclusion. $\square$

Proposition 11.2.11 provides a surjective map

$$\begin{array}{rccc} \lim_{\mathrm{left}} : & \Upsilon^{-1}(\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}) & \longrightarrow & \Upsilon^{-1}(\bar{\mathfrak{p}}_{\mathfrak{C}}) \\ & \mathfrak{z}_*^{\mathrm{left}} & \longmapsto & \mathfrak{z}_*^{\mathrm{left}} + \langle Z_{<0}, Z_{>0} \rangle. \end{array} \tag{11.2.12}$$

The notation $\lim_{\mathrm{left}}$ will be justified in Chapter 14.

11.3. Left cells

11.3.A. Definitions. — Recall the definitions given in the preamble of §III.

Definition 11.3.1. — *A **Calogero-Moser left $\mathfrak{C}$-cell** is a Calogero-Moser $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$-cell. Given $c \in \mathscr{C}$, a **Calogero-Moser left c-cell** is a Calogero-Moser $\mathfrak{r}_c^{\mathrm{left}}$-cell. A **generic Calogero-Moser left cell** is a Calogero-Moser $\mathfrak{r}^{\mathrm{left}}$-cell.*

The set of Calogero-Moser left $\mathfrak{C}$-cells is denoted by ${}^{\mathrm{CM}}\mathrm{Cell}_L^{\mathfrak{C}}(W)$. When $\mathfrak{C} = 0$ (respectively $\mathfrak{C} = \mathfrak{C}_c$, with $c \in \mathscr{C}$), this set is denoted by ${}^{\mathrm{CM}}\mathrm{Cell}_L(W)$ (respectively ${}^{\mathrm{CM}}\mathrm{Cell}_L^c(W)$).

As usual, the notion of Calogero-Moser left $\mathfrak{C}$-cell depends on the choice of the prime ideal $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$. The next proposition is immediate.

Proposition 11.3.2. — *If $\mathfrak{C}'$ is a prime ideal of $\mathbf{k}[\mathscr{C}]$ contained in $\mathfrak{C}$ and if $\mathfrak{r}_{\mathfrak{C}'}^{\mathrm{left}}$ is contained in $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$, then every Calogero-Moser left $\mathfrak{C}$-cell is a union of Calogero-Moser left $\mathfrak{C}'$-cells.*

Also, since $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}} \subset \bar{\mathfrak{r}}_{\mathfrak{C}}$, we have the following result.

Proposition 11.3.3. — *Every Calogero-Moser two-sided $\mathfrak{C}$-cell is a union of Calogero-Moser left $\mathfrak{C}$-cells.*

Finally, let $\widetilde{\mathfrak{C}}$ be the maximal homogeneous ideal contained in $\mathfrak{C}$ (i.e. $\widetilde{\mathfrak{C}} = \bigoplus_{i \geqslant 0} \mathfrak{C} \cap \mathbf{k}[\mathscr{C}]^{\mathbb{N}}[i]$). Then $\widetilde{\mathfrak{C}}$ is a prime ideal of $\mathbf{k}[\mathscr{C}]$ (see Lemma C.2.9). Let $\mathfrak{r}^{\mathrm{left}}_{\widetilde{\mathfrak{C}}}$ denote the maximal homogeneous ideal contained in $\mathfrak{r}^{\mathrm{left}}_{\mathfrak{C}}$: it is a prime ideal of R lying over $\mathfrak{q}^{\mathrm{left}}_{\widetilde{\mathfrak{C}}}$ (see Corollary C.2.12). We deduce from Proposition 6.1.5 the following result.

Proposition 11.3.4. — *We have $I^{\mathrm{left}}_{\mathfrak{C}} = I^{\mathrm{left}}_{\widetilde{\mathfrak{C}}}$. In particular, the Calogero-Moser left $\mathfrak{C}$-cells and the Calogero-Moser left $\widetilde{\mathfrak{C}}$-cells coincide.*

11.3.B. Left and two-sided cells. — We fix here a Calogero-Moser two-sided $\mathfrak{C}$-cell Γ as well as a Calogero-Moser left $\mathfrak{C}$-cell C contained in Γ. Since $\bar{D}_{\mathfrak{C}}$ stabilizes Γ (see Theorem 10.2.7(a)) and since $D^{\mathrm{left}}_{\mathfrak{C}} \subset \bar{D}_{\mathfrak{C}}$ (see Corollary 11.2.9), the group $D^{\mathrm{left}}_{\mathfrak{C}}$ stabilizes Γ (and permutes the left cells contained in Γ). Set

$$C^D = \bigcup_{d \in D^{\mathrm{left}}_{\mathfrak{C}}} d(C).$$

Let $w \in C^D$. We set $\bar{\mathfrak{q}}_{\mathfrak{C}}(\Gamma) = w^{-1}(\bar{\mathfrak{r}}_{\mathfrak{C}}) \cap Q$ and $\mathfrak{q}^{\mathrm{left}}_{\mathfrak{C}}(C^D) = w^{-1}(\mathfrak{r}^{\mathrm{left}}_{\mathfrak{C}}) \cap Q$. We also set $\bar{\mathfrak{z}}_{\mathfrak{C}}(\Gamma) = \mathrm{cop}^{-1}(\bar{\mathfrak{q}}_{\mathfrak{C}}(\Gamma))$ and $\mathfrak{z}^{\mathrm{left}}_{\mathfrak{C}}(C^D) = \mathrm{cop}^{-1}(\mathfrak{q}^{\mathrm{left}}_{\mathfrak{C}}(C^D))$. It follows from Proposition B.3.5 that $\bar{\mathfrak{z}}_{\mathfrak{C}}(\Gamma)$ depends only on Γ and not on the choice of C or w, whereas $\mathfrak{z}^{\mathrm{left}}_{\mathfrak{C}}(C^D)$ depends only on C^D and not on the choice of w. We set

$$\deg_{\mathfrak{C}}(C^D) = [k_Z(\mathfrak{z}^{\mathrm{left}}_{\mathfrak{C}}(C^D)) : k_P(\mathfrak{p}^{\mathrm{left}}_{\mathfrak{C}})].$$

We sometimes use the notation $\mathfrak{z}^{\mathrm{left}}_{\mathfrak{C}}(C)$ or $\deg_{\mathfrak{C}}(C)$ instead of $\mathfrak{z}^{\mathrm{left}}_{\mathfrak{C}}(C^D)$ or $\deg_{\mathfrak{C}}(C^D)$.

Proposition 11.3.5. — *Let $w \in C^D$. Then:*

(a) $\bar{\mathfrak{z}}_{\mathfrak{C}}(\Gamma) = \lim_{\mathrm{left}}(\mathfrak{z}^{\mathrm{left}}_{\mathfrak{C}}(C^D))$.

(b) $\deg_{\mathfrak{C}}(C^D) = \dfrac{|C^D|}{|C|} = \dfrac{|D^{\mathrm{left}}_{\mathfrak{C}}|}{|(D^{\mathrm{left}}_{\mathfrak{C}} \cap {}^w H) I^{\mathrm{left}}_{\mathfrak{C}}|}$.

(c) *The map $D^{\mathrm{left}}_{\mathfrak{C}} \backslash \Gamma \longrightarrow \lim^{-1}_{\mathrm{left}}(\bar{\mathfrak{z}}_{\mathfrak{C}}(\Gamma))$, $C^D \longmapsto \mathfrak{z}^{\mathrm{left}}_{\mathfrak{C}}(C^D)$ is bijective.*

Proof. — (a) Note that $\bar{\mathfrak{z}}_{\mathfrak{C}}(\Gamma) \in \Upsilon^{-1}(\bar{\mathfrak{p}}_{\mathfrak{C}})$, $\mathfrak{z}^{\mathrm{left}}_{\mathfrak{C}}(C^D) \in \Upsilon^{-1}(\mathfrak{p}^{\mathrm{left}}_{\mathfrak{C}})$ and $\mathfrak{z}^{\mathrm{left}}_{\mathfrak{C}}(C^D) \subset \bar{\mathfrak{z}}_{\mathfrak{C}}(\Gamma)$, whence the result by Proposition 11.2.11.

(b) Through the action of w, the extension $k_R(w^{-1}(\mathfrak{r}^{\mathrm{left}}_{\mathfrak{C}}))/k_P(\mathfrak{p}^{\mathrm{left}}_{\mathfrak{C}})$ is Galois with group ${}^{w^{-1}}D^{\mathrm{left}}_{\mathfrak{C}} / {}^{w^{-1}}I^{\mathrm{left}}_{\mathfrak{C}}$ whereas the extension $k_R(w^{-1}(\mathfrak{r}^{\mathrm{left}}_{\mathfrak{C}}))/k_Q(\mathfrak{q}^{\mathrm{left}}_{\mathfrak{C}}(C^D))$ is Galois with group $({}^{w^{-1}}D^{\mathrm{left}}_{\mathfrak{C}} \cap H)/({}^{w^{-1}}I^{\mathrm{left}}_{\mathfrak{C}} \cap H)$. Hence

$$\deg_{\mathfrak{C}}(C^D) = \frac{|D^{\mathrm{left}}_{\mathfrak{C}}|}{|(D^{\mathrm{left}}_{\mathfrak{C}} \cap {}^w H) I^{\mathrm{left}}_{\mathfrak{C}}|}.$$

Moreover, $|C^D|/|C|$ is equal to the index of the stabilizer of C in $D^{\mathrm{left}}_{\mathfrak{C}}$: but this stabilizer is exactly $(D^{\mathrm{left}}_{\mathfrak{C}} \cap {}^w H) I^{\mathrm{left}}_{\mathfrak{C}}$.

(c) follows essentially from the commutativity of the diagram

$$\begin{tikzcd} D_{\mathfrak{C}}^{\mathrm{left}}\backslash G/H \arrow[r, "\sim"] \arrow[d] & \Upsilon^{-1}(\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}) \arrow[d, "\lim_{\mathrm{left}}"] \\ \bar{D}_{\mathfrak{C}}\backslash G/H \arrow[r, "\sim"] & \Upsilon^{-1}(\bar{\mathfrak{p}}_{\mathfrak{C}}), \end{tikzcd}$$

where the left vertical arrow is the canonical map (since $D_{\mathfrak{C}}^{\mathrm{left}} \subset \bar{D}_{\mathfrak{C}}$), and the horizontal bijective maps are given by Proposition B.3.5. The additional ingredient is the equality $\bar{D}_{\mathfrak{C}}\backslash G/H = \bar{I}_{\mathfrak{C}}\backslash G/H$ (see (10.2.4)). ☐

11.3.C. Left cells and simple modules. — By Example 6.4.7, we have $\mathfrak{z}_{\mathrm{sing}} \cap P \not\subset \mathfrak{p}^{\mathrm{left}}$. Consequently, the results of § 6.4 can be applied. Let us recall here some consequences (see Theorem 6.4.6):

***Theorem 11.3.6.* —** (a) *The algebra* $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$ *is split and its simple modules have dimension* $|W|$.

(b) *Every block* $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$ *admits a unique simple module.*

Given $C \in {}^{\mathrm{CM}}\mathrm{Cell}_L^{\mathfrak{C}}(W)$, we denote by $L_{\mathfrak{C}}^{\mathrm{left}}(C)$ the unique simple $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$-module belonging to the block of $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$ associated with C. When $\mathfrak{C} = 0$ (respectively $\mathfrak{C} = \mathfrak{C}_c$, for some $c \in \mathscr{C}$), the module $L_{\mathfrak{C}}^{\mathrm{left}}(C)$ is denoted by $L^{\mathrm{left}}(C)$ (respectively $L_c^{\mathrm{left}}(C)$).

The decomposition group $D_{\mathfrak{C}}^{\mathrm{left}}$ acts on the commutative ring $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}Z^{\mathrm{left}}$ (the action factors through a faithful action of $D_{\mathfrak{C}}^{\mathrm{left}}/I_{\mathfrak{C}}^{\mathrm{left}}$) and

(11.3.7) $$\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}Z^{\mathrm{left}} = \bigl(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}Z^{\mathrm{left}}\bigr)^{D_{\mathfrak{C}}^{\mathrm{left}}}.$$

So the primitive idempotents of the left-hand side (which are in one-to-one correspondence with the simple $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}Z^{\mathrm{left}}$-modules) are in one-to-one correspondence with the $D_{\mathfrak{C}}^{\mathrm{left}}$-orbits of primitive idempotents of $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}Z^{\mathrm{left}}$. Thus we get a bijection

(11.3.8) $$\mathrm{Irr}(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}Z^{\mathrm{left}}) \stackrel{\sim}{\longleftrightarrow} \Bigl(\mathrm{Irr}(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}Z^{\mathrm{left}})\Bigr)/D_{\mathfrak{C}}^{\mathrm{left}}.$$

Similarly, the decomposition group $D_{\mathfrak{C}}^{\mathrm{left}}$ acts on the commutative ring $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$ and

(11.3.9) $$\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}} = \bigl(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}\bigr)^{D_{\mathfrak{C}}^{\mathrm{left}}}.$$

So, through the bijection (11.3.8) and the Morita equivalence of Theorem 11.1.1, we get another bijective map

(11.3.10) $$\mathrm{Irr}(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}) \stackrel{\sim}{\longleftrightarrow} \Bigl(\mathrm{Irr}(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}})\Bigr)/D_{\mathfrak{C}}^{\mathrm{left}}.$$

Using the one-to-one correspondence $D_{\mathfrak{C}}^{\mathrm{left}}\backslash W \xrightarrow{\sim} D_{\mathfrak{C}}^{\mathrm{left}}\backslash G/H \xrightarrow{\sim} \Upsilon^{-1}(\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}})$, $C^D \mapsto \mathfrak{z}_{\mathfrak{C}}^{\mathrm{left}}(C^D)$ given by Proposition B.3.5, we obtain the following commutative diagram of bijective maps.

$$\begin{tikzcd} D_{\mathfrak{C}}^{\mathrm{left}}\backslash W \arrow[r, leftrightarrow] & \Upsilon^{-1}(\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}) \arrow[r, leftrightarrow] & \mathrm{Irr}(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}} Z^{\mathrm{left}}) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathrm{Irr}(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}} \mathbf{H}^{\mathrm{left}}) \arrow[d, leftrightarrow] \\ & & \Big(\mathrm{Irr}(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}} Z^{\mathrm{left}})\Big)/D_{\mathfrak{C}}^{\mathrm{left}} \arrow[r, leftrightarrow] & \Big(\mathrm{Irr}(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}} \mathbf{H}^{\mathrm{left}})\Big)/D_{\mathfrak{C}}^{\mathrm{left}} \end{tikzcd} \tag{11.3.11}$$

Proposition 11.3.12. — *Let C° be a $D_{\mathfrak{C}}^{\mathrm{left}}$-orbit in W and let $\mathfrak{z} = \mathfrak{z}_{\mathfrak{C}}^{\mathrm{left}}(C^\circ)$. We have*

$$\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}} \otimes_{\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}} L_{\mathfrak{C}}^{\mathrm{left}}(\mathfrak{z}) \simeq \bigoplus_{\substack{C \in {}^{\mathrm{CM}}\mathrm{Cell}_L^{\mathfrak{C}}(W) \\ C \subset C^\circ}} L_{\mathfrak{C}}^{\mathrm{left}}(C).$$

Proof. — First, it is clear that if C is a Calogero-Moser left $\mathfrak{C}$-cell such that $L_{\mathfrak{C}}^{\mathrm{left}}(C)$ is contained in the $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$-module $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}} \otimes_{\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}} L_{\mathfrak{C}}^{\mathrm{left}}(\mathfrak{z})$, then $\mathfrak{z}_{\mathfrak{C}}^{\mathrm{left}}(C^D) = \mathfrak{z}$, hence $C^D \subset C^\circ$. It follows that $C \subset C^\circ$.

Since the field extension $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}/\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}$ is separable, the $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$-module $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}} \otimes_{\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}} L_{\mathfrak{C}}^{\mathrm{left}}(\mathfrak{z})$ is semisimple. As it is stable under the action of $D_{\mathfrak{C}}^{\mathrm{left}}$, it is a multiple of the right-hand side of the formula. By Theorem 11.3.6, the proof of the Proposition can be reduced to the proof of the analogous statement for the algebra $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}} Z^{\mathrm{left}}$. Since this algebra is commutative, it follows that $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}} \otimes_{\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}} L_{\mathfrak{C}}^{\mathrm{left}}(\mathfrak{z})$ is multiplicity-free. $\square$

11.4. Back to cellular characters

11.4.A. Left cells and cellular characters. — Recall that the simple $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$-modules are parametrized by ${}^{\mathrm{CM}}\mathrm{Cell}_{\mathfrak{C}}^{\mathrm{left}}(W)$. There exists a unique family of non-negative integers $(\mathrm{mult}_{C,\chi}^{\mathrm{CM}})_{C \in {}^{\mathrm{CM}}\mathrm{Cell}_{\mathfrak{C}}^{\mathrm{left}}(W), \chi \in \mathrm{Irr}(W)}$ such that

$$[\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\Delta(\chi)]_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}} = \sum_{C \in {}^{\mathrm{CM}}\mathrm{Cell}_L^{\mathfrak{C}}(W)} \mathrm{mult}_{C,\chi}^{\mathrm{CM}} \cdot [L_{\mathfrak{C}}^{\mathrm{left}}(C)]_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}}$$

for all $\chi \in \mathrm{Irr}(W)$. They can be used to define the cellular characters, since

$$\mathrm{mult}_{C,\chi}^{\mathrm{CM}} = \mathrm{mult}_{\mathfrak{z}_{\mathfrak{C}}^{\mathrm{left}}(C),\chi}^{\mathrm{CM}}. \tag{11.4.1}$$

Proof. — By construction, we have

$$[\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\Delta(\chi)]_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}} = \sum_{\mathfrak{z} \in \Upsilon^{-1}(\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}})} \mathrm{mult}_{\mathfrak{z},\chi}^{\mathrm{CM}} \cdot [\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}} \otimes_{\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}} L_{\mathfrak{C}}^{\mathrm{left}}(\mathfrak{z})]_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}}$$

for all $\chi \in \mathrm{Irr}(W)$. The result follows now from Proposition 11.3.12. $\square$

We can also prove the following family of identities, which are similar to identities for the Kazhdan-Lusztig multiplicities $\mathrm{mult}^{\mathrm{KL}}_{C,\chi}$ (see Lemma 8.6.7).

Proposition 11.4.2. — *With the above notation, we have:*

(a) *If* $\chi \in \mathrm{Irr}(W)$, *then* $\sum_{C \in {}^{\mathrm{CM}}\mathrm{Cell}^{\mathfrak{C}}_L(W)} \mathrm{mult}^{\mathrm{CM}}_{C,\chi} = \chi(1)$.

(b) *If* $C \in {}^{\mathrm{CM}}\mathrm{Cell}^{\mathfrak{C}}_L(W)$, *then* $\sum_{\chi \in \mathrm{Irr}(W)} \mathrm{mult}^{\mathrm{CM}}_{C,\chi}\ \chi(1) = |C|$.

(c) *If* $C \in {}^{\mathrm{CM}}\mathrm{Cell}^{\mathfrak{C}}_L(W)$, *if* Γ *is the unique Calogero-Moser two-sided* $\mathfrak{C}$*-cell containing* C *and if* $\chi \in \mathrm{Irr}(W)$ *is such that* $\mathrm{mult}^{\mathrm{CM}}_{C,\chi} \neq 0$, *then* $\chi \in \mathrm{Irr}^{\mathrm{CM}}_{\Gamma}(W)$.

Proof. — (a) follows from the computation of the dimension of Verma modules (see (7.1.1)).

Let us now show (b). First of all, note that, thanks to the Morita equivalence of Theorem 4.3.7, we have

$$[\,\mathbf{MH}e\,]_{\mathbf{MH}} = \sum_{w \in W} [\,\mathscr{L}_w\,]_{\mathbf{MH}}.$$

By applying $\mathrm{dec}^{\mathrm{left}}_{\mathfrak{C}}$ to this equality, we deduce that

$$[\,\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\mathbf{H}^{\mathrm{left}}e\,]_{\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\mathbf{H}^{\mathrm{left}}} = \sum_{C \in {}^{\mathrm{CM}}\mathrm{Cell}^{\mathfrak{C}}_L(W)} |C| \cdot [\,L^{\mathrm{left}}_{\mathfrak{C}}(C)]_{\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\mathbf{H}^{\mathrm{left}}}.$$

Since $\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\mathbf{H}^{\mathrm{left}}e = \mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\Delta(\mathrm{co})$, we have the following equality

(11.4.3) $$[\,\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\mathbf{H}^{\mathrm{left}}e\,]_{\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\mathbf{H}^{\mathrm{left}}} = \sum_{\chi \in \mathrm{Irr}(W)} \chi(1) \cdot [\,\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\Delta(\chi)]_{\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\mathbf{H}^{\mathrm{left}}}$$

and (b) follows.

(c) is immediate, as the reduction modulo $\bar{\mathfrak{p}}$ of the Verma module is the corresponding baby Verma module, and so is indecomposable as a $\bar{\mathbf{K}}_{\mathfrak{C}}\bar{\mathbf{H}}$-module. ☐

Given C a Calogero-Moser left $\mathfrak{C}$-cell, we set

(11.4.4) $$[\,C\,]^{\mathrm{CM}}_{\mathfrak{C}} = \sum_{\chi \in \mathrm{Irr}(W)} \mathrm{mult}^{\mathrm{CM}}_{C,\chi} \cdot \chi.$$

In other words,

(11.4.5) $$[\,C\,]^{\mathrm{CM}}_{\mathfrak{C}} = \gamma_{\mathfrak{z}^{\mathrm{CM}}_{\mathfrak{C}}(C^D)}.$$

Taking Proposition 11.4.2(c) into account, the set of irreducible characters appearing with a non-zero multiplicity in a Calogero-Moser $\mathfrak{C}$-cellular character is contained in a unique Calogero-Moser $\mathfrak{C}$-family $\mathscr{F}$: we will say that the Calogero-Moser $\mathfrak{C}$-cellular character *belongs to* $\mathscr{F}$.

If $d \in D^{\mathrm{left}}_{\mathfrak{C}}$ and C is a Calogero-Moser left $\mathfrak{C}$-cell, then $d(C)$ is also a Calogero-Moser left $\mathfrak{C}$-cell. The equality (11.4.5) shows that the Calogero-Moser $\mathfrak{C}$-cellular characters associated with C and $d(C)$ coincide:

Corollary 11.4.6. — *If $d \in D_{\mathfrak{C}}^{\mathrm{left}}$ and C is a Calogero-Moser left $\mathfrak{C}$-cell, then*

$$[\,d(C)\,]_{\mathfrak{C}}^{\mathrm{CM}} = [\,C\,]_{\mathfrak{C}}^{\mathrm{CM}}.$$

Remark 11.4.7. — The previous corollary 11.4.6 shows in particular that $d(C)$ is contained in the same Calogero-Moser two-sided $\mathfrak{C}$-cell as C, which has already been proven by a different argument (see the beginning of §11.3.B). ■

Corollary 11.4.8. — *Let C be a Calogero-Moser left $\mathfrak{C}$-cell and let $\mathfrak{z} = \mathfrak{z}_{\mathfrak{C}}^{\mathrm{left}}(C^D)$. Then $|C| = \mathrm{Length}_{Z_{\mathfrak{z}}}(Z/\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}} Z)_{\mathfrak{z}}$.*

Proof. — We have $[P_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}e] = [P_{\mathfrak{C}}^{\mathrm{left}}\Delta(\mathrm{co})] = \sum_{\chi} \chi(1)[P_{\mathfrak{C}}^{\mathrm{left}}\Delta(\chi)]$. Via the Morita equivalence of Corollary 6.4.4, the module $Z/\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}} Z$ corresponds to the module $P_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}e$, hence

$$\mathrm{Length}_{Z_{\mathfrak{z}}}(Z/\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}} Z)_{\mathfrak{z}} = \sum_{\chi \in \mathrm{Irr}(W)} \mathrm{mult}_{C,\chi}^{\mathrm{CM}} \cdot \chi(1) = |C|$$

using Proposition 11.4.2(b). □

Corollary 11.4.9. — *Assume that all the reflections of W have order 2 and let $\tau_0 = (-1, 1, \varepsilon) \in \mathbf{k}^\times \times \mathbf{k}^\times \times W^\wedge$. Let $\mathfrak{z}$ be a prime ideal of Z lying over $\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}$ and let C and C_ε be two Calogero-Moser left $\mathfrak{C}$-cells such that $\mathfrak{z}_{\mathfrak{C}}^{\mathrm{left}}(C^D) = \mathfrak{z}$ and $\mathfrak{z}_{\mathfrak{C}}^{\mathrm{left}}(C_\varepsilon^D) = \tau_0(\mathfrak{z})$. Then*

$$[\,C_\varepsilon\,]_{\mathfrak{C}}^{\mathrm{CM}} = \varepsilon \cdot [\,C\,]_{\mathfrak{C}}^{\mathrm{CM}}.$$

If moreover $w_0 = -\mathrm{Id}_V \in W$, then we can take $C_\varepsilon = C w_0$, hence

$$[\,C w_0\,]_{\mathfrak{C}}^{\mathrm{CM}} = \varepsilon \cdot [\,C\,]_{\mathfrak{C}}^{\mathrm{CM}}.$$

Proof. — The first statement follows from the fact that ${}^{\tau_0}\Delta(\chi) \simeq \Delta(\chi\varepsilon)$ whereas the second can be proven as in Corollary 10.2.9. □

Remark 11.4.10. — Corollary 11.4.8 is interesting in that it provides a numerical invariant (the cardinality) of an object (a left cell) which is defined using the Galois extension $\mathbf{M}/\mathbf{K}$ (and the ring R) in terms of an invariant which is computable inside the extension $\mathbf{L}/\mathbf{K}$ (and the ring Z). ■

11.4.B. Cellular characters and projective covers. — Note that $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\bar{\mathbf{H}}^{-}$ is a $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}$-subalgebra of $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$ of dimension $|W|^2$, whose Grothendieck group is identified with $\mathbb{Z}\operatorname{Irr}(W) = K_0(\mathbf{k}W)$.

Given C a Calogero-Moser left $\mathfrak{C}$-cell, we denote by $\mathscr{P}_{\mathfrak{C}}^{\mathrm{left}}(C)$ a projective cover of the simple $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$-module $L_{\mathfrak{C}}^{\mathrm{left}}(C)$. We denote by $\operatorname{Soc}(M)$ the largest semisimple submodule (the socle) of a module M.

Proposition 11.4.11. — *We have*

$$[\operatorname{Soc}(\operatorname{Res}_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\bar{\mathbf{H}}^{-}}^{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}} \mathscr{P}_{\mathfrak{C}}^{\mathrm{left}}(C))]_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\otimes\bar{\mathbf{H}}^{-}} = \sum_{\chi\in\operatorname{Irr}(W)} \operatorname{mult}_{C,\chi}^{\mathrm{CM}} \cdot \chi = [\,C\,]_{\mathfrak{C}}^{\mathrm{CM}}.$$

Proof. — Let $\chi \in \operatorname{Irr}(W)$. Since the algebra $\mathbf{H}$ is symmetric (see (4.4.5)), $\mathscr{P}_{\mathfrak{C}}^{\mathrm{left}}(C)$ is also an injective hull of $L_{\mathfrak{C}}^{\mathrm{left}}(C)$. So

$$\operatorname{mult}_{C,\chi}^{\mathrm{CM}} = \dim_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}} \operatorname{Hom}_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}}\big(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\Delta(\chi), \mathscr{P}_{\mathfrak{C}}^{\mathrm{left}}(C)\big).$$

As $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\Delta(\chi) = \operatorname{Ind}_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\bar{\mathbf{H}}^{-}}^{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}}(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}} \otimes E_\chi)$, we deduce that

$$\operatorname{mult}_{C,\chi}^{\mathrm{CM}} = \dim_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}} \operatorname{Hom}_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\bar{\mathbf{H}}^{-}}\big(E_\chi, \operatorname{Res}_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\otimes\bar{\mathbf{H}}^{-}}^{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}} \mathscr{P}_{\mathfrak{C}}^{\mathrm{left}}(C)\big).$$

The result follows. $\square$

11.4.C. Cellular characters and b-invariant. — The following theorem is an analogue of Theorem 9.4.1 (statements (b) and (c)).

Theorem 11.4.12. — *Let C be a Calogero-Moser left $\mathfrak{C}$-cell. Then there exists a unique irreducible character χ with minimal* **b***-invariant such that $\operatorname{mult}_{C,\chi}^{\mathrm{CM}} \neq 0$. We denote this character by χ_C. The coefficient of $\mathbf{t}^{\mathbf{b}_{\chi_C}}$ in $f_{\chi_C}(\mathbf{t})$ is equal to 1.*

Proof. — Let b_C be the primitive central idempotent of $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$ associated with C. The endomorphism algebra of $b_C\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}e$ is equal to $(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}} \otimes_P Z)b_C$ and this (commutative) algebra is local. This shows that the projective module $b_C\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}e$ admits a unique simple quotient. The proof continues as that of Theorem 9.4.1. $\square$

CHAPTER 12

DECOMPOSITION MATRICES

12.1. The general framework

Let R_1 be a commutative R-algebra which is a domain, and let $\mathfrak{r}_1$ be a prime ideal of R_1. We set $R_2 = R_1/\mathfrak{r}_1$, $K_1 = \mathrm{Frac}(R_1)$ and $K_2 = \mathrm{Frac}(R_2) = k_{R_1}(\mathfrak{r}_1)$. We will say that the pair $(R_1, \mathfrak{r}_1)$ satisfies Property $(\mathscr{D}ec)$ if the three following statements are satisfied (see the Appendix D):

(D1) R_1 is noetherian.

(D2) If $h \in R_1\mathbf{H}$ and if $\mathscr{L}$ is a simple $K_1\mathbf{H}$-module, then the characteristic polynomial of h (for its action on $\mathscr{L}$) has coefficients in R_1 (note that this assumption is automatically satisfied if R_1 is integrally closed).

(D3) The algebras $K_1\mathbf{H}$ and $K_2\mathbf{H}$ are split.

In this context, completely similar to the one of § D.3 (see the Appendix D), the decomposition map

$$\mathrm{dec}_{R_2\mathbf{H}}^{R_1\mathbf{H}} : K_0(K_1\mathbf{H}) \longrightarrow K_0(K_2\mathbf{H})$$

is well-defined (see Proposition D.3.1).

If $\mathfrak{r}_2$ is a prime ideal of R_1 containing $\mathfrak{r}_1$, set $R_3 = R_1/\mathfrak{r}_2 = R_2/(\mathfrak{r}_2/\mathfrak{r}_1)$, $K_3 = \mathrm{Frac}(R_3) = k_{R_1}(\mathfrak{r}_2) = k_{R_2}(\mathfrak{r}_2/\mathfrak{r}_1)$ and assume that $(R_2, \mathfrak{r}_2)$ satisfies $(\mathscr{D}ec)$. Then the maps $\mathrm{dec}_{R_2\mathbf{H}}^{R_1\mathbf{H}}$, $\mathrm{dec}_{R_3\mathbf{H}}^{R_1\mathbf{H}}$ and $\mathrm{dec}_{R_3\mathbf{H}}^{R_2\mathbf{H}}$ are well-defined and, by Corollary D.3.2, the diagram

(12.1.1)
$$\begin{tikzcd}
K_0(K_1\mathbf{H}) \arrow[r, "\mathrm{dec}_{R_2\mathbf{H}}^{R_1\mathbf{H}}"] \arrow[dr, swap, "\mathrm{dec}_{R_3\mathbf{H}}^{R_1\mathbf{H}}"] & K_0(K_2\mathbf{H}) \arrow[d, "\mathrm{dec}_{R_3\mathbf{H}}^{R_2\mathbf{H}}"] \\
& K_0(K_3\mathbf{H})
\end{tikzcd}$$

is commutative.

***Example 12.1.2* (Specialization).** — Let $c \in \mathscr{C}$. Recall that $\mathfrak{q}_c = \mathfrak{p}_c Q$ is prime and let $\mathfrak{r}_c$ be a prime ideal of R lying over $\mathfrak{q}_c$. Let us use the notation of Example 6.2.6 and of §5.1.A. Then $R/\mathfrak{r}_c$ is an R-algebra, with fraction field $\mathbf{M}_c$. As in the proof of Theorem 4.3.7, we deduce from Corollary 6.4.4 an isomorphism of $\mathbf{K}_c$-algebras

$$\mathbf{H}_c \xrightarrow{\sim} \mathrm{Mat}_{|W|}(\mathbf{L}_c)$$

which induces, as in the generic case (see §5.2), an isomorphism of $\mathbf{M}_c$-algebras

$$\mathbf{M}_c\mathbf{H}_c \xrightarrow{\sim} \prod_{d(D_c\cap H)\in D_c/(D_c\cap H)} \mathrm{Mat}_{|W|}(\mathbf{M}_c).$$

So the $\mathbf{M}_c$-algebra $\mathbf{M}_c\mathbf{H}_c$ is split, as well as $\mathbf{MH}$, and its simple modules are indexed by $D_c/(D_c \cap H)$: this last set is in one-to-one correspondence with W (Corollary 5.1.15). So, the decomposition map $\mathrm{dec}_{R_c\mathbf{H}}^{R\mathbf{H}}$ is well-defined, and will be denoted by dec_c. We can moreover identify $K_0(\mathbf{M}_c\mathbf{H}_c)$ with the $\mathbb{Z}$-module $\mathbb{Z}W$ and, through this identification, the diagram

$$\begin{array}{ccc} K_0(\mathbf{MH}) & \xrightarrow{\mathrm{dec}_c} & K_0(\mathbf{M}_c\mathbf{H}_c) \\ \| & & \| \\ \mathbb{Z}W & \xrightarrow[\mathrm{Id}_{\mathbb{Z}W}]{} & \mathbb{Z}W \end{array} \tag{12.1.3}$$

is commutative. This follows from the fact that the Morita equivalence between $\mathbf{H}_c$ and $\mathbf{L}_c$ is the "specialization at c" of the Morita equivalence between $\mathbf{H}$ and $\mathbf{L}$. ■

12.2. Cells and decomposition matrices

Let $\mathfrak{r}$ be a prime ideal of R. We will denote by $D_\mathfrak{r}$ the decomposition group of $\mathfrak{r}$ in G and $I_\mathfrak{r}$ its inertia group. The Galois group G (respectively the decomposition group $D_\mathfrak{r}$) acts naturally on the Grothendieck group $K_0(\mathbf{MH})$ (respectively $K_0(k_R(\mathfrak{r})\mathbf{H})$). Then:

Lemma 12.2.1. — *Assume that the $k_R(\mathfrak{r})$-algebra $k_R(\mathfrak{r})\mathbf{H}$ is split. Then:*

(a) *The decomposition map* $\mathrm{dec}_{(R/\mathfrak{r})\mathbf{H}}^{R\mathbf{H}}$ *is well-defined (it will be denoted by* $\mathrm{dec}_\mathfrak{r} : K_0(\mathbf{MH}) \longrightarrow K_0(k_R(\mathfrak{r})\mathbf{H})$*).*
(b) *The decomposition map* $\mathrm{dec}_\mathfrak{r}$ *is $D_\mathfrak{r}$-equivariant.*
(c) *The group $I_\mathfrak{r}$ acts trivially $K_0(k_R(\mathfrak{r})\mathbf{H})$.*
(d) *If w and w' are in the same Calogero-Moser $\mathfrak{r}$-cell, then* $\mathrm{dec}_\mathfrak{r}(\mathscr{L}_w) = \mathrm{dec}_\mathfrak{r}(\mathscr{L}_{w'})$*.*

Proof. — Since R is integrally closed, saying that the $k_R(\mathfrak{r})$-algebra $k_R(\mathfrak{r})\mathbf{H}$ is split is equivalent to say that $(R,\mathfrak{r})$ satisfies $(\mathscr{D}ec)$. The decomposition maps being computed by reduction of the characteristic polynomials, the statement (b) is immediate. The group $I_\mathfrak{r}$ acting trivially on $k_R(\mathfrak{r})$ by definition, (c) is clear. The statement (d) then follows from (b) and (c) because the Calogero-Moser $\mathfrak{r}$-cells are $I_\mathfrak{r}$-orbits. □

Lemma 12.2.1 says that, when restricted to an $\mathfrak{r}$-block, the decomposition map $\mathrm{dec}_\mathfrak{r}$ has rank 1.

Example 12.2.2. — Set $\mathfrak{p} = \mathfrak{r} \cap P$ and assume in this Example, and only in this Example, that $\mathfrak{z}_{\mathrm{sing}} \cap P \not\subset \mathfrak{p}$. Then Theorem 6.4.6(a) implies that the $k_R(\mathfrak{r})$-algebra $k_R(\mathfrak{r})\mathbf{H}$ is split. Consequently, the decomposition map $\mathrm{dec}_\mathfrak{r} : K_0(\mathbf{MH}) \to K_0(k_R(\mathfrak{r})\mathbf{H})$ is well-defined. Simple the simple modules of $\mathbf{MH}$ have dimension $|W|$, as well as the simple $k_R(\mathfrak{r})\mathbf{H}$-modules, the decomposition map sends the isomorphism class of a simple $\mathbf{MH}$-module on the isomorphism class of a simple $k_R(\mathfrak{r})\mathbf{H}$-module. So $\mathrm{dec}_\mathfrak{r}$ defines a surjective map

$$\mathrm{dec}_\mathfrak{r} : W \longrightarrow \mathrm{Irr}(k_R(\mathfrak{r})\mathbf{H}) \tag{12.2.3}$$

whose fibers are the Calogero-Moser $\mathfrak{r}$-cells (see Lemma 12.2.1). ■

Remark 12.2.4. — The previous Example can be applied in the case where $\mathfrak{r} = \mathfrak{r}_\mathfrak{C}^{\mathrm{left}}$ or $\mathfrak{r}_\mathfrak{C}^{\mathrm{right}}$. ■

12.3. Left, right, two-sided cells and decomposition matrices

In order to define decomposition matrices, one must check that some assumptions are satisfied (see the previous conditions (D1), (D2) and (D3)). It is the aim of the next proposition to check that these assumptions hold in the cases we are interested in:

Proposition 12.3.1. — *Let $\mathfrak{r}$ be a prime ideal of R amongst $\mathfrak{r}_\mathfrak{C}$, $\mathfrak{r}_\mathfrak{C}^{\mathrm{left}}$, $\mathfrak{r}_\mathfrak{C}^{\mathrm{right}}$ or $\bar{\mathfrak{r}}_\mathfrak{C}$. Then:*

(a) *The $k_R(\mathfrak{r})$-algebra $k_R(\mathfrak{r})\mathbf{H}$ is split.*

(b) *Assume here that $\mathfrak{r} \neq \bar{\mathfrak{r}}_\mathfrak{C}$ or $\mathfrak{C} = 0$ or $\mathfrak{C}_c$ for some $c \in \mathscr{C}$. If $\mathscr{L}$ is a simple $k_R(\mathfrak{r})\mathbf{H}$-module and if $h \in \mathbf{H}/\mathfrak{r}\mathbf{H} = (R/\mathfrak{r})\mathbf{H}$, then the characteristic polynomial of h (for its action on $\mathscr{L}$) has coefficients in $R/\mathfrak{r}$.*

Proof. — (a) has been proven for $\mathfrak{r} = \mathfrak{r}_c$ in Example 12.1.2, for $\mathfrak{r} = \mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$ in Theorem 11.3.6(a) and for $\mathfrak{r} = \bar{\mathfrak{r}}_{\mathfrak{C}}$ in Proposition 9.1.3.

Let us now show (b). First of all, if $\mathfrak{r} = \mathfrak{r}_{\mathfrak{C}}$ or $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$ or $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{right}}$, then the images in the Grothendieck group $K_0(k_R(\mathfrak{r})\mathbf{H})$ of simple $k_R(\mathfrak{r})\mathbf{H}$-modules are the images of simple $\mathbf{MH}$-modules through the decomposition map (see Example 12.2.2 and Remark 12.2.4). So, if h is the image in $\mathbf{H}/\mathfrak{r}\mathbf{H}$ of $h' \in \mathbf{H}$, then the characteristic polynomial of h' acting on a simple $\mathbf{MH}$-module has coefficients in R (because R is integrally closed) and so the characteristic polynomial of h has coefficients in $R/\mathfrak{r}$ (it is the reduction modulo $\mathfrak{r}$ of the one of h).

Now, if $\mathfrak{r} = \bar{\mathfrak{r}}$ or $\bar{\mathfrak{r}}_c$, then the simple $k_R(\mathfrak{r})\mathbf{H}$-modules are obtained by scalar extension from the simple $k_P(\mathfrak{r} \cap P)\mathbf{H}$-modules, and the result follows from the fact that $P/\bar{\mathfrak{p}} \simeq \mathbf{k}[\mathscr{C}]$ and $P/\bar{\mathfrak{p}}_c \simeq \mathbf{k}$ is integrally closed. $\square$

Taking Proposition 12.3.1 into account, we can define decomposition maps

$$\mathrm{dec}_{\mathfrak{C}} : K_0(\mathbf{MH}) \xrightarrow{\sim} K_0(\mathbf{M}_{\mathfrak{C}}\mathbf{H}),$$

$$\mathrm{dec}_{\mathfrak{C}}^{\mathrm{left}} : K_0(\mathbf{MH}) \longrightarrow K_0(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}),$$

$$\mathrm{dec}_{\mathfrak{C}}^{\mathrm{right}} : K_0(\mathbf{MH}) \longrightarrow K_0(\mathbf{M}_{\mathfrak{C}}^{\mathrm{right}}\mathbf{H}^{\mathrm{right}}),$$

$$\overline{\mathrm{dec}}_{\mathfrak{C}} : K_0(\mathbf{MH}) \longrightarrow K_0(\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}}),$$

$$\overline{\mathrm{dec}}_{\mathfrak{C}}^{\mathrm{left}} : K_0(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}) \longrightarrow K_0(\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}}),$$

$$\overline{\mathrm{dec}}_{\mathfrak{C}}^{\mathrm{right}} : K_0(\mathbf{M}_{\mathfrak{C}}^{\mathrm{right}}\mathbf{H}^{\mathrm{right}}) \longrightarrow K_0(\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}})$$

and

$$\mathrm{dec}_{\mathfrak{C}}^{\mathrm{res}} : K_0(\bar{\mathbf{M}}\bar{\mathbf{H}}) \longrightarrow K_0(\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}}).$$

As usual, the index $\mathfrak{C}$ will be omitted if $\mathfrak{C} = 0$ or will be replaced by c if $\mathfrak{C} = \mathfrak{C}_c$ (for some $c \in \mathscr{C}$). Recall that $\mathrm{dec}_{\mathfrak{C}}$ is an isomorphism (by Example 12.1.2, which extends easily to the case where $\mathfrak{C}_c$ is replaced by any prime ideal $\mathfrak{C}$ of $\mathbf{k}[\mathscr{C}]$) and that

$$K_0(\mathbf{M}_{\mathfrak{C}}\mathbf{H}) \simeq \mathbb{Z}W \qquad \text{and} \qquad K_0(\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}}) \simeq \mathbb{Z}\operatorname{Irr}(W).$$

Note however that $\mathrm{dec}_{\mathfrak{C}}^{\mathrm{res}} : K_0(\bar{\mathbf{M}}\,\bar{\mathbf{H}}) \simeq \mathbb{Z}\operatorname{Irr}(W) \longrightarrow K_0(\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}}) \simeq \mathbb{Z}\operatorname{Irr}(W)$ is not an isomorphism in general. Some transitivity formulas follow from 12.1.1.

12.4. Isomorphism classes of baby Verma modules

The Verma modules $\Delta(\chi)$ being defined over the ring P, the fundamental properties of decomposition maps show that

$$\overline{\mathrm{dec}}_{\mathfrak{C}}^{\mathrm{left}}[\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\Delta(\chi)]_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}} = [\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\Delta}(\chi)]_{\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}}}. \tag{12.4.1}$$

The multiplicities $\mathrm{mult}_{C,\chi}^{\mathrm{CM}}$ are defined from the image of $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\Delta(\chi)$ in the Grothendieck group $K_0(\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}})$. We will now be interested to the image of $\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\Delta}(\chi)$ in the Grothendieck group $K_0(\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}})$:

Fix now a Calogero-Moser two-sided $\mathfrak{C}$-cell Γ and set $L_{\mathfrak{C}}(\Gamma) = \overline{\mathrm{dec}}_{\mathfrak{C}}[\mathscr{L}_w]_{\mathbf{MH}}$, for $w \in \Gamma$. Note that $L_{\mathfrak{C}}(\Gamma)$ does not depend on the choice of $w \in \Gamma$ by Lemma 12.2.1.

Proposition 12.4.2. — *If $\chi \in \mathrm{Irr}_{\Gamma}^{\mathrm{CM}}(W)$, then*

$$[\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\Delta}(\chi)]_{\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}}} = \chi(1)L_{\mathfrak{C}}(\Gamma).$$

Remark 12.4.3. — The Proposition 12.4.2 says that, inside a given Calogero-Moser $\mathfrak{C}$-family, the decomposition matrix of baby Verma modules in the basis of simple modules has rank 1: this had been conjectured by U. Thiel **[Thi1]**. ■

Proof. — Let C be a Calogero-Moser left $\mathfrak{C}$-cell. Then $\mathrm{mult}_{C,\chi}^{\mathrm{CM}} = 0$ if C is not contained in Γ (see Proposition 11.4.2(c)). Hence, by (12.4.1), we have

$$[\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\Delta}(\chi)]_{\bar{\mathbf{M}}_{\mathfrak{C}}\bar{\mathbf{H}}} = \sum_{\substack{C \in {}^{\mathrm{CM}}\mathrm{Cell}_L^{\mathfrak{C}}(W) \\ C \subset \Gamma}} \mathrm{mult}_{C,\chi}^{\mathrm{CM}} \cdot \overline{\mathrm{dec}}^{\mathrm{left}}[L_{\mathfrak{C}}(C)]_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}}.$$

But, if $C \subset \Gamma$, then $[L_{\mathfrak{C}}(C)]_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}} = \mathrm{dec}_{\mathfrak{C}}^{\mathrm{left}}[\mathscr{L}_w]_{\mathbf{MH}}$ where $w \in C$. Then, by the transitivity of decomposition maps, we have

$$\overline{\mathrm{dec}}^{\mathrm{left}}[L_{\mathfrak{C}}(C)]_{\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}} = L_{\mathfrak{C}}(\Gamma)$$

(by Lemma 12.2.1). The result then follows from Proposition 11.4.2(a). □

We conclude by a result which compares the Calogero-Moser $\mathfrak{C}$-cellular characters for different prime ideals $\mathfrak{C}$:

Proposition 12.4.4. — *Let $\mathfrak{C}'$ be a prime ideal of $\mathbf{k}[\mathscr{C}]$ contained in $\mathfrak{C}$ and choose a prime ideal $\mathfrak{r}_{\mathfrak{C}'}^{\mathrm{left}}$ lying over $\mathfrak{q}_{\mathfrak{C}'}^{\mathrm{left}}$ and contained in $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$. Let C be a Calogero-Moser left $\mathfrak{C}$-cell and let us write $C = C_1 \coprod \cdots \coprod C_r$, where the C_i's are Calogero-Moser left $\mathfrak{C}'$-cells (see Proposition 11.3.2). Then*

$$[C]_{\mathfrak{C}}^{\mathrm{CM}} = [C_1]_{\mathfrak{C}'}^{\mathrm{CM}} + \cdots + [C_r]_{\mathfrak{C}'}^{\mathrm{CM}}.$$

Proof. — By Proposition 12.3.1, the decomposition map $\mathbf{d} : K_0(\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}'}\mathbf{H}^{\mathrm{left}}) \longrightarrow K_0(\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\mathbf{H}^{\mathrm{left}})$ is well-defined and it satisfies the transitivity property $\mathbf{d} \circ \mathrm{dec}^{\mathrm{left}}_{\mathfrak{C}'} = \mathrm{dec}^{\mathrm{left}}_{\mathfrak{C}}$. Moreover, we have

$$\mathbf{d}\,[\,\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}'}\Delta(\chi)\,]_{\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}'}\mathbf{H}^{\mathrm{left}}} = [\,\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\Delta(\chi)\,]_{\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\mathbf{H}^{\mathrm{left}}}.$$

The result then follows from the fact that $\mathbf{d}\,[\,L_{\mathfrak{C}'}(C_i)\,]_{\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}'}\mathbf{H}^{\mathrm{left}}} = [\,L^{\mathrm{left}}_{\mathfrak{C}}(C)\,]_{\mathbf{M}^{\mathrm{left}}_{\mathfrak{C}}\mathbf{H}^{\mathrm{left}}}$ for all i (see Example 12.2.2). ☐

12.5. Hecke algebras

Let $c, c' \in \mathscr{C}(\mathbf{k})$ with $\kappa(c') \in \mathscr{K}(\mathbb{Q})$. We assume F is large enough so that $e^{2i\pi k'_{\aleph,j}/|\boldsymbol{\mu}_W|} \in F$ for all $(\aleph, j) \in \boldsymbol{\aleph}^{\circ}_W$.

Consider the morphism of $\mathbf{k}$-algebras $\mathbf{k}[\widetilde{\mathscr{C}}] \to \mathbf{k}(\hbar)$, $T \mapsto \hbar^{-1}$, $C \mapsto c + \hbar^{-1}c'$ and the morphism of $\mathbf{k}$-algebras $\mathbf{k}[\mathscr{C}] \to \mathbf{k}(\hbar)$, $C \mapsto \hbar c + c'$.

There is an isomorphism of algebras $\mathbf{k}(\hbar) \otimes_{\mathbf{k}[\mathscr{C}]} \dot{\mathbf{H}} \xrightarrow{\sim} \mathbf{k}(\hbar) \otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]} \widetilde{\mathbf{H}}$ (cf §3.5.A). It induces an equivalence $\big(\mathbf{k}(\hbar) \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{k}[\widetilde{\mathscr{C}}]/(T-1)\big)\,\tilde{\mathscr{O}} \xrightarrow{\sim} \mathbf{k}(\hbar)\tilde{\mathscr{O}}$ and an isomorphism $K_0(\big(\mathbf{k}(\hbar) \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{k}[\widetilde{\mathscr{C}}]/(T$ $K_0(\mathbf{k}(\hbar)\tilde{\mathscr{O}})$.

Recall (Proposition 7.5.3) that there is an equivalence

$$(\mathbf{k}(\hbar)\dot{\mathscr{O}})^{(\mathbb{Z})} \xrightarrow{\sim} \big(\mathbf{k}(\hbar) \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{k}[\widetilde{\mathscr{C}}]/(T-1)\big)\,\tilde{\mathscr{O}}.$$

Composing with the equivalence above provides an equivalence $(\mathbf{k}(\hbar)\dot{\mathscr{O}})^{(\mathbb{Z})} \xrightarrow{\sim} \mathbf{k}(\hbar)\tilde{\mathscr{O}}$, hence an isomorphism

$$K_0(\mathbf{k}(\hbar)\dot{\mathscr{O}})[\mathbf{t}^{\pm 1}] \xrightarrow{\sim} K_0(\mathbf{k}(\hbar)\tilde{\mathscr{O}}),\ [\dot{\Delta}(E)] \mapsto [\tilde{\Delta}(E)]\langle \log(e^{C_E(c)}) - C_E(c)\rangle.$$

Consider the discrete valuation ring $\mathbf{k}[\hbar^{-1}]_{(\hbar^{-1})}$. There is a decomposition map (cf §F.1.H) $K_0(\mathbf{k}(\hbar)\tilde{\mathscr{O}}) \to K_0(\mathscr{O}(c))$. Composing with the isomorphism above provides a morphism of $\mathbb{Z}[\mathbf{t}^{\pm 1}]$-modules $K_0(\mathbf{k}(\hbar)\dot{\mathscr{O}})[\mathbf{t}^{\pm 1}] \to K_0(\mathscr{O}(c))$.

Forgetting the gradings, i.e. setting $\mathbf{t} = 1$, we obtain a morphism of abelian groups d' from $K_0(\mathbf{k}(\hbar)\dot{\mathscr{O}})$ to the Grothendieck group L of the category of finitely generated $\mathbf{H}(c)$-modules that are locally nilpotent for V.

Applying $\mathbf{k}(V)^W \otimes_{\mathbf{k}[V]^W} -$ provides a morphism from L to the Grothendieck group L' of the category of finitely generated $(\mathbf{k}(V)^W \otimes_{\mathbf{k}[V]^W} \mathbf{H}(c))$-modules that are locally nilpotent for V. Composing with d', we obtain a morphism d'' from $K_0(\mathbf{k}(\hbar)\dot{\mathscr{O}})$ to L'.

Since every $(\mathbf{k}(V)^W \otimes_{\mathbf{k}[V]^W} \mathbf{H}(c))$-module that is locally nilpotent for V is a finite extension of $\mathbf{K}^{\mathrm{left}}_c\mathbf{H}$-modules, it follows that restriction through the quotient map $\mathbf{k}(V)^W \otimes_{\mathbf{k}[V]^W} \mathbf{H}(c) \twoheadrightarrow \mathbf{K}^{\mathrm{left}}_c\mathbf{H}$ induces an isomorphism $K_0(\mathbf{K}^{\mathrm{left}}_c\mathbf{H}\text{-mod}) \xrightarrow{\sim} L'$. Composing d'' with the inverse of that map provides a morphism $d''' : K_0(\mathbf{k}(\hbar)\dot{\mathscr{O}}) \to K_0(\mathbf{K}^{\mathrm{left}}_c\mathbf{H}\text{-mod})$.

Consider the morphism of $\mathcal{O}$-algebras $\mathcal{O}[\mathcal{Q}] \to \mathcal{O}(\mathbf{q}^{\mathbf{k}})$, $\mathbf{q}_{\aleph,j} \mapsto \mathbf{q}^{k_{\aleph,-j}} e^{2i\pi k'_{H,-j}/|\mu_W|}$. Given a $\mathbf{k}(\hbar)\dot{\mathcal{O}}$-module M with $\mathbf{k}[V^{\mathrm{reg}}] \otimes_{\mathbf{k}[V]} M = 0$, we have $d''([M]) = 0$. It follows from Theorem 8.3.1 that d'' factors through the morphism $K_0(\mathbf{k}(\hbar)\dot{\mathcal{O}}) \to K_0(F(\mathbf{q}^{\mathbf{k}})\mathcal{H}\text{-mod})$ induced by KZ. This provides a morphism

$$d_{c,c'} : K_0(F(\mathbf{q}^{\mathbf{k}})\mathcal{H}\text{-mod}) \to K_0(\mathbf{K}_c^{\mathrm{left}}\mathbf{H}\text{-mod}).$$

Let us summarize these constructions in the following theorem.

Theorem 12.5.1. — *There is a (unique) morphism $d_{c,c'} : G_0(F(\mathbf{q}^{\mathbb{C}})\mathcal{H}) \to G_0(\mathbf{K}_c^{\mathrm{left}}\mathbf{H})$ such that $d_{c,c'}([E^{\mathrm{gen}}]) = [\mathbf{K}_c^{\mathrm{left}}\Delta(\chi)]$ for all $E \in \mathrm{Irr}(W)$.*

Given L a simple $F(\mathbf{q}^{\mathbf{k}})\mathcal{H}$-module, there are non-negative integers $d_{L,\mathfrak{z}}$ such that $d_{c,c'}(L) = \sum_{\mathfrak{z}\in\Upsilon^{-1}(\mathfrak{p}_c^{\mathrm{left}})} d_{L,\mathfrak{z}}[L_c^{\mathrm{left}}(\mathfrak{z})]$.

$$\begin{tikzcd}
K_0(\mathbf{k}(\hbar)\dot{\mathcal{O}})[\mathbf{t}^{\pm 1}] \arrow[r, "\sim"] \arrow[d, "\mathbf{t}=1"'] & K_0(\mathbf{k}(\hbar)\tilde{\mathcal{O}}) \arrow[r, "\text{dec. map}"] & K_0(\mathcal{O}(c)) \arrow[d, "\text{forget grading}"] \\
K_0(\mathbf{k}(\hbar)\dot{\mathcal{O}}) \arrow[rr, dashed] \arrow[d, "\mathrm{KZ}"'] & & K_0(\mathbf{H}(c)\text{-mod}_{V-\text{loc. nilp.}}) \arrow[d, "\text{localization}"] \\
K_0(F(\mathbf{q}^{\mathbf{k}})\mathcal{H}\text{-mod}) \arrow[rr, dashed] \arrow[rrd, dashed, "d_{c,c'}"'] & & K_0((\mathbf{k}(V)^W \otimes_{\mathbf{k}[V]^W} \mathbf{H}(c))\text{-mod}_{V-\text{loc. nilp.}}) \\
 & & K_0(\mathbf{K}_c^{\mathrm{left}}\mathbf{H}\text{-mod}) \arrow[u, "\sim"']
\end{tikzcd}$$

Remark 12.5.2. — The discussion above shows that the cellular multiplicites measure something like the regular part of the characteristic cycle of a Verma module $\mathbf{k}(\tilde{c})\dot{\Delta}(E)$ (with respect to the filtration $\dot{\mathbf{H}}^{\trianglelefteq}$ of §4.4.B), although this doesn't seem to quite fit with the usual characteristic cycle theory. ∎

This theorem shows that classes of projective indecomposable $(F(\mathbf{q}^{\mathbf{k}})\mathcal{H})$-modules are sums with positive coefficients of Calogero-Moser c-cellular characters.

Remark 12.5.3. — When W has a unique class of reflections and $c \neq 0$, the algebra $F(\mathbf{q}^{\mathbf{k}})\mathcal{H}$-mod is semisimple, so Theorem 12.5.1 brings no information on cellular characters. ∎

CHAPTER 13

GAUDIN ALGEBRAS

13.1. W-covering of $\mathscr{Z}$

Let $\mathscr{Z}' = \mathscr{Z}^{\mathrm{reg}} \times_{V^{\mathrm{reg}}/W} V^{\mathrm{reg}}$. So, we have a cartesian square

$$\begin{tikzcd}
\mathscr{Z}' \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{can}"'] & \mathscr{Z} \arrow[d, "\Upsilon"] \\
\mathscr{C} \times V^{\mathrm{reg}} \times V^*/W \arrow[r, "\mathrm{can}"'] & \mathscr{C} \times V^{\mathrm{reg}}/W \times V^*/W
\end{tikzcd}$$

There is an action of W on $\mathscr{Z}'$ given by $w(z,v)=(z,w(v))$ for $z \in \mathscr{Z}^{\mathrm{reg}}$ and $v \in V^{\mathrm{reg}}$. Let $Z' = \mathbf{k}[\mathscr{Z}'] = Z^{\mathrm{reg}} \otimes_{\mathbf{k}[V^{\mathrm{reg}}]^W} \mathbf{k}[V^{\mathrm{reg}}]$.

Lemma 13.1.1. — *The multiplication map gives an isomorphism $Z' \rtimes W \xrightarrow{\sim} \mathbf{H}^{\mathrm{reg}}$. The image of Z' by that map is $C_{\mathbf{H}^{\mathrm{reg}}}(V^*) = \mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}] \otimes \Theta^{-1}(\mathbf{k}[V^*])$.*

There are isomorphisms

$$\mathscr{C} \times V^{\mathrm{reg}} \times V^* \xrightarrow[\mathrm{can}]{\sim} \mathscr{C} \times \big((V^{\mathrm{reg}} \times V^*)/\Delta W\big) \times_{V^{\mathrm{reg}}/W} V^{\mathrm{reg}} \xrightarrow[\Theta^\# \times \mathrm{id}]{\sim} \mathscr{Z}'.$$

Proof. — There is a commutative diagram

$$\begin{tikzcd}
Z^{\mathrm{reg}} \otimes_{\mathbf{k}[V^{\mathrm{reg}}]^W} (\mathbf{k}[V^{\mathrm{reg}}] \rtimes W) \arrow[r, "\mathrm{mult}"] \arrow[d, "\Theta\otimes\mathrm{id}"', "\sim"] & \mathbf{H}^{\mathrm{reg}} \arrow[d, "\sim"', "\Theta"] \\
\mathbf{k}[\mathscr{C} \times (V^{\mathrm{reg}} \times V^*)/\Delta W] \otimes_{\mathbf{k}[V^{\mathrm{reg}}]^W} (\mathbf{k}[V^{\mathrm{reg}}] \rtimes W) \arrow[r, "\mathrm{mult}"'] & \mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}} \times V^*] \rtimes W
\end{tikzcd}$$

Since the canonical map $V^{\mathrm{reg}} \times V^* \to (V^{\mathrm{reg}} \times V^*)/\Delta W \times_{V^{\mathrm{reg}}/W} V^{\mathrm{reg}}$ is an isomorphism, it follows that the bottom horizontal map in the diagram is an isomorphism, hence the top horizontal map is an isomorphism as well. The other assertions of the lemma are clear. □

Recall that $\mathbf{H}^{\mathrm{reg}}e$ induces a Morita equivalence between $\mathbf{H}^{\mathrm{reg}}$ and Z^{reg} (Corollary 4.3.5). Through the isomorphisms of Lemma 13.1.1, this corresponds to the Morita equivalence between $Z' \rtimes W$ and $Z'^W = Z^{\mathrm{reg}}$ given by Z'.

Given m a (non-necessarily closed) point of $\mathscr{C} \times V^{\mathrm{reg}} \times V^*/W$, we put $L(m) = \mathbf{k}(m) \otimes_{\mathbf{k}[\mathscr{C}\times V^{\mathrm{reg}}\times V^*/W]} \mathbf{H}^{\mathrm{reg}} e$. We have $\dim_{\mathbf{k}(m)} L(m) = |W|$. The action of $\Theta^{-1}(\mathbf{k}[V^*])$ by left multiplication on $\mathbf{H}^{\mathrm{reg}} e$ induces an action on $L(m)$.

Lemma 13.1.2. — *Given $m \in \mathscr{C} \times V^{\mathrm{reg}} \times V^*/W$, the $k(m)$-algebra $Z' \otimes_{\mathbf{k}[\mathscr{C}\times V^{\mathrm{reg}}\times V^*/W]} \mathbf{k}(m)$ has dimension $|W|$ and acts faithfully on the $|W|$-dimensional $\mathbf{k}(m)$-vector space $L(m)$.*

The non-zero eigenspaces of $\Theta^{-1}(V)$ on $L(m)$ are one-dimensional.

Proof. — Note that the image of Z' in $\mathrm{End}_{\mathbf{k}(m)}(L(m))$ coincides with the image of $\Theta^{-1}(\mathbf{k}(m)[V^*])$ by Lemma 13.1.1. As a consequence, the image of the commutative algebra $\Theta^{-1}(\mathbf{k}(m)[V^*])$ in $L(m)$ has $k(m)$-dimension equal to that of $L(m)$. It follows that the non-zero eigenspaces of $\Theta^{-1}(V)$ on $L(m)$ are one-dimensional. $\square$

We now introduce the spectral variety of $\Theta^{-1}(V)$ acting on the family $\{L(m)\}_m$. Let $\mathscr{X}''$ be the closed subscheme of $\mathscr{C} \times V^{\mathrm{reg}} \times V^*/W \times V^*$ given by

$$\mathscr{X}'' = \{(c, v, u, \lambda) \mid \det_{L(c,v,u)}(\Theta^{-1}(y) - \langle y, \lambda\rangle) = 0 \ \forall y \in V\}.$$

Proposition 13.1.3. — *There is an isomorphism $\mathscr{X}'' \xrightarrow{\sim} \mathscr{X}'$, $(c, v, u, \lambda) \mapsto (\Theta^{\#}(c, v, \lambda), u)$.*

Proof. — Lemma 13.1.2 shows that characteristic and minimal polynomials of $\Theta^{-1}(V)$ on $L(c, v, u)$ agree. The proposition follows. $\square$

13.2. Gaudin operators

We consider the $\big(\Theta^{-1}(\mathbf{k}[V^*]) \otimes \mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}] \otimes \mathbf{k}[V^*]\big)$-module $\bar{L} = \mathbf{H}^{\mathrm{reg}}$, where $\Theta^{-1}(\mathbf{k}[V^*]) \otimes \mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}]$ acts by left multiplication and $\mathbf{k}[V^*]$ acts by right multiplication. Note that $\bar{L}$ is a free $(\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}] \otimes \mathbf{k}[V^*])$-module with basis W.

Given $(c, v, v^*) \in \mathscr{C} \times V^{\mathrm{reg}} \times V^*$, we put $\bar{L}(c, v, v^*) = \mathbf{k}(c, v) \otimes_{\mathbf{k}[\mathscr{C}\times V^{\mathrm{reg}}]} \mathbf{H}^{\mathrm{reg}} \otimes_{\mathbf{k}[V^*]} \mathbf{k}(v^*)$, a $\Theta^{-1}(\mathbf{k}[V^*])$-module. We denote by $(e_w)_{w\in W}$ the $\mathbf{k}$-basis of $\bar{L}(c, v, v^*)$ obtained as the image of W.

The action of $\Theta^{-1}(y)$ on $\bar{L}(c, v, v^*)$ is given by the operator

$$D_y^{c,v,v^*} : e_w \mapsto \langle y, w(v^*)\rangle e_w + \sum_{s\in \mathrm{Ref}(W)} \epsilon(s) c_s \frac{\langle y, \alpha_s\rangle}{\langle v, \alpha_s\rangle} e_{sw}.$$

Let u be the image of v^* in V^*/W. The $(\mathbf{k}[V^*] \rtimes W)$-module $\mathrm{Ind}_{\mathbf{k}[V^*]}^{\mathbf{k}[V^*]\rtimes W} \mathbf{k}(v^*)$ is isomorphic to the semisimplification of $\mathrm{Ind}_{\mathbf{k}[V^*]^W \otimes \mathbf{k}W}^{\mathbf{k}[V^*]\rtimes W} (\mathbf{k}(u) \otimes \mathbf{k}) = \mathbf{k}[V^*] \otimes_{\mathbf{k}[V^*]^W} \mathbf{k}(u)$. Consequently, $\bar{L}(c, v, v^*)$ is isomorphic to the graded module associated with a filtration of $L(c, v, u)$ (the filtration does not depend on v^*). In particular, $\bar{L}(c, v, v^*)$ depends only on the W-orbit of v^*, up to isomorphism. Also, the spectrum of $\Theta^{-1}(V)$ on $\bar{L}(c, v, v^*)$ is the same as that on $L(c, v, u)$.

We now introduce the spectral variety of $\Theta^{-1}(V)$ acting on the family $\{\bar{L}(c,v,v^*)\}_{c,v,v^*}$. Let $\bar{\mathscr{Z}}''$ be the closed subscheme of $\mathscr{C}\times V^{\mathrm{reg}}\times V^*/W\times V^*$ given by

$$\bar{\mathscr{Z}}''=\{(c,v,u,\lambda)\mid \det_{\bar{L}(c,v,v^*)}(\Theta^{-1}(y)-\langle y,\lambda\rangle)=0\ \forall y\in V\}$$

where $v^*\in V^*$ has image u in V^*/W. Forgetting λ gives a morphism $\bar{\mathscr{Z}}''\to\mathscr{C}\times V^{\mathrm{reg}}\times V^*/W$ and the fiber at (c,v,u) is the spectrum of $\Theta^{-1}(V)$.

From Proposition 13.1.3, we deduce the following description of that variety.

Proposition 13.2.1. — *There is an isomorphism $\bar{\mathscr{Z}}''\xrightarrow{\sim}\mathscr{Z}'$, $(c,v,u,\lambda)\mapsto(\Theta^{\#}(c,v,\lambda),u)$.*

13.3. Topology

We assume in §13.3 that $\mathbf{k}=\mathbb{C}$.

13.3.A. γ-cells. — Recall that given $(c,v,v^*)\in\mathscr{C}(\mathbb{C})\times V^{\mathrm{reg}}\times V^*$, we have a family of commuting operators $\{D_y^{c,v,v^*}\}_{y\in V}$ acting on $\bigoplus_{w\in W}\mathbb{C}e_w$:

$$D_y^{c,v,v^*}:e_w\mapsto\langle y,w(v^*)\rangle e_w+\sum_{s\in\mathrm{Ref}(W)}\epsilon(s)c_s\frac{\langle y,\alpha_s\rangle}{\langle v,\alpha_s\rangle}e_{sw}.$$

Let $\gamma:[0,1]\to\mathscr{C}(\mathbb{C})\times V^{\mathrm{reg}}/W\times V^*/W$ be a path with $\gamma([0,1))\subset\mathscr{P}(\mathbb{C})^{\mathrm{nr}}$ and $\gamma(0)=(0,W\cdot v_{\mathbb{C}},W\cdot v_{\mathbb{C}}^*)$ as in §6.6. We denote by $\hat{\gamma}$ a path in $\mathscr{C}(\mathbb{C})\times V\times V^*$ lifting γ with $\hat{\gamma}(0)=(0,v_{\mathbb{C}},v_{\mathbb{C}}^*)$.

Theorem 13.3.1. — *Let $w\in W$. There is a unique path $\rho_w:[0,1]\to V^*$ such that*

- *$\rho_w(0)=w^{-1}(v_{\mathbb{C}}^*)$.*
- *$\langle\rho_w(t),y\rangle$ is an eigenvalue of $D_y^{\hat{\gamma}(t)}$ for $y\in V$ and $t\in[0,1)$*

Two elements $w',w''\in W$ are in the same Calogero-Moser γ-cell if and only if $\rho_{w'}(1)=\rho_{w''}(1)$.

Proof. — Appendix §B.7 applied to the covering $\mathscr{Z}'\to\mathscr{C}\times V^{\mathrm{reg}}\times V^*/W$ shows the existence of a path $\tilde{\gamma}_w$ in $\mathscr{Z}'(\mathbb{C})$ lifting the image of $\hat{\gamma}$ in $\mathscr{C}(\mathbb{C})\times V\times V^*/W$ and such that $\tilde{\gamma}_w(0)=(z_w,w^{-1}(v_{\mathbb{C}}^*))$, where $z_w=(0,(v_{\mathbb{C}},w^{-1}(v_{\mathbb{C}}^*))\Delta W)$. Note that the image of $\tilde{\gamma}_w$ in $\mathscr{Z}(\mathbb{C})$ is the path γ_w of §6.6. Define $\rho_w(t)$ to be the λ-component of the image of $\tilde{\gamma}_w(t)$ in $\bar{\mathscr{Z}}'$", via the isomorphism of Proposition 13.2.1. It satisfies the required properties.

The last statement follows by base change via the unramified map $V^{\mathrm{reg}}\to V^{\mathrm{reg}}/W$. ∎

13.3.B. Left cells. — Let $\mathfrak{C}$ be a prime ideal of $\mathbb{C}[\mathscr{C}]$ and let $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$ be a prime ideal of R as in the preamble to §III.

Let $y_1 \in \mathscr{R}(\mathbb{C})$ be a point in the irreducible component determined by $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$ such that $\mathrm{Stab}_G(y_1) = G_{\mathfrak{r}^{\mathrm{left}}}^I$. Fix a path $\hat{\hat{\gamma}} : [0,1] \to \mathscr{R}(\mathbb{C}) \times_{V/W \times V^*/W} (V^{\mathrm{reg}} \times V^*)$ such that $\hat{\hat{\gamma}}([0,1)) \subset \mathscr{R}(\mathbb{C})^{\mathrm{nr}} \times_{V/W\times V^*/W} (V^{\mathrm{reg}} \times V^*)$, $\hat{\hat{\gamma}}(0) = (y_0, (v_{\mathbb{C}}, v_{\mathbb{C}}^*))$ and $\hat{\hat{\gamma}}(1)$ maps onto y_1. We denote by $\hat{\gamma}$ the image of $\hat{\hat{\gamma}}$ in $\mathscr{C}(\mathbb{C}) \times V \times V^*$.

Theorem 13.3.1 and Proposition 6.6.2 have the following consequence.

Theorem 13.3.2. — *Let $w \in W$. There is a unique path $\rho_w : [0,1] \to V^*$ such that*

- *$\langle \rho_w(t), y\rangle$ is an eigenvalue of $D_y^{\hat{\gamma}(t)}$ for $y \in V$ and $t \in [0,1)$*
- *$\rho_w(0) = w^{-1}(v_{\mathbb{C}}^*)$.*

Two elements $w', w'' \in W$ are in the same Calogero-Moser left $\mathfrak{C}$-cell if and only if $\rho_{w'}(1) = \rho_{w''}(1)$.

13.4. Gaudin algebra and cellular characters

13.4.A. Left specialization. — Let $Z'^{\mathrm{left}} = Z' \otimes_{\mathbf{k}[V^*]^W} \mathbf{k} = Z^{\mathrm{reg,left}} \otimes_{\mathbf{k}[V^{\mathrm{reg}}]^W} \mathbf{k}[V^{\mathrm{reg}}]$, a $\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}]$-algebra, free of rank $|W|$ as a $\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}]$-module. It is acted on by W and $(Z'^{\mathrm{left}})^W = Z^{\mathrm{reg,left}}$.

Lemma 13.1.1 shows that Θ gives an isomorphism $Z'^{\mathrm{left}} \rtimes W \xrightarrow{\sim} \mathbf{H}^{\mathrm{reg,left}}$ and the image of Z'^{left} by that map is the $\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}]$-subalgebra generated by $\Theta^{-1}(V)$.

There is a Morita equivalence between $\mathbf{H}^{\mathrm{reg,left}}$ and $Z^{\mathrm{reg,left}}$ given by the bimodule $\Delta(\mathrm{co})^{\mathrm{reg}} = \Delta(\mathrm{co}) \otimes_{\mathbf{k}[V]} \mathbf{k}[V^{\mathrm{reg}}] = \mathbf{H}^{\mathrm{reg}} e \otimes_{\mathbf{k}[V^*]^W} \mathbf{k} = \mathbf{H}^{\mathrm{reg,left}} e$. It corresponds to the Morita equivalence between $Z'^{\mathrm{left}} \rtimes W$ and $Z^{\mathrm{reg,left}}$ given by Z'^{left}. Note in particular that $Z'^{\mathrm{left}} \rtimes W$ acts faithfully on $\Delta(\mathrm{co})^{\mathrm{reg}}$.

13.4.B. Gaudin algebra. — There is a canonical isomorphism $\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}] \otimes \mathbf{k}W \xrightarrow{\sim} \mathbf{H}^{\mathrm{reg}} \otimes_{\mathbf{k}[V^*]} \mathbf{k} = \Delta^{\mathrm{reg}}(\mathbf{k}W)$. Through this isomorphism, the action of $\Theta^{-1}(y)$ (for $y \in V$) is given by

$$\mathscr{D}_y = \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) C_s \frac{\langle y, \alpha_s\rangle}{\alpha_s} s \ \in \mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}][W]$$

where $\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}][W]$ denote the group algebra of W over the algebra $\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}]$: note that, in this algebra, the elements of $\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}]$ and those of W commute. In other words, $\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}][W] = \mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}] \otimes \mathbf{k}W$ as an algebra. We denote by $\mathrm{Gau}(W)$ the $\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}]$-subalgebra of $\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}}][W]$ generated by the $\mathscr{D}_y$'s ($y \in V$). It will be called the *generic Gaudin algebra* associated with W: note that it is commutative.

Note also that the external action of W stabilizes $\mathrm{Gau}(W)$: we have ${}^w\mathscr{D}_y = \mathscr{D}_{w(y)}$ for $y \in V$ and $w \in W$. Therefore, we can consider the algebra $\mathrm{Gau}(W) \rtimes W$.

Recall that $\Delta^{\mathrm{reg}}(\mathrm{co})$ has a filtration with $\mathrm{gr}\,\Delta^{\mathrm{reg}}(\mathrm{co}) \simeq \Delta(\mathbf{k}W)$. There is a filtration of $\mathbf{H}^{\mathrm{reg,left}}$ given for $r \leqslant 0$ by

$$\mathbf{H}^{\mathrm{reg,left},\leqslant r} = \{h \in \mathbf{H}^{\mathrm{reg,left}} \mid h\Delta^{\mathrm{reg}}(\mathrm{co})^{\leqslant i} \subset \Delta^{\mathrm{reg}}(\mathrm{co})^{\leqslant i+r},\ \forall i \leqslant 0\}.$$

Via the isomorphism $Z'^{\mathrm{left}} \rtimes W \xrightarrow{\sim} \mathbf{H}^{\mathrm{reg,left}}$, we obtain a filtration with $(Z'^{\mathrm{left}} \rtimes W)^{\leqslant r} = (Z'^{\mathrm{left}})^{\leqslant r} \otimes \mathbf{k}W$. We deduce the following:

- The algebra $\mathrm{Gau}(W)$ is the image of $Z' = \Theta^{-1}(\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}} \times V^*])$ in $\mathrm{End}(\Delta(\mathbf{k}W))$
- The algebra $\mathrm{Gau}(W)^W$ is the image of $Z^{\mathrm{reg}} = \Theta^{-1}(\mathbf{k}[\mathscr{C} \times V^{\mathrm{reg}} \times V^*]^{\Delta W})$ in $\mathrm{End}(\Delta(\mathbf{k}W))$
- The algebra $\mathrm{Gau}(W) \rtimes W$ is the image of $\mathbf{H}^{\mathrm{reg,left}}$ in $\mathrm{End}(\Delta(\mathbf{k}W))$
- the kernel of the action of $\mathbf{H}^{\mathrm{reg,left}}$ on $\Delta(\mathbf{k}W)$ is a nilpotent ideal.

Thanks to Proposition E.1.2, we also deduce that $\mathrm{Gau}(W)$ induces a Morita equivalence between $\mathrm{Gau}(W) \rtimes W$ and $\mathrm{Gau}(W)^W$.

13.4.C. Cellular characters. — Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathrm{Gau}(W)$. We define a character of W

$$\gamma_{\mathfrak{m}}^{\mathrm{Gau}} = ([(\mathbf{k}(\mathfrak{C})(V) \otimes \mathbf{k}W)_{\mathfrak{m}}]_{(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathrm{Gau}(W))_{\mathfrak{m}}})^*.$$

By Proposition E.1.2, the restriction map induces a bijection

$$\big(\mathrm{Irr}(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathrm{Gau}(W))\big)/W \xrightarrow{\sim} \mathrm{Irr}(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathrm{Gau}(W)^W). \tag{13.4.1}$$

Let $\mathfrak{z}$ be the maximal ideal of $Z_{\mathfrak{C}}^{\mathrm{left}}$ corresponding to the orbit of $\mathfrak{m}$ via this bijection.

Theorem 13.4.2. — *We have $\gamma_L^{\mathrm{Gau}} = \gamma_{\mathfrak{z}}^{\mathrm{CM}}$.*

Note that, as consequence, we have $\gamma_{{}^wL}^{\mathrm{Gau}} = \gamma_L^{\mathrm{Gau}}$ for all $w \in W$.

There is a corresponding result for cellular multiplicities:

$$\mathrm{Length}_{(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathrm{Gau}(W))_{\mathfrak{m}}}((\mathbf{k}(\mathfrak{C})(V) \otimes E)_{\mathfrak{m}}) = \mathrm{mult}_{\mathfrak{z},\chi}^{\mathrm{CM}}.$$

Consider now an algebraically closed field K and a K-point p of $\mathrm{Spec}(\mathbf{k}(\mathfrak{C})[V^{\mathrm{reg}}])$ outside the ramification locus of $f : \mathrm{Spec}(\mathbf{k}(\mathfrak{C})Z'^{\mathrm{left}}) \to \mathrm{Spec}(\mathbf{k}(\mathfrak{C})) \times V^{\mathrm{reg}}$.

There is a bijection $\mathrm{Irr}(\mathbf{K}_{\mathfrak{C}}^{\mathrm{left}}\mathrm{Gau}(W)) \xrightarrow{\sim} f^{-1}(p)$. Denote by $z_{\mathfrak{m}}$ the point of $f^{-1}(p)$ corresponding to $\mathfrak{m}$. We have $\gamma_{\mathfrak{m}}^{\mathrm{Gau}} = [K(f^{-1}(p))]_{KW}$. So, the $\mathfrak{C}$-cellular characters are the generalized eigenspaces of the Gaudin operators at p.

Remark 13.4.3. — The description of Calogero-Moser cellular characters provided by Theorem 13.4.2 allows efficient computations in small groups. ■

CHAPTER 14

BIALYNICKI-BIRULA CELLS OF $\mathscr{Z}_c$

Assumption. *In this chapter §14, we assume that* $\mathbf{k} = \mathbb{C}$ *and we fix an element* $c \in \mathscr{C}$.

The group $\mathbb{C}^\times$ acts on the algebraic variety $\mathscr{Z}_c$. We shall interprete geometrically several notions introduced in this book (families, cellular characters,...) using this action (fixed points, attractive or repulsive sets...). The main result of this chapter (and maybe of this book) is concerned with the case of a family corresponding to a smooth point of $\mathscr{Z}_c$: we will show that the associated cell characters are irreducible. This result will be seen as a geometric result. Indeed, the smoothness of the fixed point implies that the attractive and repulsive sets are affine spaces which intersect properly and transversally; a computation of the intersection multiplicity will conclude the proof (see Theorem 14.4.1).

14.1. Generalities on $\mathbb{C}^\times$-actions

Let $\mathscr{X}$ be an *affine* algebraic variety endowed with a regular $\mathbb{C}^\times$-action $\mathbb{C}^\times \times \mathscr{X} \to \mathscr{X}$, $(\xi, x) \mapsto \xi \bullet x$. We will denote by $\mathscr{X}^{\mathbb{C}^\times}$ the closed subvariety consisting of the fixed points under the action of $\mathbb{C}^\times$. Given $x \in \mathscr{X}$, we say that $\lim_{\xi \to 0} \xi \bullet x$ *exists and is equal to* x_0 if there exists a morphism of varieties $\varphi : \mathbb{C} \to \mathscr{X}$ such that, if $\xi \in \mathbb{C}^\times$, then $\varphi(\xi) = \xi \bullet x$ and $\varphi(0) = x_0$. It is then clear that $x_0 \in \mathscr{X}^{\mathbb{C}^\times}$. Similarly, we will say that $\lim_{\xi \to 0} \xi^{-1} \bullet x$ *exists and is equal to* x_0 if there exists a morphism of varieties $\varphi : \mathbb{C} \to \mathscr{X}$ such that, if $\xi \in \mathbb{C}^\times$, then $\varphi(\xi) = \xi^{-1} \bullet x$ and $\varphi(0) = x_0$.

We denote by $\mathscr{X}^{\mathrm{att}}$ (respectively $\mathscr{X}^{\mathrm{rép}}$) the set of $x \in \mathscr{X}$ such that $\lim_{\xi \to 0} \xi \bullet x$ (respectively $\lim_{\xi \to 0} \xi^{-1} \bullet x$) exists. It is a closed subvariety of $\mathscr{X}$ and the maps

$$\begin{array}{rccc} \lim_{\mathrm{att}} : & \mathscr{X}^{\mathrm{att}} & \longrightarrow & \mathscr{X}^{\mathbb{C}^\times} \\ & x & \longmapsto & \lim_{\xi \to 0} \xi \bullet x \end{array}$$

and
$$\begin{array}{rccc} \lim_{\text{rép}}: & \mathscr{X}^{\text{rép}} & \longrightarrow & \mathscr{X}^{\mathbb{C}^\times} \\ & x & \longmapsto & \lim_{\xi\to 0} \xi^{-1}\bullet x \end{array}$$

are morphisms of varieties (which are of course surjective: a section is given by the closed immersion $\mathscr{X}^{\mathbb{C}^\times} \subset \mathscr{X}^{\text{att}} \cap \mathscr{X}^{\text{rép}}$), because $\mathscr{X}$ is affine by assumption (this follows from the facts that this is true for the affine space $\mathbb{C}^N$endowed with a linear action of $\mathbb{C}^\times$ and that $\mathscr{X}$ can be seen as a $\mathbb{C}^\times$-stable closed subvariety of such a $\mathbb{C}^N$). Note that this is no longer true in general if $\mathscr{X}$ is not affine, as it is shown by the example $\mathbf{P}^1(\mathbb{C})$ endowed with the action $\xi\bullet[x;y]=[\xi x;y]$.

Finally, given $x_0 \in \mathscr{X}^{\mathbb{C}^\times}$, we denote by $\mathscr{X}^{\text{att}}(x_0)$ (respectively $\mathscr{X}^{\text{rép}}(x_0)$) the inverse image of x_0 by the map $\lim_{\text{att}}$ (respectively $\lim_{\text{rép}}$). The closed subvariety $\mathscr{X}^{\text{att}}(x_0)$ (respectively $\mathscr{X}^{\text{rép}}(x_0)$) will be called the *attractive set* (respectively the *repulsive set*) of x_0: it is a closed subvariety of $\mathscr{X}$. Let us recall the following classical fact, due to Bialynicki-Birula [**Bia**]:

Proposition 14.1.1. — *If x_0 is a smooth point of $\mathscr{X}$, then there exists $N \geqslant 0$ such that $\mathscr{X}^{\text{att}}(x_0) \simeq \mathbb{C}^N$. In particular, $\mathscr{X}^{\text{att}}(x_0)$ is smooth and irreducible.*

The same statements hold for $\mathscr{X}^{\text{rép}}(x_0)$.

We will describe the notions developed in the previous chapters (families, cellular characters) via fixed points and attractive sets of the $\mathbb{C}^\times$-action on $\mathscr{Z}_c$.

14.2. Fixed points and families

The results of this section §14.2 are due to Gordon [**Gor1**]. There are several $\mathbb{C}^\times$-actions on all of our varieties ($\mathscr{P}$, $\mathscr{Z}$, $\mathscr{R}$,...). We will use the one which induces the $\mathbb{Z}$-grading of Example 3.2.1. In other words, an element $\xi \in \mathbb{C}^\times$ acts on $\mathbf{H}$ as the element $(\xi^{-1},\xi,1\rtimes 1)$ of $\mathbb{C}^\times \times \mathbb{C}^\times \times (\mathrm{Hom}(W,\mathbb{C}^\times)\rtimes \mathscr{N})$. Therefore, for the action on $\mathbf{H}$, ξ acts trivially on $\mathbb{C}[\mathscr{C}]\otimes \mathbb{C}W$, acts with non-negative weights on $\mathbb{C}[V]$, with non-positive weights on $\mathbb{C}[V^*]$. We get an action on P and Z_c, which induces regular actions of $\mathbb{C}^\times$ on the varieties $\mathscr{P}_\bullet \simeq V/W \times V^*/W$ and $\mathscr{Z}_c$ making the morphism
$$\Upsilon_c : \mathscr{Z}_c \longrightarrow \mathscr{P}_\bullet = V/W \times V^*/W$$
$\mathbb{C}^\times$-equivariant. Given $\xi \in \mathbb{C}^\times$ and $z \in \mathscr{Z}_c$, the image of z through this action of ξ will be denoted by $\xi\bullet z$. The unique fixed point of $\mathscr{P}_\bullet$ is $(0,0)$:

(14.2.1) $$\mathscr{P}_\bullet^{\mathbb{C}^\times} = (0,0).$$

Since Υ_c is a finite morphism, we deduce that

(14.2.2) $$\mathscr{Z}_c^{\mathbb{C}^\times} = \Upsilon_c^{-1}(0,0).$$

Proposition 14.2.3. — *The construction above provides a bijection between $\mathscr{X}_c^{\mathbb{C}^\times}$ and the set of Calogero-Moser $\boldsymbol{c}$-families.*

14.3. Attractive sets and cellular characters

First of all, note that

(14.3.1) $\mathscr{P}_\bullet^{\mathrm{att}} = V/W \times 0 \subset V/W \times V^*/W$ and $\mathscr{P}_\bullet^{\mathrm{rép}} = 0 \times V^*/W \subset V/W \times V^*/W$.

In other words, $\mathscr{P}_\bullet^{\mathrm{att}}$ is the irreducible subvariety of $\mathscr{P}_\bullet$ associated with the prime ideal $\mathfrak{p}_c^{\mathrm{left}}$. Moreover, since Υ_c is a finite morphism, we have

Lemma 14.3.2. — *We have $\mathscr{X}_c^{\mathrm{att}} = \Upsilon_c^{-1}(V/W \times 0)$ and $\mathscr{X}_c^{\mathrm{rép}} = \Upsilon_c^{-1}(0 \times V^*/W)$.*

Proof. — Let $\rho : Z_c \to \mathbb{C}[\mathbf{t}, \mathbf{t}^{-1}]$ be a morphism of $\mathbb{C}$-algebras such that $\rho(P_\bullet) \subset \mathbb{C}[\mathbf{t}]$. Since Z_c is integral over $P_\bullet$, it follows that $\rho(Z_c)$ is integral over $\rho(P_\bullet)$. As $\mathbb{C}[\mathbf{t}]$ is integrally closed, we deduce that $\rho(Z_c) \subset \mathbb{C}[\mathbf{t}]$. This shows that $\mathscr{P}_\bullet^{\mathrm{att}} = V/W \times 0 \subset \Upsilon_c(\mathscr{X}_c^{\mathrm{att}})$. The reverse inclusion is clear, and the other equality is proven similarly. $\square$

We have the following immediate consequence.

Proposition 14.3.3. — *There is a bijection from $\Upsilon_c^{-1}(\mathfrak{p}_c^{\mathrm{left}})$ to the set of irreducible components of $\mathscr{X}_c^{\mathrm{att}}$ sending $\mathfrak{z}$ to the corresponding irreducible closed subvariety $\mathscr{X}_c^{\mathrm{att}}[\mathfrak{z}]$.*

Since $\lim_{\mathrm{att}} : \mathscr{X}_c^{\mathrm{att}} \to \mathscr{X}_c^{\mathbb{C}^\times}$ is a morphism of varieties, the image of $\mathscr{X}_c^{\mathrm{att}}[\mathfrak{z}]$ is irreducible. As $\mathscr{X}_c^{\mathbb{C}^\times}$ is a finite set, we deduce that $\lim_{\mathrm{att}}(\mathscr{X}_c^{\mathrm{att}}[\mathfrak{z}])$ is reduced to a point. Hence, the morphism of varieties $\lim_{\mathrm{att}} : \mathscr{X}_c^{\mathrm{att}} \to \mathscr{X}_c^{\mathbb{C}^\times}$ induces a surjective map $\Upsilon_c^{-1}(\mathfrak{p}_c^{\mathrm{left}}) \longrightarrow \Upsilon_c^{-1}(\bar{\mathfrak{p}}_c)$: this is the map $\lim_{\mathrm{left}}$ defined in (11.2.12).

14.4. The smooth case

Assumption and notation. *We fix in §14.4 a point $z_0 \in \mathscr{X}_c^{\mathbb{C}^\times}$ which is assumed to be* ***smooth*** *in $\mathscr{X}_c$. We denote by χ the unique irreducible character of the associated Calogero-Moser $\boldsymbol{c}$-family. We denote by Γ the Calogero-Moser two-sided $\boldsymbol{c}$-cell associated with z_0 and we fix a Calogero-Moser left $\boldsymbol{c}$-cell C contained in Γ.*

The aim of this section is to show the following result.

Theorem 14.4.1. — *With the assumption and notation above, we have:*

(a) $|\Gamma| = \chi(1)^2$.
(b) $\bigcup_{d \in D_c^{\mathrm{left}}} {}^d C = \Gamma$.
(c) $|C| = \chi(1)$.
(d) $[\,C\,]_c^{\mathrm{CM}} = \chi$.
(e) $\deg_c(C) = \chi(1)$.

We will provide a geometrical proof of Theorem 14.4.1. A later entirely algebraic proof of (c,d,e) can be deduced from [**Bel6**, Theorem 10(3)]. In type A_n, this can also be deduced, using Gaudin operators (cf Chapter 13), from [**MuTaVa**].

NOTATION - Given A is a commutative local ring with maximal ideal $\mathfrak{m}$ and given M a finitely generated A-module, we denote by $e_{\mathfrak{m}}(M)$ the *multiplicity* of M for the ideal $\mathfrak{m}$, as it is defined in [**Ser**, Chapter V, §A.2].

Let A be a regular commutative ring (not necessarily local) and M and N two finitely generated A-modules such that $M \otimes_A N$ has finite length. Given $\mathfrak{a}$ a prime ideal of A, we put

$$\chi_{\mathfrak{a}}(M,N) = \sum_{i=0}^{\dim A} (-1)^i \operatorname{Length}_{A_{\mathfrak{a}}}(\operatorname{Tor}_i^A(M,N)_{\mathfrak{a}}),$$

as in [**Ser**, Chapter V, §B, Theorem 1] . ■

Proof. — (a) follows from Theorem 10.2.7(c).

(b) The set of irreducible components of $\lim_{\mathrm{att}}^{-1}(z_0)$ is in bijection with $D_c^{\mathrm{left}} \backslash \Gamma$ (see Proposition 11.3.5(c)). Since z_0 is smooth (and isolated), we have

$$\mathscr{X}_c^{\mathrm{att}}(z_0) \simeq \mathbb{C}^N$$

for some N, hence $\mathscr{X}_c^{\mathrm{att}}(z_0)$ is smooth and irreducible (Proposition 14.1.1). This shows that $\lim_{\mathrm{att}}^{-1}(z_0)$ is irreducible and isomorphic to an affine space. It follows that $|D_c^{\mathrm{left}} \backslash \Gamma| = 1$, which shows (b). In other words, with the notation introduced in §11.3.B, we have $C^D = \Gamma$.

(c) Let $\mathfrak{z}_L = \mathfrak{z}_c^{\mathrm{left}}(C)$: then $\mathfrak{z}_L$ is the defining ideal (in Z) of $\mathscr{X}_c^{\mathrm{att}}(z_0)$, but we will consider its image in Z_c. Let $\bar{\mathfrak{z}} = \bar{\mathfrak{z}}_c(\Gamma)$: then $\bar{\mathfrak{z}}$ is the defining ideal of the point z_0 (which will be seen as an ideal of Z_c). We define similarly $\mathfrak{z}_R$ as being the defining ideal of $\mathscr{X}_c^{\mathrm{rép}}(z_0)$: we denote by C' a Calogero-Moser *right* c-cell contained in Γ (so that $\mathfrak{z}_R = \mathfrak{z}_c^{\mathrm{right}}(C')$). Corollary 11.4.8 shows that

$$(\clubsuit) \qquad |C| = \operatorname{Length}_{Z_{c,\mathfrak{z}_L}}(Z_c/\mathfrak{p}_c^{\mathrm{left}} Z_c) \quad \text{and} \quad |C'| = \operatorname{Length}_{Z_{c,\mathfrak{z}_R}}(Z_c/\mathfrak{p}_c^{\mathrm{right}} Z_c).$$

Let $m_L = \mathrm{mult}_{C,\chi}^{\mathrm{CM}}$. We have

$$(\diamondsuit_L) \qquad |C| = m_L \chi(1) \quad \text{and} \quad [\,C\,]_c^{\mathrm{CM}} = m_L \chi.$$

By symmetry, we can define a non-negative integer m_R satisfying

$$(\diamondsuit_R) \qquad |C'| = m_R\,\chi(1).$$

Let us first compute the multiplicity of the $Z_{c,\bar{\mathfrak{z}}}$-module $Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{left}} Z_{c,\bar{\mathfrak{z}}}$ for $\bar{\mathfrak{z}} Z_{c,\bar{\mathfrak{z}}}$. The Krull dimension of this module is $n = \dim_{\mathbf{k}} V$. By the additivity formula [**Ser**, Chapter V, §A.2], we have

$$e_{\bar{\mathfrak{z}} Z_{c,\bar{\mathfrak{z}}}}(Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{left}} Z_{c,\bar{\mathfrak{z}}}) = \sum_{\mathrm{coht}(\mathfrak{z})=n} \mathrm{Length}_{Z_{c,\mathfrak{z}}}(Z_{c,\mathfrak{z}}/\mathfrak{p}_c^{\mathrm{left}} Z_{c,\mathfrak{z}})\, e_{\bar{\mathfrak{z}} Z_{c,\bar{\mathfrak{z}}}}(Z_{c,\bar{\mathfrak{z}}}/\mathfrak{z} Z_{c,\bar{\mathfrak{z}}}).$$

Here, $\mathrm{coht}(\mathfrak{z})$ denotes the coheight of the prime ideal $\mathfrak{z}$ of $Z_{c,\bar{\mathfrak{z}}}$. Since $\mathscr{X}_c^{\mathrm{att}}(z_0)$ is irreducible of dimension n, there is only one prime ideal of $Z_{c,\bar{\mathfrak{z}}}$ with coheight n which contains $\mathfrak{p}_c^{\mathrm{left}} Z_{c,\bar{\mathfrak{z}}}$ (and so such that $\mathrm{Length}_{Z_{c,\mathfrak{z}}}(Z_{c,\mathfrak{z}}/\mathfrak{p}_c^{\mathrm{left}} Z_{c,\mathfrak{z}})$ is non-zero), this is the prime ideal $\mathfrak{z}_L$. Moreover, since $Z_{c,\bar{\mathfrak{z}}}/\mathfrak{z}_L Z_{c,\bar{\mathfrak{z}}}$ is a regular ring (because $\mathscr{X}_c^{\mathrm{att}}(z_0)$ is smooth), the multiplicity $e_{\bar{\mathfrak{z}} Z_{c,\bar{\mathfrak{z}}}}(Z_{c,\bar{\mathfrak{z}}}/\mathfrak{z} Z_{c,\bar{\mathfrak{z}}})$ is equal to 1 (see [**Ser**, Chapter IV]). Hence, it follows from (♣) and $(\diamondsuit_L)$ that

$$(\heartsuit_L) \qquad e_{\bar{\mathfrak{z}} Z_{c,\bar{\mathfrak{z}}}}(Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{left}} Z_{c,\bar{\mathfrak{z}}}) = m_L\,\chi(1).$$

By symmetry,

$$(\heartsuit_R) \qquad e_{\bar{\mathfrak{z}} Z_{c,\bar{\mathfrak{z}}}}(Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{right}} Z_{c,\bar{\mathfrak{z}}}) = m_R\,\chi(1).$$

On the other hand, $P/\mathfrak{p}_c^{\mathrm{left}}$ is a polynomial algebra and $Z_c/\mathfrak{p}_c^{\mathrm{left}} Z_c$ is a free $P/\mathfrak{p}_c^{\mathrm{left}}$-module of rank $|W|$. So $Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{left}} Z_{c,\bar{\mathfrak{z}}}$ is a Cohen-Macaulay $Z_{c,\bar{\mathfrak{z}}}$-module of dimension n. Similarly, $Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{right}} Z_{c,\bar{\mathfrak{z}}}$ is a Cohen-Macaulay $Z_{c,\bar{\mathfrak{z}}}$-module of dimension n. Since Z_c has dimension $2n$, it follows from [**Ser**, Chapter V, §B, Corollary to Theorem 4] that

(♠)

$$\chi_{\bar{\mathfrak{z}}}(Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{left}} Z_{c,\bar{\mathfrak{z}}}, Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{right}} Z_{c,\bar{\mathfrak{z}}}) = \mathrm{Length}_{Z_{c,\bar{\mathfrak{z}}}}(Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{left}} Z_{c,\bar{\mathfrak{z}}} \otimes_{Z_{c,\bar{\mathfrak{z}}}} Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{right}} Z_{c,\bar{\mathfrak{z}}}) = \chi(1)^2 > 0.$$

Consequently [**Ser**, Chapter V, §B, Complement to Theorem 1],

$$e_{\bar{\mathfrak{z}} Z_{c,\bar{\mathfrak{z}}}}(Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{left}} Z_{c,\bar{\mathfrak{z}}}) \cdot e_{\bar{\mathfrak{z}} Z_{c,\bar{\mathfrak{z}}}}(Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{right}} Z_{c,\bar{\mathfrak{z}}}) \leqslant \chi_{\bar{\mathfrak{z}}}(Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{left}} Z_{c,\bar{\mathfrak{z}}}, Z_{c,\bar{\mathfrak{z}}}/\mathfrak{p}_c^{\mathrm{right}} Z_{c,\bar{\mathfrak{z}}}).$$

From this last equality and $(\heartsuit_L)$, $(\heartsuit_R)$ and (♠), we deduce that

$$m_L m_R \leqslant 1.$$

We obtain $m_L = m_R = 1$, which proves (c).

(d) follows immediately from (c).

(e) follows from (b) and from Proposition 11.3.5(b). □

CHAPTER 15

CALOGERO-MOSER VERSUS KAZHDAN-LUSZTIG

We have recalled in § 8.6 the definition of Kazhdan-Lusztig left, right or two-sided c-cells, of Kazhdan-Lusztig c-families, and of Kazhdan-Lusztig c-cellular characters, starting from the representation theory of Hecke algebras. On the other hand, the notions of Calogero-Moser left, right or two-sided c-cells, of Calogero-Moser c-families and of Calogero-Moser c-cellular characters have been defined and studied in Part III of this book. We conjecture that these notions coincide. The aim of this chapter is to state precise conjectures and to give arguments which support these conjectures.

15.1. Hecke families

The aim of this section is to recall the statement of Martino's Conjecture [**Mart1**] which relates Calogero-Moser families and Hecke families (see definition 8.5.1), to recall what is known about this conjecture, and to show some theoretical arguments which support it.

> *Let $k^\sharp = (k^\sharp_{\aleph,j})_{(\aleph,j)\in\boldsymbol{\aleph}^\circ}$ denote the element of $\mathscr{C}_\mathbb{R}$ defined by $k^\sharp_{\aleph,j} = k_{\aleph,-j}$ (the indices j being viewed modulo $e_\aleph$).* ***We assume that Assumption (Free-Sym) is satisfied*** *(see §8.1.B).*

15.1.A. Statement and known cases. — We recall here the statement given in [**Mart1**, Conjecture 2.7]:

Conjecture FAM **(Martino)**. — *If $b \in \mathrm{Idem}_{\mathrm{pr}}(\bar{Z}_c)$, then there exists a central idempotent $b^{\mathscr{H}}$ of $\mathcal{O}^{\mathrm{cyc}}[\mathbf{q}^\mathbb{R}]\mathscr{H}_W^{\mathrm{cyc}}(k^\sharp)$ such that:*

(a) $\mathrm{Irr}_{\mathbf{H}}(W,b) = \mathrm{Irr}_{\mathscr{H}}(W,b^{\mathscr{H}})$;

(b) $\dim_\mathbb{C}(\bar{Z}\,b) = \dim_{F(\mathbf{q}^\mathbb{R})}\big(F(\mathbf{q}^\mathbb{R})\mathscr{H}_W^{\mathrm{cyc}}(k^\sharp)b^{\mathscr{H}}\big)$.

In particular, every Calogero-Moser c-family is a union of Hecke $k^\sharp$-families.

This Conjecture has been checked in many cases by computing separately the Calogero-Moser families and Hecke families. At the time this book is written, no theoretical link has been made towards a proof of this Conjecture which does not rely on the Shephard-Todd classification.

***Theorem 15.1.1* (Bellamy, Chlouveraki, Gordon, Martino)**
Assume that W has type $G(de, e, n)$ and assume that, if $n = 2$, then e is odd or $d = 1$. Then the Conjecture FAM holds.

The proof of this Theorem follows from the following works:

- M. Chlouveraki has computed the Hecke families in [**Chl3**] and [**Chl5**].
- Whenever $e = 1$, the Calogero-Moser families have been computed by I. Gordon [**Gor2**] for rational values of k (using Hilbert schemes). This result has been extended to all values of k by M. Martino [**Mart2**] using purely algebraic methods.
- M. Chlouveraki's combinatoric and I. Gordon's combinatoric have been compared by M. Martino [**Mart1**] to show that Conjecture FAM holds whenever $e = 1$.
- Whenever e is any non-negative integer (satisfying the conditions of the Theorem), the Calogero-Moser families have been computed by [**Bel5**] for rational values of k, because this computation relied on I. Gordon's result. His method can nevertheless be extended to any value of k, once M. Martino's result has been established [**Mart2**].

It was also Conjectured by M. Martino that, whenever c is generic, then the Calogero-Moser c-families and the Hecke $k^\sharp$-families coincide. A counter-example has been recently found by U. Thiel [**Thi1**].

Thanks to his computations, U. Thiel has also obtained new cases of Conjecture FAM amongst the exceptional complex reflection groups [**Thi2**, Theorem 25.4]. It must be noticed that M. Chlouveraki has computed the partitions into Hecke families for exceptional groups [**Chl4**] in all cases, while U. Thiel has computed the partition into Calogero-Moser families for some (not all) exceptional groups, and mainly in the generic parameter case. We summarize his results in the generic case:

***Theorem 15.1.2* (Thiel).** — *If W has type*

$$G_4, G_5, G_6, G_7, G_8, G_9, G_{10}, G_{12}, G_{13}, G_{14}, G_{15}, G_{20}, G_{22}, G_{23}, G_{24}, G_{25}, G_{26},$$

then the Conjecture FAM holds for generic parameters.

15.1.B. Theoretical arguments. — Corollary 9.4.2 shows that:

***Proposition 15.1.3*.** — *In Conjecture FAM, the statement (a) implies the statement (b).*

Proof. — Keep the notation of Conjecture FAM (b, $b^{\mathscr{H}}$,...). Since the algebra $F(\mathbf{q}^{\mathbb{R}})\mathscr{H}_W^{\mathrm{cyc}}(k^\sharp)$ is split semisimple, we have

$$\dim_{F(\mathbf{q}^{\mathbb{R}})}\big(F(\mathbf{q}^{\mathbb{R}})\mathscr{H}_W^{\mathrm{cyc}}(k^\sharp)b^{\mathscr{H}}\big) = \sum_{\chi \in \mathrm{Irr}_{\mathscr{H}}(W,b^{\mathscr{H}})} \chi(1)^2.$$

But, on the other hand, it follows from Corollary 9.4.2 that

$$\dim_{\mathbb{C}}(\bar{Z}_c b) = \sum_{\chi \in \mathrm{Irr}_{\mathbf{H}}(W,b)} \chi(1)^2.$$

Whence the result. ☐

***Remark 15.1.4*.** — The better theoretical argument to support Conjecture FAM is (for the moment) the following. It has been proven that, if χ and χ' are in the same Calogero-Moser c-family (respectively Hecke $k^\sharp$-family), then $\Omega_\chi^c(\mathbf{eu}) = \Omega_{\chi'}^c(\mathbf{eu})$ (respectively $C_\chi(k^\sharp) = C_{\chi'}(k^\sharp)$): see Lemma 9.2.3 (respectively Lemma 8.5.2). But it follows from Corollary 7.3.2 and from the definition of $C_\chi(k^\sharp)$ that

$$\Omega_\chi^c(\mathbf{eu}_c) = C_\chi(k^\sharp). \tag{15.1.5}$$

Even though this numerical invariant is not enough for determining in general the Calogero-Moser families, it is relatively sharp. ■

A last argument is given by the next proposition, which follows from Lemma 8.5.4 and Corollary 9.3.5:

***Proposition 15.1.6*.** — *If $\mathscr{F}$ is a Calogero-Moser c-family (respectively a Hecke $k^\sharp$-family), then $\mathscr{F}\varepsilon$ is a Calogero-Moser c-family (respectively a Hecke $k^\sharp$-family).*

15.2. Kazhdan-Lusztig cells

> **Assumption.** *In §15.2, we assume that W is a Coxeter group, that $\mathbf{k} = \mathbb{C}$ and that $\mathbf{k}_{\mathbb{R}} = \mathbb{R}$. We fix an element $c \in \mathscr{C}(\mathbb{R})$.*

15.2.A. Cells and characters. — The first conjecture is concerned with two-sided cells and their associated families.

Conjecture LR. — *There exists a choice of the prime ideal $\bar{\mathfrak{r}}_c$ lying over $\bar{\mathfrak{q}}_c$ such that:*

(a) *The partition of W into Calogero-Moser two-sided c-cells coincides with the partition into Kazhdan-Lusztig two-sided c-cells.*

(b) *Assume that $c_s \geqslant 0$ for all $s \in \mathrm{Ref}(W)$. If $\Gamma \in {}^{\mathrm{CM}}\mathrm{Cell}_{LR}^c(W) = {}^{\mathrm{KL}}\mathrm{Cell}_{LR}^c(W)$, then $\mathrm{Irr}_\Gamma^{\mathrm{CM}}(W) = \mathrm{Irr}_\Gamma^{\mathrm{KL}}(W)$.*

We propose a similar conjecture for left cells and cellular characters.

Conjecture L. — *There exists a choice of the prime ideal $\mathfrak{r}_c^{\mathrm{left}}$ lying over $\mathfrak{q}_c^{\mathrm{left}}$ such that:*

(a) *The partition of W into Calogero-Moser left c-cells coincides with the partition into Kazhdan-Lusztig left c-cells.*

(b) *Assume that $c_s \geqslant 0$ for all $s \in \mathrm{Ref}(W)$. If $C \in {}^{\mathrm{CM}}\mathrm{Cell}_L^c(W) = {}^{\mathrm{KL}}\mathrm{Cell}_L^c(W)$, then $[\,C\,]_c^{\mathrm{CM}} = [\,C\,]_c^{\mathrm{KL}}$.*

A similar conjecture can be stated for right cells. Also, if Conjectures LR and L have positive answer, it should be true that $\mathfrak{r}_c^{\mathrm{left}} \subset \bar{\mathfrak{r}}_c$.

We propose a specific choice of ideals $\mathfrak{r}_0$, $\bar{\mathfrak{r}}_c$ and $\mathfrak{r}_c^{\mathrm{left}}$. We describe that choice in the equivalent setting of paths, cf §6.6.

Let $C'_{\mathbb{R}}$ be the dual chamber to $C_{\mathbb{R}}$, obtained as the image of $C_{\mathbb{R}}$ through some isomorphism of $\mathbb{R}W$-modules $\mathbb{R} \otimes_{\mathbb{Q}} V_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{R} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}^*$. We choose $(v_{\mathbb{C}}, v_{\mathbb{C}}^*) \in C_{\mathbb{R}} \times C'_{\mathbb{R}}$ and we choose the path γ contained in $\mathscr{C}(\mathbb{R})$. We conjecture that Conjectures LR and L hold with such choices.

15.2.B. Characters. — First of all, note that the set of Calogero-Moser c-families, as well as the set of Calogero-Moser c-cellular characters, do not depend on the choice of the ideal $\mathfrak{r}_c^{\mathrm{left}}$. At the level of characters, the statements (b) of Conjectures LR and L imply the following simpler statement, which does refer to the choice of a prime ideal of R.

***Conjecture C*. —** (a) *The partition of* $\mathrm{Irr}(W)$ *into Calogero-Moser* c*-families coincides with the partition into Kazhdan-Lusztig* c*-families (Gordon-Martino).*
(b) *The set of Calogero-Moser* c*-cellular characters coincides with the set of Kazhdan-Lusztig* c*-cellular characters.*

Note that (a) above has been conjectured by Gordon and Martino [**GoMa**, Conjecture 1.3(1)]. So Conjecture LR lifts Gordon-Martino's Conjecture at the level of two-sided cells.

COMMENTARY - The choices of the prime ideals $\bar{\mathfrak{r}}_c$ or $\mathfrak{r}_c^{\mathrm{left}}$ are not relevant for the Conjecture C, but they are relevant for the Conjectures LR and L. ■

15.3. Evidence

As will be explained in Part IV, the conjectures stated in § 15.2 hold when W has type A_1 or B_2: they also hold in type A_2, but we have not included the computations in this book. The case of type B_2 will be treated in § 19. However, the difficulty of the computations does not allow us for now to extend this list of examples. Note that Conjectures LR and L have been proved by the first author whenever W is dihedral of order $2m$, with m odd [**Bon5**, Corollary 6.3] (it turns out that, in this case, the Galois group G is $\mathfrak{S}_W$).

A different approach, via Gaudin operators (cf Chapter 13) has been used to settle Conjecture L for type A_n [**Wh, BrGoWh, HaKaRyWe**].

The aim of §15.3 is to give some evidence in support of these conjectures. Note however that Conjecture C, which only deals with characters (and not with the partition of W into cells), holds for some infinite series of groups (see the details below).

15.3.A. The case $c = 0$. — The following facts will be shown in § 17.

Proposition 15.3.1. — *When $c=0$, there is only one Calogero-Moser left, right or two-sided cell: it is W itself. Moreover,*

$$\mathrm{Irr}_W^{\mathrm{CM}}(W)=\mathrm{Irr}(W) \qquad \textit{and} \qquad [\,W\,]_0^{\mathrm{CM}}=[\,\mathbb{C}W\,]_{\mathbb{C}W}=\sum_{\chi\in\mathrm{Irr}(W)}\chi(1)\chi.$$

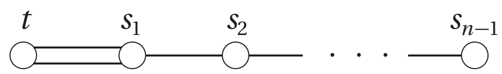
Corollary 15.3.2. — *Conjectures L and LR hold when $c=0$.*

Proof. — This follows from the comparison of [**Bon2**, Corollaries 2.13 et 2.14] with Proposition 15.3.1. □

15.3.B. Constructible characters, Lusztig families. — In the sequel of this Chapter, we will only deal with positive parameters. We do not know how to treat the case where only some parameters are equal to 0 (in order to compare with [**Bon2**, Corollaries 2.13 et 2.14]).

From now on, and until the end §15, we assume that $c_s>0$ for all $s\in\mathrm{Ref}(W)$.

CONVENTION - When (W,S) has type B_n, we write $S=\{t,s_1,s_2,\dots,s_{n-1}\}$ with the convention that t is not conjugate to some s_i, so the Dynkin diagram is



In this case, we will set $b=c_t$ and $a=c_{s_1}=c_{s_2}=\cdots=c_{s_{n-1}}$. ■

Lusztig [**Lus4**, §22] has defined a notion of *constructible characters* of W (that we will call here *c-constructible characters*). We can then define a graph $\mathcal{G}_c(W)$ as follows:

- The set of vertices of $\mathcal{G}_c(W)$ is $\mathrm{Irr}(W)$.
- Two distinct irreducible characters of W are linked in $\mathcal{G}_c(W)$ if they appear in the same c-constructible character.

We then define *Lusztig c-families* as the connected components of $\mathcal{G}_c(W)$.

Proposition 15.3.3. — *Assume that one of the following hold:*

(1) *c is constant;*
(2) *$|S|\leqslant 2$;*
(3) *(W,S) has type F_4;*
(4) *(W,S) has type B_n, $a\neq 0$ and $b/a\in\{1/2,1,3/2,2\}\cup]n-1,+\infty)$.*

Then:

(a) *The c-constructible characters and the Kazhdan-Lusztig c-cellular characters coincide.*
(b) *The Lusztig c-families and the Kazhdan-Lusztig c-families coincide.*

Proof. — Lusztig [**Lus4**, Conjectures 14.2] has proposed a series of conjectures (numbered P1, P2,..., P15) about Kazhdan-Lusztig cells and the **a**-*function*. They have been proven in the following cases:

(1) when c is constant in [**Lus4**, chapitre 15];
(2) when $|S| \leqslant 2$ in [**Lus4**, chapitre 17];
(3) when (W,S) has type F_4 in [**Ge2**];
(4) when (W,S) has type B_n and $a = 0$ or when $a \neq 0$ and $b/a \in \{1/2, 1, 3/2, 2\}$ in [**Lus4**, Chapter 16];
(4′) when (W,S) has type B_n, $a \neq 0$ and $b/a > n-1$ in [**BoIa**], [**Bon1**] et [**GeIa**].

Also, it is shown in [**Lus4**, Lemma 22.2] and [**Ge3**, §6 and §7] that these conjectures implies that the c-constructible characters and the Kazhdan-Lusztig c-cellular characters coincide. This shows (a). The statement (b) now follows from [**BoGe**, Corollary 1.8]. ☐

15.3.C. Conjectures about characters. —

Families. — The c-constructible characters (and so the Lusztig c-families) have been computed in all cases by Lusztig [**Lus4**]. We deduce from Proposition 15.3.3 that the Kazhdan-Lusztig c-families are known in the cases (1), (2), (3) and (4) of Proposition 15.3.3. But, the explicit computation of Calogero-Moser c-families for classical type has been made in the series of articles [**Bel1**], [**Bel5**], [**Gor1**], [**Gor2**], [**GoMa**], [**Mart2**]. Using all of those computations, we obtain the following theorem.

Theorem 15.3.4. — *Assume that one of the following holds:*

(1) $|S| \leqslant 2$.
(2) (W,S) *has type* A_n *or* D_n.
(3) (W,S) *has type* B_n, $a > 0$ *and* $b/a \in \{1/2, 1, 3/2, 2\} \cup]n-1, +\infty)$.

Then Conjecture C(a) holds.

Cellular characters. — If (W,S) has type A_n or if (W,S) has type B_n with $a > 0$ and $b/a \in \{1/2, 3/2\} \cup]n-1, +\infty)$, then it follows from the previous results that the Kazhdan-Lusztig c-cellular characters are irreducible. Moreover, it follows from the work of Gordon and Martino that, in those same cases, the Calogero-Moser space $\mathscr{X}_c$ is smooth. Therefore, the next theorem follows from Theorem 14.4.1.

Theorem 15.3.5. — *Assume that one of the following holds:*

(1) (W,S) *has type* A_n*;*

(2) (W,S) *has type* B_n *with* $a \neq 0$ *and* $b/a \in \{1/2, 3/2\} \cup]n-1, +\infty)$.

Then Conjecture C(b) holds (and the c-cellular characters are irreducible).

Whenever W is dihedral, the first author proved Conjecture C(b), by a direct explicit computation using Gaudin algebras [**Bon5**, Table 4.14].

Other arguments. — First of all, note that, if we assume that Lusztig's Conjectures P1, P2,..., P15 hold (see [**Lus4**, conjectures 14.2]), then the previous argument imply that Conjecture C(a) holds in type B_n and Conjecture C(b) holds in type B_n whenever $a > 0$ and $b/a \notin \{1, 2, \dots, n-1\}$ (because in this case, the c-constructible characters are irreducible and the Calogero-Moser space is smooth).

Remark 15.3.6. — If $\mathscr{F}$ is a Calogero-Moser (respectively Kazhdan-Lusztig) c-family, then $\mathscr{F}\varepsilon$ is a Calogero-Moser (respectively Kazhdan-Lusztig) c-family: see Corollary 9.3.5 and (8.6.14).

Similarly, if χ is a Calogero-Moser (respectively Kazhdan-Lusztig) c-cellular character, then $\chi\varepsilon$ is a Calogero-Moser (respectively Kazhdan-Lusztig) c-cellular character: see Corollaries 11.4.9 and (8.6.13). ■

Remark 15.3.7. — If $\mathscr{F}$ is a Calogero-Moser (respectively Lusztig) c-family, then there exists a unique character $\chi \in \mathscr{F}$ with minimal **b**-invariant: see Theorem 9.4.1(b) (respectively [**Bon4**], or [**Lus2**, Theorem 5.25 and its proof] whenever c is constant).

Similarly, if χ is a Calogero-Moser c-cellular character (respectively a c-constructible character), then there exists a unique irreducible component χ of minimal **b**-invariant: see Theorem 11.4.12. ■

15.3.D. Cells. —

Two-sided cells. — The first argument which supports Conjecture LR comes from the comparison of the cardinality of cells, and from the fact that Conjecture C(a) has been proven in many cases.

Remark 15.3.8. — Assume here (W, c) satisfies one of the assumptions of Theorem 15.3.4. Let $\mathscr{F}$ be a Calogero-Moser c-family (that is, a Kazhdan-Lusztig c-family according to Theorem 15.3.4). Let Γ_{CM} (respectively Γ_{KL}) denote the Calogero-Moser (respectively Kazhdan-Lusztig) two-sided c-cell covering $\mathscr{F}$. It follows from Theorem 10.2.7(c) that

$$|\Gamma_{\mathrm{CM}}| = \sum_{\chi \in \mathscr{F}} \chi(1)^2$$

and it follows from (8.6.6) that

$$|\Gamma_{\mathrm{KL}}| = \sum_{\chi \in \mathscr{F}} \chi(1)^2.$$

Therefore,

$$|\Gamma_{\mathrm{CM}}| = |\Gamma_{\mathrm{KL}}|.$$

This is not sufficient to show that $\Gamma_{\mathrm{CM}} = \Gamma_{\mathrm{KL}}$. However, this shows Conjecture LR whenever the Galois group G is equal to $\mathfrak{S}_W$: indeed, by replacing $\bar{\mathfrak{r}}_c$ by some $g(\bar{\mathfrak{r}}_c)$ for some $g \in G = \mathfrak{S}_W$, we can arrange that $\Gamma_{\mathrm{CM}} = \Gamma_{\mathrm{KL}}$ (for all families $\mathscr{F}$). This also shows the importance of making the choice of $\bar{\mathfrak{r}}_c$ precise in Conjecture LR. ■

Remark 15.3.9. — Let Γ_{CM} (respectively Γ_{KL}) be a Calogero-Moser (respectively Kazhdan-Lusztig) two-sided c-cell. Let w_0 denote the longest element of W. Then:

- By (8.6.12) and (8.6.15), $w_0\Gamma_{\mathrm{KL}} = \Gamma_{\mathrm{KL}} w_0$ is a Kazhdan-Lusztig c-cell and $\mathrm{Irr}^{\mathrm{KL}}_{w_0\Gamma_{\mathrm{KL}}}(W) = \mathrm{Irr}^{\mathrm{KL}}_{\Gamma_{\mathrm{KL}}}(W)\varepsilon$.
- Since all the reflections of W have order 2, it has been shown in Corollary 10.2.9 that, *if w_0 is central in W*, then $w_0\Gamma_{\mathrm{CM}} = \Gamma_{\mathrm{CM}} w_0$ is a Calogero-Moser two-sided c-cell and $\mathrm{Irr}^{\mathrm{CM}}_{w_0\Gamma_{\mathrm{CM}}}(W) = \mathrm{Irr}^{\mathrm{CM}}_{\Gamma_{\mathrm{CM}}}(W)\varepsilon$.

These results show some analogy *whenever w_0 is central in W*. For the second statement, it is not reasonable to expect that it is true whenever w_0 is not central (as is shown by the type A_2) without making a judicious choice of $\bar{\mathfrak{r}}_c$. ■

Left cells. — Let us recall that numerous Kazhdan-Lusztig left cells give rise to the same Kazhdan-Lusztig cellular character. On the Calogero-Moser side, Corollary 11.4.6 also shows that numerous Calogero-Moser left cells give rise to the same Calogero-Moser cellular character (see for instance Theorem 14.4.1 in the smooth case).

Remark 15.3.10. — Let C_{CM} (respectively C_{KL}) be a Calogero-Moser (respectively Kazhdan-Lusztig) left c-cell. Let w_0 denote the longest element of W. Then:

- It follows from (8.6.12) and (8.6.13) that $w_0 C_{\mathrm{KL}}$ and $C_{\mathrm{KL}} w_0$ are Kazhdan-Lusztig left c-cells and that $[\, w_0 C_{\mathrm{KL}}\,]_c^{\mathrm{KL}} = [\, C_{\mathrm{KL}} w_0\,]_c^{\mathrm{KL}} = [\, C_{\mathrm{KL}}\,]_c^{\mathrm{KL}} \varepsilon$.
- Since all the reflections of W have order 2, it follows from Corollary 11.4.9 that, *if w_0 is central in W*, then $w_0 C_{\mathrm{CM}} = C_{\mathrm{CM}} w_0$ is a Calogero-Moser left c-cell and that $[\, w_0 C_{\mathrm{CM}}\,]_c^{\mathrm{CM}} = [\, C_{\mathrm{CM}} w_0\,]_c^{\mathrm{CM}} = [\, C_{\mathrm{CM}}\,]_c^{\mathrm{CM}} \varepsilon$. ■

Remark 15.3.11. — Note also the analogy between the following equalities: if C is a Calogero-Moser (respectively Kazhdan-Lusztig) left c-cell and if $\chi \in \mathrm{Irr}(W)$, then

$$\begin{cases} |C| = \displaystyle\sum_{\psi \in \mathrm{Irr}(W)} \mathrm{mult}_{C,\psi}^{\mathrm{CM}}\, \psi(1), \\ \chi(1) = \displaystyle\sum_{C' \in {}^{\mathrm{CM}}\mathrm{Cell}_L(W)} \mathrm{mult}_{C',\chi}^{\mathrm{CM}} \end{cases}$$

(respectively

$$\begin{cases} |C| = \displaystyle\sum_{\psi \in \mathrm{Irr}(W)} \mathrm{mult}_{C,\psi}^{\mathrm{KL}}\, \psi(1), \\ \chi(1) = \displaystyle\sum_{C' \in {}^{\mathrm{KL}}\mathrm{Cell}_L(W)} \mathrm{mult}_{C',\chi}^{\mathrm{KL}} \end{cases}).$$

See Proposition 11.4.2 (respectively Lemma 8.6.7). It would be interesting to study if other numerical properties of Kazhdan-Lusztig left cells (as for instance [**Ge3**, lemme 4.6]) are also satisfied by Calogero-Moser left cells. ■

A final argument to support Conjectures L and LR is the following.

Theorem 15.3.12. — *Assume that we are in one of the following cases:*

(1) *(W,S) has type A_n and $c > 0$;*
(2) *(W,S) has type B_n with $a > 0$ and $b/a \in \{1/2, 3/2\} \cup]n-1, +\infty)$.*

Then there exists a bijective map $\varphi : W \to W$ such that:

(a) *If* Γ *is a Kazhdan-Lusztig two-sided* c*-cell, then* $\varphi(\Gamma)$ *is a Calogero-Moser two-sided* c*-cell and* $\mathrm{Irr}_{\Gamma}^{\mathrm{KL}}(W) = \mathrm{Irr}_{\varphi(\Gamma)}^{\mathrm{CM}}(W)$.

(b) *If* C *is a Kazhdan-Lusztig left* c*-cell, then* $\varphi(C)$ *is a Calogero-Moser left* c*-cell and* $[C]_c^{\mathrm{KL}} = [\varphi(C)]_c^{\mathrm{CM}}$.

Proof. — Under the assumptions (1) or (2), the Calogero-Moser space $\mathscr{Z}_c$ is smooth (see [**EtGi**, Theorem 1.24] in case (1) and [**Gor1**, Lemma 4.3 and its proof]) in case (2)) and so Theorem 14.4.1 can be applied to all the Calogero-Moser c-cells of W. The result follows now from a comparison of cardinalities of cells. □

CHAPTER 16

CONJECTURES ABOUT THE GEOMETRY OF $\mathscr{Z}_c$

Assumption. *We assume in this chapter that* $\mathbf{k} = \mathbb{C}$.

16.1. Cohomology

We follow some of the notations of Appendix A. Given $i \in \mathbb{Z}$, we set

$$(\mathbb{C}W)^i = \bigoplus_{\substack{w\in W \\ \operatorname{codim}_{\mathbb{C}}(V^w)=i}} \mathbb{C}w,$$

so that $\mathbb{C}W = \bigoplus_{i\in\mathbb{Z}}(\mathbb{C}W)^i$. This is of course not a grading on the algebra $\mathbb{C}W$, but the filtration by the vector subspaces $(\mathbb{C}W)^{\leqslant i} = \bigoplus_{j \leqslant i}(\mathbb{C}W)_j$ induces a structure of filtered algebra on $\mathbb{C}W$.

If A is any subalgebra of $\mathbb{C}W$, it inherits a structure of filtered algebra by setting $A^{\leqslant i} = A \cap (\mathbb{C}W)^{\leqslant i}$. As in Appendix A, we can then define its associated Rees algebra $\operatorname{Rees}(A)$ (which is contained in $\mathbb{C}[\hbar] \otimes A$), as well as its associated graded algebra $\operatorname{gr}(A)$.

If $\mathscr{X}$ is a quasi-projective complex algebraic variety, we denote by $\mathrm{H}^i(\mathscr{X})$ its i-th singular cohomology group with coefficients in $\mathbb{C}$. We denote by $\mathrm{H}^{2\bullet}(\mathscr{X})$ the graded algebra $\bigoplus_{i\in\mathbb{N}} \mathrm{H}^{2i}(\mathscr{X})$. The Euler characteristic of $\mathscr{X}$ will be denoted by $\boldsymbol{\chi}(\mathscr{X})$. If $\mathscr{X}$ is endowed with an algebraic action of $\mathbb{C}^\times$, we denote by $\mathrm{H}^i_{\mathbb{C}^\times}(\mathscr{X})$ its i-th equivariant cohomology group, with coefficients in $\mathbb{C}$. We denote by $\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{X})$ the graded algebra $\bigoplus_{i\in\mathbb{N}} \mathrm{H}^{2i}_{\mathbb{C}^\times}(\mathscr{X})$: it will be viewed as a $\mathbb{C}[\hbar]$-algebra by identifying $\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathrm{pt})$ with $\mathbb{C}[\hbar]$ in the usual way.

Recall that, given $c \in \mathscr{C}$, we have defined in §4.2.C a morphism of algebras $\boldsymbol{\Omega}^c : Z_c \longrightarrow \mathrm{Z}(\mathbb{C}W)$. We propose the following conjectures about the (equivariant) cohomology of the variety $\mathscr{Z}_c$.

Conjecture COH. — *Let* $c \in \mathscr{C}$.

(1) *If* $i \in \mathbb{N}$, *then* $\mathrm{H}^{2i+1}(\mathscr{Z}_c) = 0$.

(2) *We have an isomorphism of graded algebras* $\mathrm{H}^{2\bullet}(\mathscr{X}_c) \simeq \mathrm{gr}(\mathrm{Im}\,\boldsymbol{\Omega}^c)$.

Conjecture ECOH. — *Let* $c \in \mathscr{C}$.

(1) *If* $i \in \mathbb{N}$, *then* $\mathrm{H}^{2i+1}_{\mathbb{C}^\times}(\mathscr{X}_c) = 0$.

(2) *We have an isomorphism of graded* $\mathbb{C}[\hbar]$*-algebras* $\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{X}_c) \simeq \mathrm{Rees}(\mathrm{Im}\,\boldsymbol{\Omega}^c)$.

Example 16.1.1. — Assume here that $c = 0$. Recall (Example 4.2.8) that $\mathbf{H}_0 = \mathbb{C}[V \times V^*] \rtimes W$ and that $Z_0 = \mathbb{C}[V \times V^*]^{\Delta W}$. In particular, $\mathscr{X}_0 = (V \times V^*)/\Delta W$ and $\mathrm{Im}(\boldsymbol{\Omega}^0) = \mathbb{C}$. Therefore,

$$\mathrm{H}^i(Z_0) = \mathrm{H}^i(V \times V^*)^W = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathrm{H}^i_{\mathbb{C}^\times}(Z_0) = \mathrm{H}^i_{\mathbb{C}^\times}(V \times V^*)^W \simeq \mathrm{H}^i_{\mathbb{C}^\times}(\mathrm{pt}),$$

so Conjectures COH and ECOH hold. ■

Given $E \in \mathrm{Irr}(W)$, we denote by $e_E \in \mathrm{Z}(\mathbb{C}W)$ the corresponding primitive central idempotent (the unique one such that $\omega_E(e_E) = 1$). Given $\mathscr{F}$ is a subset of $\mathrm{Irr}(W)$, we set $e_{\mathscr{F}} = \sum_{E \in \mathscr{F}} e_E$. It follows from Lemma 9.2.3 and from (14.2.2) that

(16.1.2)
$$\mathrm{Im}\,\boldsymbol{\Omega}^c = \bigoplus_{p \in \mathscr{X}_c^{\mathbb{C}^\times}} \mathbb{C} e_{\Theta_c^{-1}(p)}.$$

In particular,

(16.1.3)
$$\dim_{\mathbb{C}}(\mathrm{Im}\,\boldsymbol{\Omega}^c) = |\mathscr{X}_c^{\mathbb{C}^\times}|$$

hence

(16.1.4)
$$\boldsymbol{\chi}(\mathscr{X}_c) = \dim_{\mathbb{C}}(\mathrm{Im}\,\boldsymbol{\Omega}^c).$$

This is compatible with Conjecture COH.

***Theorem 16.1.5* (Etingof-Ginzburg)**. — *If* $\mathscr{X}_c$ *is smooth, then Conjecture COH holds.*

Proof. — Assume that $\mathscr{X}_c$ is smooth. In [**EtGi**, Theorem 1.8], Etingof and Ginzburg proved that this implies that $\mathscr{X}_c$ has no odd cohomology, and that

$$\mathrm{H}^{2\bullet}(\mathscr{X}_c) \simeq \mathrm{gr}(\mathrm{Z}(\mathbb{C}W)).$$

By Proposition 9.6.6, the smoothness of $\mathscr{X}_c$ implies that $\Theta_c : \mathrm{Irr}(W) \to \mathscr{X}_c^{\mathbb{C}^\times}$ is bijective, so that $\mathrm{Im}\,\boldsymbol{\Omega}^c = \mathrm{Z}(\mathbb{C}W)$. □

Based on Etingof-Ginzburg's result, Peng Shan and the first author proved the following, using localization methods [**BoSh**, Theorem A]:

Theorem 16.1.6. — *If $\mathscr{Z}_c$ is smooth, then Conjecture ECOH holds.*

Apart from the smooth case and the case $c = 0$, both conjectures are known only in rank 1. For Conjecture COH, see the upcoming Chapter 18 (Theorem 18.5.8). For Conjecture ECOH, see [**BoSh**, Proposition 1.7].

16.2. Fixed points

> **Assumption and notation.** *Recall that $\mathscr{N} = N_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$. We fix in this section an element of finite order $\tau \in \mathscr{N}$. That element τ acts on $\mathscr{C}$ and $\mathscr{Z}$ and, if $c \in \mathscr{C}^\tau$, it acts on $\mathscr{Z}_c$. We denote by $\mathscr{Z}^\tau$ (respectively $\mathscr{Z}_c^\tau$, for $c \in \mathscr{C}^\tau$) the **reduced** closed subvariety of $\mathscr{Z}$ (respectively $\mathscr{Z}_c$) consisting of fixed points of τ in $\mathscr{Z}$ (respectively $\mathscr{Z}_c$): it is an affine variety whose algebra of regular functions is $Z/\sqrt{\langle \tau(z) - z, z \in Z\rangle}$ (respectively $Z_c/\sqrt{\langle \tau(z) - z, z \in Z_c\rangle}$).*

We say that a pair (V', W') is a *reflection subquotient* of (V, W) if V' is a subspace of V and if there exists a subgroup N' of the stabilizer of V' in W such that $W' = N'/N_1'$ is a reflection group on V', where N_1' is the kernel of the action of N' on V'. In this case, we denote by $\mathrm{Ref}(W')$ the set of reflections of W' for its action on V', by $\mathscr{C}(W')$ the vector space of maps $c' : \mathrm{Ref}(W') \to \mathbb{C}$ constant on W'-conjugacy classes and by $\mathscr{Z}(V', W')$ the Calogero-Moser space associated with (V', W').

Conjecture FIX. — *Given $\mathscr{X}$ an irreducible component of $\mathscr{Z}^\tau$, there exists a reflection subquotient (V', W') of (V, W) and a linear map $\varphi : \mathscr{C}^\tau \to \mathscr{C}(V', W')$ such that*

$$\mathscr{X} \simeq \mathscr{Z}(V', W') \times_{\mathscr{C}(W')} \mathscr{C}^\tau.$$

Example 16.2.1. — Assume in this example that $\tau \in \mathbb{C}^\times$ is a root of unity, acting on V by scalar multiplication. Note that $\mathscr{C}^t = \mathscr{C}$ in this case. Under this assumption, we will show in § 19.8 that Conjecture FIX holds when W is of type B_2. It is shown by the first author [**Bon5**, Theorem 7.1 and Proposition 8.3] that it also holds if W is of type G_2 (computer calculation using the `MAGMA` software [**Magma**]), or if W is dihedral of order $2m$, with m odd, and τ is a primitive m-th root of unity.

When τ is a root of unity, it is shown in [**BoMa**] that Conjecture FIX holds if W is the group denoted G_4 in Shephard-Todd classification (this uses again the `MAGMA` software). The best result about Conjecture FIX has been obtained by Ruslan Maksimau and the first author [**BoMa**]. We need some notation to state it. Let $\mathscr{C}_{\mathrm{sm}}$ denote the (open) subset of $\mathscr{C}$ consisting of elements $c \in \mathscr{C}$ such that $\mathscr{Z}_c$ is smooth (it can be empty, see Theorem 9.6.9). We put $\mathscr{Z}_{\mathrm{sm}} = \mathscr{Z} \times_{\mathscr{C}} \mathscr{C}_{\mathrm{sm}}$.

Theorem 16.2.2. — *Let $\mathscr{X}$ be an irreducible component of $\mathscr{Z}_{\mathrm{sm}}$ and assume that τ is a root of unity. Then there exists a reflection subquotient (V', W') of (V, W) and a linear map $\varphi : \mathscr{C} \to \mathscr{C}(V', W')$ such that $\varphi(\mathscr{C}_{\mathrm{sm}}) \subset \mathscr{C}_{\mathrm{sm}}(V', W')$ and*

$$\mathscr{X} \simeq \mathscr{Z}_{\mathrm{sm}}(V', W') \times_{\mathscr{C}_{\mathrm{sm}}(W')} \mathscr{C}_{\mathrm{sm}}.$$

Note also that the linear map φ of the above theorem is explicitly described in [**BoMa**]. ■

PART IV

EXAMPLES

CHAPTER 17

CASE $c = 0$

17.1. Two-sided cells, families

Recall that R_+ denotes the unique maximal bi-homogeneous ideal of R and that

$$R/R_+ \simeq \mathbf{k}$$

(see Corollary 5.3.3). Recall also that D_+ (respectively I_+) denotes its decomposition (respectively inertia) group and that

$$D_+ = I_+ = G$$

(see Corollary 5.3.4).

Proposition 17.1.1. — *R_+ is the unique prime ideal of R lying over $\bar{\mathfrak{p}}_0$.*

Proof. — Indeed, $\bar{\mathfrak{p}}_0 = P_+$ and so R_+ is a prime ideal of R lying over $\bar{\mathfrak{p}}_0$: since it is stabilized by G, the uniqueness is proven. □

So if we denote by $\bar{\mathfrak{r}}_0$ the unique prime ideal of R lying over $\bar{\mathfrak{p}}_0 = P_+$, $\bar{D}_0$ its decomposition group and $\bar{I}_0$ its inertia group, then

$$\bar{\mathfrak{r}}_0 = R_+ \qquad \text{and} \qquad \bar{D}_0 = \bar{I}_0 = G. \tag{17.1.2}$$

Hence:

Corollary 17.1.3. — *W contains only one Calogero-Moser two-sided 0-cell, namely W itself, and*

$$\mathrm{Irr}_W^{\mathrm{CM}}(W) = \mathrm{Irr}(W).$$

A feature of the specialization at 0 is that the algebra $\bar{\mathbf{H}}_0$ inherits the $(\mathbb{N}\times\mathbb{N})$-grading, and so the $\mathbb{N}$-grading. If we set

$$\bar{\mathbf{H}}_{0,+} = \bigoplus_{i \geqslant 1} \bar{\mathbf{H}}_0^{\mathbb{N}}[i],$$

then $\bar{\mathbf{H}}_{0,+}$ is a nilpotent two-sided ideal of $\bar{\mathbf{H}}_0$ and, since $\bar{\mathbf{H}}_0^{\mathbb{N}}[0] = \mathbf{k}W$, we get the following result:

Proposition 17.1.4. — $\operatorname{Rad}(\bar{\mathbf{H}}_0) = \bar{\mathbf{H}}_{0,+}$ *and* $\bar{\mathbf{H}}_0/\operatorname{Rad}(\bar{\mathbf{H}}_0) \simeq \mathbf{k}W$.

In particular,

$$[\mathscr{L}_{\bar{\mathbf{K}}_0}(\chi)]^{\mathrm{gr}}_{\mathbf{k}W} = \chi \in K_0(\mathbf{k}W)[\mathbf{t},\mathbf{t}^{-1}] \tag{17.1.5}$$

and

$$[\bar{\mathbf{K}}_0\bar{\mathscr{M}}(\chi)]_{\bar{\mathbf{H}}_0} = \chi(1)\,[\mathbf{k}W]_{\mathbf{k}W} \in \mathbb{Z}\operatorname{Irr}(W) \simeq K_0(\bar{\mathbf{H}}_0). \tag{17.1.6}$$

17.2. Left cells, cellular characters

Recall that, in §5.1.B, we have fixed a prime ideal $\mathfrak{r}_0$ of R lying over $\mathfrak{q}_0 = \mathfrak{C}_0 Q$ as well as a field isomorphism

$$\operatorname{iso}_0 : \mathbf{k}(V\times V^*)^{\Delta Z(W)} \xrightarrow{\sim} \mathbf{M}_0 = k_R(\mathfrak{r}_0)$$

whose restriction to $\mathbf{k}(V\times V^*)^{\Delta W}$ is the canonical isomorphism $\mathbf{k}(V\times V^*)^{\Delta W} \xrightarrow{\sim} \operatorname{Frac}(Z_0) \xrightarrow{\sim} \mathbf{L}_0$. Hence, $R/\mathfrak{r}_0 \subset \operatorname{iso}_0(\mathbf{k}[V\times V^*]^{\Delta Z(W)})$ and these two rings have the same fraction field, the field $\mathbf{M}_0$. Recall also that we do not know if these two rings are equal, or equivalently, if $R/\mathfrak{r}_0$ is integrally closed or not.

Proposition 17.2.1. — *There exists a unique prime ideal of R lying over $\mathfrak{p}_0^{\mathrm{left}}$ and containing $\mathfrak{r}_0$.*

Proof. — Let $\mathfrak{p}^* = \operatorname{iso}_0^{-1}(\mathfrak{p}_0^{\mathrm{left}}/\mathfrak{p}_0)$. Since $\mathbf{k}[V\times V^*]^{W\times W}/\mathfrak{p}^* \simeq \mathbf{k}[V\times 0]^{W\times W}$, there is only one prime ideal $\mathfrak{r}^*$ of $\mathbf{k}[V\times V^*]$ lying over $\mathfrak{p}^*$: it is the defining ideal of the irreducible closed subvariety $V\times 0$ of $V\times V^*$. In other words,

$$\mathbf{k}[V\times V^*]/\mathfrak{r}^* = \mathbf{k}[V\times 0].$$

Consequently, the unique prime ideal $\mathfrak{r}_0^{\mathrm{left}}$ of R lying over $\mathfrak{p}_0^{\mathrm{left}}$ and containing $\mathfrak{r}_0$ is defined by $\mathfrak{r}_0^{\mathrm{left}}/\mathfrak{r}_0 = \operatorname{iso}_0(\mathfrak{r}^* \cap \mathbf{k}[V\times V^*]^{\Delta Z(W)}) \cap (R/\mathfrak{r}_0)$. ◻

Let $\mathfrak{r}_0^{\mathrm{left}}$ be the unique prime ideal of R lying over $\mathfrak{q}_0^{\mathrm{left}}$ and containing $\mathfrak{r}_0$ (see Proposition 17.2.1) and let D_0^{left} (respectively I_0^{left}) denote its decomposition (respectively inertia) group.

Proposition 17.2.2. — (a) *$\iota(W \times W) \subset D_0^{\mathrm{left}}$ and $\iota(W \times 1) \subset I_0^{\mathrm{left}}$.*

(b) *The canonical map $\bar{\iota} : W \times W \longrightarrow D_0^{\mathrm{left}}/I_0^{\mathrm{left}}$ is surjective and its kernel contains $W \times \mathrm{Z}(W)$.*

(c) *$D_0^{\mathrm{left}}/I_0^{\mathrm{left}}$ is a quotient of $W/\mathrm{Z}(W)$.*

(d) *If $R/\mathfrak{r}_0$ is integrally closed (i.e. if $R/\mathfrak{r}_0 \simeq \mathbf{k}[V \times V^*]^{\Delta \mathrm{Z}(W)}$), then* $\mathrm{Ker}(\bar{\iota}) = W \times \mathrm{Z}(W)$ *and $D_0^{\mathrm{left}}/I_0^{\mathrm{left}} \simeq W/\mathrm{Z}(W)$.*

Proof. — The first statement of (a) follows from the uniqueness of $\mathfrak{r}_0^{\mathrm{left}}$ (see Proposition 17.2.1). For the second statement, let us use here the notation of the proof of Proposition 17.2.1, and note that $W \times 1$ acts trivially on $\mathbf{k}[V \times V^*]/\mathfrak{r}^*$.

Now, let B_0 be the inverse image of $R/\mathfrak{r}_0$ in $\mathbf{k}[V \times V^*]$ through iso_0. Then $\mathbf{k}[V \times 0]^{W \times W} \subset B_0/\mathfrak{r}^* \subset \mathbf{k}[V \times 0]^{\Delta \mathrm{Z}(W)} = \mathbf{k}[V \times 0]^{W \times \mathrm{Z}(W)} \subset \mathbf{k}[V \times 0]$. (b), (c) and (d) follow from these observations. ∎

This study of decomposition and inertia groups allows us to deduce the following result.

Corollary 17.2.3. — *W contains only one Calogero-Moser left 0-cell, namely W itself, and*

$$[W]_0^{\mathrm{CM}} = [\mathbf{k}W]_{\mathbf{k}W} = \sum_{\chi \in \mathrm{Irr}(W)} \chi(1).$$

Proof. — The first statement follows from Proposition 17.2.2(a) whereas the second one follows from Proposition 11.4.2(a). ∎

Let us conclude with an easy remark, which, combined with Proposition 17.2.2, shows that the pair $(I_0^{\mathrm{left}}, D_0^{\mathrm{left}})$ has a surprising behaviour.

Proposition 17.2.4. — *Let $\mathfrak{C}$ be a prime ideal of $\mathbf{k}[\mathscr{C}]$. Then there exists $h \in H$ such that ${}^h I_{\mathfrak{C}}^{\mathrm{left}} \subset I_0^{\mathrm{left}}$.*

Proof. — Let $\widetilde{\mathfrak{C}}$ denote the maximal homogeneous ideal of $\mathbf{k}[\mathscr{C}]$ contained in $\mathfrak{C}$. By Proposition 11.3.4, we have $I_{\mathfrak{C}}^{\mathrm{left}} = I_{\widetilde{\mathfrak{C}}}^{\mathrm{left}}$. This means that we may assume that $\mathfrak{C}$ is homogeneous. In particular, $\mathfrak{C} \subset \mathfrak{C}_0$. So $\mathfrak{q}_{\mathfrak{C}}^{\mathrm{left}} \subset \mathfrak{q}_0^{\mathrm{left}}$ and there exists $h \in H$ such that $h(\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}) \subset \mathfrak{r}_0^{\mathrm{left}}$. Therefore, ${}^h I_{\mathfrak{C}}^{\mathrm{left}} \subset I_0^{\mathrm{left}}$. ∎

It would be tempting to think, after Proposition 17.2.2, that $D_0^{\text{left}} = \iota(W \times W)$ and $I_0^{\text{left}} = \iota(W \times \mathrm{Z}(W))$. However, this would contradict Proposition 17.2.4, if we assume that Conjectures LR and L hold: indeed, I_0^{left} must contain conjugates of subgroups admitting as orbits the left cells. We will see in Chapter 18 that if $\dim_{\mathbf{k}}(V) = 1$, then $D_0^{\text{left}} = G$.

CHAPTER 18

GROUPS OF RANK 1

Assumption and notation. *In §18, we assume that* $\dim_{\mathbf{k}} V = 1$*, we fix a non-zero element* y *of* V *and we denote by* x *the element of* V^* *such that* $\langle y, x\rangle = 1$*. We also fix an integer* $d \geqslant 2$ *and we assume that* $\mathbf{k}$ *contains a primitive* d*-th root of unity* ζ*. We denote by* s *the automorphism of* V *defined by* $s(y) = \zeta y$*, so that* $s(x) = \zeta^{-1} x$*. We assume finally that* $W = \langle s\rangle$*:* s *is a reflection and* W *is cyclic of order* d*.*

18.1. The algebra $\widetilde{\mathbf{H}}$

18.1.A. Definition. — We have $\mathrm{Ref}(W) = \{s^i \mid 1 \leqslant i \leqslant d-1\}$. Given $1 \leqslant i \leqslant d-1$, we denote by C_i the indeterminate C_{s^i}, so that $\mathbf{k}[\widetilde{\mathscr{C}}] = \mathbf{k}[T, C_1, C_2, \dots, C_{d-1}]$ and $\mathbf{k}[\mathscr{C}] = \mathbf{k}[C_1, C_2, \dots, C_{d-1}]$. The $k[\widetilde{\mathscr{C}}]$-algebra $\widetilde{\mathbf{H}}$ is generated by x, y, s with the relations

(18.1.1) $$s y s^{-1} = \zeta y,\ \ s x s^{-1} = \zeta^{-1} x \text{ and } [y, x] = T + \sum_{1 \leqslant i \leqslant d-1} (\zeta^i - 1) C_i\, s^i.$$

We set $C_0 = C_{s^0} = 0$. The hyperplane arrangement $\mathscr{A}$ is reduced to one element, and $\mathscr{A}/W$ as well (we write $\mathscr{A}/W = \{\aleph\}$), we put $K_j = K_{\aleph, j}$ (for $0 \leqslant j \leqslant d-1$). Recall that the family $(K_j)_{0 \leqslant j \leqslant d-1}$ is determined by the relations

(18.1.2) $$\forall\ 0 \leqslant i \leqslant d-1,\ C_i = \sum_{j=0}^{d-1} \zeta^{i(j-1)} K_j.$$

We put

$$K_{di+j} = K_j$$

for all $i \in \mathbb{Z}$ and $j \in \{0, 1, \dots, d-1\}$. Recall that

$$K_0 + K_1 + \cdots + K_{d-1} = 0.$$

The last defining relation for $\widetilde{\mathbf{H}}$ can be rewritten as

$$[y,x] = T + d \sum_{0 \leqslant i \leqslant d-1} (K_{H,i} - K_{H,i+1})\varepsilon_i,$$

where $\varepsilon_i = d^{-1} \sum_{j=0}^{d-1} \zeta^{ij} s^j$.

18.1.B. Differential operators on $\mathbb{C}^\times$. — We have

$$D_y = T\partial_y - x^{-1} \sum_{i=1}^{d-1} \zeta^i C_i s^i = T\partial_y - d\,x^{-1} \sum_{i=0}^{d-1} K_i \varepsilon_i.$$

Given L a $\mathbb{C}[\mathscr{C}][y] \rtimes W$-module, the W-equivariant connection on $\mathscr{O}_{\mathbb{C}^\times} \otimes L$ is given by

$$\nabla(p \otimes l) = \frac{dp}{dx} \otimes l + p \otimes y \cdot l + d\,x^{-1} \sum_{i=0}^{d-1} K_i p \otimes \varepsilon_i l.$$

When $L = \mathbb{C}[\mathscr{C}][y]/(y) \otimes \det^n = \mathbb{C}[\mathscr{C}]$, we obtain $\nabla = \partial + x^{-1} K_n$.

18.1.C. The variety $(V \times V^*)/\Delta W$. — Let $X = x^d$ and $Y = y^d$. Recall that $\mathbf{eu}_0 = xy$. We have

$$\mathbf{k}[V \times V^*]^{\Delta W} = \mathbf{k}[X, Y, \mathbf{eu}_0]$$

and the relation

(18.1.3) $$\mathbf{eu}_0^d = XY$$

holds. It is easy to check that this relation generates the ideal of relations.

18.2. The algebra Z

Recall that $\mathbf{eu} = yx + \sum_{i=1}^{d-1} C_i\ s^i$ (its image in $\mathbf{H}_0$ is $\mathbf{eu}_0$) and that $\varepsilon : W \to \mathbf{k}^\times$ is the determinant, characterized by $\varepsilon(s) = \zeta$. We have $\varepsilon^d = 1$ and

$$\operatorname{Irr} W = \{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{d-1}\}.$$

The image of the Euler element by Ω_χ can be computed thanks to Corollary 7.3.2: we have

(18.2.1) $$\Omega_{\varepsilon^i}(\mathbf{eu}) = d K_{-i}$$

for all $i \in \mathbb{Z}$.

Recall that $\bar{\mathfrak{p}} = \langle X, Y \rangle_P$ and that $\bar{\mathfrak{p}} Z \subseteq \operatorname{Ker}(\Omega_\chi)$ for all $\chi \in \operatorname{Irr}(W)$. More precisely, we have

(18.2.2) $$\bigcap_{i=1}^{d} \operatorname{Ker}(\Omega_{\varepsilon^i}) = \bar{\mathfrak{p}} Z = \langle X, Y \rangle_Z.$$

Proof. — It follows from Example 9.3.6 that the generic Calogero-Moser families are reduced to one element. Given $1 \leqslant i \leqslant d$, let b_i denote the primitive central idempotent of $\mathbf{k}(\mathscr{C})\bar{\mathbf{H}}$ such that $\operatorname{Irr}\mathbf{k}(\mathscr{C})\bar{\mathbf{H}}b_i = \{L_{(0)}(\varepsilon^i)\}$. We have

$$\mathbf{k}(\mathscr{C})\bar{Z} \simeq \prod_{i=1}^{d} \mathbf{k}(\mathscr{C})\bar{Z}\, b_i$$

and, by Theorem 9.4.1,

$$\dim_{\mathbf{k}(\mathscr{C})} \mathbf{k}(\mathscr{C})\bar{Z}\, b_i = 1. \tag{18.2.3}$$

Since b_j is characterized by $\Omega_{\varepsilon^i}(b_j) = \delta_{i,j}$ (the Kronecker symbol) for all $i \in \{1,2,\dots,d\}$, the equality (18.2.2) follows. ☐

The next result is well-known.

Theorem 18.2.4. — *We have* $Z = P[\mathbf{eu}] = \mathbf{k}[C_1,\dots,C_{d-1},X,Y,\mathbf{eu}] = \mathbf{k}[K_1,\dots,K_{d-1},X,Y,\mathbf{eu}]$ *and the ideal of relations is generated by*

$$\prod_{i=1}^{d}(\mathbf{eu} - dK_i) = XY.$$

Proof. — The equality $Z = \mathbf{k}[C_1,\dots,C_{d-1},X,Y,\mathbf{eu}] = P[\mathbf{eu}]$ has been proven in Example 4.4.12. Define

$$z = \prod_{i=1}^{d}(\mathbf{eu} - dK_i).$$

Since $\Omega_{\varepsilon^i}(z) = 0$ for all i by (18.2.1), it follows from (18.2.2) that

$$z \equiv 0 \mod \langle X, Y\rangle_Z.$$

Moreover, since $\mathbf{eu}_0^d = XY$, we have

$$z \equiv XY \mod \langle C_1,\dots,C_{d-1}\rangle_Z.$$

Therefore,

$$z - XY \in \langle X,Y\rangle_Z \cap \langle C_1,\dots,C_{d-1}\rangle_Z = \langle C_1X, C_1Y, C_2X, C_2Y, \dots, C_{d-1}X, C_{d-1}Y\rangle_Z.$$

On the other hand, $z - XY$ is bi-homogeneous with bidegree (d,d), whereas C_iX and C_iY are bi-homogeneous with bidegree $(d+1,1)$ and $(1,d+1)$ respectively. Consequently, $z - XY = 0$, which is the required relation.

Since the minimal polynomial of $\mathbf{eu}$ over P has degree $|W| = d$ (Proposition 5.1.19), we deduce that

$$\prod_{i=1}^{d}(\mathbf{t} - dK_i) - XY$$

is the minimal polynomial of $\mathbf{eu}$ over P: this concludes the proof of the theorem. ☐

Corollary 18.2.5. — *The* $\mathbf{k}$*-algebra* Z *is a complete intersection.*

We denote by $F_{\mathbf{eu}}(\mathbf{t}) \in P[\mathbf{t}]$ the minimal polynomial of $\mathbf{eu}$ over P. Theorem 18.2.4 gives

$$F_{\mathbf{eu}}(\mathbf{t}) = \prod_{i=1}^{d} (\mathbf{t} - dK_i) - XY. \tag{18.2.6}$$

18.3. The ring R, the group G

18.3.A. Symmetric polynomials. — To take advantage of the fact that the minimal polynomial of the Euler element is symmetric in the variables K_i, we will recall here some classical facts about symmetric polynomials. Given $T_1, T_2, \ldots, T_d$ a family of indeterminates and given $1 \leqslant i \leqslant d$, we denote by $\sigma_i(\mathbf{T})$ the i-th elementary symmetric function

$$\sigma_i(\mathbf{T}) = \sigma_i(T_1, \ldots, T_d) = \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant d} T_{j_1} \cdots T_{j_i}.$$

Recall the well-known formula

$$\det\Big(\frac{\partial \sigma_i(\mathbf{T})}{\partial T_j}\Big)_{1 \leqslant i,j \leqslant d} = \prod_{1 \leqslant i < j \leqslant d} (T_j - T_i). \tag{18.3.1}$$

The group $\mathfrak{S}_d$ acts on $\mathbf{k}[T_1, \ldots, T_d]$ by permutation of the indeterminates. Recall the following classical result (a particular case of Theorem 2.2.1).

Proposition 18.3.2. — *The polynomials* $\sigma_1(\mathbf{T}), \ldots, \sigma_d(\mathbf{T})$ *are algebraically independent and* $\mathbf{k}[T_1, \ldots, T_d]^{\mathfrak{S}_d} = \mathbf{k}[\sigma_1(\mathbf{T}), \ldots, \sigma_d(\mathbf{T})]$*. Moreover, the* $\mathbf{k}$*-algebra* $\mathbf{k}[T_1, \ldots, T_d]$ *is a free* $\mathbf{k}[\sigma_1(\mathbf{T}), \ldots, \sigma_d(\mathbf{T})]$*-module of rank* $d!$

Recall also that $\sigma_1(\mathbf{T}) = T_1 + \cdots + T_d$.

Corollary 18.3.3. — *We have* $\big(\mathbf{k}[T_1, \ldots, T_d]/\langle \sigma_1(\mathbf{T}) \rangle\big)^{\mathfrak{S}_d} \simeq \mathbf{k}[\sigma_2(\mathbf{T}), \ldots, \sigma_d(\mathbf{T})]$ *and the* $\mathbf{k}$*-algebra* $\mathbf{k}[T_1, \ldots, T_d]/\langle \sigma_1(\mathbf{T}) \rangle$ *is a free* $\mathbf{k}[\sigma_2(\mathbf{T}), \ldots, \sigma_d(\mathbf{T})]$*-module of rank* $d!$*.*

As a consequence of Proposition 18.3.2, there exists a unique polynomial Δ_d in d variables such that

$$\prod_{1 \leqslant i < j \leqslant d} (T_j - T_i)^2 = \Delta_d(\sigma_1(\mathbf{T}), \sigma_2(\mathbf{T}), \ldots, \sigma_d(\mathbf{T})). \tag{18.3.4}$$

18.3.B. Presentation of R. — Let $\sigma_i(\mathbf{K}) = \sigma_i(K_1, \dots, K_d)$ (in particular, $\sigma_1(\mathbf{K}) = 0$). By Corollary 18.3.3, the ring $P_{\mathrm{sym}} = \mathbf{k}[\sigma_2(\mathbf{K}), \dots, \sigma_d(\mathbf{K}), X, Y]$ is the invariant ring, in P, of the group $\mathfrak{S}_d$ acting by permutation of the K_i's. Moreover,

(18.3.5) *P is a free P_{sym}-module of rank $d!$.*

Let us introduce a new family of indeterminates $E_1, \dots, E_{d-1}$, and let $E_d = -(E_1 + \dots + E_{d-1})$ and $\sigma_i(\mathbf{E}) = \sigma_i(E_1, \dots, E_d)$ (in particular $\sigma_1(\mathbf{E}) = 0$). Let $R_{\mathrm{sym}} = \mathbf{k}[E_1, \dots, E_{d-1}, X, Y] = \mathbf{k}[E_1, \dots, E_d, X, Y]/\langle \sigma_1(\mathbf{E}) \rangle$, on which the symmetric group $\mathfrak{S}_d$ acts by permutation of the E_i's. The ring $R_{\mathrm{sym}}^{\mathfrak{S}_d}$ is again a polynomial algebra equal to $\mathbf{k}[\sigma_2(\mathbf{E}), \dots, \sigma_d(\mathbf{E}), X, Y]$ (still thanks to Corollary 18.3.3).

Identification. *We identify the $\mathbf{k}$-algebras P_{sym} and $R_{\mathrm{sym}}^{\mathfrak{S}_d}$ through the equalities*

$$\begin{cases} \sigma_1(d\mathbf{K}) = \sigma_1(\mathbf{E}) = 0 \\ \forall\ 2 \leqslant i \leqslant d-1,\ \sigma_i(d\mathbf{K}) = \sigma_i(\mathbf{E}) \\ \sigma_d(d\mathbf{K}) = \sigma_d(\mathbf{E}) + (-1)^d XY \end{cases}$$

Note that $\sigma_i(d\mathbf{K}) = d^i \sigma_i(\mathbf{K})$.

As a consequence,

(18.3.6) *R_{sym} is a free P_{sym}-module of rank $d!$.*

Lemma 18.3.7. — *The ring $P \otimes_{P_{\mathrm{sym}}} R_{\mathrm{sym}}$ is an integrally closed domain.*

Proof. — First of all, note that we may, and we will, assume in this proof that $\mathbf{k}$ is integrally closed. Let $\tilde{R} = P \otimes_{P_{\mathrm{sym}}} R_{\mathrm{sym}}$. Then $\tilde{R}$ admits the following presentation:

$$(\mathscr{P}) \qquad \begin{cases} \text{Generators:} & K_1, K_2, \dots, K_d, E_1, E_2, \dots, E_d, X, Y \\ \text{Relations:} & \begin{cases} \sigma_1(d\mathbf{K}) = \sigma_1(\mathbf{E}) = 0 \\ \forall\ 2 \leqslant i \leqslant d-1,\ \sigma_i(d\mathbf{K}) = \sigma_i(\mathbf{E}) \\ \sigma_d(d\mathbf{K}) = \sigma_d(\mathbf{E}) + (-1)^d XY \end{cases} \end{cases}$$

The presentation $(\mathscr{P})$ of $\tilde{R}$ shows that $\tilde{R}$ is endowed with an $\mathbb{N}$-grading such that $\deg(K_i) = \deg(E_i) = 2$ and $\deg(X) = \deg(Y) = d$. Thus, the degree 0 component of $\tilde{R}$ is isomorphic to $\mathbf{k}$, which shows that

(♣) *$\tilde{R}$ is connected.*

Since $\tilde{R}$ is a free P_{sym}-module of finite rank, it follows that $\tilde{R}$ has pure dimension $d+1$. The presentation $(\mathscr{P})$ shows that

(♡) *$\tilde{R}$ is complete intersection.*

Let us now show that

($\spadesuit$) $\tilde{R}$ *is regular in codimension* 1.

Let $\tilde{\mathscr{R}} = \operatorname{Spec} \tilde{R}$, a closed subvariety of $\mathbb{A}^{2d+2}(\mathbf{k})$ consisting of elements $r = (k_1, \ldots, k_d, e_1, \ldots, e_d, x, y)$ satisfying the equations ($\mathscr{P}$). The Jacobian $\operatorname{Jac}(r)$ of this system of equations ($\mathscr{P}$) in $r \in \tilde{\mathscr{R}}$ is given by

$$\operatorname{Jac}(r) = \begin{pmatrix} d & \cdots & d & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & -1 & \cdots & -1 & 0 & 0 \\ \frac{\partial \sigma_2(d\mathbf{K})}{\partial K_1}(r) & \cdots & \frac{\partial \sigma_2(d\mathbf{K})}{\partial K_d}(r) & -\frac{\partial \sigma_2(\mathbf{E})}{\partial E_1}(r) & \cdots & -\frac{\partial \sigma_2(\mathbf{E})}{\partial E_d}(r) & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \frac{\partial \sigma_{d-1}(d\mathbf{K})}{\partial K_1}(r) & \cdots & \frac{\partial \sigma_{d-1}(d\mathbf{K})}{\partial K_d}(r) & -\frac{\partial \sigma_{d-1}(\mathbf{E})}{\partial E_1}(r) & \cdots & -\frac{\partial \sigma_{d-1}(\mathbf{E})}{\partial E_d}(r) & 0 & 0 \\ \frac{\partial \sigma_d(d\mathbf{K})}{\partial K_1}(r) & \cdots & \frac{\partial \sigma_{d-1}(d\mathbf{K})}{\partial K_d}(r) & -\frac{\partial \sigma_d(\mathbf{E})}{\partial E_1}(r) & \cdots & -\frac{\partial \sigma_d(\mathbf{E})}{\partial E_d}(r) & (-1)^{d+1} y & (-1)^{d+1} x \end{pmatrix}$$

Since $\tilde{\mathscr{R}}$ is of pure dimension $d+1$, its singular locus is the closed subvariety X of points r where the rank of $\operatorname{Jac}(r)$ is less than or equal to d. A point of X satisfies the equations

$$\det\Big(\frac{\partial \sigma_i(d\mathbf{K})}{\partial K_j}(k_1, \ldots, k_d)\Big)_{1 \leqslant i,j \leqslant d} = \det\Big(\frac{\partial \sigma_i(\mathbf{E})}{\partial E_j}(e_1, \ldots, e_d)\Big)_{1 \leqslant i,j \leqslant d} = 0.$$

By (18.3.1), this means that

$$\prod_{1 \leqslant i<j \leqslant d} (k_j - k_i) = \prod_{1 \leqslant i<j \leqslant d} (e_j - e_i) = 0$$

In particular,

$$\Delta_d(\sigma_1(k_1, \ldots, k_d), \ldots, \sigma_d(k_1, \ldots, k_d)) = \Delta_d(\sigma_1(e_1, \ldots, e_d), \ldots, \sigma_d(e_1, \ldots, e_d)) = 0.$$

It is well-known that $\Delta_d(0, U_2, \ldots, U_d)$ is an irreducible polynomial in the indeterminates $U_2, \ldots, U_d$. It follows that the variety of $(a_2, \ldots, a_d, x, y) \in \mathbb{A}^{d+1}(\mathbf{k})$ such that

$$(*) \qquad \Delta_d(0, a_2, \ldots, a_{d-1}, a_d) = \Delta_d(0, a_2, \ldots, a_{d-1}, a_d + (-1)^d x y) = 0.$$

has dimension $\leqslant d-1$. Consequently, X has codimension $\geqslant 2$ in $\tilde{\mathscr{R}}$.

The assertions ($\diamondsuit$) and ($\spadesuit$) imply that $\tilde{R}$ is normal (see [**Ser**, §IV.D, Theorem 11]). So it is a direct product of integrally closed domains. Since it is connected, it follows that it is an integrally closed domain. $\square$

We can now describe the ring R.

Theorem 18.3.8. — *The ring R satisfies the following properties:*

(a) *R is isomorphic to $P \otimes_{P_{\mathrm{sym}}} R_{\mathrm{sym}}$. It admits the following presentation:*

$$(\mathscr{P}) \qquad \begin{cases} \textit{Generators:} & K_1, K_2, \ldots, K_d, E_1, E_2, \ldots, E_d, X, Y \\ \textit{Relations:} & \begin{cases} \sigma_1(d\mathbf{K}) = \sigma_1(\mathbf{E}) = 0 \\ \forall\ 2 \leqslant i \leqslant d-1,\ \sigma_i(d\mathbf{K}) = \sigma_i(\mathbf{E}) \\ \sigma_d(d\mathbf{K}) = \sigma_d(\mathbf{E}) + (-1)^d XY \end{cases} \end{cases}$$

(b) *R is complete intersection and is a free P-module of rank $d!$.*

(c) *There exists a unique morphism of P-algebras* $\mathrm{cop}: Z \to R$ *such that* $\mathrm{cop}(\mathbf{eu}) = E_d$. *This morphism is injective, with Q its image.*

(d) *For the action of $\mathfrak{S}_d$ by permutation of the E_i's, we have $R^{\mathfrak{S}_d} = P$ and $R^{\mathfrak{S}_{d-1}} = Q$.*

(e) *$G = \mathfrak{S}_W \simeq \mathfrak{S}_d$; given $\sigma \in \mathfrak{S}_d$ and $1 \leqslant i \leqslant d$, we have $\sigma(E_i) = E_{\sigma(i)}$.*

(f) *G is a reflection group for its action on $R_+/(R_+)^2$.*

Proof. — Let $\tilde{R} = P \otimes_{P_{\mathrm{sym}}} R_{\mathrm{sym}}$. The relations $(\mathscr{P})$ show that, in the polynomial ring $\tilde{R}[\mathbf{t}]$, the equality

$$\prod_{i=1}^{d}(\mathbf{t} - dK_i) - XY = \prod_{i=1}^{d}(\mathbf{t} - E_i)$$

holds. It follows that $F_{\mathbf{eu}}(E_d) = 0$. By Theorem 18.2.4, we deduce that there exists a unique morphism of P-algebras $\mathrm{cop}: Z \to \tilde{R}$ such that $\mathrm{cop}(\mathbf{eu}) = E_d$. Let $\mathfrak{z} = \mathrm{Ker}(\mathrm{cop})$. We have $\mathfrak{z} \cap P = 0$ since $P \subset \tilde{R}$ and, since Z is a domain and is integral over P, this forces $\mathfrak{z} = 0$. So $\mathrm{cop}: Z \to \tilde{R}$ is injective.

Let $\tilde{\mathbf{M}}$ be the fraction field of $\tilde{R}$ (recall that $\tilde{R}$ is a domain by Lemma 18.3.7). By construction, $\tilde{R}$ is a free P-module of rank $d!$ and, by Corollary 18.3.3, $\tilde{R}^{\mathfrak{S}_d} = P$. So the extension $\tilde{\mathbf{M}}/\mathbf{K}$ is Galois, contains $\mathbf{L}$ (the fraction field of Q) and satisfies $\mathrm{Gal}(\tilde{\mathbf{M}}/\mathbf{K}) = \mathfrak{S}_d$. Moreover, $\mathrm{Gal}(\tilde{\mathbf{M}}/\mathbf{L}) = \mathfrak{S}_{d-1}$ since $\mathfrak{S}_{d-1}$ is the stabilizer of E_d in $\mathfrak{S}_d$. Since the unique normal subgroup of $\mathfrak{S}_d$ contained in $\mathfrak{S}_{d-1}$ is the trivial group, this shows that $\tilde{\mathbf{M}}/\mathbf{K}$ is a Galois closure of $\mathbf{L}/\mathbf{K}$. So $\tilde{\mathbf{M}} \simeq \mathbf{M}$.

Since $\tilde{R}$ is integrally closed (Lemma 18.3.7) and integral over P, this implies that $\tilde{R} \simeq R$. Now all the statements of Theorem 18.3.8 can be deduced from these observations. For the statement (f), we can use (b), and Proposition C.3.7 because $\mathfrak{S}_d$ acts trivially on the relations, or check it directly by noting that $R_+/(R_+)^2$ is the $\mathbf{k}$-vector space of dimension $2d$ generated by $K_1, \ldots, K_d, E_1, \ldots, E_d, X, Y$, with the relations $K_1 + \cdots + K_d = 0$ and $E_1 + \cdots + E_d = 0$: this shows that, as a representation of $\mathfrak{S}_d$, $R_+/(R_+)^2$ is the direct sum of the irreducible reflection representation and of $d+1$ copies of the trivial representation. ☐

18.3.C. Choice of the ideal $\mathfrak{r}_0$. — Let $\mathfrak{r}'$ denote the ideal of R generated by the elements $E_i - \zeta^i E_d$. We then have

$$\sigma_1(\mathbf{E}) \equiv \sigma_2(\mathbf{E}) \equiv \cdots \equiv \sigma_{d-1}(\mathbf{E}) \equiv 0 \mod \mathfrak{r}'.$$

We choose the ideal of R $\mathfrak{r}_0 = \mathfrak{r}' + \langle K_1, \dots, K_d \rangle_R$. The $\mathbf{k}$-algebra $R/\mathfrak{r}_0$ has the following presentation:

$$(\mathscr{P}_0) \qquad \begin{cases} \text{Generators:} & E_d, X, Y \\ \text{Relation:} & E_d^d = XY \end{cases}$$

Recall that $\mathrm{Z}(W) = W$. We recover the isomorphisms of P-algebras $R/\mathfrak{r}_0 \simeq Q/\mathfrak{q}_0 \simeq \mathbf{k}[V \times V^*]^{\Delta W}$ by mapping $\mathrm{eu} = E_d$ to $\mathbf{eu}_0 = yx \in \mathbf{k}[V \times V^*]^{\Delta W}$. Recall that an element $w \in W$, viewed as an element of the Galois group $G = \mathfrak{S}_W \simeq \mathfrak{S}_d$, is characterized by the equality

$$(w(\mathrm{eu}) \mod \mathfrak{r}_0) \equiv w(y)x \in \mathbf{k}[V \times V^*]^{\Delta W}.$$

Since $s^i(y) = \zeta^i y$, we have

$$s^i(\mathrm{eu}) = E_i. \tag{18.3.9}$$

For the action of $G = \mathfrak{S}_W \simeq \mathfrak{S}_d$, this corresponds to identifying the sets $\{1, 2, \dots, d\}$ and W via the bijective map $i \mapsto s^i$.

18.3.D. Choice of the ideals $\mathfrak{r}^{\mathrm{left}}$, $\mathfrak{r}^{\mathrm{right}}$ and $\bar{\mathfrak{r}}$. — Let $\mathfrak{r}''$ denote the ideal of R generated by the $E_i - dK_i$'s. Then

$$\forall\ 1 \leqslant i \leqslant d,\ \sigma_i(d\mathbf{K}) \equiv \sigma_i(\mathbf{E}) \mod \mathfrak{r}''.$$

In particular, $XY \in \mathfrak{r}''$. We choose $\mathfrak{r}^{\mathrm{left}} = \mathfrak{r}'' + \langle Y \rangle_R$, $\mathfrak{r}^{\mathrm{right}} = \mathfrak{r}'' + \langle X \rangle_R$ and $\bar{\mathfrak{r}} = \mathfrak{r}'' + \langle X, Y \rangle_R$. Then

$$\begin{cases} R/\mathfrak{r}^{\mathrm{left}} \simeq \mathbf{k}[K_1, \dots, K_{d-1}, X] = P/\mathfrak{p}^{\mathrm{left}}, \\ R/\mathfrak{r}^{\mathrm{right}} \simeq \mathbf{k}[K_1, \dots, K_{d-1}, Y] = P/\mathfrak{p}^{\mathrm{right}}, \\ R/\bar{\mathfrak{r}} \simeq \mathbf{k}[K_1, \dots, K_{d-1}] = \mathbf{k}[\mathscr{C}] = P/\bar{\mathfrak{p}}. \end{cases} \tag{18.3.10}$$

The next proposition follows easily.

Proposition 18.3.11. — $D^{\mathrm{left}} = I^{\mathrm{left}} = D^{\mathrm{right}} = I^{\mathrm{right}} = \bar{D} = \bar{I} = 1.$

18.4. Cells, families, cellular characters

> **Notation.** *We fix in this section a prime ideal $\mathfrak{C}$ of $\mathbf{k}[\mathscr{C}]$ and we denote by k_i the image of K_i in $\mathbf{k}[\mathscr{C}]/\mathfrak{C}$.*

By (18.3.10), we have

(18.4.1) $$\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}} = \mathfrak{r}^{\mathrm{left}} + \mathfrak{C}R, \quad \mathfrak{r}_{\mathfrak{C}}^{\mathrm{right}} = \mathfrak{r}^{\mathrm{right}} + \mathfrak{C}R \quad \text{and} \quad \bar{\mathfrak{r}}_{\mathfrak{C}} = \bar{\mathfrak{r}} + \mathfrak{C}R$$

and

(18.4.2) $$\begin{cases} R/\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}} = \mathbf{k}[\mathscr{C}]/\mathfrak{C} \otimes \mathbf{k}[X] = P/\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}, \\ R/\mathfrak{r}_{\mathfrak{C}}^{\mathrm{right}} = \mathbf{k}[\mathscr{C}]/\mathfrak{C} \otimes \mathbf{k}[Y] = P/\mathfrak{p}_{\mathfrak{C}}^{\mathrm{right}}, \\ R/\bar{\mathfrak{r}}_{\mathfrak{C}} = \mathbf{k}[\mathscr{C}]/\mathfrak{C} = P/\bar{\mathfrak{p}}_{\mathfrak{C}}. \end{cases}$$

We will denote by $\mathfrak{S}[\mathfrak{C}]$ the subgroup of $\mathfrak{S}_d$ consisting of permutations stabilizing the fibers of the natural map $\{1,2,\ldots,d\} \to \mathbf{k}[\mathscr{C}]/\mathfrak{C}$, $i \mapsto k_i$. In other words,

$$\mathfrak{S}[\mathfrak{C}] = \{\sigma \in \mathfrak{S}_d \mid \forall\, 1 \leqslant i \leqslant d,\ k_{\sigma(i)} = k_i\}.$$

Proposition 18.4.3. — $D_{\mathfrak{C}}^{\mathrm{left}} = I_{\mathfrak{C}}^{\mathrm{left}} = D_{\mathfrak{C}}^{\mathrm{right}} = I_{\mathfrak{C}}^{\mathrm{right}} = \bar{D}_{\mathfrak{C}} = \bar{I}_{\mathfrak{C}} = \mathfrak{S}[\mathfrak{C}]$.

Corollary 18.4.4. — *Let $i, j \in \mathbb{Z}$. Then s^i and s^j are in the same Calogero-Moser two-sided (respectively left, respectively right) $\mathfrak{C}$-cell if and only if $k_i = k_j$.*

Let us conclude with the description of families and cellular characters.

Corollary 18.4.5. — *Let $i, j \in \mathbb{Z}$. Then ε^{-i} and ε^{-j} are in the same Calogero-Moser family if and only if $k_i = k_j$.*

The map $\omega \mapsto \sum_{i\in\omega} \varepsilon^{-i}$ induces a bijective map between the set of $\mathfrak{S}[\mathfrak{C}]$-orbits in $\{1,2,\ldots,d\}$ (that is, the set of fibers of the map $i \mapsto k_i$) and the set of Calogero-Moser $\mathfrak{C}$-cellular characters.

Proof. — Since $Z = P[\mathbf{eu}]$, we deduce that ε^{-i} and ε^{-j} are in the same Calogero-Moser $\mathfrak{C}$-family if and only if $\Omega_{\varepsilon^{-i}}^{\bar{\mathbf{K}}_{\mathfrak{C}}}(\mathbf{eu}) = \Omega_{\varepsilon^{-j}}^{\bar{\mathbf{K}}_{\mathfrak{C}}}(\mathbf{eu})$. So the first statement follows from (18.2.1).

For the second statement, note that $\mathbf{eu}$ acts on $\mathscr{L}_{s^i}$ by multiplication by $s^i(\mathrm{eu}) = E_i$. So, modulo $\mathfrak{r}_{\mathfrak{C}}^{\mathrm{left}}$ (or $\bar{\mathfrak{r}}_{\mathfrak{C}}$), the element $s^i(\mathrm{eu})$ is congruent to $d k_i = \Omega_{\varepsilon^{-i}}^{\bar{\mathbf{K}}_{\mathfrak{C}}}(\mathbf{eu})$. Hence, if ω is an $\mathfrak{S}[\mathfrak{C}]$-orbit in $\{1,2,\ldots,d\}$, then $C = \{s^i \mid i \in \omega\}$ is a Calogero-Moser left, right or two-sided $\mathfrak{C}$-cell (see Corollary 18.4.4) and, as a two-sided cell, it covers the Calogero-Moser $\mathfrak{C}$-family $\{\varepsilon^{-i} \mid i \in \omega\}$. Since $\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathscr{M}^{\mathrm{left}}(\varepsilon^{-i})$ is an absolutely simple

$\mathbf{M}_{\mathfrak{C}}^{\mathrm{left}}\mathbf{H}^{\mathrm{left}}$-module (because it has dimension $|W|$), it must be isomorphic to $\mathscr{L}_{\mathfrak{C}}^{\mathrm{left}}(C)$. This shows that $[\,C\,]_{\mathfrak{C}}^{\mathrm{CM}} = \sum_{i\in\omega} \varepsilon^{-i}$. □

18.5. Complements

We will be interested here in geometric properties of $\mathscr{Z}$ (smoothness, ramification) and in the properties of the group D_c. To simplify the statements, we will make the following assumption:

Assumption and notation. *In this section, and only in this section, we will assume that* $\mathbf{k}$ *is* ***algebraically closed***. *We will identify the variety* $\mathscr{Z}$ *with*

$$\mathscr{Z} = \{(k_1,\dots,k_d,x,y,e) \in \mathbb{A}^{d+3}(\mathbf{k}) \mid k_1+\cdots+k_d = 0 \text{ and } \prod_{i=1}^{d}(e-d\,k_i) = x\,y\}.$$

Similarly, $\mathscr{P}$ *(respectively* $\mathscr{C}$*) will be identified with the affine space*

$$\mathscr{P} = \{(k_1,\dots,k_d,x,y) \in \mathbb{A}^{d+2}(\mathbf{k}) \mid k_1+\cdots+k_d = 0\}$$

(respectively

$$\mathscr{C} = \{(k_1,\dots,k_d) \in \mathbb{A}^{d}(\mathbf{k}) \mid k_1+\cdots+k_d = 0\} \quad),$$

which allows to redefine

$$\begin{array}{cccc} \Upsilon: & \mathscr{Z} & \longrightarrow & \mathscr{P} \\ & (k_1,\dots,k_d,x,y,e) & \longmapsto & (k_1,\dots,k_d,x,y). \end{array}$$

Finally, we denote by $\mathscr{Z}_{\mathrm{sing}}$ *the singular locus of* $\mathscr{Z}$ *and* $\mathscr{Z}_{\mathrm{ram}}$ *the ramification locus of* Υ.

18.5.A. Smoothness. — Let us start by the description of the singular locus of $\mathscr{Z}$:

Proposition 18.5.1. — *Given* $1 \leqslant i < j \leqslant d$, *let* $\mathscr{Z}_{i,j} = \{(k_1,\dots,k_d,x,y,e) \in \mathscr{Z} \mid e = d\,k_i = d\,k_j$ *and* $x = y = 0\}$. *Then*

$$\mathscr{Z}_{\mathrm{sing}} = \bigcup_{1\leqslant i<j\leqslant d} \mathscr{Z}_{i,j}.$$

Moreover, $\mathscr{Z}_{i,j} \simeq \mathbb{A}^{d-2}(\mathbf{k})$ *is an irreducible component of* $\mathscr{Z}_{\mathrm{sing}}$ *and* $\mathscr{Z}_{\mathrm{sing}}$ *is purely of codimension* 3.

Proof. — The variety $\mathscr{Z}$ being described as an hypersurface in the affine space $\{(k_1,\ldots,k_d,x,y,e)\in\mathbb{A}^{d+3}(\mathbf{k})\mid k_1+\cdots+k_d=0\}\simeq\mathbb{A}^{d+2}(\mathbf{k})$, a point of $z=(k_1,\ldots,k_d,x,y,e)\in\mathscr{Z}$ is singular if and only if the jacobian matrix of the equation vanishes at z. This is equivalent to the following system of equations:

$$\begin{cases} x=y=0, \\ \forall\ 1\leqslant i\leqslant d, \prod_{j\neq i}(e-d\,k_j)=0, \\ \sum_{i=1}^{d}\prod_{j\neq i}(e-d\,k_j)=0. \end{cases}$$

The last equation is implied by the second family of equations, it is then easy to check that $\mathscr{Z}_{\mathrm{sing}}$ is as expected.

The last statements are immediate. □

Corollary 18.5.2. — *Given $c\in\mathscr{C}$ and $z\in\mathscr{Z}_c$, then z is singular in $\mathscr{Z}$ if and only if it is singular in $\mathscr{Z}_c$.*

18.5.B. Ramification. — The variety $\mathscr{Z}$ being normal, the variety $\mathscr{P}$ being smooth and the morphism $\Upsilon:\mathscr{Z}\to\mathscr{P}$ being finite and flat, the purity of the branch locus [**SGA1**, Talk X, Theorem 3.1] tells us that the ramification locus of Υ is purely of codimension 1. It is in fact easily computable:

Proposition 18.5.3. — *Let $z=(k_1,\ldots,k_d,x,y,e)\in\mathscr{Z}$ and $p=(k_1,\ldots,k_d,x,y)=\Upsilon(z)\in\mathscr{P}$. Let $F_{\mathbf{eu},p}(\mathbf{t})\in\mathbf{k}[\mathbf{t}]$ denote the specialization of $F_{\mathbf{eu}}(\mathbf{t})$ at p. Then Υ is ramified at z if and only if $F'_{\mathbf{eu},p}(e)=0$, that is if and only if e is a multiple root of $F_{\mathbf{eu},p}$.*

Proof. — Since $Z=P[\mathbf{t}]/\langle F_{\mathbf{eu}}(\mathbf{t})\rangle$ (see Theorem 18.2.4), this follows immediately from [**SGA1**, Talk I, Corollary 7.2]. □

Corollary 18.5.4. — *Let $c=(k_1,\ldots,k_d)\in\mathscr{C}$ and $(x,y)\in\mathbb{A}^2(\mathbf{k})$ (so that $(c,x,y)\in\mathscr{P}$). Then $(c,x,y)\in\Upsilon(\mathscr{Z}_{\mathrm{ram}})$ if and only if $\Delta_d(0,\sigma_2(c),\ldots,\sigma_{d-1}(c),\sigma_d(c)-(-1)^d x\,y)=0$.*

18.5.C. About the group D_c. — Amongst the groups D_c, the only one we use is D_0. In this subsection, we will show that, even when $n = \dim_{\mathbf{k}}(V) = 1$, the groups D_c have a very subtle behaviour.

The material of this subsection §18.5.C has been explained to us by G. Malle (any errors being of course our responsibility). We thank him warmly for his help.

Fix $c \in \mathscr{C}$, and let $F^c_{\mathbf{eu}}(\mathbf{t}) \in \mathbf{k}[X, Y][\mathbf{t}]$ denote the specialization of $F_{\mathbf{eu}}(\mathbf{t})$ at c. Note that D_c is the Galois group of $F^c_{\mathbf{eu}}(\mathbf{t})$, viewed as a polynomial with coefficients in the field $\mathbf{k}(X, Y)$. Actually, we have $F^c_{\mathbf{eu}}(\mathbf{t}) \in \mathbf{k}[T][\mathbf{t}]$, where $T = XY$. The following result will be helpful for computing D_c.

Lemma 18.5.5. — *D_c is the Galois group of $F^c_{\mathbf{eu}}(\mathbf{t})$ viewed as a polynomial with coefficients in $\mathbf{k}(T)$.*

Proof. — If L is a splitting field of $F^c_{\mathbf{eu}}(\mathbf{t})$ over $\mathbf{k}(T)$, then the field $L(Y)$ of rational functions in one variable is a splitting field of the same polynomial over $\mathbf{k}(T, Y) = \mathbf{k}(X, Y)$. The result follows. ☐

Corollary 18.5.6. — *The subgroup D_c of $G = \mathfrak{S}_d$ contains a cycle of length d.*

Proof. — Thanks to Lemma 18.5.5, we may view $F^c_{\mathbf{eu}}(\mathbf{t})$ as an element of $\mathbf{k}[T][\mathbf{t}]$. Since $\mathbf{k}$ has characteristic zero and $\mathbf{k}[T]$ is regular of dimension 1, the inertia group at infinity I is cyclic. Since $d \geqslant 2$, the polynomial $F^c_{\mathbf{eu}}(\mathbf{t})$ is totally ramified at infinity, which implies that I acts transitively on $\{1, 2, \dots, d\}$. Whence the result. ☐

Corollary 18.5.6 gives very restrictive conditions on the group D_c. For instance, we have the following results, the first of which is due to Schur, the second one to Burnside.

Corollary 18.5.7. — (a) *If d is not prime and D_c is primitive, then D_c is 2-transitive.*
(b) *If d is prime, then D_c is 2-transitive or D_c contains a normal Sylow d-subgroup.*

We will give now some examples that show that the description of D_c in general can be rather complicated. In the following table, we assume that $\mathbf{k} = \mathbb{C}$, and $c \in \mathscr{C}$ is chosen so that $F^c_{\mathbf{eu}}(\mathbf{t}) \in \mathbb{Q}[\mathbf{t}]$. We will denote by $D_c^{(\mathbb{Q})}$ the Galois group of $F^c_{\mathbf{eu}}(\mathbf{t})$ viewed as an element of $\mathbb{Q}[\mathbf{t}]$: it might be different from D_c, as it is shown by table. We denote by Fr_{pr} the Frobenius group $(\mathbb{Z}/r\mathbb{Z}) \ltimes (\mathbb{Z}/p\mathbb{Z})$, viewed as a subgroup of $\mathfrak{S}_p$, where p is prime and r divides $p-1$.

$F^{c}_{\mathbf{eu}}(\mathbf{t})$	$D_c^{(\mathbb{Q})}$	D_c
$(\mathbf{t}^2+20\mathbf{t}+180)(\mathbf{t}^2-5\mathbf{t}-95)^4-XY$	$\mathrm{Aut}(\mathfrak{A}_6)$	$\mathrm{Aut}(\mathfrak{A}_6)$
$(\mathbf{t}+1)^4(\mathbf{t}-2)^2(\mathbf{t}^3-3\mathbf{t}-14)-XY$	$(\mathfrak{S}_3\wr\mathfrak{S}_3)\cap\mathfrak{A}_9$	$(\mathfrak{S}_3\wr\mathfrak{S}_3)\cap\mathfrak{A}_9$
$\mathbf{t}(\mathbf{t}^8+6\mathbf{t}^4+25)-XY$	$\mathfrak{A}_9$	$\mathfrak{A}_9$
$\mathbf{t}(\mathbf{t}^4+6\mathbf{t}^2+25)^2-XY$	$\mathfrak{A}_9$	$\mathfrak{A}_9$
$\mathbf{t}^9-9\mathbf{t}^7+27\mathbf{t}^5-30\mathbf{t}^3+9\mathbf{t}-XY$	$\mathfrak{S}_3\ltimes(\mathbb{Z}/3\mathbb{Z})^2$	$\mathfrak{S}_3\ltimes(\mathbb{Z}/3\mathbb{Z})^2$
$\mathbf{t}^{11}-11\mathbf{t}^9+44\mathbf{t}^7-77\mathbf{t}^5+55\mathbf{t}^3-11\mathbf{t}-XY$	Fr_{110}	Fr_{22}
$\mathbf{t}^{13}-13\mathbf{t}^{11}+65\mathbf{t}^9-156\mathbf{t}^7+182\mathbf{t}^5-91\mathbf{t}^3+13\mathbf{t}-XY$	Fr_{156}	Fr_{26}

To prove the results contained in the table, let us recall some classical facts:

(a) D_c is a normal subgroup of $D_c^{(\mathbb{Q})}$.

(b) The computation of $D_c^{\mathbb{Q}}$ in all the cases can be performed thanks to the MAGMA software [**Magma**].

(c) Since there are no non-trivial unramified coverings of the complex affine line, the group D_c is generated by its inertia subgroups.

(d) Given $z \in \mathbb{C}$, let $F^{c,z}_{\mathbf{eu}}(\mathbf{t})$ denote the specialization $T \mapsto z$ of $F^{c}_{\mathbf{eu}}(\mathbf{t})$. If $\alpha \in \mathbb{C}$ is a root of $F^{c,z}_{\mathbf{eu}}(\mathbf{t})$ with multiplicity m, then D_c contains an element of order m.

From these facts, the result in the table can be obtained as follows. Let $\Delta_c(T) \in \mathbf{k}[T]$ denote the discriminant of the polynomial $F^{c}_{\mathbf{eu}}(\mathbf{t})$.

(1) The computation of $D_c^{(\mathbb{Q})}$ in the first example is done in [**MalMat**, Theorem I.9.7]. Note that, to retrieve the polynomial of [**MalMat**, Theorem I.9.7], one must replace $\mathbf{t}$ by $2\mathbf{t}-5$, and renormalize: this operation allows to obtain a polynomial whose coefficient in $\mathbf{t}^9$ is zero, as must be the case for all the $F^{c}_{\mathbf{eu}}(\mathbf{t})$. Going from $\mathbb{Q}$ to $\mathbb{C}$ then follows, as this example comes from a rigid triple.

(2) In the second example, the MAGMA software tells us that $D_c^{(\mathbb{Q})} = (\mathfrak{S}_3\wr\mathfrak{S}_3)\cap\mathfrak{A}_9$. On the other hand, D_c is a normal subgroup of $D_c^{(\mathbb{Q})}$. Moreover, -1 is a root of $F^{c,0}_{\mathbf{eu}}(\mathbf{t})$ with multiplicity 4, so D_c contains an element of order 4 (see (d)). It also contains an element of order 9 (see Corollary 18.5.6). But $D_c^{(\mathbb{Q})}$ contains only one normal subgroup containing both an element of order 9 and an element of order 4, namely itself. So $D_c = D_c^{(\mathbb{Q})}$.

(3-4) In the third and fourth examples, the equality $D_c^{(\mathbb{Q})} = \mathfrak{A}_9$ is obtained by the MAGMA software. The fact that $D_c = \mathfrak{A}_9$ follows from the fact that D_c is normal in $D_c^{(\mathbb{Q})}$ and contains an element of order 9.

(5) Once the computation of $D_c^{(\mathbb{Q})}$ done by the MAGMA software, note that

$$F_{\mathbf{eu}}^{c,2}(\mathbf{t}) = (\mathbf{t}-2)(\mathbf{t}+1)^2(\mathbf{t}^3-3\mathbf{t}+1)^2.$$

This allows to say, thanks to (d) and Corollary 18.5.6, that D_c contains an element of order 9 and an element of order 2. So 18 divides $|D_c|$. But $D_c^{(\mathbb{Q})}$ does not contain any normal subgroup of index 3 and containing an element of order 9.

(6-7) The last two examples can be treated similarly. We will only deal with the last one. In this case, thanks to the MAGMA software, we get

$$\Delta_c(T) = 13^{13}(T-2)^6(T+2)^6.$$

This discriminant is not a square in $\mathbb{Q}(T)$, but it is a square in $\mathbb{C}(T)$. The non-trivial inertia groups in D_c, except the inertia group at infinity, lie above the ideals $\langle T-2\rangle$ and $\langle T+2\rangle$. But,

$$F_{\mathbf{eu}}^{c,2}(\mathbf{t}) = (\mathbf{t}-2)(\mathbf{t}^6+\mathbf{t}^5-5\mathbf{t}^4-4\mathbf{t}^3+6\mathbf{t}^2+3\mathbf{t}-1)^2$$

and

$$F_{\mathbf{eu}}^{c,-2}(\mathbf{t}) = (\mathbf{t}+2)(\mathbf{t}^6-\mathbf{t}^5-5\mathbf{t}^4+4\mathbf{t}^3+6\mathbf{t}^2-3\mathbf{t}-1)^2.$$

So the inertia groups have order 2, which shows that D_c is generated by its elements of order 2. So $D_c = \mathrm{Fr}_{26}$.

18.5.D. Cohomology. — We assume in this subsection that $\mathbf{k} = \mathbb{C}$. We aim to prove the following result:

Theorem 18.5.8. — *If* $\dim_{\mathbb{C}}(V) = 1$, *then Conjecture COH holds.*

Proof. — Let $c \in \mathscr{C}$ and let $k_1, \dots, k_d$ be the images of $K_1, \dots, K_d$ in $\mathbb{C}[\mathscr{C}]/\mathfrak{C}_c$. We put $r = |\mathscr{Z}_c^{\mathbb{C}^\times}| = |\{k_1, k_2, \dots, k_d\}|$. The equations defining $\mathscr{Z}_c$ show that it is rationally smooth (it has only type A singularities). So it follows from the Proposition 18.5.9 below that

$$\dim_{\mathbb{C}} \mathrm{H}^i(\mathscr{Z}_c) = \begin{cases} 1 & \text{if } i = 0, \\ r-1 & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\dim_{\mathbb{C}} \mathrm{gr}(\mathrm{Im}\,\boldsymbol{\Omega}^c)^i = \begin{cases} 1 & \text{if } i=0, \\ r-1 & \text{if } i=2, \\ 0 & \text{otherwise.} \end{cases}$$

So, both $\mathrm{H}^{2\bullet}(\mathscr{Z}_c)$ and $\mathrm{gr}(\mathrm{Im}\,\boldsymbol{\Omega}^c)$ are isomorphic to the graded algebra $k[a_1,\dots,a_{r-1}]/(a_ia_j)_{1\leqslant i,j\leqslant r-1}$, with a_i's in degree 2. ☐

In order to complete the proof of Theorem 18.5.8, we fix an infinite sequence of non-zero natural numbers $d_1, d_2, \dots$, as well as an infinite sequence $z_1, z_2, \dots$ of complex numbers such that $z_i \neq z_j$ if $i \neq j$. We set

$$\mathscr{X}(r) = \{(e,x,y) \in \mathbf{A}^3(\mathbb{C}) \mid \prod_{i=1}^{r} (e-z_i)^{d_i} = xy\}.$$

First, note that $\mathscr{X}(r)$ admits an automorphism $\sigma : \mathscr{X}(r) \to \mathscr{X}(r)$, $(e,x,y) \mapsto (e,y,x)$. It is an involution. So σ acts on the cohomology of $\mathscr{X}(r)$. We denote by $\mathbb{C}_+$ (respectively $\mathbb{C}_-$) the $\mathbb{C}\langle\sigma\rangle$-module of dimension 1 on which σ acts by multiplication by 1 (respectively -1). Finally, if $z \in \mathbb{C}$, we set

$$\mathscr{X}(r)_{\neq z} = \{(e,x,y) \in \mathscr{X}(r) \mid e \neq z\}$$

and $$\mathscr{X}(r)_{=z} = \{(e,x,y) \in \mathscr{X}(r) \mid e = z\}.$$

These are σ-stable subvarieties of $\mathscr{X}(r)$. The following result describes the cohomology of $\mathscr{X}(r)$ as a $\mathbb{C}\langle\sigma\rangle$-module. We denote by $\mathrm{H}^i_c(\mathscr{X}(r))$ the i-th cohomology group with compact support of $\mathscr{X}(r)$.

Proposition 18.5.9. — *With the above notation, we have:*

(a) *The cohomology with compact support of $\mathscr{X}(r)$ is given, as a $\mathbb{C}\langle\sigma\rangle$-module, by*

$$\begin{cases} \mathrm{H}^2_c(\mathscr{X}(r)) = \mathbb{C}_-^{r-1}, \\ \mathrm{H}^3_c(\mathscr{X}(r)) = 0, \\ \mathrm{H}^4_c(\mathscr{X}(r)) = \mathbb{C}_+. \end{cases}$$

(b) *If $r \neq 1$ or $z \neq z_1$, then the cohomology with compact support of $\mathscr{X}(r)$ is given, as a $\mathbb{C}\langle\sigma\rangle$-module, by*

$$\begin{cases} \mathrm{H}^2_c(\mathscr{X}(r)_{\neq z}) = \begin{cases} \mathbb{C}_-^r & \text{if } z \notin \{z_1, z_2, \dots, z_r\}, \\ \mathbb{C}_-^{r-1} & \text{if } z \in \{z_1, z_2, \dots, z_r\}, \end{cases} \\ \mathrm{H}^3_c(\mathscr{X}(r)_{\neq z}) = \mathbb{C}_+, \\ \mathrm{H}^4_c(\mathscr{X}(r)_{\neq z}) = \mathbb{C}_+. \end{cases}$$

(c) *Finally,*

$$\begin{cases} \mathrm{H}^2_c(\mathscr{X}(1)_{\neq z_1}) = \mathbb{C}^r_-, \\ \mathrm{H}^3_c(\mathscr{X}(1)_{\neq z_1}) = \mathbb{C}_+ \oplus \mathbb{C}_-, \\ \mathrm{H}^4_c(\mathscr{X}(1)_{\neq z_1}) = \mathbb{C}_+. \end{cases}$$

REMARK - As $\mathscr{X}(r)$ and $\mathscr{X}(r)_{\neq z}$ are affine surface, their cohomology with compact support vanishes in degree different from 2, 3 or 4. ∎

Proof. — We first gather some elementary fact which will allow a proof of this Theorem by induction. If $\lambda \in \mathbb{C}$, let

$$\mathscr{H}_\lambda = \{(x,y) \in \mathbb{C}^2 \mid xy = \lambda\}.$$

It is endowed with an action of σ given by $(x,y) \mapsto (y,x)$. Then it is well-known that

$$(\clubsuit) \qquad \begin{cases} \mathrm{H}^1_c(\mathscr{H}_\lambda) = \mathbb{C}_-, \\ \mathrm{H}^2_c(\mathscr{H}_\lambda) = \begin{cases} \mathbb{C}_+ & \text{if } \lambda \neq 0, \\ \mathbb{C}_+ \oplus \mathbb{C}_- & \text{if } \lambda = 0. \end{cases} \end{cases}$$

Also, the quotient of $\mathbb{C}^2$ by the action of σ given by $(x,y) \mapsto (y,x)$ is isomorphic to $\mathbb{C}^2$, through the map $(x,y) \mapsto (x+y, xy)$. Therefore, by setting $u = x+y$ and $v = xy$, we obtain that

$$\mathscr{X}(r)/\langle\sigma\rangle = \{(e,u,v) \in \mathbb{C}^3 \mid \prod_{i=1}^r (e-z_i)^{d_i} = v\},$$

so

$$(\diamondsuit) \qquad \mathscr{X}(r)/\langle\sigma\rangle \simeq \mathbb{C}^2.$$

On the other hand, there is an obvious isomorphism of varieties

$$(\heartsuit) \qquad \mathscr{X}(r)_{=z} \xrightarrow{\sim} \mathscr{H}_{\prod_{i=1}^r (z-z_i)^{d_i}}$$

which is σ-equivariant. Finally, if ξ denotes a complex number such that $\xi^2 = \prod_{i=1}^{r-1}(z_r - z_i)^{d_i}$, then the map

$$(\spadesuit) \qquad \begin{array}{ccc} \mathscr{X}(r)_{\neq z_r} & \longrightarrow & \mathscr{X}(r-1)_{\neq z_r} \\ (e,x,y) & \longmapsto & (e, \xi^{-1}x, \xi^{-1}y) \end{array}$$

is a σ-equivariant isomorphism of varieties.

We can now start the proof of the proposition by induction on r.

• If $r=1$, then, if we translate $e \mapsto e - z_1$, we may assume that $z_1 = 0$. Then

$$\mathscr{X}(1) = \{e,x,y) \in \mathbb{C}^3 \mid e^{d_1} = xy\} \simeq \mathbb{C}^2/\boldsymbol{\mu}_{d_1},$$

where the group $\boldsymbol{\mu}_{d-1}$ of d_1-th roots of unity in $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by by $\zeta \cdot (x,y) = (\zeta x, \zeta^{-1} y)$. So the cohomology of $\mathscr{X}(1)$ is equal to the space of invariants, under the action of $\boldsymbol{\mu}_{d_1}$, of the cohomology of $\mathbb{C}^2$. So (a) follows. Now, the "open-closed" long exact sequence gives, by using (♡), the following exact sequence of $\mathbb{C}\langle\sigma\rangle$-modules

$$\begin{array}{rcccccl} & & 0 & \longrightarrow & \mathrm{H}^1_c(\mathscr{H}_{z^{d_1}}) & \longrightarrow & \\ \mathrm{H}^2_c(\mathscr{X}(1)_{\neq z}) & \longrightarrow & \mathrm{H}^2_c(\mathscr{X}(1)) & \longrightarrow & \mathrm{H}^2_c(\mathscr{H}_{z^{d_1}}) & \longrightarrow & \\ \mathrm{H}^3_c(\mathscr{X}(1)_{\neq z}) & \longrightarrow & \mathrm{H}^3_c(\mathscr{X}(1)) & \longrightarrow & 0 & \longrightarrow & \\ \mathrm{H}^4_c(\mathscr{X}(1)_{\neq z}) & \longrightarrow & \mathrm{H}^4_c(\mathscr{X}(1)) & \longrightarrow & 0 & \longrightarrow & \end{array}$$

But, by (a), $\mathrm{H}^2_c(\mathscr{X}(1)) = \mathrm{H}^3_c(\mathscr{X}(1)) = 0$, so (b) follows from (♣).

• Let $r \geqslant 2$ and assume that the result cohomology of $\mathscr{X}(r-1)$ is given by Proposition 18.5.9. If $z \in \mathbb{C}$, let $\lambda(z) = \prod_{i=1}^r (z-z_i)^{d_i}$. The "open-closed" long exact sequence gives, by using (♡), the following exact sequence of $\mathbb{C}\langle\sigma\rangle$-modules

$$(*)\qquad\begin{array}{rcccccl} & & 0 & \longrightarrow & \mathrm{H}^1_c(\mathscr{H}_{\lambda(z)}) & \longrightarrow & \\ \mathrm{H}^2_c(\mathscr{X}(r)_{\neq z}) & \longrightarrow & \mathrm{H}^2_c(\mathscr{X}(r)) & \longrightarrow & \mathrm{H}^2_c(\mathscr{H}_{\lambda(z)}) & \longrightarrow & \\ \mathrm{H}^3_c(\mathscr{X}(r)_{\neq z}) & \longrightarrow & \mathrm{H}^3_c(\mathscr{X}(r)) & \longrightarrow & 0 & \longrightarrow & \\ \mathrm{H}^4_c(\mathscr{X}(r)_{\neq z}) & \longrightarrow & \mathrm{H}^4_c(\mathscr{X}(r)) & \longrightarrow & 0 & \longrightarrow & \end{array}$$

Assume first that $z = z_r$. Then $\lambda(z) = 0$ and, by (♠), $\mathscr{X}(r)_{\neq z_r} \simeq \mathscr{X}(r-1)_{\neq z_r}$. Using the induction hypothesis and (♣), the exact sequence $(*)$ becomes

$$0 \longrightarrow \mathbb{C}_- \longrightarrow \mathbb{C}_-^{r-1} \longrightarrow \mathrm{H}^2_c(\mathscr{X}(r)) \longrightarrow \mathbb{C}_+ \oplus \mathbb{C}_- \longrightarrow \mathbb{C}_+ \longrightarrow \mathrm{H}^3_c(\mathscr{X}(r)) \longrightarrow 0 \longrightarrow \mathbb{C}_+ \longrightarrow \mathrm{H}^4_c(\mathscr{X}(r)) \longrightarrow 0.$$

But it follows from (♢) that $\mathrm{H}^i_c(\mathscr{X}(r))^\sigma = 0$ if $i \in \{2,3\}$. Since the map $\mathbb{C}_+ \to \mathrm{H}^3_c(\mathscr{X}(r))$ is surjective, this forces $\mathrm{H}^3_c(\mathscr{X}(r)) = 0$. Also, $\mathrm{H}^2_c(\mathscr{X}(r)) = \mathbb{C}_-^l$ for some l so, taking the $\mathbb{C}_-$-isotypic component in the above exact sequence yields an exact sequence

$$0 \longrightarrow \mathbb{C}_- \longrightarrow \mathbb{C}_-^{r-1} \longrightarrow \mathbb{C}_-^l \longrightarrow \mathbb{C}_- \longrightarrow 0.$$

So $l = r-1$, as desired. This proves (a).

Now, if $z \in \{z_1, z_2, \dots, z_r\}$ then, by symmetry, we may assume that $z = z_r$ and the isomorphism (♠) yields (b) in this case by the induction hypothesis. Now, assume that $z \notin \{z_1, z_2, \dots, z_r\}$. Then $\lambda(z) \neq 0$ and it follows from (a) that the exact sequence $(*)$ becomes

$$0 \longrightarrow \mathbb{C}_- \longrightarrow \mathrm{H}_c^2(\mathscr{X}(r)_{\neq z}) \longrightarrow \mathbb{C}_-^{r-1} \longrightarrow \mathbb{C}_+ \longrightarrow \mathrm{H}_c^3(\mathscr{X}(r)_{\neq z}) \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{H}_c^2(\mathscr{X}(r)_{\neq z}) \longrightarrow \mathbb{C}_+ \longrightarrow 0.$$

So (b) follows because the maps are σ-equivariant. $\square$

CHAPTER 19

TYPE B_2

Assumption and notation. *In §19, we assume that* $\dim_{\mathbf{k}} V = 2$*, we fix a* $\mathbf{k}$*-basis* (x, y) *of* V *and we denote by* (X, Y) *its dual basis. Let* s *and* t *be the two reflections of* $\mathrm{GL}_{\mathbf{k}}(V)$ *whose matrices in the basis* (x, y) *are given by*

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \textit{and} \qquad t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We assume moreover that $W = \langle s, t \rangle$*: it is a Weyl group of type* B_2*.*

19.1. The algebra H

We set $w = st$, $w' = ts$, $s' = tst$, $t' = sts$ and $w_0 = stst = tsts = -\mathrm{Id}_V$. Then

$$W = \{1, s, t, w, w', s', t', w_0\} \qquad \text{and} \qquad \mathrm{Ref}(W) = \{s, t, s', t'\}.$$

Moreover,

$$\mathrm{Ref}(W)/W = \{\{s, s'\}, \{t, t'\}\}.$$

The matrices of the elements w, w', s', t' and w_0 in the basis (x, y) are given by

$$\begin{cases} w = st = s't' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ w' = ts = t's' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ s' = tst = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \\ t' = sts = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ w_0 = ss' = tt' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \end{cases} \tag{19.1.1}$$

We set $A = C_s$ and $B = C_t$ so that, in $\mathbf{H}$, the following relations hold:

$$(19.1.2)\qquad \begin{cases} [x,X] = -A(s+s') - 2Bt, \\ [x,Y] = A(s-s'), \\ [y,X] = A(s-s'), \\ [y,Y] = -A(s+s') - 2Bt'. \end{cases}$$

We deduce for example that

$$(19.1.3)\qquad \begin{cases} [x,X^2] = -A(s+s')X - A(s-s')Y, \\ [x,XY] = -2BtY, \\ [x,Y^2] = A(s+s')X + A(s-s')Y, \\ [y,X^2] = A(s+s')Y + A(s-s')X, \\ [y,XY] = -2Bt'X, \\ [y,Y^2] = -A(s+s')Y - A(s-s')X. \end{cases}$$

Finally, note that $\mathbf{H}$ is endowed with an automorphism η corresponding to $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathscr{N}$:

$$\eta(x) = y - x, \quad \eta(y) = x + y, \quad \eta(X) = \frac{Y-X}{2}, \quad \eta(Y) = \frac{X+Y}{2},$$
$$\eta(A) = B, \quad \eta(B) = A, \quad \eta(s) = t \quad \text{and} \quad \eta(s) = t.$$

19.2. Irreducible characters

Let ε_s (respectively ε_t) denote the unique linear character of W such that $\varepsilon_s(s) = -1$ and $\varepsilon_s(t) = 1$ (respectively $\varepsilon_t(s) = 1$ and $\varepsilon_t(t) = -1$). Note that $\varepsilon_s\varepsilon_t = \varepsilon$. Let χ denote the character of the representation V of W. Then

$$\operatorname{Irr}(W) = \{1, \varepsilon_s, \varepsilon_t, \varepsilon, \chi\}$$

and the character table of W is given by Table 19.2.1. The fake degrees are given by

$$(19.2.2)\qquad \begin{cases} f_1(\mathbf{t}) = 1, \\ f_{\varepsilon_s}(\mathbf{t}) = \mathbf{t}^2, \\ f_{\varepsilon_t}(\mathbf{t}) = \mathbf{t}^2, \\ f_{\varepsilon}(\mathbf{t}) = \mathbf{t}^4, \\ f_{\chi}(\mathbf{t}) = \mathbf{t} + \mathbf{t}^3. \end{cases}$$

g	1	w_0	s	t	w
$\|\mathrm{Cl}_W(g)\|$	1	1	2	2	2
$o(g)$	1	2	2	2	4
$\mathrm{C}_W(g)$	W	W	$\langle s, s'\rangle$	$\langle t, t'\rangle$	$\langle w\rangle$
1	1	1	1	1	1
ε_s	1	1	−1	1	−1
ε_t	1	1	1	−1	−1
ε	1	1	−1	−1	1
χ	2	−2	0	0	0

TABLE 19.2.1. Character Table of W

19.3. Computation of $(V \times V^*)/\Delta W$

Before computing the center Z of $\mathbf{H}$, we will compute its specialization Z_0 at $(A, B) \mapsto (0, 0)$. By Example 5.7.5,

$$Z_0 = \mathbf{k}[V \times V^*]^{\Delta W}.$$

Thanks to (19.2.2) and Proposition 2.5.10, the bigraded Hilbert series of Z_0 is given by

(19.3.1)
$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z_0) = \frac{1 + \mathbf{tu} + \mathbf{tu}^3 + 2\mathbf{t}^2\mathbf{u}^2 + \mathbf{t}^3\mathbf{u} + \mathbf{t}^3\mathbf{u}^3 + \mathbf{t}^4\mathbf{u}^4}{(1-\mathbf{t}^2)(1-\mathbf{t}^4)(1-\mathbf{u}^2)(1-\mathbf{u}^4)}.$$

Set

$$\sigma = x^2 + y^2, \quad \pi = x^2 y^2, \quad \Sigma = X^2 + Y^2 \quad \text{and} \quad \Pi = X^2 Y^2.$$

Then

(19.3.2)
$$\mathbf{k}[V^*]^W = \mathbf{k}[\sigma, \pi] \qquad \text{and} \qquad \mathbf{k}[V]^W = \mathbf{k}[\Sigma, \Pi].$$

So the bigraded Hilbert series of $P_\bullet = \mathbf{k}[V \times V^*]^{W\times W} = \mathbf{k}[V]^W \otimes \mathbf{k}[V^*]^W = \mathbf{k}[\sigma, \pi, \Sigma, \Pi]$ is given by

(19.3.3)
$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(P_\bullet) = \frac{1}{(1-\mathbf{t}^2)(1-\mathbf{t}^4)(1-\mathbf{u}^2)(1-\mathbf{u}^4)}.$$

Now, set

$$\mathbf{eu}_0 = xX + yY,\ \ \mathbf{eu}'_0 = (xY + yX)XY,\ \ \mathbf{eu}''_0 = xy(xY + yX),\ \ \mathbf{eu}'''_0 = xy(xX + yY)XY,$$

$$\boldsymbol{\delta}_0 = xyXY,\quad \boldsymbol{\delta}'_0 = (x^2 - y^2)(X^2 - Y^2)\quad \text{and}\quad \boldsymbol{\Delta}_0 = xy(x^2 - y^2)XY(X^2 - Y^2).$$

It is then easy to check that the family $(1, \mathbf{eu}_0, \mathbf{eu}'_0, \mathbf{eu}''_0, \mathbf{eu}'''_0, \boldsymbol{\delta}_0, \boldsymbol{\delta}'_0, \boldsymbol{\Delta}_0)$ is $P_\bullet$-linearly independent and is contained in $\mathbf{k}[V \times V^*]^{\Delta W}$. On the other hand, 1, $\mathbf{eu}_0$, $\mathbf{eu}'_0$, $\mathbf{eu}''_0$, $\mathbf{eu}'''_0$, $\boldsymbol{\delta}_0$, $\boldsymbol{\delta}'_0$ and $\boldsymbol{\Delta}_0$ have respective bidegrees $(0,0)$, $(1,1)$, $(1,3)$, $(3,1)$, $(3,3)$, $(2,2)$, $(2,2)$ and $(4,4)$. So the bigraded Hilbert series of the free $P_\bullet$-module with basis $(1, \mathbf{eu}_0, \mathbf{eu}'_0, \mathbf{eu}''_0, \mathbf{eu}'''_0, \boldsymbol{\delta}_0, \boldsymbol{\delta}'_0, \boldsymbol{\Delta}_0)$ is equal to the one of Z_0 (see (19.3.1) and (19.3.3)). Hence

(19.3.4) $$\mathbf{k}[V \times V^*]^{\Delta W} = P_\bullet \oplus P_\bullet \mathbf{eu}_0 \oplus P_\bullet \mathbf{eu}'_0 \oplus P_\bullet \mathbf{eu}''_0 \oplus P_\bullet \mathbf{eu}'''_0 \oplus P_\bullet \boldsymbol{\delta}_0 \oplus P_\bullet \boldsymbol{\delta}'_0 \oplus P_\bullet \boldsymbol{\Delta}_0.$$

The following result, already known (see for instance **[AlFo]**), describes the algebra Z_0:

Theorem 19.3.5. — *$Z_0 = \mathbf{k}[V \times V^*]^{\Delta W} = \mathbf{k}[\sigma, \pi, \Sigma, \Pi, \mathbf{eu}_0, \mathbf{eu}'_0, \mathbf{eu}''_0, \boldsymbol{\delta}_0]$ and the ideal of relations is generated by the following ones:*

$$\begin{cases} (1) & \mathbf{eu}_0\ \mathbf{eu}'_0 = \sigma\Pi + \Sigma\ \boldsymbol{\delta}_0, \\ (2) & \mathbf{eu}_0\ \mathbf{eu}''_0 = \Sigma\pi + \sigma\ \boldsymbol{\delta}_0, \\ (3) & \boldsymbol{\delta}_0\ \mathbf{eu}'_0 = \Pi\ \mathbf{eu}''_0, \\ (4) & \boldsymbol{\delta}_0\ \mathbf{eu}''_0 = \pi\ \mathbf{eu}'_0, \\ (5) & \boldsymbol{\delta}_0^2 = \pi\Pi, \\ (6) & \mathbf{eu}'^2_0 = \Pi(4\ \boldsymbol{\delta}_0 - \mathbf{eu}_0^2 + \sigma\Sigma), \\ (7) & \mathbf{eu}''^2_0 = \pi(4\ \boldsymbol{\delta}_0 - \mathbf{eu}_0^2 + \sigma\Sigma), \\ (8) & \mathbf{eu}'_0\ \mathbf{eu}''_0 = 4\pi\Pi + \sigma\Sigma\ \boldsymbol{\delta}_0 - \boldsymbol{\delta}_0\ \mathbf{eu}_0^2, \\ (9) & \mathbf{eu}_0(4\ \boldsymbol{\delta}_0 - \mathbf{eu}_0^2 + \sigma\Sigma) = \sigma\ \mathbf{eu}'_0 + \Sigma\ \mathbf{eu}''_0. \end{cases}$$

Moreover, $Z_0 = P \oplus P\mathbf{eu}_0 \oplus P\mathbf{eu}_0^2 \oplus P\boldsymbol{\delta}_0 \oplus P\boldsymbol{\delta}_0\mathbf{eu}_0 \oplus P\boldsymbol{\delta}_0\mathbf{eu}_0^2 \oplus P\mathbf{eu}'_0 \oplus P\mathbf{eu}''_0$.

Proof. — It is easily checked that

(19.3.6) $$\boldsymbol{\delta}'_0 = 2\ \mathbf{eu}_0^2 - \sigma\Sigma - 4\ \boldsymbol{\delta}_0,\quad \boldsymbol{\Delta}_0 = \boldsymbol{\delta}_0\ \boldsymbol{\delta}'_0\quad \text{and}\quad \mathbf{eu}'''_0 = \boldsymbol{\delta}_0\ \mathbf{eu}_0.$$

According to (19.3.4), these three relations imply immediately that $\mathbf{k}[V \times V^*]^{\Delta W} = \mathbf{k}[\sigma, \pi, \Sigma, \Pi, \mathbf{eu}_0, \mathbf{eu}'_0, \mathbf{eu}''_0, \boldsymbol{\delta}_0]$. This shows the first statement.

The relations given in Theorem 19.3.5 follow from direct computations. Taking (5) into account, the relation (8) can be rewritten

(8′) $$\mathbf{eu}'_0\ \mathbf{eu}''_0 = \boldsymbol{\delta}_0(4\ \boldsymbol{\delta}_0 + \sigma\Sigma - \mathbf{eu}_0^2),$$

whereas (6) and (7) imply

(10) $$\pi\ \mathbf{eu}'^2_0 = \Pi\ \mathbf{eu}''^2_0.$$

Let E, E', E'' and D be indeterminates over the field $\mathbf{k}(\sigma,\pi,\Sigma,\Pi)$ and let

$$\rho : \mathbf{k}[\sigma,\pi,\Sigma,\Pi,E,E',E'',D] \longrightarrow \mathbf{k}[V\times V^*]^{\Delta W}$$

denote the unique morphism of $\mathbf{k}$-algebras which sends the sequence $(\sigma,\pi,\Sigma,\Pi,E,E',E'',D)$ to $(\sigma,\pi,\Sigma,\Pi,\mathbf{eu}_0,\mathbf{eu}_0',\mathbf{eu}_0'',\boldsymbol{\delta}_0)$. Then ρ is surjective. Let f_i denote the element of $\mathbf{k}[\sigma,\pi,\Sigma,\Pi,E,E',E'',D]$ corresponding to the relation (i) of the Theorem (for $1\leqslant i\leqslant 9$), by subtracting the right-hand side to the left-hand side. Set

$$\mathfrak{I} = \langle f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9\rangle \subset \operatorname{Ker}\rho.$$

Let $\tilde{Z}_0 = \mathbf{k}[\sigma,\pi,\Sigma,\Pi,E,E',E'',D]/\mathfrak{I}$ and let e, e', e'' and d denote the respective images of E, E', E'' and D in $\tilde{Z}_0$. Set

$$\tilde{Z}_0' = P_\bullet + P_\bullet\, e + P_\bullet\, e^2 + P_\bullet\, e' + P_\bullet\, e'' + P_\bullet\, d + P_\bullet\, d\,e + P_\bullet\, d\,e^2.$$

Then $\tilde{Z}_0'$ is a $\mathbf{k}$-vector subspace of $\tilde{Z}_0$. The relations given by $(f_i)_{1\leqslant i\leqslant 9}$ show that $\tilde{Z}_0'$ is a $\mathbf{k}$-subalgebra of $\tilde{Z}_0$. As moreover σ, π, Σ, Π, e, e', e'' and d belong to $\tilde{Z}_0'$, we deduce that $\tilde{Z}_0 = \tilde{Z}_0'$.

Consequently, $\tilde{Z}_0$ is a quotient of the graded $\mathbf{k}$-vector space

$$\mathscr{E} = P_\bullet \oplus P_\bullet[-2] \oplus (P_\bullet[-4])^3 \oplus P_\bullet[-6] \oplus P_\bullet[-8],$$

and Z_0 is a quotient of $\tilde{Z}_0$. As the Hilbert series of the $\mathbf{k}$-vector space $\mathscr{E}$ is equal to the Hilbert series of Z_0, we deduce that

$$\tilde{Z}_0' = P_\bullet \oplus P_\bullet e \oplus P_\bullet e^2 \oplus P_\bullet e' \oplus P_\bullet e'' \oplus P_\bullet d \oplus P_\bullet d\,e \oplus P_\bullet d\,e^2$$

and that $\tilde{Z}_0 \simeq Z_0$. This shows that $\operatorname{Ker}\rho = \mathfrak{I}$, as desired. □

Corollary 19.3.7. — *The relations* (1), (2),…, (9) *is a **minimal** system of relations. In particular, the* $\mathbf{k}$*-algebra* $Z_0 = \mathbf{k}[V\times V^*]^{\Delta W}$ *is not complete intersection.*

Proof. — Let us use the notation of the proof of Theorem 19.3.5. It is sufficient to show that $(f_i)_{1\leqslant i\leqslant 9}$ is a minimal system of generators of $\mathfrak{I}$. Let $\bar{Z}_0 = Z_0/\langle\sigma,\pi,\Sigma,\Pi\rangle$ and let e, e', e'' and d denote the respective images of $\mathbf{eu}_0$, $\mathbf{eu}_0'$, $\mathbf{eu}_0''$ and $\boldsymbol{\delta}_0$ in $\bar{Z}_0$. Then it follows from (19.3.2) and from the relations (19.3.6) that

$$\bar{Z}_0 = \mathbf{k} \oplus \mathbf{k}e \oplus \mathbf{k}e^2 \oplus \mathbf{k}e' \oplus \mathbf{k}e'' \oplus \mathbf{k}d \oplus \mathbf{k}d\,e \oplus \mathbf{k}d\,e^2.$$

Let $\bar{f}_i \in \mathbf{k}[E,E',E'',D]$ denote the reduction of the polynomial f_i modulo $\langle\sigma,\pi,\Sigma,\Pi\rangle$. We only need to show that $(\bar{f}_i)_{1\leqslant i\leqslant 9}$ is a minimal system of generators of the kernel of the morphism of $\mathbf{k}$-algebras

$$\bar{\rho} : \mathbf{k}[E,E',E'',D] \longrightarrow \bar{Z}_0$$

which sends E, E', E'' and D on e, e', e'' and d respectively.

The algebra $N = k[E,E',E'',D]$ is bigraded, with E, E', E'' and D of respective bidegrees $(1,1)$, $(1,3)$, $(3,1)$ and $(2,2)$, and the elements $\bar{f}_1,\dots,\bar{f}_9$ are homogeneous of respective bidegrees $(2,4)$, $(4,2)$, $(3,5)$, $(5,3)$, $(4,4)$, $(2,6)$, $(6,2)$, $(4,4)$, $(3,3)$. We deduce that

$$\Big(\sum_{i=1}^{9}\mathbf{k}\bar{f}_i\Big)\cap\Big(\sum_{i=1}^{9}N_+\bar{f}_i\Big)\subset\Big(\sum_{i=1}^{9}\mathbf{k}\bar{f}_i\Big)\cap\Big(\sum_{i=1}^{9}\mathbf{k}E\bar{f}_i\Big)$$

Since all these spaces are bigraded, this intersection is contained in

$$(\mathbf{k}\bar{f}_3)\cap(\mathbf{k}E\bar{f}_1)+(\mathbf{k}\bar{f}_4)\cap(\mathbf{k}E\bar{f}_2)+(\mathbf{k}\bar{f}_5+\mathbf{k}\bar{f}_8)\cap(\mathbf{k}E\bar{f}_9).$$

Since E divides neither $\bar{f}_3$ nor $\bar{f}_4$, nor any non-zero element of $\mathbf{k}\bar{f}_5+\mathbf{k}\bar{f}_8$, we conclude that $\big(\sum_{i=1}^{9}\mathbf{k}\bar{f}_i\big)\cap\big(\sum_{i=1}^{9}N_+\bar{f}_i\big)=0$, so $(\bar{f}_i)_{1\leqslant i\leqslant 9}$ is a minimal system of generators of $\operatorname{Ker}\bar{\rho}$. $\square$

Corollary 19.3.8. — *The minimal polynomial of* $\mathbf{eu}_0$ *over* $P_\bullet$ *is*

$$\mathbf{t}^8-2\sigma\Sigma\,\mathbf{t}^6+\big(\sigma^2\Sigma^2+2(\sigma^2\Pi+\Sigma^2\pi-8\pi\Pi)\big)\,\mathbf{t}^4-2\sigma\Sigma(\sigma^2\Pi+\Sigma^2\pi-8\pi\Pi)\,\mathbf{t}^2+(\sigma^2\Pi-\Sigma^2\pi)^2.$$

Proof. — By multiplying the relation (9) by $\mathbf{eu}_0$ and by using the relations (1) and (2), we get

$$\mathbf{eu}_0^2(4\boldsymbol{\delta}_0+\sigma\Sigma-\mathbf{eu}_0^2)=\sigma^2\Pi+\Sigma^2\pi+2\sigma\Sigma\boldsymbol{\delta}_0.$$

We deduce immediately that

$$\boldsymbol{\delta}_0(4\mathbf{eu}_0^2-2\sigma\Sigma)=\mathbf{eu}_0^4-\sigma\Sigma\mathbf{eu}_0^2+\sigma^2\Pi+\Sigma^2\pi.$$

Taking the square of this relation and using the relation (5), we get that the polynomial of the Corollary vanishes at $\mathbf{eu}_0$. The degree of the minimal polynomial of $\mathbf{eu}_0$ over $P_\bullet$ being equal to $|W|=8$, the proof of the Corollary is complete. $\square$

19.4. The algebra Z

Recall that

$$\mathbf{eu}=xX+yY+A(s+s')+B(t+t')$$

and set

$$\begin{cases}\mathbf{eu}'=(xY+yX)XY-A(s-s')XY+BtY^2+Bt'X^2,\\ \mathbf{eu}''=xy(xY+yX)-Axy(s-s')+By^2t+Bx^2t',\\ \boldsymbol{\delta}=xyXY+Bxt'X+BytY+B^2(1+w_0)+AB(w+w').\end{cases}$$

A brute force computation shows that

(19.4.1) $$\mathbf{eu},\mathbf{eu}',\mathbf{eu}'',\boldsymbol{\delta}\in Z=\mathrm{Z}(\mathbf{H})$$

and that the following equalities hold:

$$(19.4.2)\qquad \begin{cases} \text{(Z1)} & \mathbf{eu}\,\mathbf{eu}' = \sigma\Pi + \Sigma\,\boldsymbol{\delta}, \\ \text{(Z2)} & \mathbf{eu}\,\mathbf{eu}'' = \Sigma\pi + \sigma\,\boldsymbol{\delta}, \\ \text{(Z3)} & \boldsymbol{\delta}\,\mathbf{eu}' = \Pi\,\mathbf{eu}'' + B^2\Sigma\,\mathbf{eu}, \\ \text{(Z4)} & \boldsymbol{\delta}\,\mathbf{eu}'' = \pi\,\mathbf{eu}' + B^2\sigma\,\mathbf{eu}, \\ \text{(Z5)} & \boldsymbol{\delta}^2 = \pi\Pi + B^2\,\mathbf{eu}^2, \\ \text{(Z6)} & \mathbf{eu}'^2 = \Pi(4\,\boldsymbol{\delta} - \mathbf{eu}^2 + \sigma\Sigma + 4A^2 - 4B^2) + B^2\Sigma^2, \\ \text{(Z7)} & \mathbf{eu}''^2 = \pi(4\,\boldsymbol{\delta} - \mathbf{eu}^2 + \sigma\Sigma + 4A^2 - 4B^2) + B^2\sigma^2, \\ \text{(Z8)} & \mathbf{eu}'\,\mathbf{eu}'' = \boldsymbol{\delta}(4\,\boldsymbol{\delta} - \mathbf{eu}^2 + \sigma\Sigma + 4A^2 - 4B^2) - B^2\sigma\Sigma, \\ \text{(Z9)} & \mathbf{eu}(4\,\boldsymbol{\delta} - \mathbf{eu}^2 + \sigma\Sigma + 4A^2 - 4B^2) = \sigma\,\mathbf{eu}' + \Sigma\,\mathbf{eu}''. \end{cases}$$

We immediately see that $\mathbf{eu}_0$, $\mathbf{eu}_0'$, $\mathbf{eu}_0''$ and $\boldsymbol{\delta}_0$ are the respective images, in $Z_0 = Z/\mathfrak{p}_0 Z$, of the elements $\mathbf{eu}$, $\mathbf{eu}'$, $\mathbf{eu}''$ and $\boldsymbol{\delta}$. On the other hand, the relations (1), (2),..., (9) of Theorem 19.3.5 are also the images, modulo $\mathfrak{p}_0$, of the relations (Z1), (Z2),..., (Z9).

Theorem 19.4.3. — *The* $\mathbf{k}$*-algebra* Z *is generated by* A, B, σ, π, Σ, Π, $\mathbf{eu}$, $\mathbf{eu}'$, $\mathbf{eu}''$ *and* $\boldsymbol{\delta}$. *The ideal of relations is generated by* (Z1), (Z2),..., (Z9).

Moreover, $Z = P \oplus P\mathbf{eu} \oplus P\mathbf{eu}^2 \oplus P\boldsymbol{\delta} \oplus P\boldsymbol{\delta}\mathbf{eu} \oplus P\boldsymbol{\delta}\mathbf{eu}^2 \oplus P\mathbf{eu}' \oplus P\mathbf{eu}''$.

Proof. — The proof follows exactly the same arguments as the ones of Theorem 19.3.5, based in part on comparisons of bigraded Hilbert series. □

Corollary 19.4.4. — *The relations* (Z1), (Z2),..., (Z9) *is a minimal system of relations. In particular, the* $\mathbf{k}$*-algebra* Z *is not complete intersection.*

Proof. — This follows immediately from Theorem 19.4.3, using the same arguments as in the proof of Corollary 19.3.7. □

Corollary 19.4.5. — *The minimal polynomial of* $\mathbf{eu}$ *over* P *is*

$$\mathbf{t}^8 - 2(\sigma\Sigma + 4A^2 + 4B^2)\,\mathbf{t}^6 + \big(\sigma^2\Sigma^2 + 2(\sigma^2\Pi + \Sigma^2\pi - 8\pi\Pi) + 8(A^2+B^2)\sigma\Sigma + 16(A^2-B^2)^2\big)\,\mathbf{t}^4$$
$$-2\big((\sigma\Sigma + 4A^2 - 4B^2)(\sigma^2\Pi + \Sigma^2\pi) - 8\sigma\Sigma\pi\Pi + 2B^2\sigma^2\Sigma^2\big)\,\mathbf{t}^2 + (\sigma^2\Pi - \Sigma^2\pi)^2.$$

Proof. — The proof follows exactly the same steps as the proof of Corollary 19.3.8, but by starting with the relations (Z1),..., (Z9) instead of the relations (1),..., (9). □

$z \in Z$	**eu**	**eu**$'$	**eu**$''$	$\boldsymbol{\delta}$
Ω_1	$-2(B+A)$	0	0	$2B(B+A)$
Ω_{ε_s}	$-2(B-A)$	0	0	$2B(B-A)$
Ω_{ε_t}	$2(B-A)$	0	0	$2B(B-A)$
Ω_{ε}	$2(B+A)$	0	0	$2B(B+A)$
Ω_{χ}	0	0	0	0

TABLE 19.5.1. Table central characters of $\bar{\mathbf{H}}$

Acknowledgments — The above computations (checking that **eu**, **eu**$'$, **eu**$''$ and $\boldsymbol{\delta}$ are central, and checking the relations (Z1),..., (Z9)) have been done without computer. Even though we have been very carefully, the heaviness of the computations imply that it might happen that some mistakes occur. However, U. Thiel has developed a `MAGMA` package (called `CHAMP`, see **[Thi3]**) for computing in the algebra **H**: hence, he has checked *independently* that the elements **eu**, **eu**$'$, **eu**$''$ and $\boldsymbol{\delta}$ are central and that the relations (Z1),..., (Z9) hold. We wish to thank warmly U. Thiel for this checking: he has also checked that the minimal polynomial of **eu** is given by Corollary 19.4.5. ■

19.5. Calogero-Moser families

The Table 19.5.1 gives the values of Ω_ψ (for $\psi \in \mathrm{Irr}(W)$) at the generators of the P-algebra Z. They are obtained by computing effectively the actions of **eu**, **eu**$'$, **eu**$''$ and $\boldsymbol{\delta}$ or by using Corollary 7.3.2 and using the relations (Z1),..., (Z9) (knowing that $\Omega_\psi(\sigma) = \Omega_\psi(\pi) = \Omega_\psi(\Sigma) = \Omega_\psi(\Pi) = 0$).

Now, let K be a field and fix a morphism $\mathbf{k}[\mathscr{C}] \to K$. Let a and b denote the respective images of A and B in K. The previous Table allows to compute immediately the partitions of $\mathrm{Irr}(W)$ into Calogero-Moser K-families, according to the values of a and b. The (well-known) result is given in Table 19.5.2.

Conditions	K-families
$a=b=0$	$\mathrm{Irr}(W)$
$a=0,\ b\neq 0$	$\{1,\varepsilon_s\}$, $\{\varepsilon_t,\varepsilon\}$ et $\{\chi\}$
$a\neq 0,\ b=0$	$\{1,\varepsilon_t\}$, $\{\varepsilon_s,\varepsilon\}$ et $\{\chi\}$
$a=b\neq 0$	$\{1\}$, $\{\varepsilon\}$ et $\{\varepsilon_s,\varepsilon_t,\chi\}$
$a=-b\neq 0$	$\{\varepsilon_s\}$, $\{\varepsilon_t\}$ et $\{1,\varepsilon,\chi\}$
$ab\neq 0,\ a^2\neq b^2$	$\{1\}$, $\{\varepsilon_s\}$, $\{\varepsilon_t\}$, $\{\varepsilon\}$ et $\{\chi\}$

TABLE 19.5.2. Calogero-Moser K-families

19.6. The group G

Since $w_0=-\mathrm{Id}_V$ belongs to W (and since all the reflections of W have order 2), the results of § 5.5 can be applied. In particular, if $\tau_0=(-1,1,\varepsilon)\in\mathbf{k}^\times\times\mathbf{k}^\times\times W^\wedge$, then τ_0 can be seen as the element $w_0\in W\hookrightarrow G$ and is central in G (see Proposition 5.5.2). Hence, by (5.5.3), we get

$$G\subset W_4,$$

where W_4 is the subgroup of $\mathfrak{S}_W$ consisting of permutations σ of W such that $\sigma(-x)=-\sigma(x)$ for all $x\in W$. We denote by N_4 the (normal) subgroup of W_4 consisting of permutations $\sigma\in W_4$ such that $\sigma(x)\in\{x,-x\}$ for all $x\in W$. Then, if we set $\mu_2=\{1,-1\}$,

$$N_4\simeq(\mu_2)^4.$$

Moreover,

$$|W_4|=384 \qquad\text{and}\qquad |N_4|=16.$$

Let $\varepsilon_W:\mathfrak{S}_W\to\mu_2=\{1,-1\}$ be the sign character and let $W_4'=W_4\cap\operatorname{Ker}\varepsilon_W$ and $N_4'=W_4'\cap N_4$. Then

$$N_4'=\{(\eta_1,\eta_2,\eta_3,\eta_4)\in(\mu_2)^4\mid \eta_1\eta_2\eta_3\eta_4=1\}\simeq(\mu_2)^3.$$

Moreover,

$$|W_4'|=192 \qquad\text{and}\qquad |N_4'|=8.$$

Recall that H is identified with the stabilizer, in G, of $1\in W$. Moreover, G contains the image of $W\times W$ in $\mathfrak{S}_W$. This image, isomorphic to $(W\times W)/\Delta\mathrm{Z}(W)$, has order 32 and its intersection with H, isomorphic to $\Delta W/\Delta\mathrm{Z}(W)\simeq W/\mathrm{Z}(W)$, has order 4.

The elements (s,s) and (t,t) of $W \times W$ are sent to distinct elements of N_4. So $H \cap N_4$ is a subgroup of N_4' of order 4. Since $(w_0, 1)$ is sent to an element of N_4' which does not belong to H, we deduce that

$$N_4' \subset G.$$

Let $f(\mathbf{t}) \in P[\mathbf{t}]$ denote the unique monic polynomial of degree 4 such that $f(\mathbf{eu}^2) = 0$ (it is given by Corollary 19.4.5). According to (B.6.1), we have

$$\operatorname{disc}(f(\mathbf{t}^2)) = 256 \ \operatorname{disc}(f)^2 \cdot (\sigma^2\Pi - \Sigma^2\pi)^2,$$

and so the discriminant of the minimal polynomial of **eu** is a square in P. Hence,

$$G \subseteq W_4'.$$

We will show that this inclusion is an equality.

Theorem 19.6.1. — $G = W_4'$.

Proof. — It is sufficient to show that $|G| = 192$. We already know that $N_4' \subset G \subset W_4'$, which shows that $G \cap N_4 = N_4'$. To show the Theorem, it is sufficient to show that $G/N_4' \simeq \mathfrak{S}_4$. But, $G/N_4' = G/(G \cap N_4)$ is the Galois group of the polynomial f. So we only need to show that the Galois group of f over $\mathbf{K}$ is $\mathfrak{S}_4$. Let $\bar{G}$ denote this Galois group.

Let $\mathfrak{p} = \langle \sigma - 2, \Sigma + 2, A - 1, B, \Pi - \pi \rangle$. Then $\mathfrak{p}$ is a prime ideal and $P/\mathfrak{p} \simeq \mathbf{k}[\pi]$. Let $\bar{f}$ denote the reduction of f modulo $\mathfrak{p}$. Then

$$\bar{f}(\mathbf{t}) = \mathbf{t}(\mathbf{t}^3 + (16\pi - 16\pi^2)\mathbf{t} - 64\pi^2).$$

So, by (B.6.2), we have

$$\operatorname{disc}(\bar{f}) = (64\pi^2)^2 \cdot \big(-4(16\pi - 16\pi^2)^3 - 27 \cdot (-64\pi^2)^2\big) = 2^{24}\pi^7(\pi - 4)(2\pi + 1)^2.$$

So the discriminant of $\bar{f}$ is not a square in $\mathbf{k}[\pi]$, which implies that the discriminant of f is not a square in P. So $\bar{G}$ is not contained in the alternating group $\mathfrak{A}_4$.

Since f is irreducible, $\bar{G}$ is a transitive subgroup of $\mathfrak{S}_4$. In particular, 4 divides $|\bar{G}|$. Moreover, if $c \in \mathscr{C}$ is such that $c_s = c_t = 1$, then, by Table 19.5.2 and Theorem 10.2.7, G admits a subgroup (the inertia group of $\bar{\mathfrak{r}}_c$) which admits an orbit of length 6. So 3 divides $|G|$ and so 3 divides also $|\bar{G}|$. Hence, 12 divides $|\bar{G}|$ and, since $\bar{G} \not\subset \mathfrak{A}_4$, this forces $\bar{G} = \mathfrak{S}_4$. □

Remark 19.6.2. — Recall that W_4 is a Weyl group of type B_4 and that W_4' is a Weyl group of type D_4. ■

19.7. Calogero-Moser cells, Calogero-Moser cellular characters

19.7.A. Results. — The aim of this section is to show that the Conjectures LR and L hold for W. If a and b are positive real numbers and if $c_s = a$ and $c_t = b$, then the description of Kazhdan-Lusztig left, right or two-sided c-cells, of Kazhdan-Lusztig c-families and c-cellular characters is easy and can be found, for instance, in [**Lus4**]. The different cases to be considered are $a > b$, $a = b$ and $a < b$: by using the automorphism η of W which exchanges s and t, we can assume that $a \geqslant b > 0$. The Conjectures LR and L then follow from the description of Calogero-Moser left, right or two-sided c-cells, of Calogero-Moser c-families and c-cellular characters given in Table 19.7.2:

Theorem 19.7.1. — *Let $c \in \mathscr{C}$, set $a = c_s$ and $b = c_t$ and assume that $ab \neq 0$. Then there exists a choice of prime ideals $\mathfrak{r}_c^{\mathrm{left}} \subset \bar{\mathfrak{r}}_c$ such that the Calogero-Moser left, right or two-sided c-cells, of Calogero-Moser c-families and c-cellular characters given by Table 19.7.2.*

Consequently, Conjectures LR and L hold if W has type B_2.

NOTATION - In Table 19.7.2, we have set

$$\Gamma_\chi = \{t, st, ts, sts\},\ \Gamma_\chi^+ = \{t, st\},\ \Gamma_\chi^- = \{ts, sts\},\ \Gamma_s = \{s, ts, sts\} \text{ et } \Gamma_t = \{t, st, tst\}.$$

Moreover:

- $W_3' = H$ denotes the stabilizer of $1 \in W$ in $G = W_4'$ and W_2' denotes the stabilizer of s in W_3'. Note that W_3' (respectively W_2') is a Weyl group of type $D_3 = A_3$ (respectively $D_2 = A_1 \times A_1$).
- $\mathfrak{S}_3$ denotes the subgroup of W_3' which stabilizes Γ_s (this is also the stabilizer of Γ_t): it is isomorphic to the symmetric group of degree 3.
- $\mathbb{Z}/2\mathbb{Z}$ denotes the stabilizer, in W_2', of Γ_χ^+ (or Γ_χ^-). ■

We will now concentrate on the proof of Theorem 19.7.1: we will first start by the generic case, by running over the descending chain of prime ideals $\bar{\mathfrak{p}} \supset \mathfrak{p}^{\mathrm{left}} \supset \langle \pi \rangle$. The use of the ideal $\langle \pi \rangle$ will help us to remove some ambiguity for the computation of Calogero-Moser left cells. It is natural to ask whether this method can be extended, since the prime ideal $\langle \pi \rangle$ is not chosen randomly: it is the defining ideal of a W-orbit of hyperplanes in V^*.

After studying the generic case, we will specialize our parameters to deduce Theorem 19.7.1. The most difficult step is the computation of left cells (see for instance Proposition 19.7.23).

Conditions	$\bar{D}_c = \bar{I}_c$	Two-sided cells			ψ	$\dim_{\mathbf{k}} \mathscr{L}_c(\psi)$	I_c^{left}	D_c^{left}	Left cells		
		Γ	$\lvert\Gamma\rvert$	$\mathrm{Irr}_\Gamma(W)$					C	$\lvert C\rvert$	$[C]_c^{\mathrm{CM}}$
$a^2 \neq b^2$ $ab \neq 0$	W_2'	1	1	$\mathbf{1}_W$	$\mathbf{1}_W$	8	$\mathbb{Z}/2\mathbb{Z}$	W_2'	1	1	$\mathbf{1}_W$
		w_0	1	ε	ε	8			w_0	1	ε
		s	1	ε_s	ε_s	8			s	1	ε_s
		$w_0 s$	1	ε_t	ε_t	8			$w_0 s$	1	ε_t
		Γ_χ	4	χ	χ	8			Γ_χ^+	2	χ
									Γ_χ^-	2	χ
$a = b$ $ab \neq 0$	W_3'	1	1	$\mathbf{1}_W$	$\mathbf{1}_W$	8	$\mathfrak{S}_3$	$\mathfrak{S}_3$	1	1	$\mathbf{1}_W$
		w_0	1	ε	ε	8			w_0	1	ε
		$W \setminus \{1, w_0\}$	6	$\varepsilon_s, \varepsilon_t, \chi$	ε_s	1			Γ_s	3	$\varepsilon_s + \chi$
					ε_t	1			Γ_t	3	$\varepsilon_t + \chi$
					χ	6					

TABLE 19.7.2. Calogero-Moser cells, families, cellular characters

Notation. *If $z \in Z$ (or $q \in Q$), we denote by $F_z(\mathbf{t})$ (or $F_q(\mathbf{t})$) the minimal polynomial of z (or q) over P. If $F(\mathbf{t}) \in P[\mathbf{t}]$, we will denote by $\bar{F}(\mathbf{t})$ (respectively $F^{\mathrm{left}}(\mathbf{t})$, respectively $F^{\pi}(\mathbf{t})$) the reduction of $F(\mathbf{t})$ modulo $\bar{\mathfrak{p}}$ (respectively $\mathfrak{p}^{\mathrm{left}}$, respectively $\langle \pi \rangle$).*

19.7.B. Generic two-sided cells. — We set $\mathrm{eu} = \mathrm{cop}(\mathbf{eu})$, $\mathrm{eu}' = \mathrm{cop}(\mathbf{eu}')$, $\mathrm{eu}'' = \mathrm{cop}(\mathbf{eu}'')$ and $\delta = \mathrm{cop}(\boldsymbol{\delta})$. Recall that $\bar{\mathfrak{p}} = \langle \sigma\,\pi, \Sigma, \Pi \rangle_P$, that $\bar{\mathfrak{z}} = \mathrm{Ker}(\Omega_1)$ and that $\bar{\mathfrak{q}} = \mathrm{cop}(\bar{\mathfrak{z}})$: according to Table 19.5.1, we have

(19.7.3) $$\bar{\mathfrak{q}} = \bar{\mathfrak{p}} Q + \langle \mathrm{eu} + 2(A+B), \delta - 2B(A+B), \mathrm{eu}', \mathrm{eu}'' \rangle_Q.$$

Also, $Q/\bar{\mathfrak{q}} = P/\bar{\mathfrak{p}} = \mathbf{k}[A, B]$. Recall that

(19.7.4) $$\bar{F}_{\mathbf{eu}}(\mathbf{t}) = \mathbf{t}^4(\mathbf{t} - 2(A+B))(\mathbf{t} - 2(B-A))(\mathbf{t} + 2(A+B))(\mathbf{t} + 2(B-A)).$$

Recall also that, since $w_0 = -\mathrm{Id}_V \in W$ and that W is generated by reflections of order 2, we have $\mathrm{eu}_{v w_0} = -\mathrm{eu}_v$ for all $v \in W$ (see Proposition 5.5.2).

Lemma 19.7.5. — *Let $v \in W \setminus \{1, w_0\}$. Then there exists a unique prime ideal $\bar{\mathfrak{r}}$ of R lying over $\bar{\mathfrak{q}}$ and such that $\mathrm{eu}_v \equiv 2(A-B) \mod \bar{\mathfrak{r}}$.*

Proof. — Let us first show the existence statement. Let $\bar{\mathfrak{r}}'$ be a prime ideal of R lying over $\bar{\mathfrak{q}}$: then $\mathrm{eu} \equiv -2(A+B) \mod \bar{\mathfrak{r}}'$ and $\mathrm{eu}_{w_0} \equiv 2(A+B) \mod \bar{\mathfrak{r}}'$. By (19.7.4), there exists a unique element $v' \in W \setminus \{1, w_0\}$ such that $\mathrm{eu}_{v'} \equiv 2(A-B) \mod \bar{\mathfrak{r}}$.

Recall also that H is the stabilizer of $1 \in W$ in $G = W_4' \subset \mathfrak{S}_W$: this is also the stabilizer of w_0. Then H acts transitively on $W \setminus \{1, w_0\}$ (by Theorem 19.6.1) and so there exists $\sigma \in H$ such that $\sigma(v') = v$. Let $\bar{\mathfrak{r}} = \sigma(\bar{\mathfrak{r}}')$. Then $\bar{\mathfrak{r}}$ is a prime ideal of R lying over $\bar{\mathfrak{q}}$ (since $\sigma \in H$) and $\mathrm{eu}_v \equiv 2(A-B) \mod \bar{\mathfrak{r}}$. This concludes the proof of the existence statement.

Let us now show the uniqueness statement. So let $\bar{\mathfrak{r}}$ and $\bar{\mathfrak{r}}'$ be two prime ideals of R lying over $\bar{\mathfrak{q}}$ such that $\mathrm{eu}_v - 2(A-B) \in \bar{\mathfrak{r}} \cap \bar{\mathfrak{r}}'$. Then there exists $\sigma \in H$ such that $\bar{\mathfrak{r}}' = \sigma(\bar{\mathfrak{r}})$. We then have $\mathrm{eu}_v \equiv \mathrm{eu}_{\sigma(v)} \equiv 2(B-A) \mod \bar{\mathfrak{r}}$. By (19.7.4), we know that $2(A-B)$ is a simple root of $\bar{f}(\mathbf{t})$, so $\sigma(v) = v$. Consequently, $\sigma \in I$, where I is the stabilizer of v in H. By Theorem 19.6.1, I is the Klein group acting on $W \setminus \{1, w_0, v, v w_0\}$ (note that $|I| = 4$).

Let $\bar{D}$ (respectively $\bar{I}$) denote the decomposition (respectively inertia) group of $\bar{\mathfrak{r}}$ (in G). By (19.7.4), we have $\bar{I} \subset \bar{D} \subset I$ and it remains to show that $I = \bar{I}$. But the generci two-sided cell covering the generic Calogero-Moser family $\{\chi\}$ has cardinality $\chi(1)^2 = 4$, and it is an orbit under the action of $\bar{I}$. So $|\bar{I}| \geqslant 4 = |I|$. Whence the result. ☐

As a consequence of the proof of the previous Lemma, we obtain the next result:

Corollary 19.7.6. — *Let $v \in W \setminus \{1, w_0\}$. Let $\bar{\mathfrak{r}}$ denote the unique prime ideal of R lying over $\bar{\mathfrak{q}}$ and such that $\mathrm{eu}_v \equiv 2(A-B) \mod \bar{\mathfrak{r}}$. Let $\bar{D}$ (respectively $\bar{I}$) denote the decomposition (respectively inertia) group of $\bar{\mathfrak{r}}$ in G. Then:*

(a) $\bar{D} = \bar{I} = \{\tau \in G \mid \tau(1) = 1$ *and* $\tau(v) = v\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(b) $R/\bar{\mathfrak{r}} = Q/\bar{\mathfrak{q}} = P/\bar{\mathfrak{p}} \simeq \mathbf{k}[A, B]$.

(c) *The generic Calogero-Moser two-sided cells (with respect to $\bar{\mathfrak{r}}$) are* $\{1\}$, $\{w_0\}$, $\{v\}$, $\{v w_0\}$ *and* $W \setminus \{1, w_0, v, v w_0\}$. *Moreover,* $\mathrm{Irr}_{\{1\}}(W) = \{\mathbf{1}_W\}$, $\mathrm{Irr}_{\{w_0\}}(W) = \{\varepsilon\}$, $\mathrm{Irr}_{\{v\}}(W) = \{\varepsilon_s\}$, $\mathrm{Irr}_{\{v w_0\}}(W) = \{\varepsilon_t\}$ *and* $\mathrm{Irr}_{\{1\}}(W) = \{\chi\}$.

Choice. *From now on, and until the end of this Chapter, we denote by $\bar{\mathfrak{r}}$ the unique prime ideal of R lying over $\bar{\mathfrak{q}}$ and such that* $\mathrm{eu}_s \equiv 2(A-B) \mod \bar{\mathfrak{r}}$.

With this choice,

$$\bar{D} = \bar{I} = W_2', \tag{19.7.7}$$

and the generic Calogero-Moser two-sided cells are $\{1\}$, $\{s\}$, $\{w_0 s\}$, $\{w_0\}$ and Γ_χ, and

$$\text{(19.7.8)}\qquad \begin{cases} \mathrm{Irr}^{\mathrm{CM}}_{\{1\}}(W)=\{\mathbf{1}_W\}, \\ \mathrm{Irr}^{\mathrm{CM}}_{\{s\}}(W)=\{\varepsilon_s\}, \\ \mathrm{Irr}^{\mathrm{CM}}_{\{w_0 s\}}(W)=\{\varepsilon_t\}, \\ \mathrm{Irr}^{\mathrm{CM}}_{\{w_0\}}(W)=\{\varepsilon\}, \\ \mathrm{Irr}^{\mathrm{CM}}_{\Gamma_\chi}(W)=\{\chi\}. \end{cases}$$

19.7.C. Generic cellular characters. — Recall that $\mathfrak{p}^{\mathrm{left}}=\langle\Sigma,\Pi\rangle_P$.

Lemma 19.7.9. — *We have* $\mathfrak{q}^{\mathrm{left}}=\mathfrak{p}^{\mathrm{left}}Q+\langle \mathrm{eu}+2(B+A),\mathrm{eu}'+B\Sigma,\mathrm{eu}'',\delta-2B(A+B)\rangle_Q$.

Proof. — Let $\mathfrak{q}'=\langle \mathrm{eu}+2(B+A),\mathrm{eu}'+B\Sigma,\mathrm{eu}'',\delta-2B(A+B)\rangle_Q$. First of all, note that $Q/\mathfrak{p}^{\mathrm{left}}Q$ is the $P/\mathfrak{p}^{\mathrm{left}}=\mathbf{k}[A,B,\Sigma,\Pi]$-algebra admitting the following presentation:

$$\text{(19.7.10)}\qquad \begin{cases} (\mathrm{Q1}^{\mathrm{left}}) & \mathrm{eu}\;\mathrm{eu}'=\Sigma\;\delta, \\ (\mathrm{Q2}^{\mathrm{left}}) & \mathrm{eu}\;\mathrm{eu}''=0, \\ (\mathrm{Q3}^{\mathrm{left}}) & \delta\;\mathrm{eu}'=\Pi\;\mathrm{eu}''+B^2\Sigma\;\mathrm{eu}, \\ (\mathrm{Q4}^{\mathrm{left}}) & \delta\;\mathrm{eu}''=0, \\ (\mathrm{Q5}^{\mathrm{left}}) & \delta^2=B^2\;\mathrm{eu}^2, \\ (\mathrm{Q6}^{\mathrm{left}}) & \mathrm{eu}'^2=\Pi(4\;\delta-\mathrm{eu}^2+4A^2-4B^2)+B^2\Sigma^2, \\ (\mathrm{Q7}^{\mathrm{left}}) & \mathrm{eu}''^2=0, \\ (\mathrm{Q8}^{\mathrm{left}}) & \mathrm{eu}'\;\mathrm{eu}''=\delta(4\;\delta-\mathrm{eu}^2+4A^2-4B^2), \\ (\mathrm{Q9}^{\mathrm{left}}) & \mathrm{eu}(4\;\delta-\mathrm{eu}^2+4A^2-4B^2)=\Sigma\;\mathrm{eu}''. \end{cases}$$

A straightforward computation shows that these relations hold in $Q/\mathfrak{q}'$. Let $\mathfrak{q}''=\mathfrak{p}^{\mathrm{left}}Q+\mathfrak{q}'$. Then $Q/\mathfrak{q}''\simeq\mathbf{k}[\Sigma,\Pi,A,B]\simeq P/\mathfrak{p}^{\mathrm{left}}$, so $\mathfrak{q}''$ is a prime ideal of Q, containing $\mathfrak{p}^{\mathrm{left}}$ and contained in $\bar{\mathfrak{q}}$ (by (19.7.3)). The result then follows from the uniqueness statement in Corollary 11.2.4. ∎

Recall that the Calogero-Moser cellular characters can be defined without computing the Calogero-Moser left cells, by using only prime ideals of Z (or Q) lying over $\mathfrak{p}^{\mathrm{left}}$. Note also that

$$\text{(19.7.11)}\qquad F^{\mathrm{left}}_{\mathbf{eu}}(\mathbf{t})=\mathbf{t}^4(\mathbf{t}-2(A+B))(\mathbf{t}-2(B-A))(\mathbf{t}+2(A+B))(\mathbf{t}+2(B-A)).$$

This equality allows us to construct other prime ideals of Q lying over $\mathfrak{p}^{\mathrm{left}}$:

Lemma 19.7.12. — *Set*

$$\begin{cases} \mathfrak{q}_1^{\mathrm{left}} = \mathfrak{q}^{\mathrm{left}} = \mathfrak{p}^{\mathrm{left}}Q + \langle \mathrm{eu}+2(B+A), \mathrm{eu}'-B\Sigma, \mathrm{eu}'', \delta-2B(B+A)\rangle_Q, \\ \mathfrak{q}_{\varepsilon_s}^{\mathrm{left}} = \mathfrak{p}^{\mathrm{left}}Q + \langle \mathrm{eu}+2(B-A), \mathrm{eu}'-B\Sigma, \mathrm{eu}'', \delta-2B(B-A)\rangle_Q, \\ \mathfrak{q}_{\varepsilon_t}^{\mathrm{left}} = \mathfrak{p}^{\mathrm{left}}Q + \langle \mathrm{eu}-2(B-A), \mathrm{eu}'+B\Sigma, \mathrm{eu}'', \delta-2B(B-A)\rangle_Q, \\ \mathfrak{q}_{\varepsilon}^{\mathrm{left}} = \mathfrak{p}^{\mathrm{left}}Q + \langle \mathrm{eu}-2(B+A), \mathrm{eu}'+B\Sigma, \mathrm{eu}'', \delta-2B(A+B)\rangle_Q, \\ \mathfrak{q}_{\chi}^{\mathrm{left}} = \mathfrak{p}^{\mathrm{left}}Q + \langle \mathrm{eu}, \mathrm{eu}'', \delta\rangle_Q. \end{cases}$$

Then:

(a) *If $\gamma \in \mathrm{Irr}(W)$, then $\mathfrak{q}_\gamma^{\mathrm{left}}$ is a prime ideal of Q lying over $\mathfrak{p}^{\mathrm{left}}$. The associated generic Calogero-Moser cellular character is γ.*

(b) *If $\gamma \in \mathrm{Hom}(W, \mathbf{k}^\times)$, then $\mathfrak{q}_\gamma^{\mathrm{left}} = \mathrm{Ker}(\Omega_\gamma^{\mathrm{left}})$ and $Q/\mathfrak{q}_\gamma^{\mathrm{left}} = P^{\mathrm{left}}$.*

(c) *If we denote by eu'_χ the image of eu' in $Q/\mathfrak{q}_\chi^{\mathrm{left}}$, then $Q/\mathfrak{q}_\chi^{\mathrm{left}} = (P/\mathfrak{p}^{\mathrm{left}})[\mathrm{eu}'_\chi]$ and the minimal polynomial of eu'_χ is $\mathbf{t}^2 - \Pi(4A^2-4B^2) - B^2\Sigma^2$. In particular, $[k_Q(\mathfrak{q}_\chi^{\mathrm{left}}) : k_P(\mathfrak{p}^{\mathrm{left}})] = 2$.*

(d) *If $\mathfrak{q}$ is a prime ideal Q lying over $\mathfrak{p}^{\mathrm{left}}$, then there exists $\gamma \in \mathrm{Irr}(W)$ such that $\mathfrak{q} = \mathfrak{q}_\gamma^{\mathrm{left}}$.*

Proof. — (b) is easily checked by a direct computation, and it implies (a) whenever γ is a linear character.

It follows from the presentation (19.7.10) of $Q/\mathfrak{p}^{\mathrm{left}}Q$ that $Q/\mathfrak{q}_\chi^{\mathrm{left}} = (P/\mathfrak{p}^{\mathrm{left}})[\mathrm{eu}'_\chi]$ and that the minimal polynomial of eu'_χ is $\mathbf{t}^2 - \Pi(4A^2-4B^2) - B^2\Sigma^2$. Since this last polynomial with coefficients in $P/\mathfrak{p}^{\mathrm{left}} = \mathbf{k}[\Sigma, \Pi, A, B]$ is irreducible, this implies that $\mathfrak{q}_\chi^{\mathrm{left}}$ is a prime ideal lying over $\mathfrak{p}^{\mathrm{left}}$. We deduce (c) and the first statement of (a). The last statement of (a) follows from Theorem 14.4.1.

(d) follows from the fact that the sum of the already constructed Calogero-Moser cellular characters is equal to the character of the regular representation of W. ☐

19.7.D. Generic left cells. — We will now lift the results about cellular characters to results about left cells. The first one is a consequence of Theorem 19.7.12.

Corollary 19.7.13. — *Let $\mathfrak{r}^{\mathrm{left}}$ be a prime ideal of R lying over $\mathfrak{q}^{\mathrm{left}}$ and contained in $\bar{\mathfrak{r}}$ and let D^{left} (respectively I^{left}) denote its decomposition (respectively inertia) group. Then $D^{\mathrm{left}} = W_2'$ and $|I^{\mathrm{left}}| = 2$.*

Proof. — First of all, by Corollary 11.2.9, we have $D^{\mathrm{left}} \subset \bar{D} = W_2'$. It follows from Lemma 19.7.12(c) that, if C is a generic Calogero-Moser cell contained in the generic Calogero-Moser two-sided cell Γ covering the family $\{\chi\}$, then $|C| = 2$ and $|C^D| = 4$. In particular, 2 divides $|I^{\mathrm{left}}|$ and $I^{\mathrm{left}} \subsetneq D^{\mathrm{left}}$ by Proposition 11.3.5(b). The Corollary follows. ☐

Corollary 19.7.14. — *There exists a unique prime ideal* $\mathfrak{r}^{\mathrm{left}}$ *of* R *lying over* $\mathfrak{q}^{\mathrm{left}}$ *and contained in* $\bar{\mathfrak{r}}$.

Proof. — Let $\mathfrak{r}^{\mathrm{left}}$ and $\mathfrak{r}_*^{\mathrm{left}}$ be two prime ideals of R lying over $\mathfrak{q}^{\mathrm{left}}$ and contained in $\bar{\mathfrak{r}}$. Then there exists $h \in H$ such that $\mathfrak{r}_*^{\mathrm{left}} = h(\mathfrak{r}^{\mathrm{left}})$. We deduce from Proposition 11.2.8 that $\bar{\mathfrak{r}} = g(\bar{\mathfrak{r}})$. So h belongs to the decomposition group of $\bar{\mathfrak{r}}$, which is the same as the one of $\mathfrak{r}^{\mathrm{left}}$ (by Corollary 19.7.13). So $\mathfrak{r}_*^{\mathrm{left}} = \mathfrak{r}^{\mathrm{left}}$. □

We will denote by $\mathfrak{r}^{\mathrm{left}}$ *the unique prime ideal of* R *lying over* $\mathfrak{q}^{\mathrm{left}}$ *and contained in* $\bar{\mathfrak{r}}$. *We will denote by* D^{left} *(respectively* I^{left}*) its decomposition (respectively inertia) group.*

Corollary 19.7.13 says that

(19.7.15) $$D^{\mathrm{left}} = W_2' \qquad \text{and} \qquad |I^{\mathrm{left}}| = 2.$$

Corollary 19.7.16. — $R/\mathfrak{r}^{\mathrm{left}} \simeq Q/\mathfrak{q}_\chi^{\mathrm{left}}$ *is integrally closed.*

Corollary 19.7.17. — $\{1\}$, $\{s\}$, $\{tst\}$ *and* $\{w_0\}$ *are generic Calogero-Moser left cells, and their associated generic Calogero-Moser cellular characters are given by*

$$\begin{cases} [\,1\,]^{\mathrm{CM}} = \mathbf{1}_W, \\ [\,s\,]^{\mathrm{CM}} = \varepsilon_s, \\ [\,tst\,]^{\mathrm{CM}} = \varepsilon_t, \\ [\,w_0\,]^{\mathrm{CM}} = \varepsilon. \end{cases}$$

Proof. — This follows from the fact that the subsets given in the Corollary are also generic Calogero-Moser two-sided cells and that their associated generic Calogero-Moser families are given by Corollary 19.7.6. □

Corollary 19.7.18. — *The following congruences hold in* R*:*

$$\begin{cases} \mathrm{eu} \equiv -2(B+A) \mod \mathfrak{r}^{\mathrm{left}}, \\ s(\mathrm{eu}) \equiv -2(B-A) \mod \mathfrak{r}^{\mathrm{left}}, \\ tst(\mathrm{eu}) \equiv 2(B-A) \mod \mathfrak{r}^{\mathrm{left}}, \\ w_0(\mathrm{eu}) \equiv 2(B+A) \mod \mathfrak{r}^{\mathrm{left}}, \\ t(\mathrm{eu}) \equiv st(\mathrm{eu}) \equiv ts(\mathrm{eu}) \equiv sts(\mathrm{eu}) \equiv 0 \mod \mathfrak{r}^{\mathrm{left}}. \end{cases}$$

Proof. — By (19.7.11), the following congruence holds in $R[\mathbf{t}]$:

$$(*)\ \prod_{w\in W}(\mathbf{t}-w(\mathrm{eu}))\equiv \mathbf{t}^4(\mathbf{t}-2(A+B))(\mathbf{t}-2(B-A))(\mathbf{t}+2(A+B))(\mathbf{t}+2(B-A)) \mod \mathfrak{r}^{\mathrm{left}}R[\mathbf{t}].$$

We already know that, since $\mathfrak{q}^{\mathrm{left}}\subset\mathfrak{r}^{\mathrm{left}}$, that $\mathrm{eu}\equiv -2(B+A) \mod \mathfrak{r}^{\mathrm{left}}$. This implies that $s(\mathrm{eu})$ is congruent to $-2(B-A)$, $2(B+A)$, $2(B-A)$ or 0 modulo $\mathfrak{r}^{\mathrm{left}}$. But, since $s(\mathrm{eu})\equiv -2(B-A) \mod \bar{\mathfrak{r}}$ by construction, this forces $s(\mathrm{eu})\equiv -2(B-A) \mod \mathfrak{r}^{\mathrm{left}}$.

The third and fourth congruences are obtained from the first two ones by noting that $tst(\mathrm{eu})=w_0s(\mathrm{eu})=-s(\mathrm{eu})$ and $w_0(\mathrm{eu})=-\mathrm{eu}$.

The last one follows from $(*)$. □

Corollary 19.7.19. — *The following congruences hold in R:*

$$\begin{cases} \delta\equiv 2B(B+A) \mod \mathfrak{r}^{\mathrm{left}},\\ s(\delta)\equiv 2B(B-A) \mod \mathfrak{r}^{\mathrm{left}},\\ tst(\delta)\equiv 2B(B-A) \mod \mathfrak{r}^{\mathrm{left}},\\ w_0(\delta)\equiv 2B(B+A) \mod \mathfrak{r}^{\mathrm{left}},\\ t(\delta)\equiv st(\delta)\equiv ts(\delta)\equiv sts(\delta)\equiv 0 \mod \mathfrak{r}^{\mathrm{left}} \end{cases}$$

and

$$\begin{cases} \mathrm{eu}'\equiv -B\Sigma \mod \mathfrak{r}^{\mathrm{left}},\\ s(\mathrm{eu}')\equiv -B\Sigma \mod \mathfrak{r}^{\mathrm{left}},\\ tst(\mathrm{eu}')\equiv B\Sigma \mod \mathfrak{r}^{\mathrm{left}},\\ w_0(\mathrm{eu}')\equiv B\Sigma \mod \mathfrak{r}^{\mathrm{left}},\\ t(\mathrm{eu}')^2\equiv st(\mathrm{eu}')^2\equiv ts(\mathrm{eu}')^2\equiv sts(\mathrm{eu}')^2\equiv B^2(\Sigma^2-4\Pi)+4A^2\Pi \mod \mathfrak{r}^{\mathrm{left}}. \end{cases}$$

Finally, $g(\mathrm{eu}'')\equiv 0 \mod \mathfrak{r}^{\mathrm{left}}$ for all $g\in G$.

Proof. — The equalities $(Q1^{\mathrm{left}})$,…, $(Q9^{\mathrm{left}})$ are of course also satisfied in the algebra $R/\mathfrak{p}^{\mathrm{left}}R$. Since $\mathfrak{p}^{\mathrm{left}}R$ is a G-stable ideal of R, we can apply any element of G to the equalities $(Q1^{\mathrm{left}})$,…, $(Q9^{\mathrm{left}})$, and then reduce modulo $\mathfrak{r}^{\mathrm{left}}$. We deduce for instance from $(Q7^{\mathrm{left}})$ that $g(\mathrm{eu}'')\equiv 0 \mod \mathfrak{r}^{\mathrm{left}}$ for all $g\in G$, as desired.

Whenever $g(\mathrm{eu})\not\equiv 0 \mod \mathfrak{r}^{\mathrm{left}}$, we deduce from $(Q9^{\mathrm{left}})$ that $4g(\delta)\equiv g(\mathrm{eu})^2-4A^2+4B^2 \mod \mathfrak{r}^{\mathrm{left}}$, which allows to show that the congruences of δ, $s(\delta)$, $tst(\delta)$ and $w_0(\delta)$ modulo $\mathfrak{r}^{\mathrm{left}}$ are the ones expected. Moreover, if $g\in W\setminus\{1,s,tst,w_0\}$, then $g(\mathrm{eu})\equiv 0 \mod \mathfrak{r}^{\mathrm{left}}$ and we deduce from $(Q5^{\mathrm{left}})$ that $g(\delta)\equiv 0 \mod \mathfrak{r}^{\mathrm{left}}$.

Finally, whenever $g(\mathrm{eu})\not\equiv 0 \mod \mathfrak{r}^{\mathrm{left}}$, the congruence of $g(\mathrm{eu}')$ modulo $\mathfrak{r}^{\mathrm{left}}$ is easily determined thanks to $(Q1^{\mathrm{left}})$, and is as expected. However, whenever $g(\mathrm{eu})\equiv 0 \mod \mathfrak{r}^{\mathrm{left}}$, then $g(\delta)\equiv 0 \mod \mathfrak{r}^{\mathrm{left}}$ by the previous observations, and it follows from $(\mathrm{Q}6^{\mathrm{left}})$ that $g(\mathrm{eu}')^2\equiv B^2(\Sigma^2-4\Pi)+4A^2\Pi \mod \mathfrak{r}^{\mathrm{left}}$. □

Lemma 19.7.20. — $F^{\mathrm{left}}_{\mathrm{eu}'}(\mathbf{t})=(\mathbf{t}-B\Sigma)^2(\mathbf{t}+B\Sigma)^2(\mathbf{t}^2-B^2\Sigma^2-4A^2\Pi+4B^2\Pi)^2$.

Proof. — First of all, by applying the elements of $W \times W$ to $\mathbf{eu}'_0 \in \mathbf{k}[V \times V^*]$, we note that $\mathbf{eu}'_0$ has 8 conjugates, and so the minimal polynomial of $\mathbf{eu}'_0$ over $P_\bullet$ has degree 8. Consequently, $F_{\mathrm{eu}'}(\mathbf{t})$ has degree 8: it is in fact the characteristic polynomial of the multiplication by $\mathbf{eu}'$ in the free P-module Z. Hence,

$$F_{\mathrm{eu}'}(\mathbf{t}) = \prod_{w \in W} (\mathbf{t} - w(\mathrm{eu}'))$$

and the result then follows from Corollary 19.7.19. □

As a conclusion, if $g \in \{t, st, ts, sts\}$ and if $q \in \{\mathrm{eu}, \mathrm{eu}'', \delta\}$ then

$$g(q) \equiv 0 \mod \mathfrak{r}^{\mathrm{left}} \tag{19.7.21}$$

and

$$g(\mathrm{eu}')^2 \equiv B^2(\Sigma^2 - 4\Pi) + 4A^2\Pi \mod \mathfrak{r}^{\mathrm{left}}. \tag{19.7.22}$$

The next Proposition makes the congruence (19.7.22) more precise: it is the most subtle point of this Chapter.

Proposition 19.7.23. — *The following congruences hold in R:*

$$\begin{cases} t(\mathrm{eu}') \equiv st(\mathrm{eu}') \mod \mathfrak{r}^{\mathrm{left}}, \\ ts(\mathrm{eu}') \equiv sts(\mathrm{eu}') \mod \mathfrak{r}^{\mathrm{left}}, \\ t(\mathrm{eu}') \not\equiv ts(\mathrm{eu}') \mod \mathfrak{r}^{\mathrm{left}}. \end{cases}$$

Proof. — By Corollary 19.7.19,

$$(\mathbf{t} - t(\mathrm{eu}'))(\mathbf{t} - st(\mathrm{eu}'))(\mathbf{t} - ts(\mathrm{eu}'))(\mathbf{t} - sts(\mathrm{eu}')) \equiv (\mathbf{t}^2 - B^2\Sigma^2 - 4A^2\Pi + 4B^2\Pi)^2 \mod \mathfrak{r}^{\mathrm{left}} R[\mathbf{t}].$$

This shows that there exists a unique $g_0 \in \{st, ts, sts\}$ such that $g_0(\mathrm{eu}') \equiv t(\mathrm{eu}')$ mod $\mathfrak{r}^{\mathrm{left}}$. Since $t(\mathrm{eu}') \not\equiv 0 \mod \mathfrak{r}^{\mathrm{left}}$ and $sts(\mathrm{eu}') = tw_0(\mathrm{eu}') = -t(\mathrm{eu}')$, the element g_0 is not equal to sts. So

$$g_0 \in \{st, ts\}.$$

We only need to show that

$$g_0 = st. \tag{$*$}$$

Let $F^\pi_{\mathrm{eu}'}(\mathbf{t})$ be the reduction modulo πP of the minimal polynomial of eu'. Set

$$\begin{aligned} F^\pi(\mathbf{t}) &= \mathbf{t}^4 + 2B\Sigma\mathbf{t}^3 + (-4A^2\Pi + 4B^2\Pi - \sigma\Sigma\Pi)\mathbf{t}^2 \\ &\quad -(8A^2B\Sigma\Pi + 2B^3\Sigma^3 - 8B^3\Sigma\Pi + 2B\sigma\Sigma^2\Pi - 4B\sigma\Pi^2)\mathbf{t} \\ &\quad -4A^2B^2\Sigma^2\Pi - B^4\Sigma^4 + 4B^4\Sigma^2\Pi - B^2\sigma\Sigma^3\Pi + 4B^2\sigma\Sigma\Pi^2 + \sigma^2\Pi^3. \end{aligned}$$

Then

$$F^\pi_{\mathrm{eu}'}(\mathbf{t}) = F^\pi(\mathbf{t}) \cdot F^\pi(-\mathbf{t}).$$

Let $\mathfrak{r}^\pi$ be a prime ideal of R lying over πP and contained in $\mathfrak{r}^{\text{left}}$. Let $\mathfrak{r}_0^\pi$ be a prime ideal of R lying over $\pi P + \mathfrak{p}_0$ and contained in $\mathfrak{r}^\pi$. One checks easily that $F^\pi(\mathbf{t})$ is prime to $F^\pi(-\mathbf{t})$ and is separable, so $F^\pi_{\mathrm{eu}'}(\mathbf{t})$ admits 8 different roots in $R/\mathfrak{r}^\pi$, which are the classes of the $g(\mathrm{eu}')$'s, where g runs over W. Let $F^{\text{left}}(\mathbf{t})$ denote the reduction modulo $\mathfrak{p}^{\text{left}}$ of $F^\pi(\mathbf{t})$. Then

$$F^{\text{left}}(\mathbf{t}) = (\mathbf{t} + B\Sigma)^2(\mathbf{t}^2 - B^2\Sigma^2 - 4A^2\Pi + 4B^2\Pi).$$

So it follows from Corollary 19.7.19 that eu' and $s(\mathrm{eu}')$ are roots of $F^\pi(\mathbf{t})$ in $R/\mathfrak{r}^\pi$. On the other hand, since W_2' acts transitively $\{t, st, ts, sts\}$ and stabilizes $\mathfrak{r}^{\text{left}}$, we may, by replacing $\mathfrak{r}^\pi$ by $g(\mathfrak{r}^\pi)$ for some $g \in W_2'$ if necessary, assume that $t(\mathrm{eu}')$ is a root of $F^\pi(\mathbf{t})$ modulo $\mathfrak{r}^\pi$. The other roots modulo $\mathfrak{r}^\pi$ is then $st(\mathrm{eu}')$ or $ts(\mathrm{eu}')$ (this cannot be $sts(\mathrm{eu}') = -t(\mathrm{eu}')$, as this one is a root of $F^\pi(-\mathbf{t})$ modulo $\mathfrak{r}^\pi$). So let g_1 denote the unique element of $\{st, ts\}$ such that $g_1(\mathrm{eu}')$ is a root of $F^\pi(\mathbf{t})$ modulo $\mathfrak{r}^\pi$. By reduction modulo $\mathfrak{r}_0^\pi$, we get

$$\mathrm{eu}' \cdot s(\mathrm{eu}') \cdot t(\mathrm{eu}') \cdot g_1(\mathrm{eu}') \equiv F^\pi(0) \equiv \sigma^2\Pi^3 \mod \mathfrak{r}_0^\pi.$$

Moreover, there exists $g \in G$ such that $\mathfrak{r}_0 \subset g(\mathfrak{r}_0^\pi)$. But, since $G = W_4'$, there exists signs η_1, η_2, η_3, η_4 such that $\{g(1), g(s), g(t), g(g_1)\} = \{\eta_1, \eta_2 s, \eta_3 t, \eta_4 g_1\}$ and $\eta_1\eta_2\eta_3\eta_4 = 1$. Consequently,

$$\mathrm{eu}' \cdot s(\mathrm{eu}') \cdot t(\mathrm{eu}') \cdot g_1(\mathrm{eu}') \equiv \sigma^2\Pi^3 \mod \mathfrak{r}_0.$$

The next computation can be performed directly inside $\mathbf{k}[V \times V^*]^{\Delta Z(W)} \supset R/\mathfrak{r}_0$:

$$\begin{aligned} \mathrm{eu}_0' \cdot s(\mathrm{eu}_0') \cdot t(\mathrm{eu}_0') \cdot ts(\mathrm{eu}_0') &= (xY + yX)(xX + yY)(-xY + yX)(-xX + yY)X^4Y^4 \\ &= (y^2Y^2 - x^2X^2)(y^2X^2 - x^2Y^2)\Pi^2 \\ &= ((x^4 + y^4)\Pi - \pi(X^4 + Y^4))\Pi^2 \\ &\equiv \sigma^2\Pi^3 \mod \pi\mathbf{k}[V \times V^*]. \end{aligned}$$

So $g_1 = ts$.

We have therefore proven that

$$(\mathbf{t} - \mathrm{eu}')(\mathbf{t} - s(\mathrm{eu}'))(\mathbf{t} - t(\mathrm{eu}'))(\mathbf{t} - ts(\mathrm{eu}')) \equiv F^\pi(\mathbf{t}) \mod \mathfrak{r}^\pi R[\mathbf{t}].$$

By reduction modulo $\mathfrak{r}^{\text{left}}$, we get

$$(\mathbf{t} - t(\mathrm{eu}'))(\mathbf{t} - ts(\mathrm{eu}')) \equiv \mathbf{t}^2 - B^2\Sigma^2 - 4A^2\Pi + 4B^2\Pi \mod \mathfrak{r}^{\text{left}} R[\mathbf{t}].$$

So $t(\mathrm{eu}') \equiv -ts(\mathrm{eu}') \mod \mathfrak{r}^{\text{left}}$, which shows that $g_0 \neq ts$. So $g_0 = st$. □

Corollary 19.7.24. — *For the choice of $\mathfrak{r}^{\text{left}}$ made in this section, the generic Calogero-Moser left cells are $\{1\}$, $\{s\}$, $\{tst\}$, $\{w_0\}$, $\{ts, sts\}$ and $\{t, st\}$.*

Let g_{left} denote the involution of G which leaves 1, s, tst and w_0 fixed and such that $g_{\text{left}}(t) = st$ and $g_{\text{left}}(ts) = sts$. Then $I^{\text{left}} = \langle g_{\text{left}} \rangle$.

Remark 19.7.25. — We understand better here the convention chosen for the action of $W \times W$ on $\mathbf{k}(V \times V^*)$ (see §5.1.B). Indeed, if we had chosen the other action (the one described in Remark 5.1.10), the generic Calogero-Moser *left* cells would have coincided with the Kazhdan-Lusztig *right* cells. ■

19.7.E. Proof of Theorem 19.7.1. — Keep here the notation of Theorem 19.7.1 ($a = c_s$, $b = c_t$). Let us fix for the moment a prime ideal $\mathfrak{r}_c^{\mathrm{left}}$ of R containing $\mathfrak{r}^{\mathrm{left}}$ and $\mathfrak{p}_c R$. Since $R/\bar{\mathfrak{r}} \simeq P/\bar{\mathfrak{p}}$, we deduce that $\bar{\mathfrak{r}}_c = \bar{\mathfrak{r}}+\bar{\mathfrak{p}}_c R$ is the unique prime ideal of R lying over $\bar{\mathfrak{r}}$ and containing $\mathfrak{r}_c^{\mathrm{left}}$. Let D_c^{left} (respectively I_c^{left}) be the decomposition (respectively inertia) group of $\mathfrak{r}_c^{\mathrm{left}}$ and $\bar{D}_c$ (respectively $\bar{I}_c$) be the decomposition (respectively inertia) group of $\bar{\mathfrak{r}}_c$.

It follows from Corollaries 19.7.18 and 19.7.19 and from Proposition 19.7.23 that

$$(\clubsuit)\qquad \begin{cases} \mathrm{eu} \equiv -2(b+a) \mod \mathfrak{r}_c^{\mathrm{left}}, \\ s(\mathrm{eu}) \equiv -2(b-a) \mod \mathfrak{r}_c^{\mathrm{left}}, \\ tst(\mathrm{eu}) \equiv 2(b-a) \mod \mathfrak{r}_c^{\mathrm{left}}, \\ w_0(\mathrm{eu}) \equiv 2(b+a) \mod \mathfrak{r}_c^{\mathrm{left}}, \\ t(\mathrm{eu}) \equiv st(\mathrm{eu}) \equiv ts(\mathrm{eu}) \equiv sts(\mathrm{eu}) \equiv 0 \mod \mathfrak{r}_c^{\mathrm{left}}. \end{cases}$$

$$(\diamondsuit)\qquad \begin{cases} \delta \equiv 2b(b+a) \mod \mathfrak{r}_c^{\mathrm{left}}, \\ s(\delta) \equiv 2b(b-a) \mod \mathfrak{r}_c^{\mathrm{left}}, \\ tst(\delta) \equiv 2b(b-a) \mod \mathfrak{r}_c^{\mathrm{left}}, \\ w_0(\delta) \equiv 2b(b+a) \mod \mathfrak{r}_c^{\mathrm{left}}, \\ t(\delta) \equiv st(\delta) \equiv ts(\delta) \equiv sts(\delta) \equiv 0 \mod \mathfrak{r}_c^{\mathrm{left}} \end{cases}$$

$$(\heartsuit)\qquad \begin{cases} \mathrm{eu}' \equiv -b\Sigma \mod \mathfrak{r}_c^{\mathrm{left}}, \\ s(\mathrm{eu}') \equiv -b\Sigma \mod \mathfrak{r}_c^{\mathrm{left}}, \\ tst(\mathrm{eu}') \equiv b\Sigma \mod \mathfrak{r}_c^{\mathrm{left}}, \\ w_0(\mathrm{eu}') \equiv b\Sigma \mod \mathfrak{r}_c^{\mathrm{left}}, \\ t(\mathrm{eu}') \equiv st(\mathrm{eu}') \equiv -ts(\mathrm{eu}') \equiv -sts(\mathrm{eu}') \mod \mathfrak{r}_c^{\mathrm{left}}, \\ t(\mathrm{eu}')^2 \equiv b^2\Sigma^2 + 4(a^2-b^2)\Pi \mod \mathfrak{r}_c^{\mathrm{left}}. \end{cases}$$

and

$$(\spadesuit)\qquad g(\mathrm{eu}'') \equiv 0 \mod \mathfrak{r}_c^{\mathrm{left}}$$

for all $g \in G$. Recall that we assume that $ab \neq 0$ and that two elements g and g' of W are in the same Calogero-Moser left (respectively two-sided) c-cell if and only if $g(q) \equiv g'(q) \mod \mathfrak{r}_c^{\mathrm{left}}$ (respectively $\mod \bar{\mathfrak{r}}_c$) for all $q \in \{\mathrm{eu}, \mathrm{eu}, \mathrm{eu}'', \delta\}$ (since $Q = P[\mathrm{eu}, \mathrm{eu}', \mathrm{eu}'', \delta]$).

***The case $a^2 \neq b^2$.* —** Assume here that $a^2 \neq b^2$. It follows from the congruences (♣), (♢), (♡) and (♠) that the Calogero-Moser left c-cells are $\{1\}$, $\{s\}$, $\{tst\}$, $\{w_0\}$, $\Gamma_\chi^+ = \{t, st\}$ and $\Gamma_\chi^- = \{ts, sts\}$ and that the Calogero-Moser two-sided c-cells are $\{1\}$, $\{s\}$, $\{tst\}$, $\{w_0\}$ and Γ_χ.

The results on Calogero-Moser c-families Calogero-Moser c-cellular characters given by Table 19.7.2 then follow from Corollary 10.2.8, from Proposition 12.4.4, from (19.7.8) and from Corollary 19.7.17.

Let us now determine D_c^{left} and I_c^{left}. Note that $I^{\mathrm{left}} \subset I_c^{\mathrm{left}}$ and that, according to the description of Calogero-Moser left c-cells, that is of I_c^{left}-orbits, this forces the equality. On the other hand, since $\bar{\mathfrak{r}}_c = \bar{\mathfrak{r}} + \bar{\mathfrak{p}}_c R$, we have $D^{\mathrm{left}} \subset D_c^{\mathrm{left}}$. Since the D_c^{left}-orbits are contained in the Calogero-Moser two-sided c-cells, the description of these last ones forces again the equality. We show similarly that $\bar{D}_c = \bar{D} = W_2'$ and that $\bar{I}_c = \bar{I} = W_2'$.

***The case where $a = b$.* —** In this case, the last congruence of (♡) becomes

$$t(\mathrm{eu}')^2 \equiv b^2\Sigma^2 \mod \mathfrak{r}_c^{\mathrm{left}}.$$

So $t(\mathrm{eu}') \equiv b\Sigma \mod \mathfrak{r}_c^{\mathrm{left}}$ or $t(\mathrm{eu}') \equiv -b\Sigma \mod \mathfrak{r}_c^{\mathrm{left}}$. By replacing $\mathfrak{r}_c^{\mathrm{left}}$ by $g(\mathfrak{r}_c^{\mathrm{left}})$, where $g \in W_2' = D^{\mathrm{left}}$ exchanges t and sts, we can make the following choice:

> **Choice of $\mathfrak{r}_c^{\mathrm{left}}$.** *We choose the prime ideal $\mathfrak{r}_c^{\mathrm{left}}$ so that $t(\mathrm{eu}') \equiv b\Sigma \mod \mathfrak{r}_c^{\mathrm{left}}$.*

The family of congruences (♣), (♢), (♡) and (♠) show that the Calogero-Moser left c-cells are $\{1\}$, $\{w_0\}$, $\Gamma_s = \{s, ts, sts\}$ and $\Gamma_t = \{t, st, tst\}$ and that the Calogero-Moser two-sided c-cells are $\{1\}$, $\{w_0\}$ and $W \setminus \{1, w_0\}$.

As previously, the results on Calogero-Moser c-families and Calogero-Moser c-cellular characters given by Table 19.7.2 follow from Corollary 10.2.8, from Proposition 12.4.4, from (19.7.8) and from Corollary 19.7.17.

Let us conclude by the description of D_c^{left}, I_c^{left}, $\bar{D}_c$ and $\bar{I}_c$. First of all, I_c^{left} has two orbits of length 3 (Γ_s and Γ_t) so its order is divisible by 3. Moreover, it contains I^{left} which has order 2. So its order is divisible by 6. The description of left c-cells then allows to conclude that $I_c^{\mathrm{left}} = \mathfrak{S}_3$. On the other hand, D_c^{left} permutes the left c-cells which have the same associated c-cellular character. So D_c^{left} stabilizes Γ_s and Γ_t. This forces the equality $D_c^{\mathrm{left}} = I_c^{\mathrm{left}} = \mathfrak{S}_3$.

On the two-sided cells side, recall that $\bar{D}_c = \bar{I}_c$ because $\bar{D}_c/\bar{I}_c$ is a quotient of $\bar{D}/\bar{I}$. Moreover, the inclusions $W_2' \subset \bar{I}_c$ and $I_c^{\mathrm{left}} \subset \bar{I}_c$ show that $W_3' \subset \bar{I}_c$. The equality $\bar{I}_c = W_3'$ becomes obvious.

19.8. Complement: fixed points

As announced in Example 16.2.1, we will show that Conjecture FIX holds when W is of type B_2 and τ is a root of unity. If τ is not of order dividing 4, then $\mathscr{Z}^\tau = \mathscr{Z}^{\mathbb{C}^\times}$ is a union of affine spaces isomorphism $\mathscr{C}$, so this case is easy. If τ has order dividing 2, then $\mathscr{Z}^\tau = \mathscr{Z}$, and there is again nothing to prove.

Assumption. *We assume in this section that* $\mathbf{k} = \mathbb{C}$ *and we fix a primitive* 4*-th root of unity* τ.

We identify $\mathscr{Z}$ with the set of $(a, b, s, S, p, P, e, e', e'', d) \in \mathbf{A}^{10}(\mathbb{C})$ satisfying the relations $(Z1)$, $(Z2)$,..., $(Z9)$ in (19.4.2), with A, B, σ, Σ, π, Π, $\mathbf{eu}$, $\mathbf{eu}'$, $\mathbf{eu}''$ and $\boldsymbol{\delta}$ replaced by a, b, s, S, p, P, e, e', e'' and d respectively. Then

$$\mathscr{Z}^\tau = \{(a, b, s, S, p, P, e, e', e'', d) \in \mathscr{Z} \mid s = S = e' = e'' = 0\}.$$

Therefore,

$$\mathscr{Z}^\tau \simeq \{(a, b, p, P, e, d) \in \mathbf{A}^6(\mathbb{C}) \mid \begin{cases} d^2 = e^2 + pP \\ p(4d - e^2 + 4a^2 - 4b^2) = 0 \\ P(4d - e^2 + 4a^2 - 4b^2) = 0 \\ d(4d - e^2 + 4a^2 - 4b^2) = 0 \\ e(4d - e^2 + 4a^2 - 4b^2) = 0 \end{cases} \}.$$

This shows that $\mathscr{Z}^\tau$ has two irreducible components $\mathscr{X}$ and $\mathscr{X}_0$, where

$$\mathscr{X} = \{(a, b, p, P, e, d) \in \mathbf{A}^6(\mathbb{C}) \mid d^2 = e^2 + pP \text{ and } 4d - e^2 + 4a^2 - 4b^2 = 0\}$$

and
$$\mathscr{X}_0 = \{(a, b, p, P, e, d) \in \mathbf{A}^6(\mathbb{C}) \mid p = P = d = e = 0\}\}.$$

Note that the intersection of $\mathscr{X}$ and $\mathscr{X}_0$ is not empty.

So $\mathscr{X}_0 \simeq \mathscr{C} \simeq \mathrm{pt} \times_{\mathrm{pt}} \mathscr{C}$ and Conjecture FIX holds for this easy irreducible component. On the other hand,

$$\begin{aligned} \mathscr{X} &\simeq \{(a, b, p, P, e) \in \mathbf{A}^5(\mathbb{C}) \mid (e^2 - 4a^2 + 4b^2)^2 = 16(e^2 + pP)\} \\ &= \{(a, b, p, P, e) \in \mathbf{A}^5(\mathbb{C}) \mid (e - 2(a + b))(e - 2(a - b))(e + 2(a + b))(e + 2(a - b)) = pP\}. \end{aligned}$$

Let V' be the τ-eigenspace of w and take $W' = \langle w \rangle$. Then (V', W') is a reflection subquotient of (V, W), with $\dim_{\mathbb{C}} V' = 1$. We now use the description of $\mathscr{Z}(V', W')$ given in Theorem 18.2.4:

$$\begin{aligned} \mathscr{Z}(V', W') \simeq \{(k_0, k_1, k_2, k_3, e, x, y) \in \mathbf{A}^7(\mathbb{C}) \mid &(e - 4k_0)(e - 4k_1)(e - 4k_2)(e - 4k_3) = xy \\ &\text{and } k_0 + k_1 + k_2 + k_3 = 0\}, \end{aligned}$$

where the k_i's are the coordinates in $\mathscr{C}(V', W')$. So, if we set $\varphi : \mathscr{C} \to \mathscr{C}(V', W')$, $(a, b) \mapsto \frac{1}{2}(a+b, a-b, -a-b, -a+b)$, then φ is linear and

$$\mathscr{X} \simeq \mathscr{Z}(V', W') \times_{\mathscr{C}(V',W')} \mathscr{C}. \tag{19.8.1}$$

Note that φ is well-defined only up to permutation of the four coordinates in $\mathscr{C}(V', W')$.

APPENDICES

APPENDIX A

FILTRATIONS

A.1. Filtered modules

Let R be a commutative ring. A *filtered R-module* is an R-module M together with R-submodules $M^{\leqslant i}$ for $i \in \mathbb{Z}$ such that

$$M^{\leqslant i} \subset M^{\leqslant i+1} \text{ for } i \in \mathbb{Z},\ M^{\leqslant i} = 0 \text{ for } i \ll 0 \text{ and } M = \bigcup_{i\in\mathbb{Z}} M^{\leqslant i}.$$

Given M a filtered R-module, the *associated $\mathbb{Z}$-graded* R-module $\operatorname{gr} M$ is given by

$$(\operatorname{gr} M)_i = M^{\leqslant i}/M^{\leqslant i-1}.$$

The *principal symbol map* $\xi : M \to \operatorname{gr} M$ is defined by $\xi(m) = m \mod M^{\leqslant i-1} \in (\operatorname{gr} M)_i$, where i is minimal such that $m \in M^{\leqslant i}$. The principal symbol map is injective but not additive.

The *Rees* module associated with M is the $R[\hbar]$-submodule $\operatorname{Rees}(M) = \sum_{i\in\mathbb{Z}} \hbar^i M^{\leqslant i}$ of $R[\hbar^{\pm 1}] \otimes_R M$. We have $R[\hbar^{\pm 1}] \otimes_{R[\hbar]} \operatorname{Rees}(M) = R[\hbar^{\pm 1}] \otimes_R M$. In particular, given $t \in R^\times$, we have an isomorphism of R-modules

$$R[\hbar]/\langle \hbar - t\rangle \otimes_{R[\hbar]} \operatorname{Rees}(M) \xrightarrow{\sim} M,\ \hbar^i m \mapsto t^i m.$$

There is an isomorphism of R-modules

$$R[\hbar]/\langle \hbar\rangle \otimes_{R[\hbar]} \operatorname{Rees}(M) \xrightarrow{\sim} \operatorname{gr} M,\ \hbar^i m \mapsto \begin{cases} 0 & \text{if } m \in M^{<i} \\ \xi(m) & \text{otherwise.} \end{cases}$$

A.2. Filtered algebras

Let A be an R-algebra. A (bounded below) *filtration* on A is the data of a filtered Rb-module structure on A such that

$$1 \in A^{\leqslant 0} \setminus A^{\leqslant -1} \text{ and } A^{\leqslant i} \cdot A^{\leqslant j} \subset A^{\leqslant i+j} \text{ for all } i, j \in \mathbb{Z}.$$

The associated graded R-module $\mathrm{gr} A$ is a graded R-algebra. The Rees module associated with A is a $R[\hbar]$-algebra. An immediate consequence is the following lemma.

Lemma A.2.1. — *If* $\mathrm{gr} A$ *has no* 0 *divisors, then the principal symbol map* $\xi : A \to \mathrm{gr} A$ *is multiplicative and* A *has no* 0 *divisors.*

Proof. — Let $a, b \in A$ be two non-zero elements and let i, j minimal such that $a \in A^{\leqslant i}$ and $b \in A^{\leqslant j}$. Since $\mathrm{gr} A$ has no 0 divisors, it follows that $\xi(a)\xi(b) \neq 0$, hence $ab \notin A^{<i+j}$. This shows that $\xi(ab) = \xi(a)\xi(b)$, and that $ab \neq 0$. $\square$

Let us recall some facts of commutative algebra (cf [**Mat**, Exercises 9.4-9.5]). Let R be a commutative domain with field of fractions K. An element $x \in K$ is said to be *almost integral* over R if there exists $a \in R$, $a \neq 0$, such that for all $n \geqslant 0$, we have $ax^n \in R$. If x is integral over R, then x is almost integral over R, and the converse holds if R is noetherian.

We say that R is *completely integrally closed* if the elements of K that are almost integral over R are in R.

Lemma A.2.2. — *Assume* A *is a commutative ring. If* $\mathrm{gr} A$ *is a completely integrally closed domain, then* A *is a completely integrally closed domain.*

Proof. — Lemma A.2.1 shows that A is a domain. Let K be its field of fractions. Let $x \in K$ be almost integral over R. Let $c, d \in A$ such that $x = c/d$. Let i (resp. j) be minimal such that $c \in A^{\leqslant i}$ (resp. $d \in A^{\leqslant j}$). We show by induction on i that $x \in A$.

Let $a \in A$, $a \neq 0$, such that $ax^n \in A$ for all $n \geqslant 0$. Let $\alpha_n = ax^n$. We have $d^n \alpha_n = ac^n$, hence $\xi(d)^n \xi(\alpha_n) = \xi(a)\xi(c)^n$ (cf Lemma A.2.1). It follows that $\frac{\xi(c)}{\xi(d)}$ is an element of the field of fractions of $\mathrm{gr} A$ that is almost integral over $\mathrm{gr} A$. Consequently, it is in $\mathrm{gr} A$. Since $\mathrm{gr} A$ has no zero divisors, it follows that it is homogeneous of degree $i - j$. Let $u \in A$ with $\xi(u) = \frac{\xi(c)}{\xi(d)}$. Let $x' = x - u = \frac{c - ud}{d}$. We have $c - ud \in A^{\leqslant i-1}$ and x' is almost integral over A. It follows by induction that $x' \in A$, hence $x \in A$. $\square$

A.3. Filtered modules over filtered algebras

A *filtered A-module* is an A-module M together with a structure of filtered R-module such that

$$A^{\leqslant i} \cdot M^{\leqslant j} \subset M^{\leqslant i+j} \text{ for all } i, j \in \mathbb{Z}.$$

A *filtered morphism* of A-modules is a morphism $f : M \to N$, where M and N are filtered A-modules, such that $f(M^{\leqslant i}) \subset N^{\leqslant i}$ for all $i \in \mathbb{Z}$.

Lemma A.3.1. — *Let $f: M \to N$ be a filtered morphism of A-modules. If* $\operatorname{gr} f$ *is surjective (resp. injective), then f is surjective (resp. injective).*

Proof. — Assume $\operatorname{gr} f$ is surjective. We have $N^{\leqslant i} = f(M^{\leqslant i}) + N^{\leqslant i-1}$. Since $N^{\leqslant i} = 0$ for $i \ll 0$, it follows by induction that $N^{\leqslant i} = f(M^{\leqslant i})$, hence f is surjective.

Assume $\operatorname{gr} f$ is injective. Let $m \in M - \{0\}$ and let i be minimal such $m \in M^{\leqslant i}$. We have $f(m) \notin N^{\leqslant i-1}$, hence $f(m) \neq 0$. □

Lemma A.3.2. — *Let M be a filtered A-module and E a subset of M. If $\xi(E)$ generates* $\operatorname{gr} M$ *as an A-module, then E generates M as an A-module.*

Let F be a subset of A. If $\xi(F)$ generates $\operatorname{gr} A$ *as an R-algebra, then F generates A as an R-algebra.*

Proof. — We have a canonical morphism of filtered A-modules $f : A^{(E)} \to M$. By assumption, $\operatorname{gr} f$ is surjective, hence f is surjective by lemma A.3.1.

The second assertion follows from the first one by taking $A = R$, $M = A$ and E the set of elements of A that are products of elements of F. □

Let M and N be two finitely generated filtered A-modules. We endow the R-module $\operatorname{Hom}_A(M,N)$ with the filtration given by

$$\operatorname{Hom}_A(M,N)^{\leqslant i} = \{f \in \operatorname{End}_A(M) | f(M^{\leqslant j}) \subset N^{\leqslant i+j} \ \forall j \in \mathbb{Z}\}.$$

A map $f \in \operatorname{Hom}_A(M,N)^{\leqslant i}$ induces a morphism of $\operatorname{gr} A$-modules $\operatorname{gr} M \to \operatorname{gr} N$, homogeneous of degree i, that vanishes if $f \in \operatorname{Hom}_A(M,N)^{\leqslant i-1}$.

Lemma A.3.3. — *The construction above provides an injective morphism of graded R-modules*

$$\operatorname{gr} \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_{\operatorname{gr} A}(\operatorname{gr} M, \operatorname{gr} N).$$

Lemma A.3.4. — *Let e be an idempotent of $A^{\leqslant 0} \setminus A^{\leqslant -1}$ Then* $\operatorname{gr} A \cdot \xi(e)$ *is a progenerator for* $\operatorname{gr} A$ *if and only if Ae is a progenerator for A.*

Proof. — Note that $\xi(e)$ is an idempotent of $\operatorname{gr} A$. The A-module Ae is a progenerator if and only if e generates A as an (A,A)-bimodule. The lemma follows from Lemma A.3.2. □

A.4. Symmetric algebras

Let us recall some basic facts about symmetric algebras (cf for example [**Bro1**, §2, 3]). A *symmetric R-algebra* is an R-algebra A, finitely generated and projective as an R-module, and endowed with an R-linear map $\tau_A : A \to R$ such that

- $\tau_A(ab) = \tau_A(ba)$ for all $a, b \in A$ (i.e., τ_A is a trace) and
- the morphism of (A, A)-bimodules

$$\hat{\tau}_A : A \to \mathrm{Hom}_R(A, R),\ a \mapsto (b \mapsto \tau(ab))$$

  is an isomorphism.

Such a form τ_A is called a *symmetrizing form* for A.

Consider the sequence of isomorphisms

$$A \otimes_R A \xrightarrow[\sim]{\hat{\tau}\otimes \mathrm{id}} \mathrm{Hom}_R(A, R) \otimes_R A \xrightarrow[\sim]{f\otimes a \mapsto (b\mapsto f(b)a)} \mathrm{End}_R(A).$$

The *Casimir element* is the inverse image of id_A through the composition of maps above. The *central Casimir element* cas_A is its image in A by the multiplication map $A \otimes_R A \to A$. It is an element of $Z(A)$.

Assume A is free over R, with R-basis $\mathscr{B}$ and dual basis $(b^\vee)_{b\in\mathscr{B}}$ for the bilinear form $A \otimes_R A \to R,\ a \otimes a' \mapsto \tau(aa')$. We have $\mathrm{cas}_A = \sum_{b\in\mathscr{B}} b b^\vee$.

Lemma A.4.1. — *Let (A, τ_A) be a symmetric R-algebra and G a finite group acting on the R-algebra A and such that $\tau_A(g(a)) = \tau_A(a)$ for all $g \in G$ and $a \in A$.*

Let $B = A \rtimes G$ and define an R-linear form $\tau_B : B \to R$ by $\tau_B(a \otimes g) = \tau_A(a)\delta_{1,g}$ for $a \in A$ and $g \in G$.

The form τ_B is symmetrizing for B.

Proof. — We have

$$\tau_B((a\otimes g)(a'\otimes g')) = \tau_A(ag(a'))\delta_{g^{-1},g'} = \tau_A(a'g^{-1}(a))\delta_{g^{-1},g'} = \tau_B((a'\otimes g')(a\otimes g)).$$

Given $g \in G$, let $B_g = A \otimes g$ and $C_g = \mathrm{Hom}_R(B_g, R)$. We have $B = \bigoplus_{g\in G} B_g$ and $\mathrm{Hom}_R(B, R) = \bigoplus_g C_g$. Given $g \in G$ we have $\hat{\tau}_B(B_g) \subset C_{g^{-1}}$ and $\hat{\tau}_B(a\otimes g)(a'\otimes g^{-1}) = \tau_A(ag(a'))$. It follows that the restriction of τ_B to B_g is an isomorphism $B_g \xrightarrow{\sim} C_{g^{-1}}$. □

Let now A be a filtered R-algebra, with $A^{\leqslant -1} = 0$, $A^{\leqslant d-1} \neq A$ and $A^{\leqslant d} = A$ for some $d \geqslant 0$. Let $\bar{\tau} : (\mathrm{gr} A)^d \to R$ be an R-linear form. We extend it to an R-linear form on $\mathrm{gr} A$ by setting it to 0 on $(\mathrm{gr} A)^i$ for $i < d$. We define an R-linear form τ on A as the composition

$$\tau : A \xrightarrow{\mathrm{can}} (\mathrm{gr} A)^d \xrightarrow{\bar{\tau}} R.$$

Denote by $p_i : A^{\leqslant i} \to (\mathrm{gr} A)^i$ the canonical projection. Let $x \in A^{\leqslant i}$ and $y \in A^{\leqslant j}$. We have $\tau(xy) = \bar{\tau}(p_d(xy))$. We have $p_d(xy) = 0$ if $i + j < d$. If $i + j = d$, we have $p_d(xy) = p_i(x)p_j(y)$, hence $\tau(xy) = \bar{\tau}(p_i(x)p_j(y))$. The R-module $\mathrm{Hom}_R(A, R)$ is filtered with $\mathrm{Hom}_R(A, R)^{\leqslant i} = \mathrm{Hom}_R(A/A^{\leqslant d-i-1}, R)$ and $\hat{\tau}$ is a morphism of filtered R-modules with $\mathrm{gr}(\hat{\tau}) = \hat{\bar{\tau}}$.

Lemma A.4.2. — *Let L be an R-submodule of $A^{\leqslant 1}$ such that $A = A^{\leqslant 0}(R+L)^d$, $L^{d+1} \subset A^{<d}$ and $A^{\leqslant 0}L = LA^{\leqslant 0}$.*

If $\bar{\tau}$ is a trace, then τ is a trace.

Proof. — Note that $A = A^{\leqslant 0}L^d + A^{<d}$. We have $p_d(A^{\leqslant 0}L^d L) \subset p_d(A^{<d}) = 0$ and $p_d(LA^{\leqslant 0}L^d) = p_d(A^{\leqslant 0}LL^d) = 0$. It follows that $\tau(al) = \tau(la)$ for $a \in A^{\leqslant 0}L^d$ and $l \in L$. The considerations above show that $\tau(al) = \tau(la)$ for $a \in A^{<d}$ and $l \in L$ and $\tau(ba) = \tau(ab)$ for $b \in A$ and $a \in A^{\leqslant 0}$. ☐

The next proposition is inspired by a result of Brundan and Kleshchev on degenerate cyclotomic Hecke algebras [**BrKl**, Theorem A.2].

Proposition A.4.3. — *Assume* $\mathrm{gr} A$ *is projective and finitely generated as an R-module and assume τ and $\bar{\tau}$ are traces.*

If $\bar{\tau}$ is a symmetrizing form for $\mathrm{gr} A$, *then τ is a symmetrizing form for A.*

Proof. — Note that A is a finitely generated projective R-module. Since $\hat{\bar{\tau}}$ is an isomorphism, it follows that $\hat{\tau}$ is an isomorphism (Lemma A.3.1). ☐

A.5. Weyl algebras

Let V be a finite dimensional vector space over a field $\mathbf{k}$ of characteristic 0. Let $\mathscr{D}(V) = \widetilde{\mathbf{H}}_{1,0}$ be the Weyl algebra of V. This is the quotient of the tensor algebra $\mathrm{T}_{\mathbf{k}}(V \oplus V^*)$ by the relations

$$[x, x'] = [y, y'] = 0,\ [y, x] = \langle y, x\rangle \text{ for } x, x' \in V^* \text{ and } y, y' \in V.$$

There is an isomorphism of $\mathbf{k}$-modules given by multiplication: $\mathbf{k}[V] \otimes \mathbf{k}[V^*] \xrightarrow{\sim} \mathscr{D}(V)$.

The $\mathbf{k}$-algebra $\mathscr{D}(V)$ is filtered, with $\mathscr{D}(V)^{\leqslant -1} = 0$, $\mathscr{D}(V)^{\leqslant 0} = \mathbf{k}[V]$, $\mathscr{D}(V)^{\leqslant 1} = \mathbf{k}[V] \oplus \mathbf{k}[V] \otimes V$ and $\mathscr{D}^{\leqslant i} = (\mathscr{D}^{\leqslant 1})^i$ for $i \geqslant 2$. The associated graded algebra $\mathrm{gr}\,\mathscr{D}(V)$ is $\mathbf{k}[V \oplus V^*]$. The associated Rees algebra $\mathscr{D}_\hbar(V)$ is the quotient of $\mathbf{k}[\hbar] \otimes \mathrm{T}_{\mathbf{k}}(V \oplus V^*)$ by the relations

$$[x, x'] = [y, y'] = 0,\ [y, x] = \hbar\langle y, x\rangle \text{ for } x, x' \in V^* \text{ and } y, y' \in V.$$

Consider the induced $\mathcal{D}(V)$-module $\mathcal{D}(V)\otimes_{\mathbf{k}[V^*]}\mathbf{k}$, where $\mathbf{k}[V^*]$ acts on $\mathbf{k}$ by evaluation at 0. Via the canonical isomorphism $\mathbf{k}[V]\xrightarrow{\sim}\mathcal{D}(V)\otimes_{\mathbf{k}[V^*]}\mathbf{k}$, $a\mapsto a\otimes 1$, we obtain the faithful action of $\mathcal{D}(V)$ by polynomial differential operators on $\mathbf{k}[V]$: an element $x\in V^*$ acts by multiplication, while $y\in V$ acts by $\partial_y=\frac{\partial}{\partial y}$. As a consequence of the faithfulness of the action, the centralizer of $\mathbf{k}[V]$ in $\mathcal{D}(V)$ is $\mathbf{k}[V]$.

Note that there is an injective morphism of $\mathbf{k}[\hbar]$-algebras

$$\mathcal{D}_\hbar(V)\hookrightarrow\mathbf{k}[\hbar]\otimes\mathcal{D}(V),\ V^*\ni x\mapsto x,\ V\in y\mapsto\hbar y.$$

This provides by restriction $\mathbf{k}[\hbar]\otimes\mathbf{k}[V]$ with the structure of a faithful representation of $\mathcal{D}_\hbar(V)$.

APPENDIX B

GALOIS THEORY AND RAMIFICATION

Let R be a commutative ring, G a finite group acting on R and H a subgroup of G. We set $Q = R^H$ and $P = R^G$, so that $P \subset Q \subset R$. Let $\mathfrak{r}$ be a prime ideal of R. We denote by $k_R(\mathfrak{r})$ the fraction field of $R/\mathfrak{r}$ (that is, the quotient $R_\mathfrak{r}/\mathfrak{r}R_\mathfrak{r}$) and $G_\mathfrak{r}^D$ the stabilizer of $\mathfrak{r}$ in G. This subgroup of G then acts on $R/\mathfrak{r}$ and we denote by $G_\mathfrak{r}^I$ the kernel of this action. In other words,

$$G_\mathfrak{r}^I = \{g \in G \mid \forall\ r \in R,\ g(r) \equiv r \mod \mathfrak{r}\}.$$

The group $G_\mathfrak{r}^D$ (respectively $G_\mathfrak{r}^I$) is called the *decomposition group* (respectively the *inertia group*) of G at $\mathfrak{r}$.

We fix in this chapter a prime ideal $\mathfrak{r}$ of R and we set $\mathfrak{q} = \mathfrak{r} \cap Q$ and $\mathfrak{p} = \mathfrak{r} \cap P = \mathfrak{q} \cap P$:

$$\begin{tikzcd} \mathfrak{r} \arrow[r, phantom, "\subset"] \arrow[d, dash] & R \arrow[d, dash] \\ \mathfrak{q} \arrow[r, phantom, "\subset"] \arrow[d, dash] & Q \arrow[d, dash] \\ \mathfrak{p} \arrow[r, phantom, "\subset"] & P \end{tikzcd}$$

We also set

$$\rho_G : \operatorname{Spec} R \longrightarrow \operatorname{Spec} P,$$

$$\rho_H : \operatorname{Spec} R \longrightarrow \operatorname{Spec} Q$$

and

$$\Upsilon : \operatorname{Spec} Q \longrightarrow \operatorname{Spec} P$$

the maps respectively induced by the inclusions $P \subset R$, $Q \subset R$ and $Q \subset R$. Of course, $\rho_G = \Upsilon \circ \rho_H$: in other words, the diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "\rho_H"] \arrow[rr, bend right, "\rho_G"'] & \operatorname{Spec} Q \arrow[r, "\Upsilon"] & \operatorname{Spec} P. \end{tikzcd}$$

is commutative. For instance,

$$\rho_G(\mathfrak{r})=\mathfrak{p}, \qquad \rho_H(\mathfrak{r})=\mathfrak{q} \qquad \text{and} \qquad \Upsilon(\mathfrak{q})=\mathfrak{p}.$$

Finally, we set

$$D=G_{\mathfrak{r}}^{D} \qquad \text{and} \qquad I=G_{\mathfrak{r}}^{I}.$$

B.1. Around Dedekind's Lemma

Recall that $(R,\times)$ is a monoid. If M is another monoid, we denote by $\mathrm{Hom}_{\mathrm{mon}}(M,R)$ the set of morphisms of monoids $M\to(R,\times)$. If A is a commutative ring, We denote by $\mathrm{Hom}_{\mathrm{ring}}(A,R)$ the set of morphisms of rings from A to R: it can be seen as a subset of $\mathrm{Hom}_{\mathrm{mon}}((A,\times),R)$. These are subsets of the set $\mathscr{F}(M,R)$ of maps from M to R: note that $\mathscr{F}(M,R)$ is an R-module.

Dedekind's Lemma. *If R is a **domain**, then* $\mathrm{Hom}_{\mathrm{mon}}(M,R)$ *is an R-linearly independent family of elements of $\mathscr{F}(M,R)$.*

Proof. — Let $\varphi_1,\dots,\varphi_d$ be two by two distinct elements of $\mathrm{Hom}_{\mathrm{mon}}(M,R)$ and let $\lambda_1,\dots,\lambda_d$ in R be such that

$$(*) \qquad \forall\ m\in M,\ \lambda_1\varphi_1(m)+\cdots+\lambda_d\varphi_d(m)=0.$$

We shall show by induction on d that $\lambda_1=\lambda_2=\cdots=\lambda_d=0$. If $d=1$, this is clear as $\varphi_1(1)=1$.

So assume that $d\geqslant 2$ and that there is no non-trivial R-linear relation of length $\leqslant d-1$ between elements of $\mathrm{Hom}_{\mathrm{mon}}(M,R)$. Since $\varphi_1\neq\varphi_2$, there exists $m_0\in M$ such that $\varphi_1(m_0)\neq\varphi_2(m_0)$. Consequently, it follows from $(*)$ that

$$\lambda_1\varphi_1(m_0m)+\cdots+\lambda_d\varphi_d(m_0m)=0.$$

and

$$\varphi_1(m_0)\cdot\big(\lambda_1\varphi_1(m)+\cdots+\lambda_d\varphi_d(m)\big)=0$$

for all $m\in M$. By subtracting the second equation to the first one, we get

$$\forall\ m\in M,\ \sum_{i=2}^{d}\lambda_i(\varphi_i(m_0)-\varphi_1(m_0))\varphi_i(m)=0.$$

By the induction hypothesis, we get that $\lambda_2(\varphi_2(m_0)-\varphi_1(m_0))$, so that $\lambda_2=0$ because R is a domain. Now, the induction hypothesis allows also to conclude that $\lambda_1=\lambda_3=\cdots=\lambda_d=0$. □

Corollary B.1.1. — *Let A be a commutative ring and assume that R is a **domain**. Then* $\mathrm{Hom}_{\mathrm{ring}}(A,R)$ *is an R-linearly independent family in $\mathscr{F}(A,R)$.*

Proof. — Indeed, the set $\mathrm{Hom}_{\mathrm{ring}}(A,R)$ is a subset of the set $\mathrm{Hom}_{\mathrm{mon}}((A,\times),R)$: we then apply Dedekind's Lemma. □

B.2. Decomposition group, inertia group

We recall some classical results:

Proposition B.2.1. — *The ideal $\mathfrak{r}$ is maximal if and only if $\mathfrak{p}$ is maximal.*

Proposition B.2.2. — *The group G acts transitively on the fibers of ρ_G.*

Remark B.2.3. — Of course, the statement can also be applied to H: the group H acts transitively on the fibers of ρ_H. ■

Proof. — See [**Bou**, Chapter 5, §2, Theorem 2(i)]. □

Theorem B.2.4. — *The field extension $k_R(\mathfrak{r})/k_P(\mathfrak{p})$ is normal, with Galois group D/I ($= G_{\mathfrak{r}}^D/G_{\mathfrak{r}}^I$).*

Proof. — See [**Bou**, Chapter 5, §2, Theorem 2(ii)]. □

Corollary B.2.5. — *Let $\mathfrak{r}'$ be a prime ideal of R containing $\mathfrak{r}$ and let $D' = G_{\mathfrak{r}'}^D$ and $I' = G_{\mathfrak{r}'}^I$. Then D'/I' is isomorphic to a subquotient of D/I.*

Proof. — By replacing R by $R/\mathfrak{r}$, Q by $Q/\mathfrak{q}$ and P by $P/\mathfrak{p}$, D/I may be identified with G and the Corollary follows immediately from Theorem B.2.4. □

Theorem B.2.6. — *If Q is **unramified** over P at $\mathfrak{q}$ (i.e. if $\mathfrak{p}Q_{\mathfrak{q}} = \mathfrak{q}Q_{\mathfrak{q}}$), then I is contained in H.*

Proof. — See [**Ray**, Chapter X, Theorem 1]. □

B.3. On the $P/\mathfrak{p}$-algebra $Q/\mathfrak{p}Q$

B.3.A. Double classes. — The proposition B.3.5 below, certainly well-known (and easy), will be crucial in this book. We provide a proof for the convenience of the reader. We will need some notation. If $g \in G$, the composed morphism $Q \xrightarrow{g} R \xrightarrow{\text{can}} R/\mathfrak{r}$ factors through a morphism $\bar{g} : Q/\mathfrak{p}Q \to R/\mathfrak{r}$. The following remark is obvious:

$$\text{(B.3.1)} \qquad \textit{If } h \in H \textit{ and } i \in I\textit{, then } \overline{igh} = \bar{g}.$$

We then obtain a well-defined map

$$\text{(B.3.2)} \qquad \begin{array}{ccc} I\backslash G/H & \longrightarrow & \mathrm{Hom}_{(P/\mathfrak{p})\text{-alg}}(Q/\mathfrak{p}Q, R/\mathfrak{r}) \\ IgH & \longmapsto & \bar{g} \end{array}.$$

Note that $\mathrm{Hom}_{(P/\mathfrak{p})\text{-alg}}(Q/\mathfrak{p}Q, R/\mathfrak{r}) = \mathrm{Hom}_{P\text{-alg}}(Q, R/\mathfrak{r})$. If $\varphi \in \mathrm{Hom}_{P/\mathfrak{p}-\mathrm{alg}}(Q/\mathfrak{p}Q, R/\mathfrak{r})$, then we denote by $\tilde{\varphi}$ the composition $Q \xrightarrow{\text{can}} Q/\mathfrak{p}Q \xrightarrow{\varphi} R/\mathfrak{r}$. This defines a map

$$\text{(B.3.3)} \qquad \begin{array}{ccc} \mathrm{Hom}_{(P/\mathfrak{p})\text{-alg}}(Q/\mathfrak{p}Q, R/\mathfrak{r}) & \longrightarrow & \Upsilon^{-1}(\mathfrak{p}) \\ \varphi & \longmapsto & \mathrm{Ker}\,\tilde{\varphi}. \end{array}$$

Since $R/\mathfrak{r}$ is a domain, $\mathrm{Ker}\,\tilde{\varphi}$ is a prime ideal of Q and it is clear that $\mathrm{Ker}\,\tilde{\varphi} \in \Upsilon^{-1}(\mathfrak{p})$.

If $g \in G$, then $g(\mathfrak{r}) \cap Q \in \Upsilon^{-1}(\mathfrak{p})$. Moreover, if $h \in H$ and $d \in D$, then

$$hgd(\mathfrak{r}) \cap Q = g(\mathfrak{r}) \cap Q.$$

We have then defined a map

$$\text{(B.3.4)} \qquad \begin{array}{ccc} D\backslash G/H & \longrightarrow & \Upsilon^{-1}(\mathfrak{p}) \\ DgH & \longmapsto & g^{-1}(\mathfrak{r}) \cap Q. \end{array}$$

Proposition B.3.5. — *The map $I\backslash G/H \longrightarrow \mathrm{Hom}_{(P/\mathfrak{p})\text{-alg}}(Q/\mathfrak{p}Q, R/\mathfrak{r})$ defined in B.3.2 is bijective, as well as the map $D\backslash G/H \to \Upsilon^{-1}(\mathfrak{p})$ defined in B.3.4. Moreover, the diagram*

$$\begin{tikzcd} I\backslash G/H \arrow[rr, "\sim", "IgH \mapsto \tilde{g}"'] \arrow[d, "\text{can}"'] & & \mathrm{Hom}_{(P/\mathfrak{p})\text{-alg}}(Q/\mathfrak{p}Q, R/\mathfrak{r}) \arrow[d, "\varphi \mapsto \mathrm{Ker}\,\tilde{\varphi}"] \\ D\backslash G/H \arrow[rr, "\sim", "DgH \mapsto g^{-1}(\mathfrak{r}) \cap Q"'] & & \Upsilon^{-1}(\mathfrak{p}) \end{tikzcd}$$

is commutative.

Proof. — Let us start by showing that the first map is injective. Let g and g' be two elements of G such that $\bar{g} = \bar{g}'$. This means that

$$\forall\, q \in Q,\ g(q) \equiv g'(q) \mod \mathfrak{r}.$$

Consequently,

$$\forall\ r \in R,\ \sum_{h \in H} gh(r) \equiv \sum_{h \in H} g'h(r) \mod \mathfrak{r}.$$

But, by Dedekind's Lemma, the family of morphisms of rings $R \to R/\mathfrak{r}$ is $R/\mathfrak{r}$-linearly independent. So this means that there exists $h \in H$ such that

$$\forall\ r \in R,\ g(r) \equiv g'h(r) \mod \mathfrak{r}$$

or, equivalently,

$$\forall\ r \in R,\ g'h(g^{-1}(r)) \equiv r \mod \mathfrak{r}.$$

In other words, $g'hg^{-1} \in I$ and so $g' \in IgH$.

Let us now show that it is surjective. Let $\varphi \in \mathrm{Hom}_{P/\mathfrak{p}-\mathrm{alg}}(Q/\mathfrak{p}Q, R/\mathfrak{r})$ and let $\mathfrak{q}' = \mathrm{Ker}\,\tilde{\varphi}$. Since φ is $(P/\mathfrak{p})$-linear, we have $\mathfrak{q}' \cap P = \mathfrak{p}$. Let $\mathfrak{r}'$ be a prime ideal of R lying over $\mathfrak{q}'$. Then there exists $g \in G$ such that $\mathfrak{r}' = g(\mathfrak{r})$. So the map $g \circ \tilde{\varphi} : Q \to R/\mathfrak{r}'$ has $\mathfrak{q}' = \mathfrak{r}' \cap Q$ for kernel and is Q-linear. By Theorem B.2.4, there exists $d \in G^D_{\mathfrak{r}'}$ such that $g \circ \tilde{\varphi}(q) \equiv d(q) \mod \mathfrak{r}'$ for all $q \in Q$. Hence, $\tilde{\varphi}(q) \equiv g^{-1}d(q) \mod \mathfrak{r}$, that is, $\varphi = \overline{g^{-1}d}$.

Let us now show that the second map is bijective. If $\mathfrak{q}' \in \Upsilon^{-1}(\mathfrak{p})$, then there exists $\mathfrak{r}' \in \mathrm{Spec}\,R$ such that $\mathfrak{q}' \cap Q = \mathfrak{r}'$. Also, $\mathfrak{r}' \cap P = \mathfrak{q}' \cap P = \mathfrak{p}$ and so, by Proposition B.2.2, there exists $g \in G$ such that $\mathfrak{r}' = g(\mathfrak{r})$. This shows that the bottom horizontal row is surjective. The injectivity then follows again from Proposition B.2.2.

The commutativity of the diagram follows from the previous arguments. □

B.3.B. Residue fields. — Let $g \in [D\backslash G/H]$. We set for simplification $\mathfrak{q}_g = \mathfrak{r} \cap g(Q)$. Note that $\mathfrak{q}_g \cap P = \mathfrak{p}$ and that we obtain a sequence of morphisms of rings $P/\mathfrak{p} \hookrightarrow g(Q)/\mathfrak{q}_g \hookrightarrow R/\mathfrak{r}$. So we have a sequence of inclusions of fields

$$k_P(\mathfrak{p}) \quad \subset \quad k_{g(Q)}(\mathfrak{q}_g) \quad \subset \quad k_R(\mathfrak{r}).$$

Lemma B.3.6. — *The extension $k_R(\mathfrak{r})/k_{g(Q)}(\mathfrak{q}_g)$ is normal with Galois group $(D \cap {}^gH)/(I \cap {}^gH)$.*

Remark B.3.7. — Note that $(D \cap {}^gH)/(I \cap {}^gH)$ is naturally a subgroup D/I. ■

Proof. — Indeed, this follows from the fact that $g(Q) = R^{{}^gH}$ and from Theorem B.2.4. □

Corollary B.3.8. — *Assume that, for all prime ideals $\mathfrak{q}' \in \Upsilon^{-1}(\mathfrak{p})$, we have $k_Q(\mathfrak{q}') = k_P(\mathfrak{p})$. Then $D\backslash G/H = I\backslash G/H$.*

Proof. — By Lemma B.3.6 and Theorem B.2.4, it follows from the assumption that, for all $g \in G$, we have $(D \cap {}^g H)/(I \cap {}^g H) \simeq D/I$. In other words,

$$\forall\, g \in G,\ D = I \cdot (D \cap {}^g H).$$

Now let $g \in G$ and $d \in D$. Then there exists $i \in I$ and $h \in H$ such that $d = ighg^{-1}$, that is $dg = igh$. We deduce that $DgH = IgH$. □

Lemma B.3.9. — *Write $\Upsilon^{-1}(\mathfrak{p}) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ and assume that Q is unramified over P at $\mathfrak{q}_i$ for all i. Then $I \subset \bigcap_{g \in G} {}^g H$.*

Proof. — Let $g \in G$. Then $g(\mathfrak{r}) \cap Q \in \Upsilon^{-1}(\mathfrak{p})$ and it follows from Theorem B.2.6 that ${}^g I \subset H$ (since ${}^g I$ is the inertia group of $g(\mathfrak{r})$). Hence, $I \subset {}^{g^{-1}} H$. □

Proposition B.3.10. — *If $I = \bigcap_{g \in G} {}^g H = 1$ and if the extension $k_R(\mathfrak{r})/k_P(\mathfrak{p})$ is separable, then $k_R(\mathfrak{r})/k_P(\mathfrak{p})$ is the Galois closure of the family of extensions $k_{{}^g Q}(\mathfrak{q}_g)/k_P(\mathfrak{p})$, $g \in [D\backslash G/H]$.*

Remark - If R is a domain (which implies that P and Q are domains) and if G acts faithfully, then the assumption $\bigcap_{g \in G} {}^g H = 1$ is equivalent to say that the extension $\mathrm{Frac}(R)/\mathrm{Frac}(P)$ is the Galois closure of $\mathrm{Frac}(Q)/\mathrm{Frac}(P)$.

Note also that the assumption $I = 1$ implies that G acts faithfully. ■

Proof. — By Theorem B.2.4, the extension $k_R(\mathfrak{r})/k_P(\mathfrak{p})$ is normal with Galois group D. By Lemma B.3.6, the extension $k_R(\mathfrak{r})/k_{g(Q)}(\mathfrak{q}_g)$ is normal with Galois group $D \cap {}^g H$.

Let k be the normal closure of the family of extensions $k_{g(Q)}(\mathfrak{q}_g)/k_P(\mathfrak{p})$, $g \in [H\backslash G/D]$. Then the Galois group $\mathrm{Gal}(k_R(\mathfrak{r})/k)$ is the intersection of the conjugates, in D, of the groups $D \cap {}^g H$, for g running in $[H\backslash G/D]$. So

$$\mathrm{Gal}(k_R(\mathfrak{r})/k) = \bigcap_{\substack{g \in [H\backslash G/D] \\ d \in D}} {}^d(D \cap {}^g H) = \bigcap_{\substack{g \in [D\backslash G/H] \\ d \in D}} D \cap {}^{dg} H.$$

Since ${}^h H = H$ for all $h \in H$, we then have

$$\mathrm{Gal}(k_R(\mathfrak{r})/k) = \bigcap_{\substack{g \in [D\backslash G/H] \\ d \in D \\ h \in H}} D \cap {}^{dgh} H = \bigcap_{g \in G} {}^g H = 1,$$

by assumption. Whence the result. □

Counter-example B.3.11. — If there is some ramification, then the above proposition does not hold in general, even if we assume that P, Q and R are Dedekind domains. Indeed, let $\sqrt[3]{2}$ be a cubic root of 2 in $\mathbb{C}$, ζ a primitive subic root of 1 in $\mathbb{C}$ and let R denote the integral closure of $\mathbb{Z}$ in $M = \mathbb{Q}(\sqrt[3]{2}, \zeta)$. Let $G = \mathrm{Gal}(M/\mathbb{Q}) \simeq \mathfrak{S}_3$ and $H = \mathrm{Gal}(M/\mathbb{Q}(\sqrt[3]{2})) \simeq \mathbb{Z}/2\mathbb{Z}$. Then, $P = \mathbb{Z}$ and, if $\mathfrak{r}$ is a prime ideal of R such that $\mathfrak{r} \cap \mathbb{Z} = 2\mathbb{Z}$, then $D = G$ and $|I| = 3$.

Hence, $D\backslash G/H$ is a singleton and the corresponding field extension $k_R(\mathfrak{r})/k_Q(\mathfrak{q})$ is Galois with Galois group $\mathbb{Z}/2\mathbb{Z}$ (according to Lemma B.3.6), as well as $k_R(\mathfrak{r})/k_P(\mathfrak{p})$. So $k_P(\mathfrak{p}) = k_Q(\mathfrak{q}) \simeq \mathbb{F}_2$ and $k_R(\mathfrak{r}) \simeq \mathbb{F}_4$. So $k_R(\mathfrak{r})$ is not the Galois closure of the field extension $k_Q(\mathfrak{q})/k_P(\mathfrak{p})$. ■

B.3.C. The case of fields. — Whenever R is a field, the situation becomes much simpler.

> **Assumption.** *In this subsection, and only in this subsection, we assume that R is a **field**: it will be denoted by M. We set $L = Q = M^H$ and $K = P = M^G$. We also assume that G acts **faithfully** on M. Hence, M/K is a Galois extension with Galois group G and M/L is a Galois extension with Galois group H.*

It follows from the assumption that $\mathfrak{p} = \mathfrak{q} = \mathfrak{r} = 0$ and that $D = G$ and $I = 1$. Hence, Proposition B.3.5 provides a bijective map

$$G/H \xhookrightarrow{\sim} \mathrm{Hom}_{K-\mathrm{alg}}(L, M).$$

If $g \in G$, the morphism of K-algebras $L \to M$, $q \mapsto g(q)$, extends to a morphism of M-algebras

$$\begin{array}{rccc} g_L: & M \otimes_K L & \longrightarrow & M \\ & m \otimes_K l & \longmapsto & mg(l). \end{array}$$

Proposition B.3.12. — *The morphism of M-algebras*

$$\bigoplus_{g \in [G/H]} g_L : M \otimes_K L \longrightarrow \bigoplus_{g \in [G/H]} M$$

is an isomorphism.

Proof. — Since L is a K-vector space of dimension $|G/H|$, $M \otimes_K L$ is an M-vector space of dimension $|G/H|$. It is so sufficient to show that the map $\sum_{g \in [G/H]} g_L$ is injective, which is equivalent to the M-linear independence of the family of maps $L \to M$, $q \mapsto g(q)$, whenever g runs over $[G/H]$ (see Corollary B.1.1 to the Dedekind's Lemma). □

B.4. Recollection about the integral closure

Proposition B.4.1. — *Let $f \in P[\mathbf{t}]$, let P' be a P-algebra containing P and let $g \in P'[\mathbf{t}]$. We assume that f and g are monic and that g divides f (in $P'[\mathbf{t}]$). Then the coefficients of g are integral over P.*

Proof. — See [**Bou**, Chapter 5, §1, Proposition 11]. □

Corollary B.4.2. — *If P is* ***integral and integrally closed****, with fraction field K, if A is a K-algebra and if $x \in A$ is integral over P, then the minimal polynomial of x over K belongs to $P[\mathbf{t}]$.*

Proof. — See [**Bou**, Chapter 5, §1, Corollary of Proposition 11]. □

Proposition B.4.3. — *If P is a* ***domain*** *and if $f \in P[\mathbf{t}, \mathbf{t}^{-1}]$ is integral over P, then $f \in P$.*

Proof. — Let $d \geqslant 1$ and let $p_0, p_1, \ldots, p_{d-1}$ be elements of P such that $p_0 + p_1 f + \cdots + p_{d-1} f^{d-1} = f^d$. Let δ be the $\mathbf{t}$-valuation of P and δ' its degree. Since P is integral, the degree of f^d is $d\delta'$, and so the above equality can hold only if $\delta' = 0$. Similarly, $\delta = 0$. So f is constant. □

B.5. On the calculation of Galois groups

Let K be a field and let $f(\mathbf{t}) = \mathbf{t}^d + a_{d-1}\mathbf{t}^{d-1} + \cdots + a_1\mathbf{t} + a_0 \in K[\mathbf{t}]$ be separable. We denote by M a splitting field of f (over K) and we denote by

$$\mathrm{Gal}_K(f) = \mathrm{Gal}(M/K).$$

The group $\mathrm{Gal}_K(f)$ is called the *Galois group* of f over K.

Let $t_1, \ldots, t_d$ be elements of M such that

$$f(\mathbf{t}) = \prod_{i=1}^{d} (\mathbf{t} - t_i),$$

so that

$$M = K[t_1, \ldots, t_d] = K(t_1, \ldots, t_d).$$

This numbering provides an injective morphism of groups

$$\mathrm{Gal}_K(f) \hookrightarrow \mathfrak{S}_d.$$

Assume now that P is an integrally closed domain, that K is its fraction field, and that $f \in P[\mathbf{t}]$. Let R denote the integral closure of P in M and let $G = \mathrm{Gal}(M/K)$.

Then $P = R^G$ since P is integrally closed. If $r \in R$, we denote by $\bar{r}$ its image in $R/\mathfrak{r}$. Write

$$\bar{f} = \prod_{j=1}^{l} f_j,$$

where $f_j \in k_P(\mathfrak{p})[\mathbf{t}]$ is an irreducible polynomial. Then $D/I = \mathrm{Gal}(k_R(\mathfrak{r})/k_P(\mathfrak{p}))$ by Theorem B.2.4. But, R contains $t_1,\dots,t_d$, so

$$\bar{f}(\mathbf{t}) = \prod_{i=1}^{d} (\mathbf{t} - \bar{t}_i).$$

We denote by Ω_j the subset of $\{1,2,\dots,d\}$ such that

$$f_j(\mathbf{t}) = \prod_{i \in \Omega_j} (\mathbf{t} - \bar{t}_i).$$

Let $k_j = k_P(\mathfrak{p})((\bar{t}_i)_{i\in\Omega_j})$: it is a splitting field of f_j over $k_P(\mathfrak{p})$. Let $G_j = \mathrm{Gal}(k_j/k_P(\mathfrak{p}))$, that is, the Galois group of $\bar{f}_j$. Then,

(B.5.1)

the canonical morphism $D/I = \mathrm{Gal}(k_R(\mathfrak{r})/k_P(\mathfrak{p})) \to \mathrm{Gal}(k_j/k_P(\mathfrak{p})) = G_j$ *is surjective*

for all j. Since G_j acts transitively on Ω_j, we obtain in particular that

(B.5.2) $\qquad |\Omega_j|$ *divides* $|G|$ *for all* j.

B.6. Some facts on discriminants

Let $f(\mathbf{t}) \in P[\mathbf{t}]$ be a monic polynomial of degree d. We denote by $\mathrm{disc}(f)$ its discriminant. Then

(B.6.1) $$\mathrm{disc}(f(\mathbf{t}^2)) = (-4)^d \,\mathrm{disc}(f)^2 \cdot f(0).$$

Proof. — By easy specialization arguments, we may assume that P is an algebraically closed field. Let $E_1,\dots,E_d$ be the elements of P such that

$$f(\mathbf{t}) = \prod_{i=1}^{d} (\mathbf{t} - E_i).$$

We fix a square root e_i of E_i in P. So

$$f(\mathbf{t}^2) = \prod_{\substack{1 \leqslant i \leqslant d \\ \varepsilon \in \{1,-1\}}} (\mathbf{t} - \varepsilon e_i)$$

and the discriminant of $f(\mathbf{t}^2)$ is then equal to

$$\mathrm{disc}(f(\mathbf{t}^2)) = \Big(\prod_{\substack{1 \leqslant i < j \leqslant d \\ \varepsilon,\varepsilon' \in \{1,-1\}}} (\varepsilon e_i - \varepsilon' e_j)^2 \Big) \cdot \prod_{i=1}^{d} (e_i - (-e_i))^2.$$

In other words,

$$\operatorname{disc}(f(\mathbf{t}^2)) = 4^d \cdot \Big(\prod_{1 \leqslant i<j \leqslant d} (E_i - E_j)^4 \Big) \cdot \prod_{i=1}^{d} E_i = 4^d \ \operatorname{disc}(f)^2 \cdot (-1)^d f(0),$$

as expected. ☐

Let us conclude with another easy result:

(B.6.2) $$\operatorname{disc}(\mathbf{t} f(\mathbf{t})) = \operatorname{disc}(f) \cdot f(0)^2.$$

Proof. — As in the previous proof, we may assume that P is an algebraically closed field, and we denote by $E_1, \ldots, E_d$ the elements of P such that

$$f(\mathbf{t}) = \prod_{i=1}^{n} (\mathbf{t} - E_i).$$

Then

$$\operatorname{disc}(\mathbf{t} f(\mathbf{t})) = \Big(\prod_{1 \leqslant i<j \leqslant d} (E_i - E_j)^2 \Big) \cdot \prod_{i=1}^{d} (0 - E_i)^2.$$

Whence the result. ☐

B.7. Topological version

Let Y be a locally simply connected topological space endowed with a faithful action of a finite group G. Let $X = G \setminus Y$ and let $\pi : Y \to X$ be the quotient map.

Let $Y^{\mathrm{nr}} = \{y \in Y | \mathrm{Fix}_G(y) = 1\}$ be the complement of the ramification locus and let $X^{\mathrm{nr}} = \pi(Y^{\mathrm{nr}})$. Fix $y_0 \in Y^{\mathrm{nr}}$ and let $F = G \cdot y_0$. We define a right action of $g' \in G$ on $g \cdot y_0 \in F$ by $(g \cdot y_0) \cdot g' = gg' \cdot y_0$.

Let y_1 be a point of Y in the same connected component as y_0 and let $I = \mathrm{Fix}_G(y_1)$. The right action of I on F can be described in terms of lifting of paths, as we recall below.

Fix a path $\tilde{\gamma} : [0,1] \to Y$ with $\tilde{\gamma}([0,1)) \subset Y^{\mathrm{nr}}$, $\tilde{\gamma}(0) = y_0$ and $\tilde{\gamma}(1) = y_1$. Let $\gamma = \pi(\tilde{\gamma})$.

Lemma B.7.1. — *Given $y \in F$, there is a unique path $\tilde{\gamma}_y$ in Y starting at y and lifting γ. Given $y', y'' \in F$, we have $y'' \in y' \cdot I$ if and only if $\tilde{\gamma}_{y'}(1) = \tilde{\gamma}_{y''}(1)$.*

Proof. — Let $E = \pi^{-1}(\gamma([0,1]))$. We have

$$E = \coprod_{\Omega \in G/I} \Big(\bigcup_{\omega \in \Omega} \omega(\tilde{\gamma}([0,1])) \Big)$$

where the $\bigcup_{\omega \in \Omega} \omega(\tilde{\gamma}([0,1]))$ are the connected components of E and the $\omega(\tilde{\gamma}([0,1]))$ are the irreducible components of E.

Let $y \in F$. There is a unique element $g \in G$ such that $y = g \cdot y_0$. The restricted path $g(\tilde{\gamma})_{|[0,1)}$ is the unique lift of $\gamma_{|[0,1)}$ starting at y. It follows from the description of E that $g(\tilde{\gamma})$ is the unique lift of γ starting at y. The lemma follows. $\square$

We consider now the case of a non-Galois covering. Let H be a subgroup of G and let $\bar{Y} = H \setminus Y$. We denote by $\phi : Y \to \bar{Y}$ the quotient map and by $\psi : \bar{Y} \to X$ the map such that $\pi = \psi \circ \phi$. Let $\bar{F} = \phi(F)$. The right action of I on F induces a right action on $\bar{F}$. We have a bijection $H \setminus G \xrightarrow{\sim} \bar{F}$, $Hg \mapsto \phi(g \cdot y_1)$ and the right action of I on $\bar{F}$ corresponds to the right action on $H \setminus G$ by right multiplication.

Lemma B.7.2. — *Given $\bar{y} \in \bar{F}$, there is a unique path $\bar{\gamma}_{\bar{y}}$ in $\bar{Y}$ starting at $\bar{y}$ and lifting γ. Given $\bar{y}', \bar{y}'' \in \bar{F}$, we have $\bar{y}'' \in \bar{y}' \cdot I$ if and only if $\bar{\gamma}_{\bar{y}'}(1) = \bar{\gamma}_{\bar{y}''}(1)$.*

Proof. — There is an element $g \in G$ such that $\bar{y} = \phi(g \cdot y_0)$, and Hg is uniquely determined by $\bar{y}$. The path $\phi(g(\tilde{\gamma}))$ is the unique lift of γ starting at $\bar{y}$. The lemma follows. $\square$

We assume now that R is a finitely generated commutative reduced $\mathbb{C}$-algebra and Y is the topological space $(\operatorname{Spec} R)(\mathbb{C})$, for the classical topology. We have $\bar{Y} = (\operatorname{Spec} Q)(\mathbb{C})$ and $X = (\operatorname{Spec} P)(\mathbb{C})$.

The prime ideal $\mathfrak{r}$ of R corresponds to an irreducible subvariety Z of $\operatorname{Spec} R$. There is a non-empty open subset U of Z such that given $y \in U(\mathbb{C})$, we have $\operatorname{Stab}_G(y) = G_{\mathfrak{r}}^I$.

Fix a point $y_1 \in U(\mathbb{C})$. We have $I = G_{\mathfrak{r}}^I$. Lemma B.7.2 provides a topological description of the orbits of I on the fibers of ψ.

APPENDIX C

GRADINGS AND INTEGRAL EXTENSIONS

C.1. Idempotents, radical

Let Γ be a monoid. Denote by $\Delta : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma]$, $\gamma \mapsto \gamma \otimes \gamma$ the comultiplication. Let A be a ring. Let us recall the equivalence between the notion of a Γ-grading on A and that of a coaction of $\mathbb{Z}[\Gamma]$ on A.

Put $A[\Gamma] = A \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma]$. Given $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ a Γ-graded ring structure on A, we have a morphism of rings

$$\mu = \mu_A : A \longrightarrow A[\Gamma],\ A_\gamma \ni a \mapsto a \otimes \gamma$$

such that

$$\text{(C.1.1)} \qquad \begin{cases} \mu \otimes \mathrm{Id} : A \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] \to A[\Gamma],\ a \otimes \gamma \mapsto \mu(a)\gamma \text{ is an isomorphism and} \\ (1 \otimes \Delta) \circ \mu = (\mu \otimes \mathrm{Id}) \circ \mu : A \longrightarrow A \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma]. \end{cases}$$

Conversely, consider a morphism of rings $\mu : A \to A[\Gamma]$ satisfying the two properties (C.1.1) above. Let $A_\gamma = \mu_A^{-1}(A \otimes \gamma)$. Note that given $a \in A_\gamma$ and $a' \in A_{\gamma'}$, we have $aa' \in A_{\gamma+\gamma'}$. Let $a \in A$ and write $\mu(a) = \sum_{i=1}^n a_i \otimes \gamma_i$ with $a_i \in A$ and $\gamma_i \in \Gamma$. We have $(1 \otimes \Delta) \circ \mu(a) = (\mu \otimes \mathrm{Id}) \circ \mu(a)$, hence $\sum_i a_i \otimes \gamma_i \otimes \gamma_i = \sum_i \mu(a_i) \otimes \gamma_i$. It follows that $\mu(a_i) = a_i \otimes \gamma_i$, hence $a_i \in A_{\gamma_i}$. We deduce that $A = \sum_\gamma A_\gamma$. The first property shows that μ is injective, hence $A = \bigoplus_\gamma A_\gamma$: we have obtained a Γ-graded ring structure on A.

Given $f : \Gamma \to \Gamma'$ a morphism of monoids and given a Γ-grading on A, we have a Γ'-grading on A given by $A'_{\gamma'} = \bigoplus_{\gamma \in f^{-1}(\gamma')} A_\gamma$.

Assume, up to formula (C.1.6), that $\Gamma = \mathbb{Z}$. If B is a ring containing A and if $\xi \in B^\times$ commutes with A, then there exists a unique morphism of rings

$$\mu_A^\xi : A \longrightarrow B$$

such that $\mu_A^\xi(a) = a\xi^i$ if $a \in A_i$. Note that, if A is $\mathbb{N}$-graded (that is, if $A_i = 0$ for $i < 0$), then μ_A^ξ can be defined also whenever ξ is not invertible. If $\mathbf{t}$ is an indeterminate

over A, then
$$\mu_A^{\mathbf{t}} : A \longrightarrow A[\mathbf{t},\mathbf{t}^{-1}]$$
is a morphism of rings. If we denote by $\mathrm{ev}_A^{\xi} : A[\mathbf{t},\mathbf{t}^{-1}] \to B$ the evaluation morphism at ξ, then

$$\mu_A^{\xi} = \mathrm{ev}_A^{\xi} \circ \mu_A^{\mathbf{t}}. \tag{C.1.2}$$

In particular, if $B = A$ and $\xi = 1$, then

$$\mu_A^{1} = \mathrm{Id}_A \qquad \text{and} \qquad \mathrm{ev}_A^{1} \circ \mu_A^{\mathbf{t}} = \mathrm{Id}_A. \tag{C.1.3}$$

On the other hand, the morphism $\mu_A^{\xi} : A \longrightarrow B$ can be extended to a $\mathbb{Z}[\mathbf{t},\mathbf{t}^{-1}]$-linear morphism $\boldsymbol{\mu}_A^{\xi} : A[\mathbf{t},\mathbf{t}^{-1}] \longrightarrow B[\mathbf{t},\mathbf{t}^{-1}]$ and

$$\boldsymbol{\mu}_A^{\xi} \circ \mu_A^{\mathbf{t}} = \mu_A^{\xi\mathbf{t}}. \tag{C.1.4}$$

As particular cases, one can take $B = A[\mathbf{u},\mathbf{u}^{-1}]$ and $\xi = \mathbf{u}$, where $\mathbf{u}$ is another indeterminate, or take $B = A[\mathbf{t},\mathbf{t}^{-1}]$ and $\xi = \mathbf{t}^{-1}$. We obtain the following equalities:

$$\boldsymbol{\mu}_A^{\mathbf{u}} \circ \mu_A^{\mathbf{t}} = \mu_A^{\mathbf{tu}} \qquad \text{and} \qquad \boldsymbol{\mu}_A^{\mathbf{t}^{-1}} \circ \mu_A^{\mathbf{t}}(a) = a \in A[\mathbf{t},\mathbf{t}^{-1}] \tag{C.1.5}$$

for all $a \in A$. Finally, note that

$$\mathrm{ev}_A^{1} \circ \boldsymbol{\mu}_A^{\mathbf{t}^{-1}} = \mathrm{ev}_A^{1}. \tag{C.1.6}$$

Proposition C.1.7. — *Assume that A is commutative and Γ is a torsion-free abelian group. Let e be an idempotent of A. Then $e \in A_0$.*

Proof. — Replacing A by the ring generated by the homogeneous components of e, we may assume that A is noetherian and Γ is finitely generated. Given $d > 0$, consider the ring morphism $m_d : A[\Gamma] \to A[\Gamma]$, $a \otimes \gamma \mapsto a \otimes \gamma^d$. Note that $m_d(\mu(e))$ is an idempotent of $A[\Gamma]$. If $e \notin A_0$, then $\mu(e) \notin A$, hence the $m_d(\mu(e))$ are distinct for different d. Since A is commutative and noetherian, $A[\Gamma]$ is also commutative and noetherian, and so contains only finitely many idempotents. We deduce that $e \in A_0$. □

Let us recall a basis result on the homegeneity of the radical [**Row**, Theorem 2.5.40].

Proposition C.1.8. — *If Γ is a free abelian group, then* $\mathrm{Rad}(A)$ *is a homogeneous ideal of A.*

Proof. — Let $r \in \mathrm{Rad}(A)$. Write $r = \sum_{1 \leqslant i \leqslant d} r_i$ with $r_i \in A_{\gamma_i}$ for some $\gamma_i \in \Gamma$. Fix $i \in \{1, \ldots, d\}$. Fix a group morphism $\rho : \Gamma \to \mathbb{Z}$ such that $\rho(\gamma_j) \neq \rho(\gamma_i)$ for $j \neq i$.

Let n be a positive integer with $n > |\rho(\gamma_i) - \rho(\gamma_j)|$ for all $j \neq i$. Let $\Gamma' = \mathbb{Z}/n\mathbb{Z}$ and write ζ for its generator 1, so that $A[\Gamma'] = A \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]$. We have $\mathrm{Rad}(A) = \mathrm{Rad}(A[\Gamma']) \cap A$.

Consider the group morphism $\theta_l : \Gamma \to \Gamma'$, $\gamma \mapsto \zeta^{l\rho(\gamma)}$ and denote by θ_l again the induced ring morphism $A[\Gamma] \to A[\Gamma']$. We have

$$\sum_{l=1}^{n} \zeta^{-\rho(\gamma_i)l}\, \theta_l(\mu(r)) = \sum_{1 \leqslant j \leqslant d} r_j \sum_{1 \leqslant l \leqslant n} \zeta^{l(\rho(\gamma_j)-\rho(\gamma_i))} = n r_i.$$

Since $\theta_l \circ \mu$ is a morphism of rings, it follows that $n r_i \in \mathrm{Rad}(A[\Gamma'])$, hence $n r_i \in \mathrm{Rad}(A)$. Similarly, we obtain $(n+1) r_i \in \mathrm{Rad}(A)$, hence $r_i \in \mathrm{Rad}(A)$. So, $\mathrm{Rad}(A)$ is a homogeneous ideal. □

C.2. Extension of gradings

Notation. *We fix in this section a finitely generated free abelian group Γ and a commutative Γ-graded* ***domain*** *P. Let Q be a* ***domain*** *containing P and* ***integral*** *over P.*

The aim of this section is to study the gradings on Q which extend the one on P. We first start by the uniqueness problem:

Lemma C.2.1. — *If $Q = \bigoplus_{\gamma\in\Gamma} \tilde{Q}_\gamma = \bigoplus_{\gamma\in\Gamma} \hat{Q}_\gamma$ are two gradings on Q extending the one of P (that is, $P_\gamma = \tilde{Q}_\gamma \cap P = \hat{Q}_\gamma \cap P$ for all $\gamma \in \Gamma$), then $\tilde{Q}_\gamma = \hat{Q}_\gamma$ for all $\gamma \in \Gamma$.*

Proof. — As in § C.1 (from which we keep the notation), the gradings $Q = \bigoplus_{\gamma\in\Gamma} \tilde{Q}_\gamma$ and $\bigoplus_{\gamma\in\Gamma} \hat{Q}_\gamma$ correspond to ring morphisms $\tilde{\mu}_Q : Q \to Q[\Gamma]$ and $\hat{\mu}_Q : Q \to Q[\Gamma]$ extending $\mu_P : P \to P[\Gamma]$. Consider the morphism of rings $\beta : Q[\Gamma] \to Q[\Gamma]$, $q \otimes \gamma \mapsto \hat{\mu}_Q(q)\gamma^{-1}$ and let $\alpha = \beta \circ \tilde{\mu}_Q : Q \to Q[\Gamma]$. Then α is a morphism of rings and $\alpha(p) = p$ for all $p \in P$ by (C.1.1). Therefore, if $q \in Q$, then $\alpha(q) \in Q[\Gamma]$ is integral over P, hence over Q. Since Q is a domain, this implies that $\alpha(q) \in Q$ (Proposition B.4.3). On the other hand, (C.1.1) shows that the composition

$$Q \xrightarrow{\alpha} Q[\Gamma] \xrightarrow{q\otimes\gamma\mapsto q} Q$$

is the identity, hence $\alpha(q) = q$ for all $q \in Q$. It follows that $\beta \circ \tilde{\mu}_Q = \beta \circ \hat{\mu}_Q$, hence $\tilde{\mu}_Q = \hat{\mu}_Q$ since β is an isomorphism (C.1.1). □

Corollary C.2.2. — *If $Q = \bigoplus_{\gamma\in\Gamma} Q_\gamma$ is a grading on Q extending the one on P and if G is a group acting on Q, stabilizing P and its homogeneous components, then G stabilizes the homogeneous components of Q.*

Proof. — Indeed, if $g \in G$, then $Q = \bigoplus_{\gamma\in\Gamma} g(Q_\gamma)$ is a grading on Q extending the one on P. According to Lemma C.2.1, we have $g(Q_\gamma) = Q_\gamma$ for all γ. □

Counter-example C.2.3. — The assumption that Q is a *domain* is necessary in Lemma C.2.1. Indeed, if $P = P_0$ and if $Q = P \oplus P\varepsilon$ with $\varepsilon^2 = 0$, then we can endow Q with infinitely many gradings extending the one on P, for instance by saying that ε is homogeneous of any degree. ■

Proposition C.2.4. — *Assume $Q = \bigoplus_{\gamma\in\Gamma} Q_\gamma$ is a grading on Q extending the one on P. If Γ is endowed with a structure of totally ordered group such that $P_\gamma = 0$ for all $\gamma < 0$, then $Q_\gamma = 0$ for all $\gamma < 0$.*

Proof. — See [**Bou**, Chapter 5, §1, Proposition 20 and Exercise 25]. □

We will now be interested in the question of the existence of a grading on Q extending the one on P. For this, let $K = \mathrm{Frac}(P)$, $L = \mathrm{Frac}(Q)$ and we assume that the field extension L/K is finite.

Lemma C.2.5. — *The grading on P extends to a grading of its integral closure in K.*

Proof. — See [**Bou**, Chapter 5, §1, Proposition 21 and Exercise 25]. □

We will now show how the question of the existence can be read on a normal closure of the field extension L/K. Let M denote a normal closure of the field extension L/K and let R be the integral closure of P in M.

Lemma C.2.6. — *Let $\gamma \in \Gamma$. Define a grading on $P[\mathbf{x}]$ by giving to $\mathbf{x}$ the degree γ. Let $F \in P[\mathbf{x}]$ be a monic polynomial, which is **homogeneous** for this grading. If $F = F_1 \cdots F_r$, with $F_i \in P[\mathbf{x}]$ monic, then F_i is homogeneous for all i.*

Proof. — We have $\mu_{P[\mathbf{x}]}(\mathbf{x}a) = \mathbf{x}\mu_P(a)\gamma$ for $a \in P$. Let $\gamma_F \in \Gamma$ denote the *total* degree of F. We have

$$\mu_{P[\mathbf{x}]}(F) = F\gamma_F = \mu_{P[\mathbf{x}]}(F_1)\cdots\mu_{P[\mathbf{x}]}(F_r).$$

Since $P[\mathbf{x}]$ is a domain, with fraction field $K(\mathbf{x})$, the fact that $K(\mathbf{x})[\Gamma]$ is a unique factorization domain implies that there exists $G_1, \dots, G_r \in K(\mathbf{x})$ and $\gamma_1, \dots, \gamma_r \in \Gamma$ such that

$$\mu_{P[\mathbf{x}]}(F_i) = G_i\gamma_i$$

for all i. This forces F_i to be homogeneous of degree γ_i, and $F_i = G_i$. □

Corollary C.2.7. — *Let $\gamma \in \Gamma$. Define a grading on $P[\mathbf{x}]$ by giving to $\mathbf{x}$ the degree $\gamma \in \Gamma$. Let $F \in P[\mathbf{x}]$ be a monic polynomial, which is **homogeneous** for this grading. We assume that M is the splitting field of F over K. Then R admits a grading extending the one on P.*

Proof. — By Lemma C.2.5, we may assume that P is integrally closed. Let δ denote the degree of F *in the variable* $\mathbf{x}$. We shall show the result by induction on δ, the case where $\delta = 1$ being trivial (because $P = R$ in this case).

So assume that $\delta \geqslant 2$ and let F_1 be a monic irreducible polynomial of $K[\mathbf{x}]$ dividing F. By Proposition B.4.1, we have $F_1 \in P[\mathbf{x}]$. Set $K' = K[\mathbf{x}]/ < F_1 >$ and let x be the image of $\mathbf{x}$ in K'. Then K' is a field which contains the ring $P' = P[\mathbf{x}]/ < F_1 >$. In fact, K' is the fraction field of P'. Since F_1 is homogeneous by Lemma C.2.6, P' is graded (with x homogeneous of degree γ). By Lemma C.2.5, the integral closure P'' of P' in K' inherits a grading. On the other hand, $K' \subset M$ and M is the splitting field of F over K'. In $P''[\mathbf{x}]$, we have

$$F(\mathbf{x}) = (\mathbf{x} - x)F_0(\mathbf{x}),$$

with $F_0(\mathbf{x}) \in P''[\mathbf{x}]$ homogeneous, and whose degree *in the variable* $\mathbf{x}$ is equal to $\delta - 1$. Since the splitting field of F over K is equal to the splitting field of F_0 over K', the result follows from the induction hypothesis. □

Proposition C.2.8. — *Assume that P and Q are integrally closed. If the grading on P extends to a grading on Q, then this grading also extends to a grading on R.*

Proof. — Let $q_1, \dots, q_r$ be elements of Q, homogeneous of respective degrees $\gamma_1, \dots, \gamma_r$ and such that $L = K[q_1, \dots, q_r]$. We denote by $F_i \in K[\mathbf{t}]$ the minimal polynomial of q_i over K: in fact, $F_i \in P[\mathbf{t}]$ according to Corollary B.4.2. Then M is the splitting field of $F_1 \cdots F_r$. By an easy induction argument, we may assume that $r = 1$: we then write $q = q_1$, $\gamma = \gamma_1$ and $F = F_1$.

If we give to the variable $\mathbf{t}$ the degree γ, then we check easily that F becomes homogeneous (for the total degree on $P[\mathbf{t}]$). The existence of an extension of the grading then follows from Corollary C.2.7. □

Lemma C.2.9. — *Let $\mathfrak{p}$ be a prime ideal of P and let $\tilde{\mathfrak{p}}$ be the maximal homogeneous ideal of P contained in $\mathfrak{p}$ (that is $\tilde{\mathfrak{p}} = \bigoplus_{\gamma \in \Gamma} \mathfrak{p} \cap P_\gamma$). Then $\tilde{\mathfrak{p}}$ is prime.*

Proof. — Indeed, $(P/\mathfrak{p})[\Gamma]$ is a domain and $\tilde{\mathfrak{p}}$ is the kernel of the morphism obtained by composition $P \xrightarrow{\mu_P} P[\Gamma] \xrightarrow{\mathrm{can}} (P/\mathfrak{p})[\Gamma]$. □

Lemma C.2.10. — *Let $\mathfrak{q}$ be a prime ideal of Q and let $\mathfrak{p} = \mathfrak{q} \cap P$. Assume that the grading on P extends to a grading on Q. Then $\mathfrak{p}$ is homogeneous if and only if $\mathfrak{q}$ is homogeneous.*

Proof. — If $\mathfrak{q}$ is homogeneous, then $\mathfrak{p}$ is clearly homogeneous. Conversely, assume that $\mathfrak{p}$ is homogeneous. Let $\mathfrak{q}' = \bigoplus_{\gamma\in\Gamma}(\mathfrak{q}\cap Q_\gamma)$. Then $\mathfrak{q}'$ is a homogeneous ideal of Q contained in $\mathfrak{q}$ and $\mathfrak{q}'\cap P = \mathfrak{p} = \mathfrak{q}\cap P$. By Lemma C.2.9, $\mathfrak{q}'$ is a prime ideal, so $\mathfrak{q}' = \mathfrak{q}$ since Q is integral over P. □

Lemma C.2.11. — *Assume $\Gamma = \mathbb{Z}$. Let $\mathfrak{p}$ be a prime ideal of P and let P' be the largest graded subring of $P_\mathfrak{p}$. Assume that the composition $P'_i \subset P_\mathfrak{p} \xrightarrow{\text{can}} P/\mathfrak{p}$ is bijective for all $i \in \mathbb{Z}$.*

Let q be a homogeneous element of Q and let $F \in P[X]$ be its minimal polynomial. Then the image of F in $(P/\mathfrak{p})[X]$ is the minimal polynomial of $q\otimes 1 \in Q\otimes_P P/\mathfrak{p}$.

Proof. — Let $F' = \sum_{i=0}^n a_i X^i \in (P/\mathfrak{p})[X]$ be the minimal polynomial of $q\otimes 1$, with $a_n = 1$. Note that $\deg F' \leqslant \deg F$.

Let d be the homogeneous degree of q and let $G = \sum_{i=0}^n f^{-1}_{d(n-i)}(a_i) \in P'[X]$, where $f_j : P'_j \xrightarrow{\sim} P/\mathfrak{p}$ is the canonical bijection. We have $G(q) \in P'_{dn} \cap \mathfrak{p}P_\mathfrak{p} = 0$. It follows that $G = F$. □

We conclude this section with some results about the homogeneization of prime ideals of P or Q.

Corollary C.2.12. — *Assume that the grading on P extends to Q. Let $\mathfrak{q}$ be a prime ideal of Q and let $\mathfrak{p} = \mathfrak{q}\cap P$. Let $\tilde{\mathfrak{p}}$ (respectively $\tilde{\mathfrak{q}}$) be the maximal homogeneous ideal of P (respectively Q) contained in $\mathfrak{p}$ (respectively $\mathfrak{q}$). Then $\tilde{\mathfrak{p}} = \tilde{\mathfrak{q}}\cap P$.*

Proof. — This follows from the proof of Lemma C.2.10 and from the fact that the diagram

$$\begin{array}{ccccc} P & \xrightarrow{\mu_P} & P[\Gamma] & \xrightarrow{\text{can}} & (P/\mathfrak{p})[\Gamma] \\ \big\downarrow & & & & \big\downarrow \\ Q & \xrightarrow{\mu_Q} & Q[\Gamma] & \xrightarrow{\text{can}} & (Q/\mathfrak{q})[\Gamma] \end{array}$$

is commutative. □

Corollary C.2.13. — *Assume that the grading on P extends to Q and that there exists a finite group G acting on Q, stabilizing P and preserving the grading. Let $\mathfrak{q}$ be a prime ideal of Q and let $\tilde{\mathfrak{q}}$ be the maximal homogeneous ideal of Q contained in $\mathfrak{q}$. Let $D_\mathfrak{q}$ (respectively $D_{\tilde{\mathfrak{q}}}$) be the decomposition group of $\mathfrak{q}$ (respectively $\tilde{\mathfrak{q}}$) in G and $I_\mathfrak{q}$ (respectively $I_{\tilde{\mathfrak{q}}}$) be the inertia group of $\mathfrak{q}$ (respectively $\tilde{\mathfrak{q}}$) in G. Then*

$$D_\mathfrak{q} \subset D_{\tilde{\mathfrak{q}}} \qquad \text{and} \qquad I_\mathfrak{q} = I_{\tilde{\mathfrak{q}}}.$$

Proof. — The first inclusion is immediate, since G preserves the grading (see Corollary C.2.2). Moreover, $Q/\mathfrak{q}$ is a quotient of $Q/\tilde{\mathfrak{q}}$, so $I_{\tilde{\mathfrak{q}}} \subset I_{\mathfrak{q}}$. Conversely, if $g \in I_{\mathfrak{q}} \subset D_{\mathfrak{q}} \subset D_{\tilde{\mathfrak{q}}}$ and if $q \in \mathfrak{q} \cap Q_\gamma$, then $g(q) - q \in \mathfrak{q} \cap Q_\gamma \subset \tilde{\mathfrak{q}}$. So $g \in I_{\tilde{\mathfrak{q}}}$. $\square$

C.3. Gradings and reflection groups

Notation. *In this section, we fix a field k of characteristic zero and a commutative $\mathbb{N}$-graded k-algebra $R = \bigoplus_{i\in\mathbb{N}} R_i$. We assume moreover that R is a* ***domain****, that $R_0 = k$ and R is a finitely generated k-algebra. We also fix a finite group G acting* ***faithfully*** *on R by automorphisms of graded k-algebras and we set $P = R^G$. Let $R_+ = \bigoplus_{i>0} R_i$: it is the unique graded maximal ideal of R. We fix a G-stable, graded, vector subspace E^* of R such that $R_+ = R_+^2 \oplus E^*$ (such a subspace exists because kG is semisimple) and we denote by E the k-dual of E^*.*

The group G acts on the vector space E and the aim of this section is to give some criterion allowing to determine whether G is a reflection subgroup of $\mathrm{GL}_k(E)$. Our results are inspired by [**BeBoRo**].

First of all, the grading on E^* induces a grading on E and a grading on $k[E]$, the algebra of polynomial functions on E (that is, the symmetric algebra of E^*). Similarly, $k[E]$ inherits an action of G, which preserves the grading. We denote by $k[E]_+$ the unique graded maximal ideal of $k[E]$. The inclusion $E^* \hookrightarrow R$ induces a G-equivariant morphism of graded k-algebras

$$\pi : k[E] \longrightarrow R.$$

It is easily checked that

(C.3.1) *the minimal number of generators of the k-algebra R is* $\dim_k E$

(see for instance [**BeBoRo**, lemme 2.1]). We put

$$I = \operatorname{Ker} \pi,$$

so that

$$R \simeq k[E]/I. \tag{C.3.2}$$

In particular, G acts faithfully on E. Since I is homogeneous, it follows from the graded Nakayama Lemma that

(C.3.3) *the minimal number of generators of the ideal I is* $\dim_k I/k[E]_+I$

Moreover, it is also easy to check that

(C.3.4) $\quad I = k[E]I^G$ *if and only if* G *acts trivially on* $I/k[E]_+I$.

(see for instance [**BeBoRo**, lemme 3.1]). Finally, since kG is semisimple, we have

(C.3.5) $$P \simeq k[E]^G/I^G.$$

We will also need the next lemma:

Lemma C.3.6. — *If* R *is a free* P*-module, then the rank of the* P*-module* R *is* $|G|$.

Proof. — Let d be the P-rank of R. Since R is a domain, P is also a domain and, if we set $K = \mathrm{Frac}(P)$ and $M = \mathrm{Frac}(R)$, then $K = L^G$ (and so $[L:K] = |G|$) and $L = K \otimes_P R$ (and so $[L:K] = d$). Therefore, $d = |G|$. □

The main result of this section is the following (compare with [**BeBoRo**, Theorem 3.2], from which we borrow the proof).

Proposition C.3.7. — *We assume that* P *is regular and that* R *is a free* P*-module. Then the following are equivalent:*

(1) R *is complete intersection and* G *acts trivially on* $I/k[E]_+I$.
(2) G *is a subgroup of* $\mathrm{GL}_k(E)$ *generated by reflections.*

Remark C.3.8. — If P is regular and since we are working with graded objects, the following statements are equivalent:

- R is a free P-module.
- R is a flat P-module.
- R is Cohen-Macaulay.

Moreover, if R is complete intersection, then R is Cohen-Macaulay. ■

Proof. — Let $e = \dim_k E$, $i = \dim_k I/k[E]_+I$ and let d denote the Krull dimension of R (which is also the one of P). Moreover, e is the Krull dimension of $k[E]$ and of $k[E]^G$.

Let us first show (1) ⇒ (2). So assume that R is complete intersection and that G acts trivially on $I/k[E]_+I$. Since R is complete intersection, (C.3.2) and (C.3.3) show that

$$d = e - i.$$

Moreover, since G acts trivially on $I/k[E]_+I$, the ideal I of $k[E]$ can be generated by i homogeneous G-invariant elements $f_1, \dots, f_i$ and so the ideal I^G of $k[E]^G$ is generated by $f_1, \dots, f_i$. Since P is regular of Krull dimension d, the algebra $P = k[E]^G/I^G$ can be generated by d elements $\pi(g_1), \dots, \pi(g_d)$ where $g_j \in k[E]^G$ is homogeneous.

Therefore, the k-algebra $k[E]^G$ is generated by $f_1, \dots, f_i, g_1, \dots, g_d$, that is, it is generated by $i+d=e$ elements. Since the Krull dimension of $k[E]^G$ is also equal to e, this shows that $k[E]^G$ is a polynomial algebra, and so G is a reflection subgroup of $\mathrm{GL}_k(E)$ by Theorem 2.2.1.

Conversely, let us now show (2) $\Rightarrow$ (1). So assume that G is a reflection subgroup of $\mathrm{GL}_k(E)$. Then $k[E]$ is a free $k[E]^G$-module of rank $|G|$ (by Theorem 2.2.1) and so $(k[E]^G/I^G) \otimes_{k[E]^G} k[E]$ is a free P-module of rank $|G|$ (see (C.3.5)). Moreover, $k[E]/I = R$ is a free P-module of rank $|G|$ (see Lemma C.3.6). So the canonical surjection $(k[E]^G/I^G) \otimes_{k[E]^G} k[E] \twoheadrightarrow k[E]/I$ (between two free P-modules of the same rank) is an isomorphism, hence I is generated by I^G and so that G acts trivially on $I/k[E]_+I$ (by (C.3.4)).

On the other hand, since $k[E]^G$ and $k[E]^G/I^G = P$ are both polynomial algebras (see Theorem 2.2.1 for $k[E]^G$), P is complete intersection and so I^G can be generated by $e-d$ elements. We deduce from (C.3.3) and (C.3.4) that $i \leqslant e-d$ and so necessarily $i = e-d$ and R is complete intersection. $\square$

APPENDIX D

BLOCKS, DECOMPOSITION MATRICES

> **Assumption and notation.** *We fix in this appendix a commutative ring R, which will be assumed* ***noetherian****,* ***integral*** *and* ***integrally closed****. Let $\mathfrak{r}$ denote a prime ideal of R. We also fix an R-algebra $\mathcal{H}$ which will be assumed to be finitely generated and free as an R-module. We denote by $\mathrm{Z}(\mathcal{H})$ the center of $\mathcal{H}$. We set $k = \mathrm{Frac}(R/\mathfrak{r}) = k_R(\mathfrak{r})$ and $K = \mathrm{Frac}(R) = k_R(0)$. Finally, the image of an element $h \in \mathcal{H}$ in $k\mathcal{H}$ will be denoted by $\bar{h}$.*

D.1. Blocks of $k\mathcal{H}$

If A is a ring (not necessarily commutative), we denote by $\mathrm{Idem}_{\mathrm{pr}}(A)$ the set of its primitive idempotents. For instance, $\mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(\mathcal{H}))$ is the set of primitive central idempotents of $\mathcal{H}$. Since $\mathrm{Z}(\mathcal{H})$ is noetherian,

(D.1.1)
$$1 = \sum_{e \in \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(\mathcal{H}))} e.$$

Moreover, the morphism $\mathcal{H} \to k\mathcal{H}$ induces a morphism $\pi_{\mathrm{Z}} : k\mathrm{Z}(\mathcal{H}) \to \mathrm{Z}(k\mathcal{H})$ (which might be neither injective nor surjective). However, the following result has been proven by Müller [**Mül**, Theorem 3.7]:

***Proposition D.1.2* (Müller).** — (a) *If $e \in \mathrm{Idem}_{\mathrm{pr}}(k\mathrm{Z}(\mathcal{H}))$, then $\pi_{\mathrm{Z}}(e) \in \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(k\mathcal{H}))$.*
(b) *The map $\mathrm{Idem}_{\mathrm{pr}}(k\mathrm{Z}(\mathcal{H})) \to \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(k\mathcal{H}))$, $e \mapsto \pi_{\mathrm{Z}}(e)$ is bijective.*

In a finite dimensional commutative k-algebra $\mathcal{A}$ (for instance, $\mathrm{Z}(k\mathcal{H})$ or $k\mathrm{Z}(\mathcal{H})$), the prime ideals are maximal and are in one-to-one correspondence with the set of primitive idempotents of $\mathcal{A}$: if $\mathfrak{m} \in \mathrm{Spec}\,\mathcal{A}$ and $e \in \mathrm{Idem}_{\mathrm{pr}}(\mathcal{A})$, then e and $\mathfrak{m}$ are associated through this bijective map if and only if $e \notin \mathfrak{m}$ (that is, if and only if

$\mathfrak{m} = \mathrm{Rad}(\mathscr{A}e) + (1-e)\mathscr{A})$. The Proposition D.1.2 shows that $\mathrm{Spec}\, k\mathrm{Z}(\mathscr{H})$ is in one-to-one correspondence with $\mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(k\mathscr{H}))$, that is, with the set of primitive central idempotents of $k\mathscr{H}$.

Moreover, the natural (and injective) morphism $R \hookrightarrow \mathrm{Z}(\mathscr{H})$ induces a morphism $\Upsilon : \mathrm{Spec}\,\mathrm{Z}(\mathscr{H}) \to \mathrm{Spec}\, R$. The map $\mathrm{Z}(\mathscr{H}) \to k\mathrm{Z}(\mathscr{H})$ induces a bijective map between the sets $\mathrm{Spec}\, k\mathrm{Z}(\mathscr{H})$ and $\Upsilon^{-1}(\mathfrak{r})$. Recall that

$$\Upsilon^{-1}(\mathfrak{r}) = \{\mathfrak{z} \in \mathrm{Spec}\,\mathrm{Z}(\mathscr{H}) \mid \mathfrak{z} \cap R = \mathfrak{r}\}.$$

Finally, we obtain a bijective map

$$\Xi_{\mathfrak{r}} : \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(k\mathscr{H})) \xrightarrow{\sim} \Upsilon^{-1}(\mathfrak{r}) \tag{D.1.3}$$

which is characterized by the following property:

Lemma D.1.4. — *If $e \in \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(k\mathscr{H}))$ and if $\mathfrak{z} \in \Upsilon^{-1}(\mathfrak{r})$, then the following are equivalent:*

(1) $\mathfrak{z} = \Xi_{\mathfrak{r}}(e)$.
(2) $e \notin \pi_{\mathrm{Z}}(k\mathfrak{z})$.
(3) $\mathfrak{z}$ *is the preimage, in* $\mathrm{Z}(\mathscr{H})$, *of* $\pi_{\mathrm{Z}}^{-1}(\mathrm{Rad}(\mathrm{Z}(k\mathscr{H})e) + (1-e)\mathrm{Z}(k\mathscr{H}))$.

Now, by localization at $\mathfrak{r}$, $\Upsilon^{-1}(\mathfrak{r})$ is in one-to-one correspondence with $\Upsilon_{\mathfrak{r}}^{-1}(\mathfrak{r}R_{\mathfrak{r}})$, where $\Upsilon_{\mathfrak{r}} : \mathrm{Spec}\, R_{\mathfrak{r}}\mathrm{Z}(\mathscr{H}) \to \mathrm{Spec}\, R_{\mathfrak{r}}$ is the map induced by the inclusion $R_{\mathfrak{r}} \hookrightarrow R_{\mathfrak{r}}\mathrm{Z}(\mathscr{H})$. The bijective maps, in both directions, between $\Upsilon^{-1}(\mathfrak{r})$ and $\Upsilon_{\mathfrak{r}}^{-1}(\mathfrak{r}R_{\mathfrak{r}})$ are given by

$$\begin{array}{ccc} \Upsilon^{-1}(\mathfrak{r}) & \longrightarrow & \Upsilon_{\mathfrak{r}}^{-1}(\mathfrak{r}R_{\mathfrak{r}}) \\ \mathfrak{z} & \longmapsto & R_{\mathfrak{r}}\mathfrak{z} \end{array}$$

and

$$\begin{array}{ccc} \Upsilon_{\mathfrak{r}}^{-1}(\mathfrak{r}R_{\mathfrak{r}}) & \longrightarrow & \Upsilon^{-1}(\mathfrak{r}) \\ \mathfrak{z} & \longmapsto & \mathfrak{z} \cap \mathrm{Z}(\mathscr{H}). \end{array}$$

The center of the algebra $R_{\mathfrak{r}}\mathscr{H}$ is equal to $R_{\mathfrak{r}}\mathrm{Z}(\mathscr{H})$ and the canonical morphism $\mathscr{H} \to k\mathscr{H}$ extends to a morphism $R_{\mathfrak{r}}\mathscr{H} \to k\mathscr{H}$, which will still be denoted by $h \mapsto \bar{h}$. Finally, we denote by $R_{\mathfrak{r}}\mathrm{Z}(\mathscr{H}) \to k\mathrm{Z}(\mathscr{H})$, $z \mapsto \hat{z}$, the canonical morphism (so that $\bar{z} = \pi_{\mathrm{Z}}(\hat{z})$ if $z \in R_{\mathfrak{r}}\mathrm{Z}(\mathscr{H})$).

To summarize, we obtain a diagram of natural bijective maps
(D.1.5)

$$\begin{tikzcd} \Upsilon^{-1}(\mathfrak{r}) \arrow[r, leftrightarrow, "\sim"] & \Upsilon_{\mathfrak{r}}^{-1}(\mathfrak{r}R_{\mathfrak{r}}) \arrow[r, leftrightarrow, "\sim"] & \mathrm{Spec}\, k\mathrm{Z}(\mathscr{H}) \arrow[r, leftrightarrow, "\sim"] \arrow[d, leftrightarrow, "\wr"] & \mathrm{Spec}\,\mathrm{Z}(k\mathscr{H}) \arrow[d, leftrightarrow, "\wr"] \\ & & \mathrm{Idem}_{\mathrm{pr}}(k\mathrm{Z}(\mathscr{H})) \arrow[r, leftrightarrow, "\sim"] & \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(k\mathscr{H})). \end{tikzcd}$$

D.2. Blocks of $R_{\mathfrak{r}}\mathcal{H}$

> **Assumption.** *From now on, and until the end of this Appendix, we will assume that the K-algebra $K\mathcal{H}$ is* ***split****.*

The question of lifting idempotents whenever the ring $R_{\mathfrak{r}}$ is complete for the $\mathfrak{r}$-adic topology is classical. We propose here another version, valid only whenever the K-algebra $K\mathcal{H}$ is split (we only need that R is integrally closed: no assumption on the Krull dimension of R or on its completeness is necessary).

D.2.A. Central characters. — If V is a simple $K\mathcal{H}$-module, and if $z \in K\mathrm{Z}(\mathcal{H})$, then z acts on V by multiplication by a scalar $\omega_V(z) \in K$ (indeed, since $K\mathcal{H}$ is split, we have $\mathrm{End}_{K\mathcal{H}}(V) = K$). This defines a morphism of K-algebras

$$\omega_V : K\mathrm{Z}(\mathcal{H}) \longrightarrow K$$

whose restriction to $\mathrm{Z}(\mathcal{H})$ has valued in R (since $\mathrm{Z}(\mathcal{H})$ is integral over R and R is integrally closed). Hence, this defines a morphism of R-algebras

$$\omega_V : \mathrm{Z}(\mathcal{H}) \longrightarrow R.$$

By composition with the canonical projection $R \to R/\mathfrak{r}$, we obtain a morphism of R-algebras

$$\omega_V^{\mathfrak{r}} : \mathrm{Z}(\mathcal{H}) \longrightarrow R/\mathfrak{r}.$$

Since $\omega_V(1) = 1$ and $R/\mathfrak{r}$ is integral, $\mathrm{Ker}\,\omega_V$ is a prime ideal of $\mathrm{Z}(\mathcal{H})$ such that $\mathrm{Ker}\,\omega_V \cap R = \mathfrak{r}$. So

(D.2.1) $$\mathrm{Ker}\,\omega_V^{\mathfrak{r}} \in \Upsilon^{-1}(\mathfrak{r}).$$

This defines a map

$$\begin{array}{rccc} \mathcal{K}er_{\mathfrak{r}} : & \mathrm{Irr}(K\mathcal{H}) & \longrightarrow & \Upsilon^{-1}(\mathfrak{r}) \\ & V & \longmapsto & \mathrm{Ker}\,\omega_V^{\mathfrak{r}} \end{array}.$$

Definition D.2.2. — *The fibers of the map $\mathcal{K}er_{\mathfrak{r}}$ are called the* $\mathfrak{r}$***-blocks*** *of $\mathcal{H}$.*

The $\mathfrak{r}$-blocks of $\mathcal{H}$ are subsets of $\mathrm{Irr}(K\mathcal{H})$, of which they form a partition. Note that, since $\mathrm{Z}(\mathcal{H}) = R + \mathrm{Ker}(\omega_V^{\mathfrak{r}})$, the central character $\omega_V^{\mathfrak{r}}$ is determined by its kernel. Hence, two simple $K\mathcal{H}$-modules V and V' belong to the same $\mathfrak{r}$-block if and only if $\omega_V^{\mathfrak{r}} = \omega_{V'}^{\mathfrak{r}}$.

D.2.B. Lifting idempotents. — The main result of this section is the following:

Proposition D.2.3. — *We have:*

(a) *If* $e \in \mathrm{Idem}_{\mathrm{pr}}(R_\mathfrak{r}\mathrm{Z}(\mathscr{H}))$*, then* $\hat{e} \in \mathrm{Idem}_{\mathrm{pr}}(k\mathrm{Z}(\mathscr{H}))$.

(b) *The map* $\mathrm{Idem}_{\mathrm{pr}}(R_\mathfrak{r}\mathrm{Z}(\mathscr{H})) \to \mathrm{Idem}_{\mathrm{pr}}(k\mathrm{Z}(\mathscr{H}))$, $e \mapsto \hat{e}$ *is bijective.*

Proof. — Let

$$\begin{array}{rccc} \Omega: & R_\mathfrak{r}\mathrm{Z}(\mathscr{H}) & \longrightarrow & \prod_{V \in \mathrm{Irr}(K\mathscr{H})} R_\mathfrak{r} \\ & z & \longmapsto & (\omega_V(z))_{V \in \mathrm{Irr}(K\mathscr{H})}. \end{array}$$

Then Ω is a morphism of $R_\mathfrak{r}$-algebras, whose kernel I is equal to $R_\mathfrak{r}\mathrm{Z}(\mathscr{H}) \cap \mathrm{Rad}(K\mathscr{H})$ and whose image will be denoted by A.

Consequently, I is nilpotent and so Ω induces a bijective map $\mathrm{Idem}_{\mathrm{pr}}(R_\mathfrak{r}\mathrm{Z}(\mathscr{H})) \stackrel{\sim}{\longleftrightarrow} \mathrm{Idem}_{\mathrm{pr}}(A)$. Moreover, by Corollary D.4.3 (which will be proven in § D.4), the reduction modulo $\mathfrak{r}$ induces a bijective map $\mathrm{Idem}_{\mathrm{pr}}(A) \stackrel{\sim}{\longleftrightarrow} \mathrm{Idem}_{\mathrm{pr}}(kA)$. It then remains to show that the kernel of the natural map $k\mathrm{Z}(\mathscr{H}) \twoheadrightarrow kA$ is nilpotent: it is obvious as it is the image of I in $k\mathrm{Z}(\mathscr{H})$. □

Corollary D.2.4. — *The map* $\mathscr{K}er_\mathfrak{r} : \mathrm{Irr}(K\mathscr{H}) \to \Upsilon^{-1}(\mathfrak{r})$ *is surjective. Its fibers are of the form* $\mathrm{Irr}(K\mathscr{H}e)$*, where* $e \in \mathrm{Idem}_{\mathrm{pr}}(R_\mathfrak{r}\mathrm{Z}(\mathscr{H}))$.

Proof. — The first statement follows from (D.4.4) below and the second from the proof of Proposition D.2.3. □

By combining Propositions D.1.2 and D.2.3, we get the next corollary:

Corollary D.2.5. — *We have:*

(a) *If* $e \in \mathrm{Idem}_{\mathrm{pr}}(R_\mathfrak{r}\mathrm{Z}(\mathscr{H}))$*, then* $\bar{e} \in \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(k\mathscr{H}))$.

(b) *The map* $\mathrm{Idem}_{\mathrm{pr}}(R_\mathfrak{r}\mathrm{Z}(\mathscr{H})) \to \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(k\mathscr{H}))$, $e \mapsto \bar{e}$ *is bijective.*

Therefore, we get a bijective map

$$\Upsilon^{-1}(\mathfrak{r}) \stackrel{\sim}{\longleftrightarrow} \mathrm{Idem}_{\mathrm{pr}}(R_\mathfrak{r}\mathrm{Z}(\mathscr{H})). \tag{D.2.6}$$

If $\mathfrak{z} \in \Upsilon_\mathfrak{r}^{-1}(\mathfrak{r}R_\mathfrak{r})$ and if $e \in \mathrm{Idem}_{\mathrm{pr}}(R_\mathfrak{r}\mathrm{Z}(\mathscr{H}))$, then

(D.2.7) *e and $\mathfrak{z}$ are associated through this bijective map if and only if $e \notin R_\mathfrak{r}\mathfrak{z}$.*

To summarize, we obtain a diagram of natural bijective maps

(D.2.8)

$$\begin{tikzcd}
\Upsilon^{-1}(\mathfrak{r}) \arrow[r, leftrightarrow, "\sim"] \arrow[dr, dashed, leftrightarrow, "\sim"'] & \Upsilon_{\mathfrak{r}}^{-1}(\mathfrak{r}R_{\mathfrak{r}}) \arrow[r, leftrightarrow, "\sim"] \arrow[d, dashed, leftrightarrow, "\wr"] & \operatorname{Spec} k\mathrm{Z}(\mathscr{H}) \arrow[r, leftrightarrow, "\sim"] \arrow[d, leftrightarrow, "\wr"] & \operatorname{Spec} \mathrm{Z}(k\mathscr{H}) \arrow[d, leftrightarrow, "\wr"] \\
 & \mathrm{Idem}_{\mathrm{pr}}(R_{\mathfrak{r}}\mathrm{Z}(\mathscr{H})) \arrow[r, dashed, leftrightarrow, "\sim"] & \mathrm{Idem}_{\mathrm{pr}}(k\mathrm{Z}(\mathscr{H})) \arrow[r, leftrightarrow, "\sim"] & \mathrm{Idem}_{\mathrm{pr}}(\mathrm{Z}(k\mathscr{H})),
\end{tikzcd}$$

where the maps with dashed arrows exist only whenever the K-algebra $K\mathscr{H}$ is split.

Let

$$\begin{array}{ccc}
\Upsilon^{-1}(\mathfrak{r}) & \longrightarrow & \mathrm{Idem}_{\mathrm{pr}}(R_{\mathfrak{r}}\mathrm{Z}(\mathscr{H})) \\
\mathfrak{z} & \longmapsto & e_{\mathfrak{z}}
\end{array}$$

denote the bijective map of diagram D.2.8. We get a partition of $\mathrm{Irr}(K\mathscr{H})$ thanks to the action of the central idempotents $e_{\mathfrak{z}}$:

(D.2.9) $$\mathrm{Irr}(K\mathscr{H}) = \coprod_{\mathfrak{z}\in\Upsilon^{-1}(\mathfrak{r})} \mathrm{Irr}(K\mathscr{H}e_{\mathfrak{z}}).$$

The subsets $\mathrm{Irr}(K\mathscr{H}e_{\mathfrak{z}})$ are the $\mathfrak{r}$-blocks of $\mathscr{H}$.

Example D.2.10. — Whenever $\mathfrak{r}$ is the zero ideal, then $R_{\mathfrak{r}} = k = K$, $\Upsilon^{-1}(\mathfrak{r}) \simeq \operatorname{Spec} K\mathrm{Z}(\mathscr{H})$, $\mathrm{Idem}_{\mathrm{pr}}(R_{\mathfrak{r}}\mathscr{H}) = \mathrm{Idem}_{\mathrm{pr}}(K\mathscr{H})$ and $\omega_V^{\mathfrak{r}} = \omega_V$. ■

D.2.C. Ramification locus. — The following proposition is certainly classical (but is valid because R is integrally closed):

Proposition D.2.11. — *Assume that the algebra $K\mathscr{H}$ is **split**. Then there exists a (unique) radical ideal $\mathfrak{a}$ of R satisfying the following two properties:*

(1) $\operatorname{Spec}(R/\mathfrak{a})$ *is empty or purely of codimension* 1 *in* $\operatorname{Spec}(R)$*;*
(2) *If $\mathfrak{r}$ is a prime ideal of R, then* $\mathrm{Idem}_{\mathrm{pr}}(R_{\mathfrak{r}}\mathrm{Z}(\mathscr{H})) = \mathrm{Idem}_{\mathrm{pr}}(K\mathrm{Z}(\mathscr{H}))$ *if and only if $\mathfrak{a} \not\subset \mathfrak{r}$.*

Assume $\mathfrak{a} \neq R$ and let $\mathfrak{r}$ a prime ideal of R. A subset of $\mathrm{Irr}(K\mathscr{H})$ *is an $\mathfrak{r}$-block if and only if it is minimal for the property of being a $\mathfrak{p}$-block for all height one prime ideals $\mathfrak{p}$ of R with $\mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{r}$.*

Proof. — Let $(b_1, \dots, b_n)$ be an R-basis of $\mathscr{H}$ and let $\mathrm{Idem}_{\mathrm{pr}}(K\mathrm{Z}(\mathscr{H})) = \{e_1, \dots, e_l\}$ with $l = |\mathrm{Idem}_{\mathrm{pr}}(K\mathrm{Z}(\mathscr{H}))|$. We write

$$e_i = \sum_{j=1}^{n} k_{ij}\, b_j$$

with $k_{ij} \in K$.

Let us now fix a prime ideal $\mathfrak{r}$ of R. Then $\mathrm{Idem}_{\mathrm{pr}}(R_{\mathfrak{r}}\mathrm{Z}(\mathscr{H})) = \mathrm{Idem}_{\mathrm{pr}}(K\mathrm{Z}(\mathscr{H}))$ if and only if

$$(*) \qquad \forall\ 1 \leqslant i \leqslant l,\ \forall\ 1 \leqslant j \leqslant n,\ k_{ij} \in R_{\mathfrak{r}}.$$

If $k \in K$, we set $\mathfrak{a}_k = \{r \in R \mid rk \in R\}$. Then $\mathfrak{a}_k$ is an ideal of R and, if $\mathfrak{r}$ is a prime ideal of R, then $k \in R_{\mathfrak{r}}$ if and only if $\mathfrak{a}_k \not\subset \mathfrak{r}$. Define $\mathfrak{a}$ to be the radical of

$$\prod_{\substack{1 \leqslant i \leqslant l \\ 1 \leqslant j \leqslant n}} \mathfrak{a}_{k_{ij}}.$$

Now $(*)$ becomes equivalent to $\mathfrak{a} \not\subset \mathfrak{r}$. This proves the statement (2).

Let us now show that $\mathrm{Spec}(R/\mathfrak{a})$ is empty or purely of codimension 1 in $\mathrm{Spec}(R)$. For this, it is sufficient to prove that $\mathrm{Spec}(R/\mathfrak{a}_k)$ is empty or purely of codimension 1 in $\mathrm{Spec}(R)$. If $k \in R$, then $\mathfrak{a}_k = R$ and $\mathrm{Spec}(R/\mathfrak{a}_k)$ is empty. Assume that $k \notin R$, let us show that $\mathrm{Spec}(R/\mathfrak{a}_k)$ is then purely of codimension 1 in $\mathrm{Spec}(R)$. Let $\mathfrak{p}$ be a minimal prime ideal of R containing $\mathfrak{a}_k$. Then $k \notin R_{\mathfrak{p}}$. We need to prove that $\mathfrak{p}$ has height 1. But, since R is integrally closed, the same holds for $R_{\mathfrak{p}}$: so $R_{\mathfrak{p}}$ is the intersection of the localized rings $R_{\mathfrak{p}'}$, where $\mathfrak{p}'$ runs over the set of prime ideals of height 1 of R contained in $\mathfrak{p}$ (see [**Mat**, Theorem 11.5]). So there exists a prime ideal $\mathfrak{p}'$ of R of height 1 contained in $\mathfrak{p}$ and such that $k \notin R_{\mathfrak{p}'}$. Hence $\mathfrak{a}_k \subset \mathfrak{p}' \subset \mathfrak{p}$ and the minimality of $\mathfrak{p}$ implies that $\mathfrak{p} = \mathfrak{p}'$, which implies that $\mathfrak{p}$ has height 1.

Assume now $\mathfrak{a} \neq R$. Let I be a subset of $\mathrm{Irr}(K\mathscr{H})$ that is a union of $\mathfrak{p}$-blocks for all height one prime ideals $\mathfrak{p}$ of R with $\mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{r}$. There is an idempotent e of $R_{\mathfrak{a}}\mathrm{Z}(\mathscr{H})$ such that given $V \in \mathrm{Irr}(K\mathscr{H})$, we have $eV \neq 0$ if and only if $V \in I$. The discussion above shows that $e \in R_{\mathfrak{p}}\mathrm{Z}(\mathscr{H})$ for all height one prime ideals $\mathfrak{p}$ of R that do not contain $\mathfrak{a}$. So, the coefficients of e in the K-basis $(b_1, \ldots, b_n)$ of $K\mathscr{H}$ are in $\bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all height one prime ideals of R. Since that intersection is R, we deduce that $e \in \mathrm{Z}(\mathscr{H})$. This shows that I is a union of $\mathfrak{r}$-blocks. $\square$

D.3. Decomposition matrices

Let R_1 be a commutative R-algebra and let $\mathfrak{r}_1$ be a prime ideal of R_1. We set $R_2 = R_1/\mathfrak{r}_1$, $K_1 = \mathrm{Frac}(R_1)$ and $K_2 = \mathrm{Frac}(R_2) = k_{R_1}(\mathfrak{r}_1)$. Let $\mathscr{F}(\mathscr{H}, K_1[\mathbf{t}])$ denote the set of maps $\mathscr{H} \to K_1[\mathbf{t}]$. If V is a $K_1\mathscr{H}$-module of finite type and if $h \in \mathscr{H}$, we denote by $\mathrm{Char}_{K_1}^V(h)$ the characteristic polynomial of h for its action on the finite dimensional K_1-vector space V. Therefore, $\mathrm{Char}_{K_1}^V \in \mathscr{F}(\mathscr{H}, K_1[\mathbf{t}])$. Also, $\mathrm{Char}_{K_1}^V$ depends only on the class of V in the Grothendieck group $K_0(K_1\mathscr{H})$. This defines a map

$$\mathrm{Char}_{K_1} : K_0^+(K_1\mathscr{H}) \longrightarrow \mathscr{F}(\mathscr{H}, K_1[\mathbf{t}]),$$

where $K_0^+(K_1\mathscr{H})$ denotes the submonoid of $K_0(K_1\mathscr{H})$ consisting of the isomorphism classes of $K_1\mathscr{H}$-modules of finite type. It is well-known that Char_{K_1} is injective [**GeRo**, proposition 2.5].

We will say that the pair $(R_1, \mathfrak{r}_1)$ satisfies the property $(\mathscr{D}ec)$ if the following three statements are fulfilled:

(D1) R_1 if noetherian and integral.
(D2) If $h \in R_1\mathbf{H}$ and if V is a simple $K_1\mathbf{H}$-module, then $\mathrm{Char}_{K_1}^V(h) \in R_1[\mathbf{t}]$ (note that this property is automatically satisfied if R_1 is integrally closed).
(D3) The algebras $K_1\mathbf{H}$ and $K_2\mathbf{H}$ are split.

Let $\mathrm{red}_{\mathfrak{r}_1} : \mathscr{F}(\mathscr{H}, R_1[\mathbf{t}]) \longrightarrow \mathscr{F}(\mathscr{H}, R_2[\mathbf{t}])$ denote the reduction modulo $\mathfrak{r}_1$. By assumption (D3), if K_2' is an extension of K_2, the scalar extension induces an isomorphism $K_0(K_2\mathscr{H}) \xrightarrow{\sim} K_0(K_2'\mathscr{H})$, and we will identified these two Grothendieck groups.

***Proposition D.3.1* (Geck-Rouquier).** — *If $(R_1, \mathfrak{r}_1)$ satisfies $(\mathscr{D}ec)$, then there exists a unique map* $\mathrm{dec}_{R_2\mathscr{H}}^{R_1\mathscr{H}} : K_0(K_1\mathscr{H}) \longrightarrow K_0(K_2\mathscr{H})$ *which makes the following diagram*

$$\begin{tikzcd} K_0(K_1\mathscr{H}) \arrow[r, "\mathrm{Char}_{K_1}"] \arrow[d, "\mathrm{dec}_{R_2\mathscr{H}}^{R_1\mathscr{H}}"'] & \mathscr{F}(\mathscr{H}, R_1[\mathbf{t}]) \arrow[d, "\mathrm{red}_{\mathfrak{r}_1}"] \\ K_0(K_2\mathscr{H}) \arrow[r, "\mathrm{Char}_{K_2}"] & \mathscr{F}(\mathscr{H}, K_2[\mathbf{t}]) \end{tikzcd}$$

commutative. If $\mathcal{O}_1$ is a subring of K_1 containing R_1, if $\mathfrak{m}_1$ is a prime ideal of $\mathcal{O}_1$ such that $\mathfrak{m}_1 \cap R_1 = \mathfrak{r}_1$, and if $\mathscr{L}$ is an $\mathcal{O}_1\mathscr{H}$-module which is free of finite rank over $\mathcal{O}_1$, then $k_{\mathcal{O}_1}(\mathfrak{m}_1)$ is an extension of K_2 and

$$\mathrm{dec}_{R_1\mathscr{H}}^{R_2\mathscr{H}} [\, K_1\mathscr{L} \,]_{K_1\mathscr{H}} = [\, k_{\mathcal{O}_1}(\mathfrak{m}_1)\mathscr{L} \,]_{k_{\mathcal{O}_1}(\mathfrak{m}_1)\mathscr{H}}.$$

Proof. — This proposition is proven in [**GeRo**, Proposition 2.11] whenever R_1 is integrally closed. We will show how to deduce our proposition from this case. If we assume only that (D2) holds, let R_1' denote the integral closure of R_1 in K_1. Since R_1' is integral over R_1, there exists a prime ideal $\mathfrak{r}_1'$ of R_1' such that $\mathfrak{r}_1' \cap R_1 = \mathfrak{r}_1$. Set $R_2' = R_1'/\mathfrak{r}_1'$. Then $k_{R_1'}(\mathfrak{r}_1')$ is an extension of $k_{R_1}(\mathfrak{r}_1)$ so $k_{R_1'}(\mathfrak{r}_1)\mathscr{H}$ is split, which means that, by [**GeRo**, Proposition 2.11], $\mathrm{dec}_{R_2'\mathscr{H}}^{R_1'\mathscr{H}} : K_0(K_1\mathscr{H}) \longrightarrow K_0(k_{R_1'}(\mathfrak{r}_1')\mathscr{H})$ is well-defined and satisfies the desired properties. We then define $\mathrm{dec}_{R_2\mathscr{H}}^{R_1\mathscr{H}}$ by using the isomorphism $K_0(k_{R_1'}(\mathfrak{r}_1')\mathscr{H}) \simeq K_0(K_2\mathscr{H})$ and it is easy to check that this map satisfies the expected properties. $\square$

It then follows a transitivity property [**GeRo**, proposition 2.12]:

***Corollary D.3.2* (Geck-Rouquier)**. — *Let R_1 be an R-algebra, let $\mathfrak{r}_1$ be a prime ideal of R_1 and let $\mathfrak{r}_2$ be a prime ideal of $R_2 = R_1/\mathfrak{r}_1$. We assume that $(R_1, \mathfrak{r}_1)$ and $(R_2, \mathfrak{r}_2)$ both satisfy $(\mathscr{D}ec)$ and we set $R_3 = R_2/\mathfrak{r}_2$. Then*

$$\operatorname{dec}_{R_3\mathscr{H}}^{R_1\mathscr{H}} = \operatorname{dec}_{R_3\mathscr{H}}^{R_2\mathscr{H}} \circ \operatorname{dec}_{R_2\mathscr{H}}^{R_1\mathscr{H}}.$$

D.4. Idempotents and central characters

The aim of this section is to complete the proof of Corollary D.2.4. Let $\mathscr{O}$ be a local noetherian ring and let A be a $\mathscr{O}$-subalgebra of $\mathscr{O}^d = \mathscr{O} \times \mathscr{O} \times \cdots \times \mathscr{O}$ (d times). Let $\mathfrak{m} = \operatorname{Rad}(\mathscr{O})$, $k = \mathscr{O}/\mathfrak{m}$ and, if $r \in \mathscr{O}$, let $\bar{r}$ denote its image in k.

If $1 \leqslant i \leqslant d$, let $\pi_i : \mathscr{O}^d \to \mathscr{O}$ denote the i-th projection and

$$\omega_i : A \longrightarrow \mathscr{O}$$

denotes the restriction of π_i to A. We set

$$\begin{array}{rccc} \bar{\omega}_i : & A & \longrightarrow & k \\ & a & \longmapsto & \overline{\omega_i(a)}. \end{array}$$

On the set $\{1, 2, \ldots, d\}$, we denote by $\smile$ the equivalence relation defined by

$$i \smile j \quad \text{if and only if} \quad \bar{\omega}_i = \bar{\omega}_j.$$

Finally, we set

$$e_i = (0, \ldots, 0, \underbrace{1}_{i\text{-th position}}, 0, \ldots, 0) \in \mathscr{O}^d.$$

Then:

Lemma D.4.1. — *Let $I \in \{1, 2, \ldots, d\}/\smile$. Then $\sum_{i \in I} e_i \in A$.*

Proof. — By reordering if necessary the idempotents, we may assume that $I = \{1, 2, \ldots, d'\}$ with $d' \leqslant d$. We proceed in several steps:

(♣) *If $i \in I$ and $j \notin I$, then there exists $a_{ij} \in A$ such that $\omega_i(a_{ij}) = 1$ and $\omega_j(a_{ij}) = 0$.*

> *Proof of* (♣). Since $i \not\smile j$, there exists $a \in A$ such that $\bar{\omega}_i(a) \neq \bar{\omega}_j(a)$. Let $r = \omega_j(a)$ and $u = \omega_i(a) - \omega_j(a)$. Then $u \in \mathscr{O}^\times$ because $\mathscr{O}$ is local and $a_{ij} = u^{-1}(a - r \cdot 1_A) \in A$ satisfies the two conditions. ■

($\diamondsuit$) *There exists $a_1 \in A$ such that $\omega_1(a_1) = 1$ and $\omega_j(a_1) = 0$ if $j \notin I$.*

Proof of ($\diamondsuit$). By ($\clubsuit$), there exists, for all $i \in I$ and $j \notin I$, $a_{ij} \in A$ such that $\omega_i(a_{ij}) = 1$ and $\omega_j(a_{ij}) = 0$. Note that, if $i' \in I$, then $\omega_{i'}(a_{ij}) \equiv 1 \mod \mathfrak{m}$ because $\bar{\omega}_i = \bar{\omega}_{i'}$. Set $a = \prod_{i\in I, j\notin I} a_{ij}$. Then it is clear that $\omega_j(a) = 0$ if $j \notin I$ and $\omega_i(a) \equiv 1 \mod \mathfrak{m}$ if $i \in I$. It is then sufficient to take $a_1 = \omega_1(a)^{-1}a$. ■

We then define by induction the sequence $(a_i)_{1 \leqslant i \leqslant d'}$ as follows:

$$a_{i+1} = a_i^2(1 + \omega_{i+1}(a_i)^{-2}(1 - a_i^2)).$$

We will show by induction on $i \in \{1, 2, \dots, d'\}$ the following two facts:

($\heartsuit_i$) *The element a_i is well-defined and belongs to A.*

($\spadesuit_i$) *If $1 \leqslant i' \leqslant i$ and $j \notin I$, then $\omega_{i'}(a_i) = 1$ and $\omega_j(a_i) = 0$.*

Proof of ($\heartsuit_i$) *and* ($\spadesuit_i$). This is obvious if $i = 1$. Let us now assume that ($\heartsuit_i$) and ($\spadesuit_i$) hold (for some $i \leqslant d' - 1$). Let us prove that this implies that ($\heartsuit_{i+1}$) and ($\spadesuit_{i+1}$) also hold.

Then $i \smile i+1$ and so $\omega_{i+1}(a_i) \equiv \omega_i(a_i) = 1 \mod \mathfrak{m}$. So $\omega_{i+1}(a_i)$ is invertible and so a_{i+1} is well-defined and belongs to A (this is exactly ($\heartsuit_{i+1}$)).

Now, let us set $r = \omega_{i+1}(a_i)$ for simplifying. Then:

- If $1 \leqslant i' \leqslant i$, we have $\omega_{i'}(a_{i+1}) = 1 \cdot (1 + r^{-2}(1 - 1^2)) = 1$.
- $\omega_{i+1}(a_{i+1}) = r^2(1 + r^{-2}(1 - r^2)) = 1$.
- If $j \notin I$, then $\omega_j(a_{i+1}) = 0 \cdot (1 + r^{-2}(1 - 0^2)) = 0$.

So ($\spadesuit_{i+1}$) holds. ■

Therefore, $a_{d'} = \sum_{i\in I} e_i \in A$. □

Corollary D.4.2. — *The map*

$$\begin{array}{ccc} \{1, 2, \dots, d\}/\smile & \longrightarrow & \mathrm{Idem}_{\mathrm{pr}}(A) \\ I & \longmapsto & \displaystyle\sum_{i\in I} e_i \end{array}$$

is well-defined and bijective.

Proof. — The Lemma D.4.1 shows that, if $I \in \{1,2,\ldots,d\}/\smile$, then $e_I = \sum_{i \in I} e_i \in A$. If e_I is not primitive, this means that, since $\mathcal{O}$ is local, there exists two non-empty subsets I_1 and I_2 of I such that e_{I_1}, $e_{I_2} \in A$, and $I = I_1 \coprod I_2$. But, if $i_1 \in I_1$ and $i_2 \in I_2$, then $\bar{\omega}_{i_1}(e_{I_1}) = 1 \neq 0 = \bar{\omega}_{i_2}(e_{I_1})$, which is impossible because $i_1 \smile i_2$. So the map $I \mapsto e_I$ is well-defined. It is now clear that it is bijective. $\square$

If $a \in A$, let $\hat{a}$ denote its image in $kA = k \otimes_{\mathcal{O}} A$.

Corollary D.4.3. — *With this notation, we have:*

(a) *If* $e \in \mathrm{Idem}_{\mathrm{pr}}(A)$, *then* $\hat{e} \in \mathrm{Idem}_{\mathrm{pr}}(kA)$.

(b) *The map* $\mathrm{Idem}_{\mathrm{pr}}(A) \longrightarrow \mathrm{Idem}_{\mathrm{pr}}(kA)$, $e \mapsto \hat{e}$ *is bijective.*

Proof. — (a) Let $e \in \mathrm{Idem}_{\mathrm{pr}}(A)$ and assume that $\hat{e} = e_1 + e_2$, where e_1 and e_2 are two orthogonal idempotents of kA. The ring $\mathcal{O}$ being noetherian, kA is a finite dimensional commutative k-algebra. So there exists two morphisms of k-algebras ρ_1, $\rho_2 : kA \longrightarrow k'$ (where k' is a finite extension of k) such that $\rho_i(e_j) = \delta_{i,j}$. Let $\tilde{\rho}_i$ denote the composition $A \longrightarrow kA \xrightarrow{\rho_i} k'$.

Set $\mathfrak{a}_i = \mathrm{Ker}(\tilde{\rho}_i)$. The image of ρ_i being a subfield k', $\mathfrak{a}_i$ is a maximal ideal of A. Since $\mathcal{O}^d$ is integral over A, there exists a maximal ideal $\mathfrak{m}_i$ of $\mathcal{O}^d$ such that $\mathfrak{a}_i = \mathfrak{m}_i \cap A$. Since $\mathcal{O}$ is local, $\mathfrak{m}_i$ is of the form $\mathcal{O} \times \cdots \times \mathcal{O} \times \mathfrak{m} \times \mathcal{O} \times \cdots \times \mathcal{O}$, where $\mathfrak{m}$ is in t_i-th position (for some $t_i \in \{1,2,\ldots,d\}$), which implies that $\tilde{\rho}_i = \bar{\omega}_{t_i}$.

Since $\rho_1 \neq \rho_2$ and $\rho_i(e_j) = \delta_{i,j}$, we get $\bar{\omega}_{t_1} \neq \bar{\omega}_{t_2}$ and $\bar{\omega}_{t_1}(e) = \rho_1(e_1 + e_2) = 1 = \rho_2(e_1 + e_2) = \bar{\omega}_{t_2}(e)$. This contradicts Corollary D.4.2. $\square$

During this proof, the following result has been proven: if k' is a finite extension of k and if $\rho : kA \longrightarrow k'$ is a morphism of k-algebras, then

(D.4.4) *there exists* $i \in \{1,2,\ldots,d\}$ *such that* $\rho(\hat{a}) = \bar{\omega}_i(a)$ *for all* $a \in A$.

APPENDIX E

INVARIANT RINGS

Let $\mathbf{k}$ be a field. Let A be a $\mathbf{k}$-algebra acted on by a finite group G whose order is invertible in $\mathbf{k}$. Let $e = \frac{1}{|G|}\sum_{g\in G} g$, a central idempotent of $\mathbf{k}[G]$. Let $R = A \rtimes G$. The aim of this appendix is to relate the representation theory of R and that of A^G. We are mainly interested in the case where A is the algebra of regular functions on an affine scheme.

E.1. Morita equivalence

The following lemma is clear.

Lemma E.1.1. — *There is an isomorphism of R-modules $A \xrightarrow{\sim} Re$, $a \mapsto ae$ that restricts to an isomorphism of* $\mathbf{k}$*-algebras $A^G \xrightarrow{\sim} eRe$.*

Let M be an A-module whose isomorphism class is stable under the action of a subgroup H of G. There are isomorphisms of A-modules $\phi_h : h^*(M) \xrightarrow{\sim} M$ for $h \in H$, unique up to left multiplication by $\mathrm{Aut}_A(M)$. Consequently, the elements $\phi_h \in N_{\mathrm{Aut}_{A^G}(M)}(\mathrm{Aut}_A(M))$ define a morphism of groups $H \to \mathrm{Aut}_{A^G}(M)/\mathrm{Aut}_A(M)$.

Proposition E.1.2. — *The following assertions are equivalent:*

(1) *Re is a progenerator for R*
(2) *Re induces a Morita equivalence between R and A^G*
(3) *$R = ReR$*
(4) *for every simple R-module S, we have $S^G \neq 0$.*
(5) *for every simple A-module T whose isomorphism class is stable under the action of a subgroup H of G and for every non-zero direct summand U of* $\mathrm{Ind}_A^{A\rtimes H} T$, *we have $U^H \neq 0$.*

Proof. — Note that Re is a direct summand of R, as a left R-module, hence Re is a finitely generated projective R-module. The equivalence between (1) and (2) follows from Lemma E.1.1. If Re is a progenerator, then R is isomorphic to a quotient of a multiple of Re. Since the image of a morphism $Re \to R$ is contained in ReR, we deduce that if (1) holds, then (3) holds. Conversely, assume (3). There are $r_1, \ldots, r_n \in R$ such that $1 \in Rer_1 + \cdots + Rer_n$, hence the morphism $(Re)^n \to R$, $(a_1, \ldots, a_n) \mapsto a_1 r_1 + \cdots + a_n r_n$ is surjective and (1) follows.

We have $R/ReR = 0$ if and only if R/ReR has no simple module, hence if and only if e does not act by 0 on any simple R-module. This shows the equivalence of (3) and (4).

Let S be a simple R-module. There is a simple A-module T such that S is a direct summand of $\mathrm{Ind}_A^R(T)$. Let H be the stabilizer of the isomorphism class of T. There is a simple $(A \rtimes H)$-module U such that S is a direct summand of $\mathrm{Ind}_{A\rtimes H}^R(U)$. We have $S^G \neq 0$ if and only if $U^H \neq 0$. This shows the equivalence of (4) and (5). ☐

Corollary E.1.3. — *If $ReR = R$, then $\mathrm{Z}(R) = \mathrm{Z}(A^G)$.*

Proof. — By Proposition E.1.2, the rings R and $eRe \simeq A^G$ are Morita equivalent thanks to the bimodule Re, so $\mathrm{Z}(R) \simeq \mathrm{Z}(A^G)$, the isomorphism being determined by the action on the bimodule Re (Lemma E.1.4 below). The result follows. ☐

Lemma E.1.4. — *Let A and B be two rings and M an (A, B)-bimodule such that the canonical maps give isomorphisms $B \xrightarrow{\sim} \mathrm{End}_A(M)$ and $A \xrightarrow{\sim} \mathrm{End}_{B^{\mathrm{opp}}}(M)^{\mathrm{opp}}$. Then we have an isomorphism $\mathrm{Z}(A) \xrightarrow{\sim} \mathrm{Z}(B)$.*

In particular, if e is an idempotent of a ring A and if left multiplication gives an isomorphism $A \xrightarrow{\sim} \mathrm{End}_{(eAe)^{\mathrm{opp}}}(Ae)^{\mathrm{opp}}$, then there is an isomorphism $\mathrm{Z}(A) \xrightarrow{\sim} \mathrm{Z}(eAe)$, $a \mapsto ae$.

Proof. — The left multiplication on M induces induces a ring morphism $\alpha : \mathrm{Z}(A) \to \mathrm{Z}(B)$ such that $zm = m\alpha(z)$ for all $z \in \mathrm{Z}(A)$ and $m \in M$. Similarly, the right multiplication induces a ring morphism $\beta : \mathrm{Z}(B) \to \mathrm{Z}(A)$ such that $mz = \beta(z)m$ for all $z \in \mathrm{Z}(B)$ and $m \in M$. Hence, if $z \in \mathrm{Z}(A)$ and $m \in M$, then $zm = \beta(\alpha(z))m$, and so $\beta \circ \alpha = \mathrm{Id}_{\mathrm{Z}(A)}$ since the action of A on M is faithful by assumption. Similarly $\alpha \circ \beta = \mathrm{Id}_{\mathrm{Z}(B)}$. ☐

E.2. Geometric setting

We assume now that $A = \mathbf{k}[X]$, where X is an affine scheme of finite type over $\mathbf{k}$, i.e., A is a finitely generated commutative $\mathbf{k}$-algebra. Then Proposition E.1.2 has the following consequence.

Corollary E.2.1. — *If G acts freely on X, then Re induces a Morita equivalence between R and A^G.*

Let $X^{\mathrm{reg}} = \{x \in X | \mathrm{Stab}_G(x) = 1\}$ and let $R^{\mathrm{reg}} = \mathbf{k}[X^{\mathrm{reg}}] \rtimes G$. We assume X^{reg} is dense in X, i.e., the pointwise stabilizer of an irreducible component of X is trivial. The following proposition gives a sufficient condition for a double centralizer theorem.

Proposition E.2.2. — *Assume that X is a normal variety, i.e., all localizations of A at prime ideals are integral and integrally closed.*

(1) *The canonical morphism of algebras $R \to \mathrm{End}_{(A^G)^{\mathrm{opp}}}(A)^{\mathrm{opp}}$ is injective.*
(2) *If the codimension of $X \setminus X^{\mathrm{reg}}$ is $\geqslant 2$ in each connected component of X, then the morphism above is an isomorphism and $Z(R) = Z(A^G)$.*

Proof. — It follows from Corollary E.2.1 that given $f \in A^G$ such that $D(f) \subset X^{\mathrm{reg}}$, then the canonical morphism $A[f^{-1}] \rtimes G \to \mathrm{End}_{A^G[f^{-1}]}(A[f^{-1}])^{\mathrm{opp}}$ is an isomorphism. In particular, the morphism of the proposition $R \to \mathrm{End}_{(A^G)^{\mathrm{opp}}}(A)^{\mathrm{opp}}$ is an injective morphism of A-modules, since X^{reg} is dense in X.

Let K be the cokernel of the canonical morphism $R \to \mathrm{End}_{(A^G)^{\mathrm{opp}}}(A)^{\mathrm{opp}}$. We have $K \otimes_A Ak[X^{\mathrm{reg}}] = 0$, hence the support of K has codimension $\geqslant 2$. Since A is normal, it has depth $\geqslant 2$, hence $\mathrm{Ext}^1_A(K, A) = 0$. We deduce that K is a direct summand of the torsion free A-module $\mathrm{End}_{(A^G)^{\mathrm{opp}}}(A)^{\mathrm{opp}}$, hence $K = 0$. The last statement follows from Lemma E.1.4. ☐

The statement on the center of R can be obtained more directly.

Lemma E.2.3. — *If A is an integral domain and G acts faithfully on X, then $Z(R) = A^G$.*

Proof. — Let $a = \sum_{g \in G} a_g g \in Z(R)$ with $a_g \in A$ for $g \in G$. Given g_0 a non-trivial element of G, there is $x \in A$ such that $g_0 x g_0^{-1} \neq x$. We have

$$0 = [x, a] = \sum_{g \in G} (x - g x g^{-1}) a_g g,$$

hence $(x - g_0 x g_0^{-1}) a_{g_0} = 0$. Since A is integral, it follows that $a_{g_0} = 0$. We have shown that $a = a_1 \in A$. Since $[a, g] = 0$ for all $g \in G$, we deduce that $a \in A^G$. ☐

We conclude by a description of the simple R-modules whenever $R = ReR$. In this case, using the Morita equivalence between R and A^G induced by the bimodule Re, we obtain a bijective map

(E.2.4)
$$\begin{array}{ccc} \mathrm{Irr}(R) & \xrightarrow{\sim} & \mathrm{Irr}(A^G) \\ S & \longmapsto & eS. \end{array}$$

Since A is commutative, $\mathrm{Irr}(A)$ (respectively $\mathrm{Irr}(A^G)$) is in one-to-one correspondence with the maximal ideals of A (respectively of A^G), so we obtain a bijective map

$$\text{(E.2.5)} \qquad \mathrm{Irr}(A)/G \xleftrightarrow{\sim} \mathrm{Irr}(A^G)$$

(see Propositions B.2.1 and B.2.2). By composing the two previous bijective maps, we obtain a third bijective map

$$\text{(E.2.6)} \qquad \mathrm{Irr}(A)/G \xleftrightarrow{\sim} \mathrm{Irr}(R)$$

We will describe more concretely this last map. In order to do that, let Ω be a G-orbit of (isomorphism classes of) simple A-modules. The A-module $S_\Omega = A/\cap_{T\in\Omega}\mathrm{Ann}_A(T)$ inherits an action of G, hence it becomes an R-module.

Proposition E.2.7. — *Assume that $R = ReR$ and that A is commutative and finitely generated. Then:*

(a) *If $\Omega \in \mathrm{Irr}(A)/G$, then S_Ω is a simple R-module.*
(b) *The map $\mathrm{Irr}(A)/G \longrightarrow \mathrm{Irr}(R)$, $\Omega \mapsto S_\Omega$ is bijective (and coincides with the bijective map E.2.5).*
(c) *If S is a simple A-module, then $\mathrm{Res}_A^R(S)$ is semisimple and multiplicity-free, and two simple A-modules occurring in $\mathrm{Res}_A^R(S)$ are in the same G-orbit.*
(d) *If S and S' are two simple R-modules, then $S \simeq S'$ if and only if $\mathrm{Res}_A^R(S)$ and $\mathrm{Res}_A^R(S')$ have a common irreducible submodule.*

Proof. — (a) By construction, we have a well-defined injective morphism of A-modules $S_\Omega \hookrightarrow \bigoplus_{T\in\Omega} T$ (here, we identify T and $A/\mathrm{Ann}_A(T)$. So, if S is a non-zero R-submodule of S_Ω, then it is a non-zero A-submodule of S_Ω. Therefore, S contains some submodule isomorphic to $T \in \Omega$. Since the action of G stabilizes S, it follows that $S = S_\Omega$ and that

$$(*) \qquad \mathrm{Res}_A^R(S_\Omega) = \bigoplus_{T\in\Omega} T.$$

This proves (a).

(b) It follows from $(*)$ that the map $\mathrm{Irr}(A)/G \longrightarrow \mathrm{Irr}(R)$, $\Omega \mapsto S_\Omega$ is injective. Now, let $\Omega \in \mathrm{Irr}(A)/G$, let $T \in \Omega$ and let $\mathfrak{m} = \mathrm{Ann}_A(T)$. We denote by H the stabilizer of $\mathfrak{m}$ in G (that is, the decomposition group of $\mathfrak{m}$). Then $eS = S_\Omega^G \simeq T^H = (A/\mathfrak{m})^H$. But, by Theorem B.2.4, $(A/\mathfrak{m})^H = A^G/(\mathfrak{m} \cap A^G)$. This proves that eS_Ω is the simple A^G-module associated with the maximal ideal $\mathfrak{m} \cap A^G$ of A^G or, in other words, is the simple A^G-module associated with Ω through the bijective map E.2.5. This completes the proof of (b).

(c) and (d) now follow from (a), (b) and $(*)$. $\square$

APPENDIX F

HIGHEST WEIGHT CATEGORIES

We fix in Appendix F a commutative noetherian ring k.

F.1. General theory

F.1.A. Definitions and first properties. — We say that a poset Δ is *locally finite* if given any $D, D' \in \Delta$, then there are only finitely many $D'' \in \Delta$ such that $D < D'' < D'$. We say that a subset Γ of Δ is an *ideal* if given $D \in \Delta$ and $D' \in \Gamma$ with $D < D'$, then $D \in \Gamma$. Given $D \in \Delta$, we define $\Delta_{\leqslant D} = \{D' \in \Delta \mid D' \leqslant D\}$, and we define similarly $\Delta_{<D}$, $\Delta_{\geqslant D}$ and $\Delta_{>D}$. We say that an ideal Γ is *finitely generated* if there are $D_1, \ldots, D_n \in \Delta$ such that $\Gamma = \Delta_{\leqslant D_1} \cup \cdots \cup \Delta_{\leqslant D_n}$.

Let $\mathscr{C}$ be a k-linear abelian category. We assume that given $M \in \mathscr{C}$ and given I a family of subobjects of M, then there exists a subobject $\sum_{N \in I} N$ of M. This is a subobject L of M containing all subobjects in I and such that given any map $f : M \to M'$ and any subobject M'' of M' such that $f(N) \subset M''$ for all $N \in I$, then $f(L) \subset M''$.

Let Δ be a family of isomorphism classes of objects of $\mathscr{C}$ (the *standard objects*). We assume Δ is endowed with a locally finite poset structure.

Given $\Gamma \subset \Delta$, we denote by

- $\mathscr{C}[\Gamma]$ the full subcategory of $\mathscr{C}$ of objects M such that $\mathrm{Hom}(D, M) = 0$ for all $D \in \Delta \setminus \Gamma$
- $\mathscr{C}^\Gamma$ the full subcategory of $\mathscr{C}$ of objects M that have a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that $M_i/M_{i-1} \in \Gamma$ for $1 \leqslant i \leqslant r$
- $i(\mathscr{C}^\Gamma)$ the full subcategory of $\mathscr{C}$ of objects that are direct summands of objects of $\mathscr{C}^\Gamma$.

We extend the definition of (split) highest weight categories over k of **[Rou]** to the case of a non-necessarily finite Δ.

Definition F.1.1. — *We say that $\mathscr{C}$, endowed with the poset of standard objects Δ, is a highest weight category if*

(i) *for all $D \in \Delta$, we have $\operatorname{End}(D) = k$*
(ii) *given $D_1, D_2 \in \Delta$ such that $\operatorname{Hom}(D_1, D_2) \neq 0$, we have $D_1 \leqslant D_2$*
(iii) *every object of $\mathscr{C}$ is the quotient of an object of $\mathscr{C}^\Delta$*
(iv) *for all $M \in \mathscr{C}$, $D, D' \in \Delta$, there is a surjection $R \twoheadrightarrow D$ with kernel in $\mathscr{C}^{\Delta_{>D}}$ such that $\operatorname{Hom}(R, D')$ is a finitely generated projective k-module and $\operatorname{Ext}^1(R, M) = 0$.*

Remark F.1.2. — Assumption (iv) is aimed at making sense of the existence of an approximation of a projective module, and of the requirement that objects of Δ are finitely generated and projective over k. ∎

We assume from now on that $\mathscr{C}$ is a highest weight category.

Note first that Definition F.1.1 (iv) admits a version where D is replaced by an arbitrary object of $\mathscr{C}^\Delta$.

Lemma F.1.3. — *Let $N \in \mathscr{C}^\Delta$, $D' \in \Delta$ and $M \in \mathscr{C}$. Then, there exists a surjection $R \twoheadrightarrow N$ with kernel in $\mathscr{C}^\Delta$ such that $\operatorname{Hom}(R, D')$ is a finitely generated projective k-module and $\operatorname{Ext}^1(R, M) = 0$.*

Proof. — Fix a filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_r = N$ such that $N_i/N_{i-1} \in \Delta$ for $1 \leqslant i \leqslant r$. Given i, there exists a surjection $f_i : R_i \twoheadrightarrow N_i/N_{i-1}$ such that $\ker f_i \in \mathscr{C}^\Delta$, $\operatorname{Hom}(R_i, D')$ is a finitely generated projective k-module and $\operatorname{Ext}^1(R_i, N_{i-1} \oplus M) = 0$. So, f_i lifts to a map $g_i : R_i \to N_i$ and the sum $g = \sum_i g_i : \bigoplus_i R_i \to N$ is surjective. Furthermore, $L = \ker g$ has a filtration $0 = L_0 \subset L_1 \subset \cdots \subset L_r = L$ such that $L_i/L_{i-1} \simeq \ker f_i$ for $1 \leqslant i \leqslant r$. It follows that $L \in \mathscr{C}^\Delta$. Note finally that $\operatorname{Ext}^1(\bigoplus_i R_i, M) = 0$ and $\operatorname{Hom}(\bigoplus_i R_i, D')$ is a finitely generated projective k-module. □

Lemma F.1.4. — *Let $0 \to M \to L \to N \to 0$ be an exact sequence in $\mathscr{C}$ with $L, N \in i(\mathscr{C}^\Delta)$. Then $M \in i(\mathscr{C}^\Delta)$.*

Proof. — It is enough to prove the lemma for $L, N \in \mathscr{C}^\Delta$. By Lemma F.1.3, there is a surjection $R \twoheadrightarrow N$ with kernel $N' \in \mathscr{C}^\Delta$ and with $\operatorname{Ext}^1(R, M) = 0$. Let L' be the kernel of the canonical map $L \oplus R \twoheadrightarrow N$. The kernel of the canonical map $L' \twoheadrightarrow L$ is isomorphic to N', hence $L' \in \mathscr{C}^\Delta$. The kernel of the canonical map $L' \twoheadrightarrow R$ is isomorphic to M. Since $\operatorname{Ext}^1(R, M) = 0$, it follows that M is a direct summand of L', hence $M \in i(\mathscr{C}^\Delta)$. □

Lemma F.1.5. — *Let Γ be an ideal of $\mathscr{C}$ and let $M \in \mathscr{C}[\Gamma]$. Let $D \in \Delta$ and let $i \geqslant 0$. If $\operatorname{Ext}^i(D, M) \neq 0$, then there exists $D_0, \ldots, D_i \in \Gamma$ with $D = D_0 < D_1 < \cdots < D_i$.*

Proof. — When $i = 0$, there is $D' \in \Gamma$ such that $\operatorname{Hom}(D, D') \neq 0$, hence $D \leqslant D'$, so $D \in \Gamma$. We proceed now by induction on $i \geqslant 1$.

There exists $M' \in \mathscr{C}$, $\zeta \in \mathrm{Ext}^1(D, M')$ and $\xi \in \mathrm{Ext}^{i-1}(M', M)$ such that $\xi \circ \zeta \neq 0$. There exists a surjection $R \twoheadrightarrow D$ with kernel L in $\mathscr{C}^{\Delta_{>D}}$ such that $\mathrm{Ext}^1(R, M') = 0$. So, ζ factors through a map $f : L \to M'$. Since $\xi \circ \zeta \neq 0$, it follows that $\xi \circ f \neq 0$. So, there exists $D' > D$ such that $\mathrm{Ext}^{i-1}(D', M) \neq 0$. By induction, we deduce that there exists $D'_0, \ldots, D'_{i-1} \in \Gamma$ such that $D' = D'_0 < D'_1 < \cdots < D'_{i-1}$. It follows that the lemma holds for D, M and i. □

Lemma F.1.6. — *Let $D, D' \in \Delta$. The k-module $\mathrm{Ext}^1(D, D')$ is finitely generated. If it is non-zero, then $D < D'$.*

Proof. — Note first that by Definition F.1.1 (iv), the k-module $\mathrm{Hom}(D, D')$ is finitely generated for all $D, D' \in \Delta$. Consequently, $\mathrm{Hom}'s$ in $\mathscr{C}^{\Delta}$ are finitely generated k-modules. Fix a surjection $R \twoheadrightarrow D$ with kernel $L \in \mathscr{C}^{\Delta_{>D}}$ such that $\mathrm{Ext}^1(R, D') = 0$. Since $\mathrm{Hom}(L, D')$ is a finitely generated k-module, we deduce that $\mathrm{Ext}^1(D, D')$ is a finitely generated k-module. If $\mathrm{Ext}^1(D, D') \neq 0$, then $\mathrm{Hom}(L, D') \neq 0$, hence there is $D'' > D$ such that $D'' \leqslant D'$. So, $D < D'$. □

Lemma F.1.7. — *Let I be a finite subset of Δ. Fix a total order $\prec$ on I such that $i < j$ implies $i \prec j$.*

Let $M \in \mathscr{C}$ with a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that $M_i/M_{i-1} \in I$ for $1 \leqslant i \leqslant r$. Then, M has another filtration $0 = M'_0 \subset M'_1 \subset \cdots \subset M'_r = M$ such that $M'_i/M'_{i-1} \in I$ for $1 \leqslant i \leqslant r$ and $M'_i/M'_{i-1} \preceq M'_{i+1}/M'_i$ for $1 \leqslant i < r$.

Proof. — We prove the result by induction on r. Take $D \in I$ maximal for $\prec$ such that $D \simeq M_i/M_{i-1}$ for some i, and consider i minimal with this property. We have $D \not\leq M_j/M_{j-1}$ for $j < i$, hence $\mathrm{Ext}^1(D, M_j/M_{j-1}) = 0$ for $j < i$ by Lemma F.1.6. It follows that $\mathrm{Ext}^1(D, M_{i-1}) = 0$, hence M_i has a subobject L isomorphic to D such that $M_i = L \oplus M_{i-1}$. The object M/L has a filtration with $(M/L)_j = M_j$ for $j < i$ and $(M/L)_j = M_{j+1}/L$ for $j \geqslant i$. That filtration has length $r-1$, and the subquotients are in I. By induction, $N = M/L$ has another filtration $0 = N'_0 \subset \cdots \subset N'_{r-1} = N$ with $N'_i/N'_{i-1} \in I$ for $1 \leqslant i < r$ and $N'_i/N'_{i-1} \preceq N'_{i+1}/N'_i$ for $1 \leqslant i < r-1$. We obtain an appropriate filtration of M by taking $M'_1 = L$ and M'_i the inverse image of N'_{i-1} for $i > 1$. □

We define a partial order $\lessdot$ on Δ as the one generated by $D \lessdot D'$ if $\mathrm{Ext}^i(D, D') \neq 0$ for some $i \in \{0, 1\}$.

The following proposition shows that $\lessdot$ is the coarsest order on Δ that makes $(\mathscr{C}, \Delta)$ into a highest weight category.

Proposition F.1.8. — *The partial order $\lessdot$ is coarser than $<$. The category $\mathscr{C}$ equipped with the poset $(\Delta, \lessdot)$ is a highest weight category.*

Proof. — The first statement follows from Lemma F.1.5.

Consider $M \in \mathscr{C}$ and $D, D' \in \Delta$. There is a surjection $R \twoheadrightarrow D$ with kernel $N \in \mathscr{C}^{\Delta_{>D}}$ and such that $\mathrm{Hom}(R, D')$ is a finitely generated projective k-module and $\mathrm{Ext}^1(R, M) = 0$. By Lemma F.1.7, there is a filtration $0 = R_0 \subset R_1 \subset \cdots \subset R_{r-2} \subset R_{r-1} = N \subset R_r = R$ with subquotients in Δ and there is an integer $i \in \{1, \ldots, r-1\}$ such that given $j \in \{1, \ldots, r-1\}$, we have $D \lessdot (R_j/R_{j-1})$ if and only if $j > i$.

Consider $j > i$ and $l \in \{1, \ldots, r-1\}$ such that $\mathrm{Ext}^1(R_j/R_{j-1}, R_l/R_{l-1}) \neq 0$. We have $(R_j/R_{j-1}) \lessdot (R_l/R_{l-1}) \neq 0$, hence $D \lessdot (R_l/R_{l-1})$. It follows that $l > i$. We deduce that $\mathrm{Ext}^1(R/R_i, R_i) = 0$, hence $R \simeq R/R_i \oplus R_i$. The k-module $\mathrm{Hom}(R/R_i, D')$ is finitely generated and projective and $\mathrm{Ext}^1(R/R_i, M) = 0$. So (iv) holds for $(\mathscr{C}, \Delta, \lessdot)$. The conditions (i)–(iii) are clear. This completes the proof of the proposition. ☐

F.1.B. Ideals and Serre subcategories. —

Proposition F.1.9. — *Let Γ be an ideal of Δ.*

(i) *Every object of $\mathscr{C}^{\Delta}$ has a subobject in $\mathscr{C}^{\Delta\setminus\Gamma}$ whose quotient is in $\mathscr{C}^{\Gamma}$.*

(ii) *An object of $\mathscr{C}$ is in $\mathscr{C}[\Gamma]$ if and only if it is a quotient of an object of $\mathscr{C}^{\Gamma}$.*

(iii) *$\mathscr{C}[\Gamma]$ is a Serre subcategory of $\mathscr{C}$. It is a highest weight category with poset of standard objects Γ.*

(iv) *The inclusion functor $\mathscr{C}[\Gamma] \hookrightarrow \mathscr{C}$ has a left (resp. right) adjoint sending an object $M \in \mathscr{C}$ to its largest quotient (resp. subobject) in $\mathscr{C}[\Gamma]$.*

Proof. — Statement (i) follows immediately from Lemma F.1.7.

Let $f : P \twoheadrightarrow M$ be a surjection with $P \in \mathscr{C}^{\Gamma}$ and $M \in \mathscr{C}$. Let $D \in \Delta \setminus \Gamma$. There exists a surjection $g : R \twoheadrightarrow D$ with kernel in $\mathscr{C}^{\Delta_{>D}}$ such that $\mathrm{Ext}^1(R, \ker f) = 0$.

Consider now $h : D \to M$ and let $h' = h \circ g : R \to M$. The map h' factors through a map $h'' : R \to P$. Since $R \in \mathscr{C}^{\Delta\setminus\Gamma}$ and $P \in \mathscr{C}^{\Gamma}$, we deduce that $h'' = 0$, hence $h = 0$. So, $M \in \mathscr{C}[\Gamma]$. We have shown that every quotient of an object of $\mathscr{C}^{\Gamma}$ is in $\mathscr{C}[\Gamma]$.

Let $M \in \mathscr{C}[\Gamma]$. Consider a surjection $f : R \twoheadrightarrow M$ with $R \in \mathscr{C}^{\Delta}$. By (i), there is $R' \leqslant R$ such that $R' \in \mathscr{C}^{\Delta\setminus\Gamma}$ and $R/R' \in \mathscr{C}^{\Gamma}$. By induction on the length of the filtration, we see that $\mathrm{Hom}(N, M) = 0$ for all $N \in \mathscr{C}^{\Delta\setminus\Gamma}$. In particular, $\mathrm{Hom}(R', M) = 0$. So, f factors through a surjection $R/R' \twoheadrightarrow M$, hence M is a quotient of an object of $\mathscr{C}^{\Gamma}$. This shows (ii).

Note that $\mathscr{C}[\Gamma]$ is closed under subobjects and extensions, while the category of quotients of objects of $\mathscr{C}^{\Gamma}$ is closed under taking quotients. It follows that $\mathscr{C}[\Gamma]$ is a Serre subcategory.

Let $M \in \mathscr{C}[\Gamma]$ and $D, D' \in \Gamma$. We fix a surjection $f : R \twoheadrightarrow D$ as in Definition F.1.1(iv). By (i), there is $R' \leqslant \ker f$ such that $R' \in \mathscr{C}^{\Delta\setminus\Gamma}$ and $\ker f/R' \in \mathscr{C}^{\Gamma_{>D}}$. The map f factors through a surjection $R/R' \twoheadrightarrow D$.

We have $\mathrm{Hom}(R', D') = 0$, hence $\mathrm{Hom}(R/R', D') \simeq \mathrm{Hom}(R, D')$ is a finitely generated projective k-module. Since $\mathrm{Ext}^1(R, M) = 0$ and $\mathrm{Hom}(R', M) = 0$, we deduce that

$\mathrm{Ext}^1(R/R', M) = 0$. So, the surjection $R/R' \twoheadrightarrow D$ satisfies Definition F.1.1(iv) for $\mathscr{C}[\Gamma]$. We deduce that $\mathscr{C}[\Gamma]$ is a highest weight category, hence (iii) holds.

Let $M \in \mathscr{C}$. There exists a surjection $f : R \twoheadrightarrow M$ with $R \in \mathscr{C}^\Delta$. As above, there is $R' \leqslant R$ such that $R' \in \mathscr{C}^{\Delta \setminus \Gamma}$ and $R/R' \in \mathscr{C}^\Gamma$. Let $M' = f(R')$. Note that M/M' is a quotient of R/R', so $M/M' \in \mathscr{C}[\Gamma]$. Consider now $N \in \mathscr{C}[\Gamma]$. Since $\mathrm{Hom}(R', N) = 0$, we have $\mathrm{Hom}(M', N) = 0$, hence every map $M \to N$ factors through M/M'. We deduce that M/M' is the largest quotient of M that is in $\mathscr{C}[\Gamma]$. The functor $M \mapsto M/M'$ is left adjoint to the inclusion functor.

Consider now the family I of subobjects of M that are quotients of objects of $\mathscr{C}[\Gamma]$ and let M'' be their sum. Given $N \in \mathscr{C}[\Gamma]$, we have an isomorphism $\mathrm{Hom}_{\mathscr{C}[\Gamma]}(N, M'') \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(N, M)$. We deduce that the inclusion functor $\mathscr{C}[\Gamma] \hookrightarrow \mathscr{C}$ has a right adjoint, sending an object M to M''. This shows (iv). ☐

Corollary F.1.10. — *The category $\mathscr{C}$ is the union of the full subcategories $\mathscr{C}[\Gamma]$, where Γ runs over finitely generated ideals of Δ.*

F.1.C. Projective objects. — Note first that since every object of $\mathscr{C}$ is a quotient of an object of $\mathscr{C}^\Delta$, it follows that every projective object of $\mathscr{C}$ is a direct summand of an object of $\mathscr{C}^\Delta$.

We start with a projectivity criterion.

Lemma F.1.11. — *Let $N \in \mathscr{C}^\Delta$ such that $\mathrm{Ext}^1(N, D) = 0$ for all $D \in \Delta$. Then, N is projective.*

Proof. — Let $M \in \mathscr{C}$. By Lemma F.1.3, there exists a surjection $f : R \twoheadrightarrow N$ such that $\ker f \in \mathscr{C}^\Delta$ and $\mathrm{Ext}^1(R, M) = 0$. We have $\mathrm{Ext}^1(N, \ker f) = 0$, hence f is a split surjection and $\mathrm{Ext}^1(N, M) = 0$. It follows that N is projective. ☐

Lemma F.1.12. — *Let D be an object of Δ such that $\Delta_{>D}$ is finite. Then, there is a projective object P of $\mathscr{C}$ and a surjective map $P \twoheadrightarrow D$ whose kernel is in $\mathscr{C}^{\Delta_{>D}}$.*

Proof. — Fix $r > 0$ and an increasing bijection $\phi : \Delta_{>D} \xrightarrow{\sim} \{1, 2, \ldots, r\}$. Let $P_0 = D$. We construct by induction on $i \in \{1, 2, \ldots, r\}$ a family of objects $P_1, \ldots, P_r$ in $\mathscr{C}^{\Delta_{>D}}$ and surjections $f_i : P_i \twoheadrightarrow P_{i-1}$ such that $\mathrm{Ext}^1(P_i, \phi^{-1}(i)) = 0$ and $\ker f_i$ is a finite multiple of $\phi^{-1}(i)$.

Assume P_i has been constructed. Since $\mathrm{Ext}^1(P_i, \phi^{-1}(i+1))$ is a finitely generated k-module (Lemma F.1.6), there exists an object P_{i+1} of $\mathscr{C}$ and a surjection $f_{i+1} : P_{i+1} \twoheadrightarrow P_i$ such that the canonical map $\mathrm{Ext}^1(P_i, \phi^{-1}(i+1)) \to \mathrm{Ext}^1(P_{i+1}, \phi^{-1}(i+1))$ vanishes and $\ker f_{i+1}$ is a finite multiple of $\phi^{-1}(i+1)$. Since $\mathrm{Ext}^1(\phi^{-1}(i+1), \phi^{-1}(i+1)) = 0$, it follows that $\mathrm{Ext}^1(P_{i+1}, \phi^{-1}(i+1)) = 0$.

We put $P = P_r$ and $g_i = f_i \circ \cdots \circ f_r : P \twoheadrightarrow P_{i-1}$. Note that $\ker g_i \in \mathscr{C}^{\phi^{-1}(\{i, \ldots, r\})}$.

Given $D' \in \Delta$ with $D \not< D'$, we have also $\phi^{-1}(i) \not< D'$ for all i, hence $\mathrm{Ext}^1(P, D') = 0$.

Let $i \in \{1,2,\ldots,r\}$. We have $\mathrm{Ext}^1(\ker g_{i+1}, \phi^{-1}(i)) = 0$ and $\mathrm{Ext}^1(P_i, \phi^{-1}(i)) = 0$, hence $\mathrm{Ext}^1(P, \phi^{-1}(i)) = 0$. So, $\mathrm{Ext}^1(P, D') = 0$ for all $D' \in \Delta$. We deduce from Lemma F.1.11 that P is projective. $\square$

Let us now provide a criterion for the existence of enough projective objects.

Proposition F.1.13. — *If $\Delta_{>D}$ is finite for all $D \in \Delta$, then $\mathscr{C}$ has enough projective objects. More precisely, fix P_D a projective object with quotient D for every $D \in \Delta$. Then, $\{P_D\}_{D\in\Delta}$ is a generating family of projective objects. Furthermore, every object of $\mathscr{C}^\Delta$ has a finite projective resolution.*

Proof. — Given $D \in \Delta$, Lemma F.1.12 shows there is a projective object P_D and a surjection $P_D \twoheadrightarrow D$. Let $M \in \mathscr{C}^\Delta$ with a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that $M_i/M_{i-1} \in \Delta$ for $1 \leqslant i \leqslant r$. We have a surjection $P_{M_i/M_{i-1}} \twoheadrightarrow M_i/M_{i-1}$. It lifts to a map $f_i : P_{M_i/M_{i-1}} \to M_i$ and the sum $\sum f_i : \bigoplus P_{M_i/M_{i-1}} \to M$ is surjective. So, every object of $\mathscr{C}^\Delta$ is a quotient of a projective object. The last statement follows from Lemma F.1.5. $\square$

Proposition F.1.14. — *Let Γ be an ideal of Δ. Given $M, N \in \mathscr{C}[\Gamma]$, we have $\mathrm{Ext}^i_{\mathscr{C}[\Gamma]}(M,N) = \mathrm{Ext}^i_{\mathscr{C}}(M,N)$ for all $i \geqslant 0$.*

Proof. — Note that the statement of the proposition is clear when $i \leqslant 1$ since $\mathscr{C}[\Gamma]$ is a Serre subcategory of $\mathscr{C}$ (Proposition F.1.9).

⋆ Assume first Γ is a finitely generated ideal.

• Assume further that $M \in \mathscr{C}^\Delta$ is projective in $\mathscr{C}[\Gamma]$. Assume $\mathrm{Ext}^i_{\mathscr{C}}(M,N) \neq 0$ for some $i > 1$. There is $N' \in \mathscr{C}$ and $\zeta \in \mathrm{Ext}^1_{\mathscr{C}}(M,N')$ and $\xi \in \mathrm{Ext}^{i-1}_{\mathscr{C}}(N',N)$ with $\xi \circ \zeta \neq 0$. By Lemma F.1.3, there exists a surjection $f : P \twoheadrightarrow M$ with $P \in \mathscr{C}^\Delta$ such that $\mathrm{Ext}^1(P,N') = 0$. By Proposition F.1.9(i), there exists $P' \leqslant P$ such that $P' \in \mathscr{C}^{\Delta\setminus\Gamma}$ and $P/P' \in \mathscr{C}^\Gamma$. Since $\mathrm{Hom}(P',M) = 0$, we deduce that f factors through a surjection $P/P' \twoheadrightarrow M$. As M is projective in $\mathscr{C}[\Gamma]$, that last surjection splits, hence there is $\tilde{P} \leqslant P$ such that f restricts to a surjection $g : \tilde{P} \twoheadrightarrow M$ with kernel P'. Since the composition $\zeta \circ f$ vanishes, we deduce that $\zeta \circ g$ vanishes, hence ζ factors through a map $h : P' \to N'$ and $\xi \circ h \neq 0$. By Lemma F.1.5, we have $\mathrm{Ext}^{i-1}(P',N) = 0$, hence a contradiction. So, $\mathrm{Ext}^i_{\mathscr{C}}(M,N) = 0$ for all $i > 0$.

• Consider now an arbitrary $M \in \mathscr{C}[\Gamma]$. By Proposition F.1.13, there exist $P \in \mathrm{Proj}(\mathscr{C}[\Gamma])$ and a surjection $f : P \twoheadrightarrow M$. Given $i > 1$, there are isomorphisms $\mathrm{Ext}^{i-1}_{\mathscr{C}}(\ker f, N) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathscr{C}}(M,N)$ (by the discussion above) and $\mathrm{Ext}^{i-1}_{\mathscr{C}[\Gamma]}(\ker f, N) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathscr{C}[\Gamma]}(M,N)$. By induction on i, we have $\mathrm{Ext}^{i-1}_{\mathscr{C}[\Gamma]}(\ker f, N) = \mathrm{Ext}^{i-1}_{\mathscr{C}}(\ker f, N)$. It follows that $\mathrm{Ext}^i_{\mathscr{C}[\Gamma]}(M,N) = \mathrm{Ext}^i_{\mathscr{C}}(M,N)$.

⋆ Consider finally the case of an arbitrary ideal Γ. There exists a finitely generated ideal Γ' contained in Γ and such that $M,N \in \mathscr{C}[\Gamma']$. We have $\mathrm{Ext}^i_{\mathscr{C}[\Gamma']}(M,N) = \mathrm{Ext}^i_{\mathscr{C}[\Gamma]}(M,N)$ and $\mathrm{Ext}^i_{\mathscr{C}[\Gamma']}(M,N) = \mathrm{Ext}^i_{\mathscr{C}}(M,N)$ and the proposition follows. □

Lemma F.1.15. — *Let $M,N \in \mathscr{C}$.*

(i) *If $M \in \mathrm{Proj}(\mathscr{C})$ and $N \in \mathscr{C}^\Delta$, then the k-module $\mathrm{Hom}(M,N)$ is projective.*
(ii) *Given $i \geqslant 0$, the k-module $\mathrm{Ext}^i(M,N)$ is finitely generated.*
(iii) *If $M \in \mathscr{C}^\Delta$, then $\mathrm{Ext}^i(M,N) = 0$ for $i \gg 0$.*
(iv) *Let Γ be an ideal of Δ such that M is a quotient of an object of $\mathscr{C}^\Gamma$. If $\mathrm{Ext}^i(D,M) \neq 0$ for some $D \in \Delta$ and $i \geqslant 0$, then $D \in \Gamma$.*

Proof. — Assume $M \in \mathrm{Proj}(\mathscr{C})$ and $N \in \Delta$. By Lemma F.1.3, there exists a surjection $R \twoheadrightarrow M$ such that $\mathrm{Hom}(R,N)$ is a finitely generated projective k-module. Since $\mathrm{Hom}(M,N)$ is a direct summand of $\mathrm{Hom}(R,N)$, we deduce that it is a finitely generated projective k-module as well.

When $M \in \mathrm{Proj}(\mathscr{C})$ and $N \in \mathscr{C}^\Delta$, it follows by induction on the length of a filtration of N that $\mathrm{Hom}(M,N)$ is a finitely generated projective k-module. This shows (i).

When $M \in \mathrm{Proj}(\mathscr{C})$ and $N \in \mathscr{C}$, there exists $N' \in \mathscr{C}^\Delta$ such that N is a quotient of N'. We deduce that $\mathrm{Hom}(M,N)$ is a finitely generated k-module.

Let us now prove (ii) and (iii). Thanks to Proposition F.1.14, we can assume that Δ is finitely generated as an ideal, hence $\mathscr{C}$ has enough projectives by Proposition F.1.13.

Consider a surjection $f : P \twoheadrightarrow M$ with P projective. Since $\mathrm{Hom}(P,N)$ is a finitely generated k-module, so is $\mathrm{Hom}(M,N)$. We prove that the statement of the lemma by induction on $i > 0$. We have a surjection $\mathrm{Ext}^{i-1}(\ker f, N) \twoheadrightarrow \mathrm{Ext}^i(M,N)$. By induction, $\mathrm{Ext}^{i-1}(\ker f, N)$ is finitely generated, hence so is $\mathrm{Ext}^i(M,N)$. This shows (ii).

It is enough to prove (ii) for $M \in \Delta$. We proceed by induction: we assume that given $D \in \Delta_{D>M}$, we have $\mathrm{Ext}^i(D,N) = 0$ for $i \gg 0$. We can assume that $\ker f \in \mathscr{C}^{\Delta_{>M}}$, hence $\mathrm{Ext}^i(\ker f, N) = 0$ for $i \gg 0$. So, $\mathrm{Ext}^i(M,N) = 0$ for $i \gg 0$. This shows (iii).

Let us show (iv). We can assume that Γ is finitely generated. Assume $D \notin \Gamma$. Let $\Gamma' = \Delta_{\leqslant D} \cup \Gamma$. Since D is projective in $\mathscr{C}^{\Gamma'}$, we have $\mathrm{Ext}^i_{\mathscr{C}[\Gamma']}(D,M) = \mathrm{Ext}^i_{\mathscr{C}}(D,M) = 0$ if $i > 0$ (Proposition F.1.14). So we have $i = 0$. There is $M' \in \mathscr{C}^\Gamma$ and a surjection $M' \twoheadrightarrow M$. Since $\mathrm{Hom}(D,M) \neq 0$, it follows that $\mathrm{Hom}(D,M') \neq 0$, a contradiction. So (iv) holds. □

Proposition F.1.16. — *Assume Δ is finite. Then, $(\mathscr{C},\Delta)$ is a highest category over k as in* [**Rou**, Definition 4.11]. *There is a split quasi-hereditary k-algebra A* [**CPS2**, Definition 3.2] *and an equivalence $A\text{-mod} \simeq \mathscr{C}$.*

Proof. — It follows from Proposition F.1.13 that $\mathscr{C}$ has a progenerator P and from Lemma F.1.15 that $A = \mathrm{End}(P)$ is finitely generated and projective as a k-module.

So, we have an equivalence $\mathrm{Hom}(P,-):\mathscr{C}\xrightarrow{\sim}A$-mod. By Lemma F.1.15, $\mathrm{Hom}(P,D)$ is a finitely generated projective k-module for all $D\in\Delta$. Finally, Lemma F.1.12 shows that given $D\in\Delta$, there is $P\in\mathrm{Proj}(\mathscr{C})$ and a surjection $P\twoheadrightarrow D$ with kernel in $\mathscr{C}^{\Delta_{>D}}$. It follows that $(\mathscr{C},\Delta)$ is a highest category over k as in [**Rou**, Definition 4.11]. The statement about quasi-hereditary algebras is [**Rou**, Theorem 4.16]. $\square$

Given $\mathscr{A}$ an abelian category, M an object of $\mathscr{A}$ and L a simple object of $\mathscr{A}$, we denote by $[M:L]\in\mathbb{Z}_{\geqslant 0}\cup\{\infty\}$ the maximum of the set of integers n such that M has a filtration $0=M_{-1}\subset M_0\subset\cdots\subset M_{2n}=M$ with $M_{2i-1}/M_{2i-2}\simeq L$ for $1\leqslant i\leqslant n$.

Proposition F.1.17. — *Assume k is a field. Then,*

- *any object $D\in\Delta$ has a unique simple quotient $L(D)$ and $[D:L(D)]=1$*
- *every simple object of $\mathscr{C}$ is isomorphic to $L(D)$ for a unique $D\in\Delta$*
- *Let $D,D'\in\Delta$ such that $[D:L(D')]\neq 0$. Then, we have $D'\leqslant D$ and $[D:L(D')]<\infty$.*

Assume furthermore that Δ is finitely generated as an ideal. Then every object of $\mathscr{C}$ has a projective cover. In particular, given $D\in\Delta$, the simple object $L(D)$ has a projective cover $P(D)$ and $[M:L(D)]=\dim_k\mathrm{Hom}_{\mathscr{C}}(P(D),M)<\infty$ for all $M\in\mathscr{C}$.

Proof. — Let L be a simple object of $\mathscr{C}$. There is $D\in\Delta$ such that $\mathrm{Hom}(D,L)\neq 0$, since L is a quotient of an object of $\mathscr{C}^{\Delta}$. Consider now $D'\in\Delta$ such that $\mathrm{Hom}(D',L)\neq 0$. Assume $D\not\leqslant D'$ and let Γ be the ideal of Δ generated by D and D'. Then D is projective in $\mathscr{C}[\Gamma]$ hence a surjection $D\twoheadrightarrow L$ factors through a surjection $D'\twoheadrightarrow L$. Since $\mathrm{Hom}(D,D')=0$, we obtain a contradiction. It follows that there is a unique $D\in\Delta$ such that L is a quotient of D.

Fix now $D\in\Delta$ and assume there are simple objects $L,L'\in\mathscr{C}$ and a surjective map $f:D\twoheadrightarrow L\oplus L'$. We have $L,L'\in\mathscr{C}[\Delta_{\leqslant D}]$ and D is projective in $\mathscr{C}[\Delta_{\leqslant D}]$. It follows that composition with f induces a surjection $\mathrm{End}(D)\twoheadrightarrow\mathrm{Hom}(D,L\oplus L')$. Since $\mathrm{End}(D)=k$, we obtain a contradiction: D has at most one simple quotient.

Consider now I the family of subobjects of D that are in $\mathscr{C}[\Delta_{<D}]$ and let $M=\sum_{N\in I}N\in\mathscr{C}[\Delta_{<D}]$. Since $\mathrm{Hom}(D,M)=0$, we deduce that $D/M\neq 0$. Since $\mathscr{C}[\Delta_{<D}]$ is a Serre subcategory of $\mathscr{C}$, we have $\mathrm{Hom}(N,D/M)=0$ for all $N\in\mathscr{C}[\Delta_{<D}]$. Let L be a non-zero subobject of D/M. Since $L\notin\mathscr{C}[\Delta_{<D}]$, there exists a non-zero map $D\to L$. That map lifts to a non-zero map $D\to D$, hence an isomorphism since $\mathrm{End}(D)=k$. So, $L=D/M$, hence D/M is simple.

We have shown that every $D\in\Delta$ has a unique simple quotient $L(D)$, that $L(D)\simeq L(D')$ implies $D\simeq D'$ and every simple object of $\mathscr{C}$ is isomorphic to $L(D)$ for some $D\in\Delta$.

Since D is projective in $\mathscr{C}_{\leqslant D}$ and $\mathrm{End}(D)=k$, we deduce that $[D:L(D)]=1$.

Let $D'\in\Delta$ such that $[D:L(D')]\neq 0$. Let Γ be the ideal of Δ generated by D and D'. By Lemma F.1.11, there is $P\in\mathrm{Proj}(\mathscr{C}[\Gamma])$ and a surjection $P\twoheadrightarrow D'$ with kernel in

$\mathscr{C}^{\Gamma_{>D'}}$. We have $\dim_k \operatorname{Hom}(P,D) < \infty$ (Lemma F.1.15), hence $[D : L(D')] < \infty$. Also, $\operatorname{Hom}(P,D) \neq 0$, hence $D' \leqslant D$.

Assume now Δ is finitely generated as an ideal. Let $D \in \Delta$. There is a projective object P of $\mathscr{C}$ and a surjection $P \twoheadrightarrow L(D)$. Since $\operatorname{End}_{\mathscr{C}}(P)$ is finite-dimensional, it follows that there is $P(D)$ an indecomposable direct summand of P and a surjection $f : P(D) \twoheadrightarrow L(D)$. As $\operatorname{End}_{\mathscr{C}}(P(D))$ is a local k-algebra, it follows that f is a projective cover. ☐

F.1.D. Ideals and quotients. — Let Γ be an ideal of Δ. Recall that $\mathscr{C}[\Gamma]$ is a Serre subcategory of $\mathscr{C}$ (Proposition F.1.9). We put $\mathscr{C}(\Delta \setminus \Gamma) = \mathscr{C}/\mathscr{C}[\Gamma]$.

Proposition F.1.18. — *Let Γ be an ideal of Δ. Then $\mathscr{C}(\Delta \setminus \Gamma)$ is a highest weight category with poset of standard objects $\Delta \setminus \Gamma$ and given Γ' an ideal of Δ, there is a canonical equivalence*

$$(\mathscr{C}[\Gamma'])\big(\Gamma' \setminus (\Gamma \cap \Gamma')\big) \xrightarrow{\sim} \big(\mathscr{C}(\Delta \setminus \Gamma)\big)[\Gamma' \setminus (\Gamma \cap \Gamma')].$$

Proof. — Given $M \in \mathscr{C}^{\Delta \setminus \Gamma}$ and $N \in \mathscr{C}$, we have an isomorphism $\operatorname{Hom}_{\mathscr{C}}(M,N) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}(\Delta\setminus\Gamma)}(M,N)$ since $\operatorname{Hom}_{\mathscr{C}}(M,M') = \operatorname{Ext}^1_{\mathscr{C}}(M,M') = 0$ for all $M' \in \mathscr{C}[\Gamma]$ (Lemma F.1.5).

We deduce that $\operatorname{Hom}_{\mathscr{C}}(D,D') \simeq \operatorname{Hom}_{\mathscr{C}/\mathscr{C}[\Gamma]}(D,D')$ for all $D, D' \in \Delta \setminus \Gamma$. It follows that (i) and (ii) in Definition F.1.1 hold for $\mathscr{C}(\Delta \setminus \Gamma)$.

Let $M \in \mathscr{C}$. By Proposition F.1.9(iv), there is $N \subset M$ such that M/N is the largest quotient of M in $\mathscr{C}[\Gamma]$. There is a surjection $f : R \twoheadrightarrow N$ with $R \in \mathscr{C}^{\Delta}$. By Proposition F.1.9(i), there is a subobject $R' \subset \mathscr{C}$ with $R' \in \mathscr{C}^{\Delta\setminus\Gamma}$ and $R/R' \in \mathscr{C}^{\Gamma}$. Let $N' = f(R')$. We have a surjection $R/R' \twoheadrightarrow N/N'$, hence $N/N' \in \mathscr{C}[\Gamma]$. It follows that $N/N' = 0$, hence N is a quotient of R'. The image in $\mathscr{C}(\Delta \setminus \Gamma)$ of the map $R' \to M$ is a surjection. So, (iii) in Definition F.1.1 holds for $\mathscr{C}(\Delta \setminus \Gamma)$.

Let $\mathscr{I}$ (resp. $\mathscr{J}$) be the thick subcategory of $D^b(\mathscr{C})$ generated by Γ (resp. $\Delta \setminus \Gamma$). Since $\operatorname{Ext}^i_{\mathscr{C}}(D,D') = 0$ for all $i \geqslant 0$, $D \in \Delta \setminus \Gamma$ and $D' \in \Gamma$ (Lemma F.1.5), it follows that $\operatorname{Hom}_{D^b(\mathscr{C})}(C,C') = 0$ for $C \in \mathscr{J}$ and $C' \in \mathscr{I}$. We deduce that $\operatorname{Hom}_{D^b(\mathscr{C})}(M,M') \xrightarrow{\sim} \operatorname{Hom}_{D^b(\mathscr{C})/\mathscr{I}}(M,M')$ for all $M \in \mathscr{J}$ and $M' \in D^b(\mathscr{C})$. Since $\mathscr{C}(\Delta \setminus \Gamma)$ is the heart of the canonical quotient t-structure on $D^b(\mathscr{C})/\mathscr{I}$, we deduce that $\operatorname{Ext}^1_{\mathscr{C}}(M,M') \simeq \operatorname{Ext}^1_{\mathscr{C}(\Delta\setminus\Gamma)}(M,M')$ for all $M \in \mathscr{C}^{\Delta\setminus\Gamma}$ and $M' \in \mathscr{C}$. In particular, if $M \in \mathscr{C}^{\Delta\setminus\Gamma}$ is projective in $\mathscr{C}$, then it is projective in $\mathscr{C}(\Delta \setminus \Gamma)$. Also, $\mathscr{C}^{\Delta\setminus\Gamma} \xrightarrow{\sim} \big(\mathscr{C}(\Delta \setminus \Gamma)\big)^{\Delta\setminus\Gamma}$.

Let us now assume that Δ is finitely generated as an ideal. Let $M \in \mathscr{C}$. We have seen that there exists $R \in \mathscr{C}^{\Delta\setminus\Gamma}$ and a map $R \to M$ that becomes surjective in $\mathscr{C}(\Delta\setminus\Gamma)$. In the proof of Proposition F.1.13, we saw that there exists a surjection $P \twoheadrightarrow R$ with $P \in \operatorname{Proj}(\mathscr{C}) \cap \mathscr{C}^{\Delta\setminus\Gamma}$. The composition $P \to R \to M$ is surjective in $\mathscr{C}(\Delta \setminus \Gamma)$ and P is projective in $\mathscr{C}(\Delta \setminus \Gamma)$. We deduce that the quotient functor induces an equivalence

$\mathrm{Proj}(\mathscr{C})\cap\mathscr{C}^{\Delta\setminus\Gamma}\xrightarrow{\sim}\mathrm{Proj}(\mathscr{C}(\Delta\setminus\Gamma))$ and that $\mathscr{C}(\Delta\setminus\Gamma)$ has enough projectives. So, (iv) in Definition F.1.1 holds for $\mathscr{C}(\Delta\setminus\Gamma)$.

Let us consider again an arbitrary Δ. We have $\mathscr{C}[\Gamma']^{\Gamma'\setminus(\Gamma\cap\Gamma')}\xrightarrow{\sim}\big(\mathscr{C}[\Gamma'](\Gamma'\setminus(\Gamma\cap\Gamma'))\big)^{\Gamma'\setminus(\Gamma\cap\Gamma')}$ and $\mathscr{C}^{\Gamma'\setminus(\Gamma\cap\Gamma')}=\mathscr{C}[\Gamma']^{\Gamma'\setminus(\Gamma\cap\Gamma')}\xrightarrow{\sim}\big(\mathscr{C}(\Delta\setminus\Gamma)\big)^{\Gamma'\setminus(\Gamma\cap\Gamma')}$. So, we have an equivalence

$$\big(\mathscr{C}[\Gamma'](\Gamma'\setminus(\Gamma\cap\Gamma'))\big)^{\Gamma'\setminus(\Gamma\cap\Gamma')}\xrightarrow{\sim}\big(\mathscr{C}(\Delta\setminus\Gamma)\big)^{\Gamma'\setminus(\Gamma\cap\Gamma')}.$$

We have shown that every object of $(\mathscr{C}[\Gamma'])(\Gamma'\setminus(\Gamma\cap\Gamma'))$ is the cokernel of a morphism in $\mathscr{C}[\Gamma']^{\Gamma'\setminus(\Gamma\cap\Gamma')}$, hence the canonical functor $(\mathscr{C}[\Gamma'])(\Gamma'\setminus(\Gamma\cap\Gamma'))\to\mathscr{C}(\Delta\setminus\Gamma)$ is fully faithful.

Let $D,D'\in\Delta\setminus\Gamma$ and $M\in\mathscr{C}$. There is a finitely generated ideal Γ' of Γ containing D, D' and such that $M\in\mathscr{C}[\Gamma']$.

There exists a surjection $P\twoheadrightarrow D$ with kernel in $\mathscr{C}^{\Gamma'_{>D}}$ and $P\in\mathrm{Proj}(\mathscr{C}[\Gamma'])$. Since $(\mathscr{C}[\Gamma'])(\Gamma'\setminus(\Gamma\cap\Gamma'))$ is a highest weight category, we deduce that (iv) in Definition F.1.1 holds for $\mathscr{C}(\Delta\setminus\Gamma)$. The last statement of the proposition follows. ☐

Lemma F.1.19. — *Let Γ be an ideal of Δ. The quotient functor $q_\Gamma:\mathscr{C}\to\mathscr{C}(\Delta\setminus\Gamma)$ has a left adjoint ${}^\vee q_\Gamma$.*

Assume Δ is finitely generated as an ideal. Given $M\in\mathscr{C}$, there is a finitely generated ideal Γ of Δ such that $\Delta\setminus\Gamma$ is finite and such that the canonical map ${}^\vee q_\Gamma q_\Gamma(M)\to M$ is an isomorphism.

Proof. — Thanks to Corollary F.1.10 and Proposition F.1.18, it is enough to prove the lemma when Δ is finitely generated as an ideal, and we make now that assumption.

Let $\mathscr{P}_\Gamma$ be the full subcategory of projective objects of $\mathscr{C}$ that are in $i(\mathscr{C}^{\Delta\setminus\Gamma})$. The quotient functor q_Γ restricts to an equivalence ϕ from $\mathscr{P}_\Gamma$ to the category of projective objects of $\mathscr{C}(\Delta\setminus\Gamma)$ Let $N\in\mathscr{C}(\Delta\setminus\Gamma)$. Fix a projective presentation $P\xrightarrow{f}Q\to N\to 0$. Define ${}^\vee q_\Gamma(N)=\mathrm{coker}(\phi^{-1}(f))$. It is easy to check that this defines a functor $\mathscr{C}(\Delta\setminus\Gamma)\to\mathscr{C}$ that is left adjoint to q_Γ.

Let $M\in\mathscr{C}$. Let $P\to Q\to M\to 0$ be a projective presentation of M. There is an ideal Γ of Δ such that $\Delta\setminus\Gamma$ is finite and $P,Q\in i(\mathscr{C}^{\Delta\setminus\Gamma})$. It follows that the canonical map ${}^\vee q_\Gamma q_\Gamma(M)\to M$ is an isomorphism. ☐

F.1.E. Base change. — Let $\mathscr{D}$ be an additive category. We denote by $\mathscr{D}$-Mod the abelian category of additive functors $\mathscr{D}^{\mathrm{opp}}\to\mathbb{Z}$-Mod and by $\mathscr{D}$-mod its full subcategory of functors that are quotients of representable functors. The Yoneda functor defines a fully faithful embedding

$$\mathscr{D}\hookrightarrow\mathscr{D}\text{-mod},\ M\mapsto\mathrm{Hom}(-,M).$$

The family of representable functors is a generating family of projective objects of $\mathscr{D}$-Mod and we will identify $\mathscr{D}$ with the corresponding full subcategory of $\mathscr{D}$-Mod.

The full subcategory $\mathscr{D}$-mod of $\mathscr{D}$-Mod is closed under extensions and quotients. We say that $\mathscr{D}$ is *locally noetherian* if $\mathscr{D}$-mod is closed under taking subobjects in $\mathscr{D}$-Mod. When $\mathscr{D}$ is locally noetherian, the subcategory $\mathscr{D}$-mod of $\mathscr{D}$-Mod is a Serre subcategory.

Lemma F.1.20. — *Let $\mathscr{A}$ be an abelian category with enough projectives. The canonical functor $\mathscr{A} \to \mathrm{Proj}(\mathscr{A})\text{-Mod}$, $M \mapsto \mathrm{Hom}(-, M)$ is exact and fully faithful and it takes values in* $\mathrm{Proj}(\mathscr{A})$-mod.

The following conditions are equivalent

(i) *given $M \in \mathscr{A}$ and given I a family of subobjects of M, there exists a subobject $\sum_{N\in I} N$ of M*

(ii) $\mathrm{Proj}(\mathscr{A})$ *is locally noetherian*

(iii) *the Yoneda functor gives an equivalence* $\mathscr{A} \xrightarrow{\sim} \mathrm{Proj}(\mathscr{A})\text{-mod}$.

Proof. — The first statement is clear.

Assume (i). Let $P \in \mathrm{Proj}(\mathscr{A})$ and let Ψ be a subobject of $\mathrm{Hom}(-, P)$. There is a family I of objects of $\mathrm{Proj}(\mathscr{A})$ and maps $f_Q : \mathrm{Hom}(-, Q) \to \Psi$ such that $\sum f_Q : \bigoplus_{Q\in I} \mathrm{Hom}(-, Q) \to \Psi$ is surjective. Let $g_Q : Q \to P$ such that $\mathrm{Hom}(-, g_Q)$ is the composition of f_Q with the inclusion $\Psi \hookrightarrow \mathrm{Hom}(-, P)$. By assumption, there is a subobject $L = \sum \mathrm{Im}(g_Q)$ of P. We have $\Psi \subset \mathrm{Hom}(-, L)$. Since $\sum \mathrm{Hom}(-, f_Q) : \bigoplus_{Q\in I} \mathrm{Hom}(-, Q) \to \mathrm{Hom}(-, L)$ is surjective, it follows that $\Psi = \mathrm{Hom}(-, L)$. There is $P' \in \mathrm{Proj}(\mathscr{A})$ and a surjective map $P' \twoheadrightarrow L$. It follows that Ψ is a quotient of $\mathrm{Hom}(-, P')$, hence $L \in \mathrm{Proj}(\mathscr{A})$-mod. This shows that (ii) holds.

Assume (ii). Every object M of $\mathrm{Proj}(\mathscr{A})$-mod is isomorphic to the cokernel of a map $f : \mathrm{Hom}(-, P) \to \mathrm{Hom}(-, Q)$ with $P, Q \in \mathrm{Proj}(\mathscr{A})$. There is $g \in \mathrm{Hom}(P, Q)$ such that $f = \mathrm{Hom}(-, g)$. We have $M \simeq \mathrm{coker} f = \mathrm{Hom}(-, \mathrm{coker} g)$ and (iii) follows.

Assume (iii). Let $M \in \mathscr{A}$ and let I be a family of subobjects of M. Since sums of subobjects exist in $\mathbb{Z}$-Mod, they exist in $\mathrm{Proj}(\mathscr{A})$-Mod. So, there is a subobject $\Psi = \sum_{Q\in I} \mathrm{Hom}(-, Q)$ of $\mathrm{Hom}(-, M)$. Let $\Phi = \mathrm{Hom}(-, M)/\Psi$. Since $\Phi \in \mathrm{Proj}(\mathscr{A})$-mod, there is $N \in \mathscr{A}$ and an isomorphism $\Phi \simeq \mathrm{Hom}(-, N)$. There is a map $f \in \mathrm{Hom}(M, N)$ such that $\mathrm{Hom}(-, f)$ corresponds to the quotient map $\mathrm{Hom}(-, M) \twoheadrightarrow \Phi$. We have $\Psi = \ker(\mathrm{Hom}(-, f)) \simeq \mathrm{Hom}(-, \ker f)$. We deduce that $\ker f = \sum_{Q\in I} Q$. So, (i) holds. $\square$

Let $\mathscr{D}$ be a k-linear category. Note that the forgetful functor from the category of k-linear functors $\mathscr{D}^{\mathrm{opp}} \to k$-Mod to the category $\mathscr{D}$-Mod is an isomorphism of categories.

Let k' be a commutative k-algebra. We denote by $k'\mathscr{D}$ the k'-linear category with set of objects $\{k'M\}$ where M runs over the set of objects of $\mathscr{D}$ and with $\mathrm{Hom}_{k'\mathscr{D}}(k'M, k'N) = k' \otimes \mathrm{Hom}_{\mathscr{D}}(M, N)$. There is a base change functor $k' \otimes - : \mathscr{D} \to$

$k'\mathscr{D}$, $M \mapsto k'M$ that is compatible, via the Yoneda embedding, with the base change functor $k' \otimes - : \mathscr{D}\text{-Mod} \to (k'\mathscr{D})\text{-Mod}$, $F \mapsto (k'N \mapsto k' \otimes F(N))$.

Lemma F.1.21. — *Assume k' is a localization of k. Let $M \in (k'\mathscr{D})\text{-mod}$. There exists $\tilde{M} \in \mathscr{D}\text{-mod}$ such that $k'\tilde{M} \simeq M$.*

Proof. — Let $Q \in \mathscr{D}$ and N a subobject of $k'Q$ in $(k'\mathscr{D})\text{-Mod}$ such that $M \simeq (k'Q)/N$.

Let $\phi : k \to k'$ be the canonical algebra map and $\phi_Q = \phi \otimes \mathrm{Id}_Q : Q \to k' \otimes Q$. Let $L = \phi_Q^{-1}(N) \subset Q$. We have a canonical isomorphism $k'L \xrightarrow{\sim} N$, hence $M \simeq k'(Q/L)$. □

Lemma F.1.22. — *Assume k' is a finitely generated module over a localization of k.*

If $\mathscr{D}$ is locally noetherian, then $k'\mathscr{D}$ is locally noetherian.

Proof. — Let $M \in \mathscr{D}$ and let N be a subobject of $k'M$ in $(k'\mathscr{D})\text{-Mod}$.

Assume k' is a localization of k. The proof of Lemma F.1.21 shows that there exists a subobject L of M such that $k'L \simeq N$. Since $L \in \mathscr{D}\text{-mod}$, it follows that $N \in (k'\mathscr{D})\text{-mod}$.

Assume k' is a finitely generated k-module. The restriction of $k'M$ to $\mathscr{D}$-Mod is a quotient of a finite direct sum of copies of M, hence the restriction N_0 of N to $\mathscr{D}$-Mod is the quotient of an object of $\mathscr{D}$-mod. So, there exists $P \in \mathscr{D}$ and a surjective map $P \twoheadrightarrow N_0$ in $\mathscr{D}$-mod. By adjunction, we obtain a surjective map $k'P \twoheadrightarrow N$ in $(k'\mathscr{D})\text{-Mod}$. So $N \in (k'\mathscr{D})\text{-mod}$.

The general case follows by transitivity. □

Proposition F.1.23. — *Let A be a k-algebra and assume $\mathscr{D}$-Mod is a full abelian subcategory of A-Mod (i.e., it is closed under quotients and subobjects).*

Then $(k'\mathscr{D})$-Mod is equivalent to the full subcategory of $(k'A)$-Mod whose objects are quotients of direct sums objects of the form $k' \otimes M$, for $M \in \mathscr{D}$. That subcategory is closed under taking quotients and subobjects.

Proof. — Let $M, N \in \mathscr{D}\text{-Mod}$. There is a commutative diagram with canonical maps, where the two rightmost horizontal maps are adjunction maps

$$\begin{tikzcd}
k'\operatorname{Hom}_A(M,N) \arrow[r] \arrow[d, "\sim"'] & \operatorname{Hom}_{k'A}(k'M,k'N) \arrow[r, "\sim"] & \operatorname{Hom}_A(M,k'N) \arrow[d] \\
k'\operatorname{Hom}_{\mathscr{D}\text{-Mod}}(M,N) \arrow[r] & \operatorname{Hom}_{(k'\mathscr{D})\text{-Mod}}(k'M,k'N) \arrow[r, "\sim"'] & \operatorname{Hom}_{\mathscr{D}\text{-Mod}}(M,k'N)
\end{tikzcd}$$

If $M \in \mathscr{D}$, then the bottom leftmost horizontal map is an isomorphism, while the rightmost vertical map is injective. We deduce that the rightmost vertical map is an isomorphism. So, this map is an isomorphism when M is a direct sum of objects of $\mathscr{D}$, hence when M is the cokernel of a map between direct sum of object of $\mathscr{D}$. Consequently, the canonical functor $(k'\mathscr{D})\text{-Mod} \to (k'A)\text{-Mod}$ is fully faithful and its image is the full subcategory of $(k'A)$-Mod of objects that are quotients of direct sums objects of the form $k'\otimes M$, for $M\in\mathscr{D}$.

Let $M\in(k'\mathscr{D})$-Mod and $N\in(k'A)$-Mod a subobject of M. Let M' and N' be the restrictions of M and N to A. We have $M'\in\mathscr{D}$-Mod, hence $N'\in\mathscr{D}$-Mod. Since N is a quotient of $k'\otimes N'$, we deduce that $N\in(k'\mathscr{D})$-Mod. So, $(k'\mathscr{D})$-Mod is closed under taking subobjects. □

Assume k is a discrete valuation ring with residue field $\bar{k}$ and field of fractions K.

Lemma F.1.24. — *Assume* Hom*-spaces in $K\mathscr{D}$ are finite-dimensional vector spaces over K and $\mathscr{D}$ is locally noetherian. Then $K\mathscr{D}$ and $\bar{k}\mathscr{D}$ are locally noetherian.*

Let $M\in K\mathscr{D}$. There exists $\tilde{M}\in\mathscr{D}$ such that $K\tilde{M}\simeq M$ and $\mathrm{Hom}(P,M)$ *is a free k-module of finite rank for every $P\in\mathscr{D}$.*

Given any such $\tilde{M}$, the class $[\bar{k}\tilde{M}]\in K_0((\bar{k}\mathscr{D})\text{-mod})$ depends only on $[M]\in K_0((K\mathscr{D})\text{-mod})$.

Proof. — Lemma F.1.21 ensures the existence of $N\in\mathscr{D}$-mod such that $k'N\simeq M$. Let N' be the torsion subobject of N and $\tilde{M}=N/N'$. We have $k'\tilde{M}\simeq M$. Let $P\in\mathscr{D}$. Since $\mathrm{Hom}(P,M)$ is a torsion-free k-module such that $K\otimes\mathrm{Hom}(P,M)$ is a finite-dimensional K-vector space, it follows that $\mathrm{Hom}(P,M)$ is a free k-module of finite rank.

Let $\tilde{M}_1$ and $\tilde{M}_2$ be two objects of $\mathscr{D}$-mod with $K\tilde{M}_1\simeq K\tilde{M}_2$ and such that $\mathrm{Hom}(P,\tilde{M}_i)$ is a finite rank free k-module for all $P\in\mathscr{D}$ and $i\in\{1,2\}$. There exists $f\in\mathrm{Hom}(\tilde{M}_1,\tilde{M}_2)$ injective such that $\pi^n\tilde{M}_2\subset\mathrm{Im}(f)$ for some $n\geqslant 0$, where π is a generator of the maximal ideal of k. We proceed by induction on n to show that $[\bar{k}\tilde{M}_1]=[\bar{k}\tilde{M}_2]$.

Assume $n=1$. Let $L=\mathrm{coker} f$. There is an exact sequence

$$0\to\mathrm{Tor}_1^k(\bar{k},L)\to\bar{k}\tilde{M}_1\to\bar{k}\tilde{M}_2\to\bar{k}L\to 0.$$

Since $\pi L=0$, we have $\mathrm{Tor}_1^k(\bar{k},L)\simeq\bar{k}L$, hence $[\bar{k}\tilde{M}_1]=[\bar{k}\tilde{M}_2]$.

In the general case, let $\tilde{M}_3=\mathrm{Im}(f)+\pi^{n-1}\tilde{M}_2$. We have $\pi\tilde{M}_3\subset\mathrm{Im}(f)$, hence $[\bar{k}\tilde{M}_1]=[\bar{k}\tilde{M}_3]$. We have $\mathfrak{m}^{n-1}\tilde{M}_2\subset\tilde{M}_3$, so it follows by induction that $[\bar{k}\tilde{M}_2]=[\bar{k}\tilde{M}_3]$. This completes the proof of the lemma. □

The previous lemma provides a decomposition map

$$d:K_0((K\mathscr{D})\text{-mod})\to K_0((\bar{k}\mathscr{D})\text{-mod})$$

with the property that $d([KM])=[\bar{k}M]$ when $M\in\mathscr{D}$-mod and $\mathrm{Hom}(P,M)$ is a projective k-module for all $P\in\mathscr{D}$.

Proposition F.1.13 and Lemma F.1.20 imply the following result.

Lemma F.1.25. — *Assume Δ is finitely generated as an ideal. The Yoneda functor induces an equivalence $\mathscr{C}\xrightarrow{\sim}\mathrm{Proj}(\mathscr{C})$-mod.*

Given two ideals $\Gamma \subset \Gamma'$, we have a functor $F_{\Gamma\subset\Gamma'} : \mathscr{C}[\Gamma'] \to \mathscr{C}[\Gamma]$ sending an object of $\mathscr{C}[\Gamma']$ to its largest quotient in $\mathscr{C}[\Gamma]$ (Proposition F.1.9). This functor sends projective objects to projective objects.

When Γ and Γ' are finitely generated, we have a commutative diagram

$$\begin{tikzcd} \mathscr{C}[\Gamma] \arrow[r, "M\mapsto \mathrm{Hom}(-,M)", "\sim"'] \arrow[d, hook] & \mathrm{Proj}(\mathscr{C}[\Gamma])\text{-mod} \arrow[d, "-\circ F_{\Gamma\subset\Gamma'}"] \\ \mathscr{C}[\Gamma'] \arrow[r, "\sim", "M\mapsto \mathrm{Hom}(-,M)"'] & \mathrm{Proj}(\mathscr{C}[\Gamma'])\text{-mod} \end{tikzcd}$$

and the vertical arrow $-\circ F_{\Gamma\subset\Gamma'}$ is fully faithful.

Given $M \in \mathscr{C}$, there is a finitely generated ideal Γ' of Δ such that $M \in \mathscr{C}[\Gamma']$ (Corollary F.1.10) and $\mathrm{Hom}(-,M)$ defines an object of $\mathrm{Proj}(\mathscr{C}[\Gamma'])$-mod. So, we obtain a functor

$$\mathscr{C} \to \mathrm{colim}_\Gamma\ \mathrm{Proj}(\mathscr{C}[\Gamma])\text{-mod},$$

where the colimit is taken using the system of strictly transitive transition functors $-\circ F_{\Gamma\subset\Gamma'}$. Lemma F.1.25 shows the following result.

Corollary F.1.26. — *The Yoneda functor gives an equivalence*

$$\mathscr{C} \xrightarrow{\sim} \mathrm{colim}_\Gamma\ \mathrm{Proj}(\mathscr{C}[\Gamma])\text{-mod},$$

where Γ runs over finitely generated ideals of Δ.

Let k' be a noetherian commutative k-algebra. Given Γ a finitely generated ideal of Δ, we put $(k'\mathscr{C})_\Gamma = \big(k'\,\mathrm{Proj}(\mathscr{C}[\Gamma])\big)$-mod. We define $k'\mathscr{C} = \mathrm{colim}_\Gamma (k'\mathscr{C})_\Gamma$, where Γ runs over finitely generated ideals of Δ. Note that, in this colimit, given $\Gamma_1 \subset \Gamma_2$, the functors $(k'\mathscr{C})_{\Gamma_1} \to (k'\mathscr{C})_{\Gamma_2}$ are fully faithful.

The base change functor $k'\otimes - : \mathrm{Proj}(\mathscr{C}[\Gamma])\text{-mod} \to \big(k'\,\mathrm{Proj}(\mathscr{C}[\Gamma])\big)$-mod induces a base change functor $k'\otimes - : \mathscr{C} \to k'\mathscr{C}$.

Proposition F.1.27. — *Assume k' is a finitely generated module over a localization of k. The category $k'\mathscr{C}$ is a highest weight category with set of standard objects $k'\Delta = \{k'D\}_{D\in\Delta}$.*

Proof. — It follows from Lemma F.1.22 that $(k'\mathscr{C})_\Gamma$ is a Serre subcategory of $(k'\,\mathrm{Proj}(\mathscr{C}[\Gamma]))$-Mod. We deduce that $k'\mathscr{C}$ is an abelian category that admits sums of subobjects.

Let $D \in \Delta$. The object D is projective in $\mathscr{C}[\Delta_{\leqslant D}]$, hence

$$\mathrm{End}_{k'\mathscr{C}}(k'D) = \mathrm{End}_{(k'\mathscr{C})_{\Delta_{\leqslant D}}}(k'D) = k'\,\mathrm{End}_{\mathscr{C}[\Delta_{\leqslant D}]}(D) = k'.$$

So, (i) in Definition F.1.1 holds for $k'\mathscr{C}$.

Let $D_1, D_2 \in \Delta$ and let Γ be the ideal of Δ generated by D_1 and D_2. If $D_1 \not\leqslant D_2$, then D_1 is projective in $\mathscr{C}[\Gamma]$, hence as above we have $\mathrm{Hom}(k'D_1, k'D_2) \simeq k'\,\mathrm{Hom}(D_1, D_2) = 0$. We deduce that (ii) holds.

By assumption, every object of $k'\mathscr{C}$ is a quotient of an object of the form $k'P$, where P is a projective object of $\mathscr{C}[\Gamma]$ for some finitely generated Γ. Note that $k'P \in (k'\mathscr{C})^{k'\Delta}$ and (iii) holds.

Let Γ be a finitely generated ideal of Δ. Let $M \in (k'\mathscr{C})_\Gamma$. Let $D \in \Delta\setminus\Gamma$ and consider $g : k'D \to M$. Let $P \in \mathrm{Proj}(\mathscr{C}[\Gamma])$ and $f : k'P \twoheadrightarrow M$ be a surjection. Let $\Gamma' = \Gamma \cup \Delta_{\leqslant D}$. Since D is projective in $\mathscr{C}[\Gamma']$ (Lemma F.1.12), it follows that $k'D$ is projective in $(k'\mathscr{C})_{\Gamma'}$, hence there is $h : k'D \to k'P$ such that $g = f \circ h$. On the other hand, $\mathrm{Hom}_{(k'\mathscr{C})_{\Gamma'}}(k'D, k'P) \simeq k'\mathrm{Hom}_{\mathscr{C}}(D, P) = 0$. So, $g = 0$. We deduce that $\mathrm{Hom}(k'D, M) = 0$, hence $(k'\mathscr{C})_\Gamma \subset (k'\mathscr{C})[k'\Gamma]$.

Let $D, D' \in \Delta$ and $M \in k'\mathscr{C}$. Let Γ be a finitely generated ideal of Δ containing D and D' and such that $M \in (k'\mathscr{C})_\Gamma$. By Lemma F.1.12, there is $P \in \mathrm{Proj}(\mathscr{C}[\Gamma])$ and a surjective map $P \twoheadrightarrow D$ whose kernel is in $\mathscr{C}^{\Delta_{>D}}$. So, we have a surjection $k'P \twoheadrightarrow k'D$ whose kernel is in $(k'\mathscr{C})^{k'\Delta_{>D}}$. We have $\mathrm{Hom}_{(k'\mathscr{C})_\Gamma}(k'P, k'D') \simeq k'\mathrm{Hom}_{\mathscr{C}[\Gamma]}(P, D')$ and $\mathrm{Hom}_{\mathscr{C}}(P, D')$ is a finitely generated projective k-module (Lemma F.1.15). It follows that $\mathrm{Hom}_{k'\mathscr{C}}(k'P, k'D')$ is a finitely generated projective k'-module.

Consider an exact sequence $0 \to M \to N \to k'P \to 0$ with $N \in k'\mathscr{C}$. There is a finitely generated ideal Γ' of Δ containing Γ and such that $N \in (k'\mathscr{C})_{\Gamma'}$. Now, there is $R \in \mathrm{Proj}(\mathscr{C}[\Gamma'])$ and a surjection $f : k'R \twoheadrightarrow N$. By Proposition F.1.9(i), there is $R' \leqslant R$ such that $R' \in \mathscr{C}^{\Gamma\setminus\Gamma'}$ and $R/R' \in \mathscr{C}^{\Gamma}$. We have $\mathrm{Hom}(k'R', k'P) = 0$ and $\mathrm{Hom}(k'R', M) = 0$, hence f factors through a surjection $k'(R/R') \twoheadrightarrow N$. We have $k'(R/R') \in (k'\mathscr{C})_\Gamma$, hence $N \in (k'\mathscr{C})_\Gamma$, so the surjection $N \twoheadrightarrow k'P$ splits. This shows that $\mathrm{Ext}^1_{k'\mathscr{C}}(k', M) = 0$, hence (iv) holds. This completes the proof of the proposition. $\square$

F.1.F. Grothendieck groups. — The next lemma follows from [**We**, Lemma II.6.2.7].

Lemma F.1.28. — *We have* $K_0(\mathscr{C}) = \mathrm{colim}_\Gamma K_0(\mathscr{C}[\Gamma])$, *where* Γ *runs over finitely generated ideals of* Δ.

Let $V \in k$-mod. Given $M \in \mathscr{D}$, the object $V \otimes_k \mathrm{Hom}(-, M)$ is representable by an object $V \otimes_k M$: given $k^r \xrightarrow{f} k^s \to V \to 0$ an exact sequence in k-mod, we have $V \otimes_k M = \mathrm{coker}(f \otimes M : M^r \to M^s)$.

Lemma F.1.29. — *Let* $M \in \mathscr{C}^\Delta$ *and let* $0 \to V_1 \to V \to V_2 \to 0$ *be an exact sequence in* k-mod. *We have an exact sequence* $0 \to V_1 \otimes_k M \to V \otimes_k M \to V_2 \otimes_k M \to 0$.

Proof. — We can assume that Δ is finitely generated as an ideal, so that $\mathscr{C}$ has enough projectives. Let $P \in \mathrm{Proj}(\mathscr{C})$. Since $\mathrm{Hom}(P, M)$ is projective over k, it follows that we have an exact sequence $0 \to V_1 \otimes_k \mathrm{Hom}(P, M) \to V \otimes_k \mathrm{Hom}(P, M) \to V_2 \otimes_k \mathrm{Hom}(P, M) \to 0$, hence an exact sequence $0 \to \mathrm{Hom}(P, V_1 \otimes_k M) \to \mathrm{Hom}(P, V \otimes_k M) \to \mathrm{Hom}(P, V_2 \otimes_k M) \to 0$. The lemma follows. $\square$

Lemma F.1.30. — *Let Γ be an ideal of Δ such that $\Delta\backslash\Gamma$ is finite. We have an exact sequence*

$$0\to K_0(\mathscr{C}[\Gamma])\to K_0(\mathscr{C})\to K_0(\mathscr{C}(\Delta\setminus\Gamma))\to 0.$$

Proof. — Without assumption on $\Delta\setminus\Gamma$, there is an exact sequence [**We**, Theorem II.6.4]

$$K_0(\mathscr{C}[\Gamma])\to K_0(\mathscr{C})\to K_0(\mathscr{C}(\Delta\setminus\Gamma))\to 0.$$

Assume $\Delta\backslash\Gamma$ has a single element D_0. It follows from Lemma F.1.29 that the functor $D_0\otimes_k -: k\text{-mod}\to\mathscr{C}$ is exact. Let $M\in\mathscr{C}$. Let N be the cone of the adjunction map $D_0\otimes_k \operatorname{Hom}_{\mathscr{C}}(D_0,M)\to M$. We have $\operatorname{Hom}_{D^b(\mathscr{C})}(D_0,N[i])=0$ for all i. Since D_0 is projective, it follows that $\operatorname{Hom}_{\mathscr{C}}(D_0,H^i(N))=0$ for all i. So, N is a bounded complex with cohomology contained in $\mathscr{C}[\Gamma]$. It follows that $M\mapsto N$ provides a left adjoint to the inclusion functor of the thick subcategory of $D^b(\mathscr{C})$ of complexes with cohomology in $\mathscr{C}[\Gamma]$. As a consequence, the canonical map $K_0(\mathscr{C}[\Gamma])\to K_0(\mathscr{C})$ is a split injection. So, the lemma holds when $|\Delta\setminus\Gamma|=1$. The general case follows by induction on $|\Delta\setminus\Gamma|$. ☐

It follows from Lemma F.1.29 that given $V\in k$-mod and $D\in\Delta$, the class $[V\otimes_k D]\in K_0(\mathscr{C})$ depends only on $[V]\in K_0(k\text{-mod})$ and D. We denote it by $[V]\cdot[D]$. This provides $K_0(\mathscr{C})$ with a structure of $K_0(k\text{-mod})$-module.

Lemma F.1.31. — *The morphism of $K_0(k\text{-mod})$-modules*

$$K_0(k\text{-mod})^{(\Delta)}\to K_0(\mathscr{C}),\ (a_D)_{D\in\Delta}\mapsto\sum_D a_D\cdot[D]$$

is injective.

If Δ is finitely generated as an ideal and $\operatorname{Spec} k$ is connected, then the canonical map $K_0(\mathscr{C}\text{-proj})\to K_0(\mathscr{C})$ is injective with image the free $\mathbb{Z}$-submodule generated by $\{[D]\}_{D\in\Delta}$.

Proof. — When Δ is finite, the lemma follow from Lemma F.1.30. When Δ is finitely generated, given a finite family I of Δ, there is an ideal Γ of Δ such that $\Delta\backslash\Gamma$ is finite and contains I. Since the lemma holds for $\mathscr{C}/\mathscr{C}[\Gamma]$, we deduce that the lemma holds for $\mathscr{C}$. The general case follows from Lemma F.1.28. ☐

Definition F.1.32. — *We say that $\mathscr{C}$ is* separated *if $D^b(\mathscr{C})$ is generated as a triangulated category by $\{V\otimes_k D\}$, where $V\in k$-mod and $D\in\Delta$.*

Lemma F.1.33. — *If all the finitely generated ideals of Γ are finite, then $\mathscr{C}$ is separated.*

Proof. — When Δ is finite, the statement follows by induction as in the proof of Lemma F.1.30.

Let M be a bounded complex of objects of $\mathscr{C}$. There is a finitely generated ideal Γ of Δ such that M is a bounded complex of objects of $\mathscr{C}[\Gamma]$. Since the lemma holds for $\mathscr{C}[\Gamma]$, we are done. ☐

Lemma F.1.34. — *If $\mathscr{C}$ is separated, then $K_0(\mathscr{C})$ is a free $K_0(k\text{-mod})$-module with basis $\{[D]\}_{D\in\Delta}$.*

If in addition Δ is finite and $\operatorname{Spec} k$ *is connected, then there is an isomorphism*

$$K_0(k\text{-mod})\otimes_{\mathbb{Z}} K_0(\mathscr{C}\text{-proj}) \xrightarrow{\sim} K_0(\mathscr{C}),\ a\otimes[P]\mapsto a\cdot[P].$$

Proof. — The first statement follows immediately from Lemma F.1.31. The second statement follows by induction from Lemma F.1.30. ☐

Remark F.1.35. — If $D^b(\mathscr{C})$ is generated, as a triangulated category closed under taking direct summands, by $\{V\otimes_k D\}$, where $V\in k$-mod and $D\in\Delta$, then it is separated: this is shown by the proof of Lemma F.1.30 when Δ is finite, and the general case follows. ■

F.1.G. Completed Grothendieck groups. — We define the *completed Grothendieck group* of $\mathscr{C}$ as $\hat{K}_0(\mathscr{C})=\operatorname{colim}_\Gamma \lim_{\Omega\subset\Gamma} K_0(\mathscr{C}[\Gamma]/\mathscr{C}[\Omega])$ where Γ runs over finitely generated ideals of Δ and Ω runs over ideals of Γ such that $\Gamma\setminus\Omega$ is finite.

Note that given $\Omega\subset\Omega'$, the transition map $K_0(\mathscr{C}[\Gamma]/\mathscr{C}[\Omega])\to K_0(\mathscr{C}[\Gamma]/\mathscr{C}[\Omega'])$ is surjective, while given $\Gamma'\subset\Gamma$, the transition map $\lim_{\Omega\subset\Gamma} K_0(\mathscr{C}[\Gamma]/\mathscr{C}[\Omega])\to\lim_{\Omega\subset\Gamma'} K_0(\mathscr{C}[\Gamma']/\mathscr{C}[\Omega])$ is injective.

There is a canonical morphism of groups

$$K_0(\mathscr{C})\to\hat{K}_0(\mathscr{C}),\ [M]\mapsto[[M]]$$

since $\mathscr{C}=\operatorname{colim}_\Gamma\mathscr{C}[\Gamma]$, where Γ runs over finitely generated ideals of Δ.

When Γ is finitely generated, we have $\hat{K}_0(\mathscr{C})=\lim_{\Omega\subset\Gamma} K_0(\mathscr{C}[\Gamma]/\mathscr{C}[\Omega])$ where Ω runs over ideals of Δ such that $\Delta\setminus\Omega$ is finite.

When Δ is finite, we have a canonical isomorphism $K_0(\mathscr{C})\xrightarrow{\sim}\hat{K}_0(\mathscr{C})$.

Let $\operatorname{Map}^{\mathrm{fg}}(\Delta,K_0(k\text{-mod}))$ be the abelian group of maps $\chi:\Delta\to K_0(k\text{-mod})$ such that $\{D\in\Delta\mid\chi(D)\neq0\}$ is contained in a finitely generated ideal of Δ.

Lemma F.1.36. — *There is an isomorphism*

$$\sigma:\operatorname{Map}^{\mathrm{fg}}(\Delta,K_0(k\text{-mod}))\xrightarrow{\sim}\hat{K}_0(\mathscr{C}),\ \chi\mapsto\sum_{D\in\Delta}[\chi(D)]\cdot[[D]]$$

and an isomorphism

$$\hat{K}_0(\mathscr{C})\xrightarrow{\sim}\operatorname{Map}^{\mathrm{fg}}(\Delta,K_0(k\text{-mod})),\ [[M]]\mapsto(D\mapsto\sum_{i\geqslant0}(-1)^i[\operatorname{Ext}^i(D,M)]).$$

Proof. — Assume first Δ is finite. Consider the morphisms

$$f:K_0(k\text{-mod})^\Delta\xrightarrow{\sim}K_0(\mathscr{C}),\ ([V_D])_{D\in\Delta}\mapsto\sum_{D\in\Delta}[V_D]\cdot[D]$$

and

$$g:K_0(\mathscr{C})\xrightarrow{\sim}K_0(k\text{-mod})^\Delta,\ [M]\mapsto(D\mapsto\sum_{i\geqslant0}(-1)^i[\operatorname{Ext}^i(D,M)]).$$

Lemma F.1.34 shows that f is an isomorphism. Since $g \circ f$ has a triangular matrix with entries 1 on the diagonal, it follows that g is an isomorphism.

Consider now a general Δ. We have $\mathrm{Map}^{\mathrm{fg}}(\Delta, K_0(k\text{-mod})) = \mathrm{colim}_\Gamma \lim_\Omega K_0(k\text{-mod})^\Omega$ where Γ runs over finitely generated ideals of Δ and Ω runs over ideals of Γ such that $\Gamma \setminus \Omega$ is finite. The lemma follows from the case where Δ is finite. $\square$

Lemmas F.1.34 and F.1.36 have the following consequence.

Proposition F.1.37. — *If $\mathscr{C}$ is separated, then we have a canonical injection $K_0(\mathscr{C}) \hookrightarrow \hat{K}_0(\mathscr{C})$.*

Lemma F.1.38. — *Assume k is a field. The map $M \mapsto ([M : L(D)])_{D\in\Delta}$ induces an injection $\hat{K}_0(\mathscr{C}) \hookrightarrow \mathbb{Z}^\Delta$.*

Proof. — Let $M \in \mathscr{C}$ and $D \in \Delta$. There is a finitely generated ideal Γ of Δ and an ideal Ω of Γ such that $\Gamma \setminus \Omega$ is finite and contains D and such that $M \in \mathscr{C}[\Gamma]$. We have $[M : L(D)] = [M' : L(D)]$, where M' is the image of M in $\mathscr{C}(\Gamma \setminus \Omega)$. It follows that $[M : L(D)]$ depends only on the class of M in $\hat{K}_0(\mathscr{C})$.

When Δ is finite, $\mathscr{C}$ is equivalent to the category of finite-dimensional modules over a finite-dimensional k-algebra, hence the class of a module in K_0 is determined by the multiplicities of simple modules in a composition series. So, we obtained the injectivity when Δ is finite, and the general case follows. $\square$

Let k' be a noetherian commutative flat k-algebra. There is a commutative diagram

$$\begin{tikzcd}
K_0(\mathscr{C}) \arrow[r] \arrow[d] & \hat{K}_0(\mathscr{C}) \arrow[r, "\sigma^{-1}", "\sim"'] \arrow[d] & \mathrm{Map}^{\mathrm{fg}}(\Delta, K_0(k\text{-mod})) \arrow[d] \\
K_0(k'\mathscr{C}) \arrow[r] & \hat{K}_0(k'\mathscr{C}) \arrow[r, "\sim", "\sigma^{-1}"'] & \mathrm{Map}^{\mathrm{fg}}(\Delta, K_0(k'\text{-mod}))
\end{tikzcd}$$

where the vertical maps are induced by the functor $k' \otimes_k -$.

F.1.H. Decomposition maps. — Assume k is a discrete valuation ring with residue field $\bar{k}$ and field of fractions K. It follows from §F.1.E that there is a decomposition map

$$d : K_0(K\mathscr{C}) \to K_0(\bar{k}\mathscr{C})$$

with the property that $d([KM]) = [\bar{k}M]$ when M is an object of $\mathscr{C}[\Gamma]$ where Γ is a finitely generated ideal of Δ and $\mathrm{Hom}(P, M) \in k$-proj for all $P \in \mathrm{Proj}(\mathscr{C}[\Gamma])$.

We construct now decomposition maps over more general local rings k, using completed Grothendieck groups.

Assume k is a local integral ring with residue field $\bar{k}$ and field of fractions K. We have canonical isomorphisms $\dim_K : K_0(K\text{-mod}) \xrightarrow{\sim} \mathbb{Z}$ and $\dim_{\bar{k}} : K_0(\bar{k}\text{-mod}) \xrightarrow{\sim} \mathbb{Z}$.

We define $\hat{d} : K_0(K\mathscr{C}) \xrightarrow{\sim} K_0(\bar{k}\mathscr{C})$ as the map making the following diagram commutative

$$\begin{tikzcd} \hat{K}_0(K\mathscr{C}) \arrow[r, "\sigma^{-1}", "\sim"'] \arrow[d, "\hat{d}"'] & \mathrm{Map}^{\mathrm{fg}}(\Delta, \mathbb{Z}) \arrow[d, equal] \\ \hat{K}_0(k'\mathscr{C}) \arrow[r, "\sim", "\sigma^{-1}"'] & \mathrm{Map}^{\mathrm{fg}}(\Delta, \mathbb{Z}) \end{tikzcd}$$

Proposition F.1.39. — *Let $M \in \mathscr{C}$ and let Γ be a finitely generated ideal of Δ such that $M \in \mathscr{C}[\Gamma]$. Assume* $\mathrm{Hom}(P, M) \in k$-proj *for all* $P \in \mathrm{Proj}(\mathscr{C}[\Gamma])$. *Then* $\hat{d}([[KM]]) = [[kM]]$.

Proof. — Assume first Δ is finite. There are objects $\bar{D}$ of $\mathscr{C}$ for $D \in \Delta$ such that $\mathrm{Ext}^i(D, \bar{D}') = \delta_{0i}\delta_{D,D'}$ for all $D, D' \in \Delta$ and $i \geqslant 0$ (Proposition F.1.16 and [**Rou**, Proposition 4.19]). The assumption on M guarantees that it has a finite projective resolution $0 \to P^{-n} \to \cdots \to P^0 \to M \to 0$ [**Rou**, Proposition 4.23]. As a consequence, $[[M]] = \sigma(D \mapsto \sum_{i \geqslant 0}(-1)^i[\mathrm{Hom}(P^i, \bar{D})])$. Since $\dim_K[\mathrm{Hom}_{K\mathscr{C}}(KP^i, K\bar{D})] = \dim_k[\mathrm{Hom}_{k\mathscr{C}}(kP^i, k\bar{D})]$, the proposition follows.

Consider the general case. Let Γ be a finitely generated ideal of Δ such that $M \in \mathscr{C}[\Gamma]$. Let Ω be a finitely generated ideal of Γ such that $\Gamma \setminus \Omega$ is finite. There exists a projective object of $\mathscr{C}[\Gamma]$ whose image in $\mathscr{C}[\Gamma]/\mathscr{C}[\Omega]$ is a progenerator, and $\mathrm{Hom}(P, M) \simeq \mathrm{Hom}(q_\Omega(P), q_\Omega(M))$ (cf proof of Lemma F.1.19). Since the proposition holds for $q_\Omega(M)$, we deduce that it holds for M. $\square$

When k is a discrete valuation ring, there is a commutative diagram

$$\begin{tikzcd} K_0(K\mathscr{C}) \arrow[r] \arrow[d, "d"'] & \hat{K}_0(K\mathscr{C}) \arrow[r, "\sigma^{-1}", "\sim"'] \arrow[d, "\hat{d}"'] & \mathrm{Map}^{\mathrm{fg}}(\Delta, \mathbb{Z}) \arrow[d, equal] \\ K_0(\bar{k}\mathscr{C}) \arrow[r] & \hat{K}_0(k'\mathscr{C}) \arrow[r, "\sim", "\sigma^{-1}"'] & \mathrm{Map}^{\mathrm{fg}}(\Delta, \mathbb{Z}) \end{tikzcd}$$

F.1.I. Blocks. — We assume in §F.1.I that $\mathrm{Spec}\, k$ is connected.

We define the equivalence relation $\sim$ on Δ as the one generated by $D \sim D'$ when $\mathrm{Ext}^i_{\mathscr{C}}(D, D') \neq 0$ for some $i \in \{0, 1\}$. This is the equivalence relation generated by the partial order $<$ (cf Proposition F.1.8).

Proposition F.1.40. — *Given $I \in \Delta/\sim$, the full subcategory $\mathscr{C}[I]$ of $\mathscr{C}$ is an indecomposable Serre subcategory whose objects are the quotients of objects of $\mathscr{C}^I$. It is a highest weight category with poset of standard objects I. We have $\mathscr{C} = \bigoplus_{I \in \Delta/\sim} \mathscr{C}[I]$.*

Proof. — Let $I, J \in \Delta/\sim$ with $I \neq J$. Given $M \in \mathscr{C}^I$ and $N \in \mathscr{C}^J$, we have $\mathrm{Ext}^1(M, N) = 0$ and $\mathrm{Hom}(M, N) = 0$. It follows that $\mathscr{C}^\Delta = \bigoplus_{I \in \Delta/\sim} \mathscr{C}^I$. We deduce that a quotient of an object of $\mathscr{C}^I$ is in $\mathscr{C}[I]$.

Let $L\in\mathscr{C}$. There is an exact sequence $M\xrightarrow{f}N\to L\to 0$ with $M,N\in\mathscr{C}^{\Delta}$. We have decompositions $M=\bigoplus_{I\in\Delta/\sim}M^I$, $N=\bigoplus_{I\in\Delta/\sim}N^I$ and $f=\sum_I f^I$ with $f^I:M^I\to N^I$ and $M^I,N^I\in\mathscr{C}^I$. It follows that $L=\bigoplus_I \operatorname{coker} f^I$ and $\operatorname{coker} f^I$ is a quotient of an object of $\mathscr{C}^I$. Assume $L\in\mathscr{C}[I]$. Consider $J\neq I$. We have $\operatorname{Hom}(N^J,L)=0$, hence $\operatorname{coker} f^J=0$. It follows that $L=\operatorname{coker} f^I$ is the quotient of an object of $\mathscr{C}^I$. We have shown that $\mathscr{C}=\bigoplus_{I\in\Delta/\sim}\mathscr{C}[I]$.

Let e be an idempotent of the center of $\mathscr{C}[I]$. Note that e acts by 0 or 1 on an object of Δ. Let I_e be the subset of I of objects on which e acts by 0. Given $D,D'\in I$ with $\operatorname{Ext}^i(D,D')\neq 0$ for some $i\in\{0,1\}$, we have $D,D'\in I_e$ or $D,D'\in I\setminus I_e$. We deduce that $I=I_e$ or $I_e=\emptyset$. Since every object of $\mathscr{C}[I]$ is the quotient of an object of $\mathscr{C}^I$, it follows that $e=0$ or $e=1$. So, $\mathscr{C}[I]$ is indecomposable.

Since $(\mathscr{C},\Delta,\lessdot)$ is a highest weight category (Proposition F.1.8), it follows that $\mathscr{C}[I]$ is a highest weight category with set of standard I and partial order $\lessdot$, hence it is a highest weight category with the partial order $<$. □

Lemma F.1.41. — *Let $D,D'\in\Delta$.*

We have $D\sim D'$ if and only if there is a finitely generated ideal Γ of Δ and an ideal Ω of Γ such that $\Gamma\setminus\Omega$ is a finite set containing D and D' and such that $q_\Omega(D)$ and $q_\Omega(D')$ are in the same block of $\mathscr{C}[\Gamma](\Omega)$.

Proof. — Let Γ be a finitely generated ideal of Δ and Ω an ideal of Γ such that $\Gamma\setminus\Omega$ is a finite set containing D and D'. We have $\operatorname{Ext}^i_{\mathscr{C}}(D,D')\simeq\operatorname{Ext}^i_{\mathscr{C}[\Gamma](\Omega)}(q_\Omega(D),q_\Omega(D'))$ for $i\in\{0,1\}$ (cf proof of Proposition F.1.18).

We have $D\sim D'$ if and only if there exists $D_0=D,D_1,\ldots,D_n=D'$ in Δ such that $\operatorname{Ext}^*(D_i,D_{i+1})\neq 0$ or $\operatorname{Ext}^*(D_{i+1},D_i)\neq 0$ for some $*\in\{0,1\}$, for all $i\in\{0,\ldots,n-1\}$. So, $D\sim D'$ if and only if there is a finitely generated ideal Γ of Δ, an ideal Ω of Γ and $D_0=D,D_1,\ldots,D_n=D'$ in $\Gamma\setminus\Omega$ such that $\operatorname{Ext}^*_{\mathscr{C}[\Gamma](\Omega)}(q_\Omega(D_i),q_\Omega(D_{i+1}))\neq 0$ or $\operatorname{Ext}^*_{\mathscr{C}[\Gamma](\Omega)}(q_\Omega(D_{i+1}),q_\Omega(D_i))\neq 0$ for some $*\in\{0,1\}$, for all $i\in\{0,\ldots,n-1\}$: this is equivalent to the requirement that $q_\Omega(D)$ and $q_\Omega(D')$ are in the same block of $\mathscr{C}[\Gamma](\Omega)$ by Proposition F.1.40. □

Given Γ a finitely generated ideal of Δ and Ω an ideal of Γ such that $\Gamma\setminus\Omega$ is a finite set containing D and D', there is a finite family $B_{\Gamma,\Omega}$ of prime ideals of k such that given a prime ideal $\mathfrak{q}$ of k, we have $k_{\mathfrak{q}}\otimes_k D\sim k_{\mathfrak{q}}\otimes_k D'$ if and only if $\mathfrak{p}\subset\mathfrak{q}$.

Lemma F.1.42. — *Let $D,D'\in\Delta$ and let k' be a commutative noetherian k-algebra that is a finitely generated module over a localization of k. If $k'\otimes_k D\sim k'\otimes_k D'$ in $k'\Delta$ then $D\sim D'$.*

Proof. — Assume first Δ is finite. In that case, the result is classical: we have $R\operatorname{Hom}^\bullet_{k'\mathscr{C}}(k'D,k'D')\simeq k'\otimes^{\mathbb{L}}R\operatorname{Hom}^\bullet_{\mathscr{C}}(D,D')$. So, if $\operatorname{Ext}^*_{k'\mathscr{C}}(k'D,k'D')\neq 0$ for some $*\in\{0,1\}$, then $\operatorname{Ext}^*_{\mathscr{C}}(D,D')\neq 0$ for some $*\geqslant 0$.

Consider $D=D_0,\dots,D_n=D'$ in Δ with $\mathrm{Ext}^*_{k'\mathscr{C}}(k'D_i,k'D_{i+1})\neq 0$ or $\mathrm{Ext}^*_{k'\mathscr{C}}(k'D_{i+1},k'D_i)\neq 0$ for some $*\in\{0,1\}$ for all $i\in\{0,n-1\}$. We have $\mathrm{Ext}^*_{\mathscr{C}}(D_i,D_{i+1})\neq 0$ or $\mathrm{Ext}^*_{\mathscr{C}}(D_{i+1},D_i)\neq 0$ for some $*\geqslant 0$ for all $i\in\{0,n-1\}$. It follows that D and D' are in the same block of $\mathscr{C}$, hence $D\sim D'$ by Proposition F.1.40.

The general case follows from Proposition F.1.27 and its proof. □

Proposition F.1.43. — *Assume k is integral and integrally closed. One of the following holds:*

- *Given $\mathfrak{p}$ a prime ideal of k, a subset of Δ corresponds to a block of $k_{\mathfrak{p}}\mathscr{C}$ if and only if it is a block of $K\mathscr{C}$, where K is the field of fractions of k.*
- *There exists a family $\mathscr{F}$ of height one prime ideals of k with the following property: given $\mathfrak{p}$ a prime ideal of k, a subset of Δ corresponds to a block of $k_{\mathfrak{p}}\mathscr{C}$ if and only if it is a union of blocks of $k_{\mathfrak{q}}\mathscr{C}$ for any $\mathfrak{q}\in\mathscr{F}$ with $\mathfrak{q}\subset\mathfrak{p}$.*

In the second case, $\mathscr{F}$ is a union over the set of finite subsets of Δ, of finite sets.

Proof. — Assume Δ is finite. We have $\mathscr{C}\simeq A$-mod where A is a k-algebra that is finitely generated and projective as a k-module. The proposition follows from Proposition F.1.40 and Proposition D.2.11

The general case follows from Lemma F.1.41. The set $\mathscr{F}$ is the union of the finite sets associated with the categories $\mathscr{C}[\Gamma](\Omega)$, where Γ runs over finitely generated ideals of Δ, Ω runs over ideals of Γ such that $\Gamma\backslash\Omega$ is finite and generates Γ as an ideal of Δ. □

Remark F.1.44. — Proposition F.1.42 shows that if $D\not\sim D'$, then $k_{\mathfrak{q}}\otimes_k D\not\sim k_{\mathfrak{q}}\otimes_k D'$ for all prime ideals $\mathfrak{q}$ of k. ■

F.2. Triangular algebras

The construction of a highest weight category from representations of triangular algebras in [**GGOR**, §2] is done under the presence of a grading coming from an inner derivation. The method used there does not actually use that the gradings are inner, and we describe here constructions following [**GGOR**, §2], with a more general setting.

F.2.A. Definition. — Let A be a graded k-algebra with three graded subalgebras B_+, B_- and H such that

(i) H, B_- and B_+ are flat k-modules
(ii) the multiplication map $\mu: B_+\otimes H\otimes B_-\to A$ is an isomorphism of k-modules
(iii) $\mu(B_+\otimes H)=\mu(H\otimes B_+)$ and $\mu(B_-\otimes H)=\mu(H\otimes B_-)$
(iv) $B_-^{>0}=B_+^{<0}=0$, $B_-^0=B_+^0=k$ and $H=H^0$

Given $\epsilon \in \{+,-\}$, the canonical map $H \to (B_\epsilon H)/(B_\epsilon H)^{<0}$ is an isomorphism. Composing its inverse with the quotient map, we obtain a canonical morphism of graded k-algebras $B_\epsilon H \twoheadrightarrow H$.

We identify the category of H-modules with the category of graded H-modules that are concentrated in degree 0.

F.2.B. Induced modules. —

F.2.B.1. Given F a graded (B_-H)-module, we put $\Delta(F) = A \otimes_{B_-H} F$. Note that Δ is an exact functor that is left adjoint to the restriction functor from the category of graded A-modules to the category of graded (B_-H)-modules.

There is a canonical isomorphism of graded (B_+H)-modules

$$B_+H \otimes_H F \xrightarrow{\sim} \Delta(F)$$

and a canonical isomorphism of graded B_+-modules

$$B_+ \otimes F \xrightarrow{\sim} \Delta(F).$$

When E is an H-module, we view E as a graded (B_-H)-module concentrated in degree 0 through the canonical map $p : B_-H \twoheadrightarrow H$ and put $\Delta(E) = \Delta(p^*(E))$. There is a canonical isomorphism of graded (B_+H)-modules

$$B_+H \otimes_H E \xrightarrow{\sim} \Delta(E).$$

Let $n \geqslant 0$. We put

$$\Delta_n(E) = \Delta\Big(\big((B_-H/(B_-H)^{<-n}\big) \otimes_H E\Big).$$

Note that $\Delta_0(E) = \Delta(E)$ and $\Delta_n(E)$ has a filtration with subquotients

$$\Delta(E), \Delta\big((B_-H)^{-1} \otimes_H E\big), \ldots, \Delta\big((B_-H)^{-n} \otimes_H E\big).$$

F.2.B.2. Given M a B_--module and n a non-positive integer, let $\mathrm{Ann}_{B_-^{<n}}(M) = \{m \in M \mid B_-^{<n} m = 0\}$, a B_--submodule of M. We put $M_{\mathrm{ln}} = \bigcup_{n<0} \mathrm{Ann}_{B_-^{<n}}(M)$ and we say that M is *locally nilpotent* for B_- if $M = M_{\mathrm{ln}}$. The functor $M \mapsto M_{\mathrm{ln}}$ is right adjoint to the inclusion functor from the category of locally nilpotent B_--modules to the category of B_--modules.

Note that

- When M is a graded B_--module, M_{ln} is a graded B_--submodule of M.
- If M is a (B_-H)-module, then M_{ln} is a (B_-H)-submodule of M, since $HB_-^{<n} = (HB_-)^{<n} = (B_-H)^{<n} = B_-^{<n}H$.

Assume now M is an A-module. Since $(B_-)_{<i}H(B_+)_j \subset A_{<i+j} \subset A(B_-)_{<i+j}$ for $i+j<0$, it follows that M_{ln} is A-submodule of M. We deduce that $M \mapsto M_{\mathrm{ln}}$ is right adjoint to the inclusion functor from the category of A-modules (resp. graded A-modules) that are locally nilpotent as B_--modules to the category of A-modules (resp. graded A-modules).

Let $M \in A$-modgr and $E \in H$-modgr. We have an isomorphism of k-modules

$$\mathrm{Hom}_{A\text{-modgr}}(\Delta_n(E), M) \xrightarrow{\sim} \mathrm{Hom}_{H\text{-modgr}}(E, \mathrm{Ann}_{B_-^{<-n}}(M)),\ f \mapsto f_{|1\otimes E}$$

If E is concentrated in degree d, then

$$\mathrm{Hom}_{H\text{-modgr}}(E, \mathrm{Ann}_{B_-^{<-n}}(M)) = \mathrm{Hom}_H(E, \big(\mathrm{Ann}_{B_-^{<-n}}(M)\big)_d) \subset \mathrm{Hom}_H(E, M_d).$$

Note that if M is a finitely generated graded A-module that is locally nilpotent for B_-, then $M_{<i} = 0$ for $i \ll 0$.

Lemma F.2.1. — *Let $M \in A$-modgr, let E be an H-module, let $d \in \mathbb{Z}$ and let $i \in \mathbb{Z}$ such that $M_{<i} = 0$. Then the canonical map $\mathrm{Hom}_{A\text{-modgr}}(\Delta_n(E)\langle -d\rangle, M) \to \mathrm{Hom}_H(E, M_d)$ is an isomorphism for $n \geqslant d - i$.*

Proof. — We have $\big(\mathrm{Ann}_{B_-^{<-n}}(M)\big)_d = M_d$ since $(B_-)^{<-n} M_d \subset M_{<d-n} = 0$ and the result follows. □

Lemma F.2.2. — *Let $E, E' \in H$-mod and let $d \in \mathbb{Z}$.*
If $d < 0$, then $\mathrm{Hom}_{A\text{-modgr}}(\Delta(E), \Delta(E'\langle d\rangle)) = 0$
The functor Δ induces an isomorphism

$$\mathrm{Hom}_H(E, E') \xrightarrow{\sim} \mathrm{Hom}_{A\text{-modgr}}(\Delta(E), \Delta(E')).$$

Proof. — We have

$$\mathrm{Hom}_{\text{-modgr}}(\Delta(E), \Delta(E'\langle d\rangle)) \simeq \mathrm{Hom}_{B_- H\text{-modgr}}(E, \Delta(E')\langle d\rangle)$$

If $d < 0$ then $(\Delta(E')\langle d\rangle)_0 = 0$, hence $\mathrm{Hom}_{B_- H\text{-modgr}}(E, \Delta(E')\langle d\rangle) = 0$. This shows the first statement.

Note that $E' = \Delta(E')_0$ is a $(B_- H)$-submodule of $\Delta(E')$. It follows that

$$\mathrm{Hom}_{B_- H\text{-modgr}}(E, \Delta(E')) \simeq \mathrm{Hom}_H(E, E')$$

and the second statement follows. □

F.2.C. Category $\mathcal{O}^{gr}$. — We fix a set I of isomorphism classes of H-modules that are finitely generated as k-modules.

We assume that

(v) given $E \in I$, every submodule of E is a quotient of a finite multiple of E
(vi) $\mathrm{End}_H(E) = k$ for all $E \in I$
(vii) $\mathrm{Hom}_H(E, F) = 0$ for all $E, F \in I$ with $E \not\simeq F$
(viii) the H-modules $\big((B_+H)^{\geqslant n}/(B_+H)^{>n}\big) \otimes_H E$ and $\big((B_-H)^{\leqslant -n}/(B_-H)^{<-n}\big) \otimes_H E$ are direct summands of finite direct sums of objects of I for all $E \in I$ and $n \geqslant 0$
(ix) the A-module $\Delta(E)$ is noetherian for all $E \in I$.

We denote by $\mathcal{O}_H$ the full subcategory of H-Mod with objects the quotients of finite direct sums of objects in I. Note that $\mathcal{O}_H$ is an abelian subcategory of H-Mod closed under quotients and subobjects, and I is a set of projective generators for $\mathcal{O}_H$. There is an equivalence of categories

$$\bigoplus_{E\in I}\mathrm{Hom}(E,-)\colon \mathcal{O}_H \xrightarrow{\sim} (k\text{-mod})^{(I)}.$$

Example F.2.3. — Assume there is a field k_0 that is a subalgebra of k and a split semisimple k_0-algebra H_0 such that $H = H_0 \otimes_{k_0} k$. Let $I = \{L \otimes_{k_0} k\}$, where L runs over the set of isomorphism classes of simple H_0-modules. The assumptions (v)-(viii) above are satisfied. ■

Remark F.2.4. — Note that Assertion (ix) is satisfied if B_+ is noetherian: since E is a finitely generated k-module, it follows that every B_+-submodule of $\Delta(E) \simeq B_+ \otimes E$ is finitely generated. ■

We denote by $\mathcal{O}^{gr}$ the category of finitely generated graded A-modules M that are locally nilpotent for B_- and satisfy $M^i \in \mathcal{O}_H$ for all $i \in \mathbb{Z}$. Note that if $\mathcal{O}_H$ is closed under extensions, then $\mathcal{O}^{gr}$ will also be closed under extensions.

Lemma F.2.5. — *Let M be a graded A-module. The following conditions are equivalent*

(i) $M \in \mathcal{O}^{gr}$

(ii) *there exists a finite family S of objects of I, $d_E \in \mathbb{Z}$ and $n_E \in \mathbb{Z}_{\geqslant 0}$ for $E \in S$ such that M is a quotient of $\bigoplus_{E\in S} \Delta_{n_E}(E\langle d_E\rangle)$.*

Proof. — Assume (i). There is a finite subset J of $\mathbb{Z}$ such that M is generated by $\bigoplus_{j\in J} M^j$ as an A-module. Given $j \in J$, there is a finite family S_j of objects of I and a surjective morphism of H-modules $f_j : \bigoplus_{E\in S_j} E \twoheadrightarrow M^j$. By adjunction, we obtain a morphism of graded (B_-H)-modules $g_j : \bigoplus_{E\in S_j} B_-H \otimes_H E\langle -j\rangle \to M$. Let $E \in S_j$. Since $f_j(E)$ is a finitely generated H-module, there is a non-negative integer n_E such that $(B_-)^{\leqslant -n_E} f_j(E) = 0$. So, g_j factors through a morphism of graded (B_-H)-modules $h_j : \bigoplus_{E\in S_j}\big((B_-H)/(B_-H)^{\leqslant -n_E}\big) \otimes_H E\langle -j\rangle \to M$. Let

$$h'_j : \bigoplus_{E\in S_j} A \otimes_{B_-H} \big((B_-H)/(B_-H)^{\leqslant -n_E}\big) \otimes_H E\langle -j\rangle \to M$$

be the morphism of graded A-modules obtained from h_j by adjunction. The map $\sum_{j\in J} h'_j$ is surjective and this shows (ii) holds.

Assume (ii). Let $E \in I$. Note that $(\Delta_n(E))_{<-n} = 0$, hence $B_-^{<-n-i}(\Delta_n(E))_i = 0$. It follows that $\Delta_n(E)$ is locally nilpotent for B_-. There is an isomorphism of H-modules $(B_-H)/(B_-H)^{<-n} \simeq \bigoplus_{i=0}^n (B_-H)^{\leqslant -i}/(B_-H)^{<-i}$, hence

$$\Delta_n(E)^i \simeq \bigoplus_j (B_+H)^{i+j} \otimes_H \big((B_-H)^{-j} \otimes_H E\big)$$

is isomorphic to a direct summand of a finite direct sum of objects of I. We deduce that $\Delta_n(E)\in\mathcal{O}^{gr}$.

Since $\mathcal{O}^{gr}$ is closed under taking finite direct sums, quotients and shifts, it follows that (i) holds. □

Lemma F.2.6. — *Let $E\in I$, $n\in\mathbb{Z}_{\geqslant 0}$ and let $M\in\mathcal{O}^{gr}$. If $M^{<-n}=0$, then $\mathrm{Ext}^1_{\mathcal{O}^{gr}}(\Delta_n(E),M)=0$.*

In particular, given $E'\in I$ and $d\leqslant n$, we have $\mathrm{Ext}^1_{A\text{-modgr}}(\Delta_n(E),\Delta(E'\langle d\rangle))=0$.

Proof. — We have

$$\mathrm{Hom}_{\text{-modgr}}(A\otimes_{B\_H}(B\_H)^{<-n}\otimes_H E,M)\simeq\mathrm{Hom}_{(B\_H)\text{-modgr}}((B\_H)^{<-n}\otimes_H E,M)=0$$

since $((B\_H)^{<-n}\otimes_H E)^i=0$ for $i\geqslant -n$, while $M^i=0$ for $i<-n$.

There is an exact sequence of graded A-modules

$$0\to A\otimes_{B\_H}(B\_H)^{<-n}\otimes_H E\to A\otimes_H E\to\Delta_n(E)\to 0.$$

Since $A\otimes_H E$ is projective in the category of graded A-modules whose restriction to H is in $\mathcal{O}_H$, we deduce that $\mathrm{Ext}^1_{\mathcal{O}^{gr}}(\Delta_n(E),M)=0$. □

Let $\Delta^{gr}=\{\Delta(E\langle n\rangle)\}_{E\in I,\ n\in\mathbb{Z}}$, a set of objects of $\mathcal{O}^{gr}$ (cf Lemma F.2.5). We put a partial order on Δ^{gr}: given $E,F\in I$ and $i,j\in\mathbb{Z}$, we put $E\langle i\rangle<F\langle j\rangle$ if $i<j$.

Theorem F.2.7. — *$\mathcal{O}^{gr}$ is a highest weight category with poset of standard objects Δ^{gr}.*

Proof. — Since $\mathcal{O}_H$ is closed under taking quotients, we deduce that that $\mathcal{O}^{gr}$ is a full subcategory of A-modgr closed under taking quotients. Let $E\in I$. By (ix), every A-submodule of $\Delta_n(E)$ is finitely generated. So, all subobjects of objects of $\mathcal{O}^{gr}$ are finitely generated as B_+-modules by Lemma F.2.5. We deduce that $\mathcal{O}^{gr}$ is closed under taking subobjects, since $\mathcal{O}_H$ has that property.

We check now the conditions of Definition F.1.1. Conditions (i) and (ii) are given by Lemma F.2.2, and (iii) by Lemma F.2.5.

Let $E,E'\in I$, $d\in\mathbb{Z}$ and $M\in\mathcal{O}^{gr}$. By Lemma F.2.5, there is $m\geqslant 0$ such that $M^{<-m}=0$. It follows from Lemma F.2.6 that $\mathrm{Ext}^1_{\mathcal{O}^{gr}}(\Delta_n(E),M)=0$ for $n\geqslant m$. There is $m'\geqslant 0$ such that $\Delta(E')_d$ is killed by $B_-^{<-m'}$. Consequently given $n\geqslant m'$, we have

$$\mathrm{Hom}_{\mathcal{O}^{gr}}(\Delta_n(E),\Delta(E'\langle d\rangle))\simeq\mathrm{Hom}_{H\text{-modgr}}(E,\mathrm{Ann}_{B_-^{<-n}}(\Delta(E'\langle d\rangle)))=\mathrm{Hom}_{H\text{-modgr}}(E,\Delta(E')_d).$$

Since $\Delta(E')_d$ is a direct summand of a finite direct sum of objects of I, we deduce that $\mathrm{Hom}_{\mathcal{O}^{gr}}(\Delta_n(E),\Delta(E'\langle d\rangle))\in k$-proj. So, (iv) holds. □

F.2.D. Inner grading. — We assume in §F.2.D that k is a $\mathbb{Q}$-algebra and that there is $h\in A$ such that $A^i=\{a\in A\mid ha-ah=ia\}$ for all $i\in\mathbb{Z}$. We assume that $A\neq A^0$.

Lemma F.2.8. — *There is a unique decomposition $h=h'+h_0$ where $h'\in B_+\otimes H\otimes B_-^{<0}$ and $h_0\in Z(H)$.*

Proof. — We have $h \in A^0$. Write $h = h' + h_0$ where $h' \in B_+ \otimes H \otimes B_-^{<0}$ and $h_0 \in B_+ \otimes H$. Since $h_0 \in A^0$, it follows that $h_0 \in H$. Let $a \in H$. We have $0 = [h, a] = [h', a] + [h_0, a]$. Since $[h', a] \in B_+ \otimes H \otimes B_-^{<0}$ and $[h_0, a] \in H$, we deduce that $[h_0, a] = 0$. This shows that $h_0 \in Z(H)$. □

Let $E \in I$. The action of h_0 on E is given by multiplication by an element C_E of k. Note that h acts by $C_E + i$ on $\Delta(E)^i$.

We put $L = \bigcup_{E \in I}(C_E + \mathbb{Z}) \subset k$. Let $\sim$ be the equivalence relation on L generated by $\lambda \sim \lambda'$ if $\lambda - \lambda' \notin k^\times$. We assume in section §F.2.D that

(x) $C_E \not\sim C_E + i$ for $i \in \mathbb{Z} - \{0\}$.

We define a relation on I as the transitive closure of the relation $E > F$ if there exists a maximal ideal $\mathfrak{m}$ of k such that the image of $C_E - C_F$ in $k/\mathfrak{m}$ is contained in $\mathbb{Z}_{>0}$. Our assumption above ensures that $>$ is a partial order on I.

Let $\mathcal{O}$ be the category of finitely generated A-modules with are locally nilpotent for B_- and whose restriction to H is the quotient of a (possibly infinite) direct sum of objects of I.

Let $M \in A$-Mod. Given $\lambda \in k$, we put

$$\mathscr{W}_\lambda(M) = \{m \in M \mid (h - \lambda)^n m = 0 \text{ for } n \gg 0\}.$$

Given $\alpha \in L/\sim$, we put $\mathscr{W}_\alpha(M) = \sum_{\lambda \in \alpha} \mathscr{W}_\lambda(M)$.

We denote by $\mathcal{O}^{gr,\alpha}$ the full subcategory of $\mathcal{O}^{gr}$ of objects M such that $M^i \subset \mathscr{W}_{i+\alpha}(M)$ for all $i \in \mathbb{Z}$. We denote by $\mathcal{O}^{\alpha+\mathbb{Z}}$ the full subcategory of $\mathcal{O}$ of objects M such that $M = \sum_{i \in \mathbb{Z}} \mathscr{W}_{i+\alpha}(M)$.

Proposition F.2.9. — *We have $\mathcal{O}^{gr} = \bigoplus_{\alpha \in L/\sim} \mathcal{O}^{gr,\alpha}$ and $\mathcal{O} = \bigoplus_{\alpha+\mathbb{Z} \in (L/\sim)/\mathbb{Z}} \mathcal{O}^{\alpha+\mathbb{Z}}$. Furthermore, given $\alpha \in L/\sim$, the forgetful functor gives an equivalence $\mathcal{O}^{gr,\alpha} \xrightarrow{\sim} \mathcal{O}^{\alpha+\mathbb{Z}}$.*

Fix an element $\tilde{\beta} \in L/\sim$ for each $\beta \in (L/\sim)/\mathbb{Z}$. There is an equivalence of graded categories $\mathcal{O}^{(\mathbb{Z})} \xrightarrow{\sim} \mathcal{O}^{gr}$ sending $\Delta(E)$ to $\Delta(E)\langle \widetilde{C_E + \mathbb{Z}} - C_E \rangle$.

Proof. — Let $M \in \mathcal{O}$. Let us first show that $M = \sum_{\lambda \in L} \mathscr{W}_\lambda(M)$. This is clear when $M = \Delta(E)$ for some E. It follows that it holds also when $M = \Delta_n(E)$, hence for M a direct sum of $\Delta_n(E)$'s. One shows as in Lemma F.2.5 that every object $M \in \mathcal{O}$ is a quotient of a finite direct sum of $\Delta_n(E)$'s, hence the result holds for M. So, we have an H-module decomposition $M = \bigoplus_{\alpha \in L/\sim} \mathscr{W}_\alpha(M)$.

Let $M(\alpha + \mathbb{Z}) = \bigoplus_{i \in \mathbb{Z}} \mathscr{W}_{\alpha+i}(M)$: this is an A-submodule of M in $\mathcal{O}^{\alpha+\mathbb{Z}}$ and $M = \bigoplus_{\alpha+\mathbb{Z} \in (L/\sim)/\mathbb{Z}} M(\alpha + \mathbb{Z})$. This gives the required decomposition of $\mathcal{O}$.

Assume now $M \in \mathcal{O}^{gr}$. Let $M(\alpha)^i = \mathscr{W}_{i+\alpha}(M) \cap M_i$ and $M(\alpha) = \bigoplus_{i \in \mathbb{Z}} M(\alpha)^i$, a graded A-submodule of M contained in $\mathcal{O}^{gr,\alpha}$. We have $M = \bigoplus_{\alpha \in L/\sim} M(\alpha)$, and this provides the required decomposition of $\mathcal{O}^{gr}$.

Let $M \in \mathscr{O}^{\alpha+\mathbb{Z}}$. Let $M^i = \mathscr{W}_{\alpha+i}(M)$. This defines a structure of graded A-module on M, and gives an object M' of $\mathscr{O}^{gr,\alpha}$. The construction $M \mapsto M'$ is inverse to the forgetful functor $\mathscr{O}^{gr,\alpha} \to \mathscr{O}^{\alpha+\mathbb{Z}}$.

Consider now an element $\tilde{\beta} \in L/\sim$ for each $\beta \in (L/\sim)/\mathbb{Z}$. We have constructed an equivalence $F : \mathscr{O}^{\beta} \xrightarrow{\sim} \mathscr{O}^{gr,\tilde{\beta}}$. These functors extend uniquely to the required equivalence of graded categories $\mathscr{O}^{(\mathbb{Z})} \xrightarrow{\sim} \mathscr{O}^{gr}$. $\square$

From Proposition F.2.9 and Theorem F.2.7, we deduce the following result ([**GGOR**, Theorem 2.19] when k is a field). Let $\Delta = \{\Delta(E)\}_{E \in I}$, with the poset structure of I.

Theorem F.2.10. — *$\mathscr{O}$ is a highest weight category with poset of standard objects Δ.*

Example F.2.11. — Let $\mathfrak{g}$ be a finite dimensional reductive Lie algebra over $k = \mathbb{C}$. Let $\mathfrak{b}_+$ be a Borel subalgebra and $\mathfrak{h} \subset \mathfrak{b}_+$ a Cartan subalgebra. Let $\mathfrak{b}_-$ be the opposite Borel subalgebra. Let $A = U(\mathfrak{g})$, $B_\pm = U(\mathfrak{b}_\pm)$ and $H = U(\mathfrak{h})$. Let $h \in \mathfrak{h}$ be the sum of the simple coroots. We consider the inner grading on $\mathfrak{h}$, hence on A defined by $\mathrm{ad}(h)$. We have $\mathfrak{b}_+^{<0} = \mathfrak{b}_-^{>0} = \mathfrak{b}^{\neq 0} = 0$ and $\mathfrak{b}_-^0 = \mathfrak{b}_+^0 = \mathbb{C}$.

We take for I the set of isomorphism classes of simple $\mathfrak{h}$-modules, so that $\mathscr{O}_H$ is the category of semisimple $\mathfrak{h}$-modules. Then $\mathscr{O}$ is the usual BGG category. ■

Remark F.2.12. — If the grading on A is not inner, then $\mathscr{O}$ is not a highest weight category in general.

PRIME IDEALS AND GEOMETRY

$$\begin{tikzcd}
 & R \arrow[r, hook] & \mathbf{M} \\
Z \arrow[r, "\sim", "\mathrm{cop}"'] & Q \arrow[u, hook] \arrow[r, hook] & \mathbf{L} \arrow[u, hook] \arrow[u, bend right, "H"'] \\
 & P \arrow[u, hook] \arrow[r, hook] & \mathbf{K} \arrow[u, hook] \arrow[uu, bend right=60, "G"'] \\
 & \mathbf{k}[\boldsymbol{\mathscr{C}}] \arrow[u, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{\mathscr{R}} \arrow[d, "\rho_H"'] \arrow[dd, bend left=60, "\rho_G"] \\
\boldsymbol{\mathscr{Q}} \arrow[d, "\Upsilon"'] \\
\mathscr{P} = \boldsymbol{\mathscr{C}} \times V/W \times V^*/W \arrow[d, "\pi"'] \\
\boldsymbol{\mathscr{C}}
\end{tikzcd}$$

$$\begin{tikzcd}
R \arrow[r, two heads] & R_c = R/\mathfrak{r}_c \arrow[r, hook] & \mathbf{M}_c = k_R(\mathfrak{r}_c) \\
Q \arrow[u, hook] \arrow[r, two heads] & Q_c = Q/\mathfrak{q}_c \arrow[u, hook] \arrow[r, hook] & \mathbf{L}_c = k_R(\mathfrak{q}_c) \arrow[u, hook] \arrow[u, bend right, "D_c \cap H"'] \\
P \arrow[u, hook] \arrow[r, two heads] & P_c = P/\mathfrak{p}_c \arrow[u, hook] \arrow[r, hook] & \mathbf{K}_c = k_R(\mathfrak{p}_c) \arrow[u, hook] \arrow[uu, bend right=60, "D_c"'] \\
\mathbf{k}[\boldsymbol{\mathscr{C}}] \arrow[u, hook] \arrow[r, two heads] & \mathbf{k} = \mathbf{k}[\boldsymbol{\mathscr{C}}]/\mathfrak{C}_c \arrow[u, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{\mathscr{R}}_c \arrow[r] \arrow[d] & \boldsymbol{\mathscr{R}} \arrow[d] \\
\boldsymbol{\mathscr{Q}}_c \arrow[r] \arrow[d] & \boldsymbol{\mathscr{Q}} \arrow[d, "\Upsilon"] \\
\{c\} \times V/W \times V^*/W \arrow[r] \arrow[d] & \mathscr{P} = \boldsymbol{\mathscr{C}} \times V/W \times V^*/W \arrow[d] \\
\{c\} \arrow[r] & \boldsymbol{\mathscr{C}}
\end{tikzcd}$$

$$\begin{tikzcd}
P \arrow[r, two heads] \arrow[d] & P_{\mathfrak{C}}^{\mathrm{left}} = P/\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}} \arrow[r, hook] \arrow[d] & \mathbf{K}_{\mathfrak{C}}^{\mathrm{left}} = k_P(\mathfrak{p}_{\mathfrak{C}}^{\mathrm{left}}) \\
\mathbf{k}[\boldsymbol{\mathscr{C}}] \arrow[r, two heads] & \mathbf{k}[\boldsymbol{\mathscr{C}}]/\mathfrak{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{\mathscr{C}}(\mathfrak{C}) \arrow[r, hook] & \boldsymbol{\mathscr{C}}
\end{tikzcd}$$

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