Journal of Algebra 558 (2020) 102–128

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

An asymptotic cell category for cyclic groups

Cédric Bonnafé [a,*,1], Raphaël Rouquier [b,2]

[a] *Institut Montpelliérain Alexander Grothendieck (CNRS: UMR 5149), Université Montpellier 2, Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex, France*

[b] *UCLA Mathematics Department Los Angeles, CA 90095-1555, USA*

ARTICLE INFO

Article history:
Received 12 October 2018
Available online 15 January 2020
Communicated by R. Boltje

Keywords:
Tensor categories
Fusion datum
Fourier transform

ABSTRACT

In his theory of unipotent characters of finite groups of Lie type, Lusztig constructed modular categories from two-sided cells in Weyl groups. Broué, Malle and Michel have extended parts of Lusztig's theory to complex reflection groups. This includes generalizations of the corresponding fusion algebras, although the presence of negative structure constants prevents them from arising from modular categories. We give here the first construction of braided pivotal monoidal categories associated with non-real reflection groups (later reinterpreted by Lacabanne as super modular categories). They are associated with cyclic groups, and their fusion algebras are those constructed by Malle.

To Michel Broué, for his spetsial intuitions

Broué, Malle and Michel [1] have constructed a combinatorial version of Lusztig's theory of unipotent characters of finite groups of Lie type for certain complex reflection

* Corresponding author.
E-mail addresses: cedric.bonnafe@umontpellier.fr (C. Bonnafé), rouquier@math.ucla.edu (R. Rouquier).

1 The first author is partly supported by the ANR (Project No ANR-16-CE40-0010-01 GeRepMod).
2 The second author was partly supported by the NSF (grant DMS-1161999 and DMS-1702305) and by a grant from the Simons Foundation (#376202).

https://doi.org/10.1016/j.jalgebra.2019.12.015

groups ("spetsial groups"). The case of spetsial imprimitive complex reflection groups has been considered by Malle in [10]: Malle defines a combinatorial set which generalizes the one defined by Lusztig to parametrize *unipotent characters* of the associated finite reductive group when W is a Weyl group. Malle generalizes also the partition of this set into Lusztig *families*. To each family, he associates a $\mathbb{Z}$-fusion datum: a $\mathbb{Z}$-fusion datum is a structure similar to a usual fusion datum (which we will call a $\mathbb{Z}_+$-*fusion datum*) except that the structure constants of the associated fusion ring might be negative.

It is a classical problem to find a tensor category with suitable extra-structures (pivot, twist) corresponding to a given $\mathbb{Z}_+$-fusion datum. The aim of this paper is to provide an *ad hoc* categorification of the $\mathbb{Z}$-fusion datum associated with the non-trivial family of the cyclic complex reflection group of order d: it is provided by a quotient category of the representation category of the Drinfeld double of the *Taft algebra* of dimension d^2 (the Taft algebra is the positive part of a Borel subalgebra of small quantum $\mathfrak{sl}_2$ at a d-th root of unity).

The constructions of this paper have recently been extended by Lacabanne [8], [9] to some families of the complex reflection groups $G(d, 1, n)$. His construction involves super-categories instead of triangulated categories, as suggested by Etingof. In the particular case studied in our paper, Lacabanne's construction gives a reinterpretation as well as some clarifications of our results.

In the first section, we recall some basic properties of the Taft algebra and its Drinfeld double. The second section is devoted to recalling some of the structure of its category of representations: simple modules, blocks and structure of projective modules. We summarize some elementary facts on the tensor structure in the third section: invertible objects and tensor product of simple objects by the defining two-dimensional representation. The fourth section provides generators and characters of the Grothendieck group of $D(B)$, a commutative ring.

Our original work starts in the fifth section, with the study of the stable module category of $D(B)$ and a further quotient and the determination of their Grothendieck rings. In the sixth section, we define and study a pivotal structure on $D(B)$-mod and determine characters of its Grothendieck group associated to left or right traces. This gives rise to positive and negative S and T-matrices. We proceed similarly for the small quotient triangulated category. These are our fusion data.

We recall in the final section the construction of Malle's fusion datum associated with cyclic groups and we check that it coincides with the fusion data we defined in the previous section.

In Appendix A, we provide a description of S-matrices in the setting of pivotal tensor categories.

Acknowledgments

We wish to thank warmly Abel Lacabanne for the fruitful discussions we had with him and his very careful reading of a preliminary version of this work.

1. The Drinfeld double of the Taft algebra

From now on, $\otimes$ will denote the tensor product $\otimes_{\mathbb{C}}$. We fix a natural number $d \geqslant 2$ as well as a primitive d-th root of unity $\zeta \in \mathbb{C}^\times$. We denote by $\boldsymbol{\mu}_d = \langle \zeta \rangle$ the group of d-th roots of unity.

Given $n \geqslant 1$ a natural number and $\xi \in \mathbb{C}$, we set

$$(n)_\xi = 1 + \xi + \cdots + \xi^{n-1}$$

and

$$(n)!_\xi = \prod_{i=1}^{n} (i)_\xi .$$

We also set $(0)!_\xi = 1$.

1.1. The Taft algebra

We denote by B the $\mathbb{C}$-algebra defined by the following presentation:

- Generators: K, E.
- Relations: $\begin{cases} K^d = 1, \\ E^d = 0, \\ KE = \zeta EK. \end{cases}$

It follows from [7, Proposition IX.6.1] that:

(Δ) There exists a unique morphism of algebras $\Delta : B \to B \otimes B$ such that

$$\Delta(K) = K \otimes K \qquad \text{and} \qquad \Delta(E) = (1 \otimes E) + (E \otimes K).$$

(ε) There exists a unique morphism of algebras $\varepsilon : B \to \mathbb{C}$ such that

$$\varepsilon(K) = 1 \qquad \text{and} \qquad \varepsilon(E) = 0.$$

(S) There exists a unique anti-automorphism S of B such that

$$S(K) = K^{-1} \qquad \text{and} \qquad S(E) = -EK^{-1}.$$

With Δ as a coproduct, ε as a counit and S as an (invertible) antipode, B becomes a Hopf algebra, called the *Taft algebra* [6, Example 5.5.6]. It is easily checked that

$$B = \bigoplus_{i,j=0}^{d-1} \mathbb{C} K^i E^j = \bigoplus_{i,j=0}^{d-1} \mathbb{C} E^i K^j. \tag{1.1}$$

1.2. Dual algebra

Let K^* and E^* denote the elements of B^* such that

$$K^*(E^i K^j) = \delta_{i,0}\zeta^j \qquad \text{and} \qquad E^*(E^i K^j) = \delta_{i,1}.$$

Recall that B^* is naturally a Hopf algebra [7, Proposition III.3.3] and it follows from [7, Lemma IX.6.3] that

$$(E^{*i} K^{*j})(E^{i'} K^{j'}) = \delta_{i,i'}(i)!_\zeta \zeta^{j(i+j')}. \tag{1.2}$$

We deduce easily that $(E^{*i}K^{*j})_{0 \leqslant i,j \leqslant d-1}$ is a $\mathbb{C}$-basis of B^*.

We will give explicit formulas for the coproduct, the counit and the antipode in the next subsection. We will in fact use the Hopf algebra $(B^*)^{\mathrm{cop}}$, which is the Hopf algebra whose underlying space is B^*, whose product is the same as in B^* and whose coproduct is opposite to the one in B^*.

1.3. Drinfeld double

We denote by $D(B)$ the *Drinfeld quantum double* of B, as defined for instance in [7, Definition IX.4.1] or [6, Definition 7.14.1]. Recall that $D(B)$ contains B and $(B^*)^{\mathrm{cop}}$ as Hopf subalgebras and that the multiplication induces an isomorphism of vector spaces $(B^*)^{\mathrm{cop}} \otimes B \xrightarrow{\sim} D(B)$. A presentation of $D(B)$, with generators E, E^*, K, K^* is given for instance in [7, Proposition IX.6.4]. We shall slightly modify it by setting

$$z = K^{*-1}K \qquad \text{and} \qquad F = \zeta E^* K^{*-1}.$$

Then [7, Proposition IX.6.4] can be rewritten as follows:

Proposition 1.3. *The $\mathbb{C}$-algebra $D(B)$ admits the following presentation:*

- *Generators:* E, F, K, z;
- *Relations:* $\begin{cases} K^d = z^d = 1, \\ E^d = F^d = 0, \\ [z,E] = [z,F] = [z,K] = 0, \\ KE = \zeta EK, \\ KF = \zeta^{-1}FK, \\ [E,F] = K - zK^{-1}. \end{cases}$

The next corollary follows from an easy induction argument:

Corollary 1.4. *If $i \geqslant 1$, then*

$$[E, F^i] = (i)_\zeta F^{i-1}(\zeta^{1-i}K - zK^{-1})$$

and

$$[F, E^i] = (i)_\zeta E^{i-1}(\zeta^{1-i}zK^{-1} - K).$$

The algebra $D(B)$ is endowed with a structure of Hopf algebra, where the comultiplication, the counit and the antipode are still denoted by Δ, ε and S respectively (as they extend the corresponding objects for B). We have [7, Proposition IX.6.2]:

$$\begin{cases} \Delta(K) = K \otimes K, \\ \Delta(z) = z \otimes z, \\ \Delta(E) = (1 \otimes E) + (E \otimes K), \\ \Delta(F) = (F \otimes 1) + (zK^{-1} \otimes F), \end{cases} \qquad \begin{cases} S(K) = K^{-1}, \\ S(z) = z^{-1}, \\ S(E) = -EK^{-1}, \\ S(F) = -\zeta^{-1}FKz^{-1}, \end{cases} \tag{1.5}$$

$$\varepsilon(K) = \varepsilon(z) = 1 \qquad \text{and} \qquad \varepsilon(E) = \varepsilon(F) = 0. \tag{1.6}$$

1.4. Morphisms to $\mathbb{C}$

Given $\xi \in \boldsymbol{\mu}_d$, we denote by $\varepsilon_\xi : D(B) \to \mathbb{C}$ the unique morphism of algebras such that

$$\varepsilon_\xi(K) = \xi, \quad \varepsilon_\xi(z) = \xi^2 \quad \text{and} \quad \varepsilon_\xi(E) = \varepsilon_\xi(F) = 0.$$

It is easily checked that the ε_ξ's are the only morphisms of algebras $D(B) \to \mathbb{C}$. Note that $\varepsilon_1 = \varepsilon$ is the counit.

1.5. Group-like elements

It follows from (1.5) that K and z are group-like, so that $K^i z^j$ is group-like for all i, $j \in \mathbb{Z}$. The converse also holds (and is certainly already well-known).

Lemma 1.7. *If $g \in D(B)$ is group-like, then there exist i, $j \in \mathbb{Z}$ such that $g = K^i z^j$.*

Proof. Let $g \in D(B)$ be a group-like element. Let us write

$$g = \sum_{i,j,k,l=0}^{d-1} \alpha_{i,j,k,l} K^i z^j E^k F^l.$$

We denote by (k_0, l_0) the largest pair (for the lexicographic order) such that there exist i, $j \in \{0, 1, \ldots, d-1\}$ such that $\alpha_{i,j,k_0,l_0} \neq 0$. The coefficient of $K^i z^j E^{k_0} F^{l_0} \otimes K^i z^j E^{k_0} F^{l_0}$ in $g \otimes g$ is equal to α^2_{i,j,k_0,l_0}, so it is different from 0.

But, if we compute the coefficient of $K^i z^j E^{k_0} F^{l_0} \otimes K^i z^j E^{k_0} F^{l_0}$ in

$$g \otimes g = \Delta(g) = \sum_{i,j,k,l=0}^{d-1} \alpha_{i,j,k,l} \Delta(K)^i \Delta(z)^j \Delta(E)^k \Delta(F)^l$$

using the formulas (1.5), we see that it is equal to 0 if $(k_0, l_0) \neq (0, 0)$. Therefore $(k_0, l_0) = (0, 0)$, and so g belongs to the linear span of the family $(K^i z^j)_{i,j \in \mathbb{Z}}$. Now the result follows from the linear independence of group-like elements. □

1.6. Braiding

For $0 \leqslant i, j \leqslant d-1$, we set

$$\beta_{i,j} = \frac{E^{*i}}{d \cdot (i)!_\zeta} \sum_{k=0}^{d-1} \zeta^{-k(i+j)} K^{*k}.$$

It follows from (1.2) that $(\beta_{i,j})_{0 \leqslant i,j \leqslant d-1}$ is a dual basis to $(E^i K^j)_{0 \leqslant i,j \leqslant d-1}$. We set now

$$R = \sum_{i,j=0}^{d-1} E^i K^j \otimes \beta_{i,j} \in D(B) \otimes D(B).$$

Note that R is a universal R-matrix for $D(B)$ and it endows $D(B)$ with a structure of braided Hopf algebra [7, Theorem IX.4.4]). Using our generators E, F, K, z, we have:

$$R = \frac{1}{d} \sum_{i,j,k=0}^{d-1} \frac{\zeta^{(i-k)(i+j)-i(i+1)/2}}{(i)!_\zeta} E^i K^j \otimes z^{-k} F^i K^k. \tag{1.8}$$

1.7. Twist

Let us define

$$\begin{aligned} \tau : \quad D(B) \otimes D(B) &\longrightarrow D(B) \otimes D(B) \\ a \otimes b &\longmapsto b \otimes a. \end{aligned}$$

Following [7, §VIII.4], we set

$$u = \sum_{i,j=0}^{d-1} S(\beta_{ij}) E^i K^j \in D(B).$$

Recall that u is called the *Drinfeld element* of $D(B)$. It satisfies several properties (see for instance [7, Proposition VIII.4.5]). For instance, u is invertible and we will recall only three equalities:

$$\varepsilon(u) = 1, \quad \Delta(u) = (\tau(R)R)^{-1}(u \otimes u) \quad \text{and} \quad S^2(b) = ubu^{-1} \tag{1.9}$$

for all $b \in D(B)$. A straightforward computation shows that

$$S^2(b) = KbK^{-1} \tag{1.10}$$

for all $b \in D(B)$. We now set

$$\theta = K^{-1}u.$$

The following proposition is a consequence of (1.9) and (1.10).

Proposition 1.11. *The element θ is central and invertible in $D(B)$ and satisfies*

$$\varepsilon(\theta) = 1 \qquad \textit{and} \qquad \Delta(\theta) = (\tau(R)R)^{-1}(\theta \otimes \theta).$$

Let us give a formula for θ:

$$\theta = \frac{1}{d} \sum_{i,j,k=0}^{d-1} (-1)^i \frac{\zeta^{(i-k)(i+j)-i}}{(i)!_\zeta} z^{k-i} F^i E^i K^{i+j-k-1}. \tag{1.12}$$

Corollary 1.13. *We have $S(\theta) = z\theta$.*

Proof. Let $g = S(\theta)\theta^{-1}$. Since $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$ and $(S \otimes S)(R) = R$ (see for instance [7, Theorems III.3.4 and VIII.2.4]), it follows from Proposition 1.11 that g is central and group-like. Hence, by Lemma 1.7, there exists $l \in \mathbb{Z}$ such that $S(\theta) = \theta z^l$. So, by (1.12), we have

$$S(\theta)E^{d-1} = \theta z^l E^{d-1} = \frac{1}{d} \sum_{j,k \in \mathbb{Z}/d\mathbb{Z}}^{d-1} \zeta^{-jk} z^{k+l} K^{j-k-1} E^{d-1}. \tag{$\sharp$}$$

Let us now compute $S(\theta)E^{d-1}$ by using directly (1.12). We get

$$\begin{aligned}
S(\theta)E^{d-1} = E^{d-1}S(\theta) &= \frac{1}{d}\sum_{j,k\in\mathbb{Z}/d\mathbb{Z}} \zeta^{-jk}E^{d-1}z^{-k}K^{1+k-j} \\
&= \frac{1}{d}\sum_{j,k\in\mathbb{Z}/d\mathbb{Z}} \zeta^{-jk}\zeta^{1+k-j}z^{-k}K^{1+k-j}E^{d-1} \\
&= \frac{1}{d}\sum_{j,k\in\mathbb{Z}/d\mathbb{Z}} \zeta^{(1-j)(1+k)}z^{-k}K^{1+k-j}E^{d-1}.
\end{aligned}$$

So, if we set $j' = 1-j$ and $k' = -1-k$, we get

$$S(\theta)E^{d-1} = \frac{1}{d}\sum_{j',k'\in\mathbb{Z}/d\mathbb{Z}} \zeta^{-j'k'}z^{k'+1}K^{j'-k'-1}E^{d-1}.$$

Comparing with (♯), we get that $z^l = z$. $\square$

2. $D(B)$-modules

Most of the results of this section are due to Chen [2] or Erdmann, Green, Snashall and Taillefer [4], [5]. By a $D(B)$-module, we mean a finite dimensional left $D(B)$-module. We denote by $D(B)$-mod the category of (finite dimensional left) $D(B)$-modules. Given $\alpha_1, \ldots, \alpha_{l-1} \in \mathbb{C}$, we set

$$J_l^+(\alpha_1, \ldots, \alpha_{l-1}) = \begin{pmatrix} 0 & \alpha_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & \alpha_{l-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

and

$$J_l^-(\alpha_1, \ldots, \alpha_{l-1}) = {}^tJ_l^+(\alpha_1, \ldots, \alpha_{l-1}).$$

Given M a $D(B)$-module and $b \in D(B)$, we denote by $b|_M$ the endomorphism of M induced by b. For instance, $E|_M$ and $F|_M$ are nilpotent and $K|_M$ and $z|_M$ are semisimple.

2.1. Simple modules

Given $1 \leqslant l \leqslant d$ and $p \in \mathbb{Z}/d\mathbb{Z}$, we denote by $M_{l,p}$ the $D(B)$-module with $\mathbb{C}$-basis $\mathscr{M}^{(l,p)} = (e_i^{(l,p)})_{1\leqslant i\leqslant l}$ where the action of z, K, E and F in the basis $\mathscr{M}^{(l,p)}$ its given by the following matrices:

$$\begin{gathered}
z|_{M_{l,p}} = \zeta^{2p+l-1}\,\mathrm{Id}_{M_{l,p}}, \\
K|_{M_{l,p}} = \zeta^p\,\mathrm{diag}(\zeta^{l-1}, \zeta^{l-2}, \ldots, \zeta, 1),
\end{gathered}$$

$$E|_{M_{l,p}} = \zeta^p J_l^+((1)_\zeta(\zeta^{l-1}-1), (2)_\zeta(\zeta^{l-2}-1), \dots, (l-1)_\zeta(\zeta-1)),$$
$$F|_{M_{l,p}} = J_l^-(1, \dots, 1).$$

It is readily checked from the relations given in Proposition 1.3 that this defines a $D(B)$-module of dimension l. The next result is proved in [2, Theorem 2.5].

Theorem 2.1 *(Chen). The map*

$$\begin{array}{ccc} \{1,2,\dots,d\} \times \mathbb{Z}/d\mathbb{Z} & \longrightarrow & \mathrm{Irr}(D(B)) \\ (l,p) & \longmapsto & M_{l,p} \end{array}$$

is bijective.

2.2. Blocks

We put $\Lambda(d) = \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$, a set in canonical bijection with $\{1,2,\dots,d\} \times \mathbb{Z}/d\mathbb{Z}$, which parametrizes the simple $D(B)$-modules. Given $\lambda \in \Lambda(d)$, we denote by M_λ the corresponding simple $D(B)$-module. We also set $(\mathbb{Z}/d\mathbb{Z})^\# = (\mathbb{Z}/d\mathbb{Z}) \setminus \{0\}$ and $\Lambda^\#(d) = (\mathbb{Z}/d\mathbb{Z})^\# \times \mathbb{Z}/d\mathbb{Z}$. Finally, let $\Lambda^0(d) = \{0\} \times \mathbb{Z}/d\mathbb{Z}$ be the complement of $\Lambda^\#(d)$ in $\Lambda(d)$.

Define

$$\begin{array}{rccc} \iota : & \Lambda(d) & \longrightarrow & \Lambda(d) \\ & (l,p) & \longmapsto & (-l, p+l). \end{array}$$

We have $\iota^2 = \mathrm{Id}_{\Lambda(d)}$ and $\Lambda^0(d)$ is the set of fixed points of ι. Given $\mathscr{L}$ a ι-stable subset of $\Lambda(d)$, we denote by $[\mathscr{L}/\iota]$ a set of representatives of ι-orbits in $\mathscr{L}$. The next result is proved in [4, Theorem 2.26].

Theorem 2.2 *(Erdmann-Green-Snashall-Taillefer). Let λ, $\lambda' \in \Lambda(d)$. Then M_λ and $M_{\lambda'}$ belong to the same block of $D(B)$ if and only if λ and λ' are in the same ι-orbit.*

We have constructed in §1.7 a central element, namely θ. Note that

$$\textit{The element } \theta \textit{ acts on } M_{l,p} \textit{ by multiplication by } \zeta^{(p-1)(l+p-1)}. \tag{2.3}$$

Proof. It is sufficient to compute the action of θ on the vector $e_1^{(l,p)}$. Note that $E^i e_1^{(l,p)} = 0$ as soon as $i \geqslant 1$. Therefore, for computing $\theta e_1^{(l,p)}$ using the formula (1.12), only the terms corresponding to $i = 0$ remain. Consequently,

$$\begin{aligned} \omega_{l,p}(\theta) &= \frac{1}{d} \sum_{j,k=0}^{d-1} \zeta^{-jk} \zeta^{(2p+l-1)k} \zeta^{(p+l-1)(j-k-1)} \\ &= \frac{\zeta^{1-l-p}}{d} \sum_{k=0}^{d-1} \zeta^{pk} \Big(\sum_{j=0}^{d-1} \zeta^{(p+l-1-k)j} \Big) \end{aligned}$$

The term inside the big parenthesis is equal to d if $p+l-1-k \equiv 0 \mod d$, and is equal to 0 otherwise. The result follows. $\square$

2.3. Projective modules

Given $\lambda \in \Lambda(d)$, we denote by P_λ a projective cover of M_λ. The next result is proved in [4, Corollary 2.25].

Theorem 2.4 *(Erdmann-Green-Snashall-Taillefer). Let $\lambda \in \Lambda(d)$.*

(a) *If $\lambda \in \Lambda^\#(d)$, then $\dim_{\mathbb{C}}(P_\lambda) = 2d$, $\operatorname{Rad}^3(P_\lambda) = 0$ and the Loewy structure of P_λ is given by:*

$$\begin{aligned} P_\lambda / \operatorname{Rad}(P_\lambda) &\simeq \qquad M_\lambda \\ \operatorname{Rad}(P_\lambda)/\operatorname{Rad}^2(P_\lambda) &\simeq M_{\iota(\lambda)} \oplus M_{\iota(\lambda)} \\ \operatorname{Rad}^2(P_\lambda) &\simeq \qquad M_\lambda. \end{aligned}$$

(b) *$P_{d,p} = M_{d,p}$ has dimension d.*

3. Tensor structure

We mainly refer here to the work of Erdmann, Green, Snashall and Taillefer [4], [5]. Since $D(B)$ is a finite dimensional Hopf algebra, the category $D(B)$-mod inherits a structure of a tensor category. We will compute here some tensor products between simple modules. We will denote by M_l the simple module $M_{l,0}$.

3.1. Invertible modules

We denote by $V_\xi = \mathbb{C}v_\xi$ the one-dimensional $D(B)$-module associated with the morphism $\varepsilon_\xi : D(B) \to \mathbb{C}$ defined in §1.4:

$$bv_\xi = \varepsilon_\xi(b)v_\xi$$

for all $b \in D(B)$. We have

$$V_{\zeta^p} \simeq M_{1,p}. \tag{3.1}$$

An immediate computation using the comultiplication Δ shows that

$$M_{l,p} \otimes V_{\zeta^q} \simeq V_{\zeta^q} \otimes M_{l,p} \simeq M_{l,p+q} \tag{3.2}$$

as $D(B)$-modules. The V_ξ's are (up to isomorphism) the only invertible objects in the tensor category $D(B)$-mod.

3.2. Tensor product with M_2

We set $e_i = e_i^{(2,0)}$ for $i \in \{1,2\}$, so that (e_1, e_2) is the standard basis of M_2. The next result is a particular case of [4, Theorem 4.1].

Theorem 3.3 *(Erdmann-Green-Snashall-Taillefer). Let (l,p) be an element of $\{1,2,\dots, d\} \times \mathbb{Z}/d\mathbb{Z}$.*

(a) *If $l \leqslant d-1$, then $M_2 \otimes M_{l,p} \simeq M_{l+1,p} \oplus M_{l-1,p+1}$.*
(b) *$M_2 \otimes M_{d,p} \simeq P_{d-1,p}$.*

4. Grothendieck rings

We denote by $\mathrm{Gr}(D(B))$ the Grothendieck ring of the category of (left) $D(B)$-modules.

4.1. Structure

Since $D(B)$ is a braided Hopf algebra (with universal R-matrix R),

$$\textit{the ring } \mathrm{Gr}(D(B)) \textit{ is commutative.} \tag{4.1}$$

Given M a $D(B)$-module, we denote by $[\,M\,]$ the class of M in $\mathrm{Gr}(D(B))$. We set

$$\mathbf{m}_\lambda = [\,M_\lambda\,], \qquad \mathbf{m}_l = [\,M_{l,0}\,] \qquad \text{and} \qquad \mathbf{v}_\xi = [\,V_\xi\,] \in \mathrm{Gr}(D(B)).$$

Recall that $\mathbf{v}_{\zeta^p} = \mathbf{m}_{1,p}$. It follows from (3.2) and Theorem 3.3 that

$$\mathbf{v}_{\zeta^q}\mathbf{m}_{l,p} = \mathbf{m}_{l,p+q} \quad \text{and} \quad \mathbf{m}_2\mathbf{m}_{l,p} = \begin{cases} \mathbf{m}_{l+1,p} + \mathbf{m}_{l-1,p+1} & \text{if } l \leqslant d-1, \\ 2(\mathbf{m}_{d-1,p} + \mathbf{m}_{1,p-1}) & \text{if } l = d. \end{cases} \tag{4.2}$$

Proposition 4.3. *The Grothendieck ring $\mathrm{Gr}(D(B))$ is generated by $\mathbf{v}_\zeta$ and $\mathbf{m}_2$.*

Proof. We will prove by induction on l that $\mathbf{m}_{l,p} \in \mathbb{Z}[\mathbf{v}_\zeta, \mathbf{m}_2]$. Since $\mathbf{m}_{1,p} = (\mathbf{v}_\zeta)^p$, this is true for $l = 1$. Since $\mathbf{m}_{2,p} = (\mathbf{v}_\zeta)^p\mathbf{m}_2$, this is also true for $l = 2$. Now the induction proceeds easily by using (4.2). □

4.2. Some characters

If $b \in D(B)$ is *group-like*, then the map

$$\begin{aligned} \mathrm{Gr}(D(B)) &\longrightarrow \quad \mathbb{C} \\ [\,M\,] \quad &\longmapsto \mathbf{Tr}(b|_M) \end{aligned}$$

is a morphism of rings. Here, **Tr** denotes the usual trace (not the quantum trace) of an endomorphism of a finite dimensional vector space. Recall from Lemma 1.7 that the only group-like elements of $D(B)$ are the $K^i z^j$, where $(i,j) \in \Lambda(d)$. We set

$$\begin{array}{rccc} \chi_{i,j} : & \mathrm{Gr}(D(B)) & \longrightarrow & \mathbb{C} \\ & [\,M\,] & \longmapsto & \mathbf{Tr}(K^i z^j|_M). \end{array}$$

An easy computation yields

$$\chi_{i,j}(\mathbf{m}_{l,p}) = \zeta^{pi+(2p+l-1)j} \cdot (l)_{\zeta^i}. \tag{4.4}$$

Note that the $\chi_{i,j}$'s are not necessarily distinct:

Lemma 4.5. *Let λ and λ' be two elements of $\Lambda(d)$. Then $\chi_\lambda = \chi_{\lambda'}$ if and only if λ and λ' are in the same ι-orbit.*

Proof. Let us write $\lambda = (i,j)$ and $\lambda' = (i',j')$. The "if" part follows directly from (4.4). Conversely, assume that $\chi_{i,j} = \chi_{i',j'}$. By applying these two characters to $\mathbf{v}_\zeta$ and $\mathbf{m}_2$, we get:

$$\begin{cases} \zeta^{i+2j} = \zeta^{i'+2j'}, \\ \zeta^j(1+\zeta^i) = \zeta^{j'}(1+\zeta^{i'}). \end{cases}$$

It means that the pairs (ζ^j, ζ^{i+j}) and $(\zeta^{j'}, \zeta^{i'+j'})$ have the same sum and the same product, so $(\zeta^j, \zeta^{i+j}) = (\zeta^{j'}, \zeta^{i'+j'})$ or $(\zeta^j, \zeta^{i+j}) = (\zeta^{i'+j'}, \zeta^{j'})$. In other words, $(i',j') = (i,j)$ or $(i',j') = \iota(i,j)$, as expected. □

5. Triangulated categories

5.1. Stable category

As B is a Hopf algebra, it is selfinjective, i.e., B is an injective B-module. Recall that the stable category B-stab of B is the additive category quotient of B-mod by the full subcategory B-proj of projective modules. Since B is selfinjective, the category B-stab has a natural triangulated structure. Similarly, the category $D(B)$-stab is triangulated. Note that a B-module (resp. a $D(B)$-module) is projective if and only if it is injective. Since the tensor product of a projective $D(B)$-module by any $D(B)$-module is still projective [6, Proposition 4.2.12], $D(B)$-stab inherits a structure of monoidal category (such that the canonical functor $D(B)$-mod $\to D(B)$-stab is monoidal). In particular, its Grothendieck group (as a triangulated category), which will be denoted by $\mathrm{Gr}^{\mathrm{st}}(D(B))$, is a ring and the natural map

$$\begin{array}{ccc} \mathrm{Gr}(D(B)) & \longrightarrow & \mathrm{Gr}^{\mathrm{st}}(D(B)) \\ \mathbf{m} & \longmapsto & \mathbf{m}^{\mathrm{st}} \end{array}$$

is a morphism of rings. Given M a $D(B)$-module, we denote by $[M]_{\mathrm{st}}$ its class in $\mathrm{Gr}^{\mathrm{st}}(D(B))$.

It follows from Theorem 2.4 that

$$\mathbf{m}^{\mathrm{st}}_{d,p} = 0 \qquad \text{and} \qquad 2(\mathbf{m}^{\mathrm{st}}_{l,p} + \mathbf{m}^{\mathrm{st}}_{d-l,p+l}) = 0 \tag{5.1}$$

if $l \leqslant d-1$.

5.2. A further quotient

We denote by $D(B)\text{-proj}_{\mathrm{B}}$ the full subcategory of $D(B)$-mod whose objects are the $D(B)$-modules M such that $\mathrm{Res}_B^{D(B)} M$ is a projective B-module. Since $D(B)$ is a free B-module (of rank d^2), $D(B)$-proj is a full subcategory of $D(B)\text{-proj}_{\mathrm{B}}$. We denote by $D(B)\text{-stab}_{\mathrm{B}}$ the additive quotient of the category $D(B)$-mod by the full subcategory $D(B)\text{-proj}_{\mathrm{B}}$: it is also the quotient of $D(B)$-stab by the image of $D(B)\text{-proj}_{\mathrm{B}}$ in $D(B)$-stab.

Lemma 5.2. *The image of $D(B)\text{-proj}_{\mathrm{B}}$ in $D(B)$-stab is a thick triangulated subcategory. In particular, $D(B)\text{-stab}_{\mathrm{B}}$ is triangulated.*

Proof. Given M a $D(B)$-module, we denote by $\pi_M : P(M) \twoheadrightarrow M$ (resp. $i_M : M \hookrightarrow I(M)$) a projective cover (resp. injective hull) of M. We need to prove the following facts:

(a) If M belongs to $D(B)\text{-proj}_{\mathrm{B}}$, then $\mathrm{Ker}(\pi_M)$ and $I(M)/\mathrm{Im}(i_M)$ also belong to $D(B)\text{-proj}_{\mathrm{B}}$.
(b) If $M \oplus N$ belongs to $D(B)\text{-proj}_{\mathrm{B}}$, then M and N also belong to $D(B)\text{-proj}_{\mathrm{B}}$.
(c) If M and N belong to $D(B)\text{-proj}_{\mathrm{B}}$ and $f : M \to N$ is a morphism of $D(B)$-modules, then the cone of f also belong to $D(B)\text{-proj}_{\mathrm{B}}$.

(a) Assume that M belongs to $D(B)\text{-proj}_{\mathrm{B}}$. Since it is a projective B-module, there exists a morphism of B-modules $f : M \to P(M)$ such that $\pi_M \circ f = \mathrm{Id}_M$. In particular, $P(M) \simeq \mathrm{Ker}(\pi_M) \oplus M$, as a B-module. So $\mathrm{Ker}(\pi_M)$ is a projective B-module.

On the other hand, $I(M)$ is a projective $D(B)$-module since $D(B)$ is selfinjective so it is a projective B-module and so it is an injective B-module. So, again, $I(M) \simeq M \oplus I(M)/\mathrm{Im}(i_M)$, so $I(M)/\mathrm{Im}(i_M)$ is a projective B-module. This proves (a).

(b) is obvious.

(c) Let M and N belong to $D(B)\text{-proj}_{\mathrm{B}}$ and $f : M \to N$ be a morphism of $D(B)$-modules. Let $\Delta_f : M \to I(M) \oplus N$, $m \mapsto (i_M(m), f(m))$. Then the cone of f

is isomorphic in $D(B)$-stab to $(I(M) \oplus N)/\operatorname{Im}(\Delta_f)$. But Δ_f is injective, M is an injective B-module and so $I(M) \simeq M \oplus (I(M) \oplus N)/\operatorname{Im}(\Delta_f)$ as a B-module, which shows that $(I(M) \oplus N)/\operatorname{Im}(\Delta_f)$ is a projective B-module. $\square$

We denote by $\mathrm{Gr}^{\mathrm{st_B}}(D(B))$ the Grothendieck group of $D(B)$-stab$_\mathrm{B}$, viewed as a triangulated category. If M belongs to $D(B)$-proj$_\mathrm{B}$ and N is any $D(B)$-module, then $M \otimes N$ and $N \otimes M$ are projective B-modules [6, Proposition 4.2.12], so $D(B)$-stab$_\mathrm{B}$ inherits a structure of monoidal category, compatible with the triangulated structure. This endows $\mathrm{Gr}^{\mathrm{st_B}}(D(B))$ with a ring structure. The natural map $\mathrm{Gr}(D(B)) \to \mathrm{Gr}^{\mathrm{st_B}}(D(B))$ will be denoted by $\mathbf{m} \mapsto \mathbf{m}^{\mathrm{st_B}}$: it is a surjective morphism of rings that factors through $\mathrm{Gr}^{\mathrm{st}}(D(B))$.

If $\lambda \in \Lambda^{\#}(d)$, then it follows from [5, Property 1.4] that there exists a $D(B)$-module P_λ^B which is projective as a B-module and such that there is an exact sequence

$$0 \longrightarrow M_{\iota(\lambda)} \longrightarrow P_\lambda^B \longrightarrow M_\lambda \longrightarrow 0.$$

It then follows that

$$\mathbf{m}_\lambda^{\mathrm{st_B}} + \mathbf{m}_{\iota(\lambda)}^{\mathrm{st_B}} = 0. \tag{5.3}$$

Also, we still have

$$\mathbf{m}_{d,p}^{\mathrm{st_B}} = 0. \tag{5.4}$$

The next theorem follows from (5.3), (5.4) and Proposition 4.3.

Theorem 5.5. *The ring* $\mathrm{Gr}^{\mathrm{st_B}}(D(B))$ *is generated by* $\mathbf{v}_\zeta^{\mathrm{st_B}}$ *and* $\mathbf{m}_2^{\mathrm{st_B}}$*. Moreover,*

$$\mathrm{Gr}^{\mathrm{st_B}}(D(B)) = \bigoplus_{\lambda \in [\Lambda^{\#}(d)/\iota]} \mathbb{Z}\mathbf{m}_\lambda^{\mathrm{st_B}}$$

and $\mathrm{Gr}^{\mathrm{st_B}}(D(B))$ *is a free* $\mathbb{Z}$*-module of rank* $d(d-1)/2$*.*

Recall that Lemma 4.5 shows that, through the $\chi_{i,j}$'s, only $d(d+1)/2$ different characters of the ring $\mathrm{Gr}(D(B))$ have been defined. It is not clear if $\mathbb{C}\,\mathrm{Gr}(D(B))$ is semisimple in general but, for $d = 2$, it can be checked that it is semisimple (of dimension 4), so that there is a fourth character $\mathrm{Gr}(D(B)) \to \mathbb{C}$ which is not obtained through the $\chi_{i,j}$'s.

Now, a character $\chi : \mathrm{Gr}(D(B)) \to \mathbb{C}$ factors through $\mathrm{Gr}^{\mathrm{st_B}}(D(B))$ if and only if its kernel contains the $\mathbf{m}_{\lambda_0}$'s (where λ_0 runs over $\Lambda^0(d)$) and the $\mathbf{m}_\lambda + \mathbf{m}_{\iota(\lambda)}$'s (where λ runs over $\Lambda^{\#}(d)$). This implies the following result.

Theorem 5.6. *The character* $\chi_\lambda : \mathrm{Gr}(D(B)) \to \mathbb{C}$ *factors through* $\mathrm{Gr}^{\mathrm{st_B}}(D(B))$ *if and only if* $\lambda \in \Lambda^{\#}(d)$*. So the* $(\chi_\lambda)_{\lambda \in [\Lambda^{\#}(d)/\iota]}$ *are all the characters of* $\mathrm{Gr}^{\mathrm{st_B}}(D(B))$ *and the* $\mathbb{C}$*-algebra* $\mathbb{C}\,\mathrm{Gr}^{\mathrm{st_B}}(D(B))$ *is semisimple.*

5.3. Complements

Given $\mathscr{C}$ a monoidal category, we denote by $\mathbf{Z}(\mathscr{C})$ its *Drinfeld center* (see [7, §XIII.4]) and we denote by $\mathbf{For}_{\mathscr{C}} : \mathbf{Z}(\mathscr{C}) \to \mathscr{C}$ the forgetful functor.

There is an equivalence between $\mathbf{Z}(B\text{-mod})$ and $D(B)$-mod such that the forgetful functor becomes the restriction functor $\mathrm{Res}_B^{D(B)}$. The canonical functors between these categories will be denoted by $\mathrm{can}_{\mathrm{st}}^{D(B)} : D(B)\text{-mod} \to D(B)\text{-stab}$, $\mathrm{can}_{\mathrm{st}}^{B} : B\text{-mod} \to B\text{-stab}$ and $\mathrm{can}_{\mathrm{st_B}} : D(B)\text{-mod} \to D(B)\text{-stab}_{\mathrm{B}}$. The functor

$$\mathrm{can}_{\mathrm{st}}^{B} \circ \mathrm{Res}_B^{D(B)} : D(B)\text{-mod} \longrightarrow B\text{-stab}$$

factors through $\mathbf{Z}(B\text{-stab})$ (this triangulated category needs to be defined in a homotopic setting, for example that of stable ∞-categories). We obtain a commutative diagram of functors

$$\begin{tikzcd}
D(B)\text{-mod} \arrow[r, "\mathrm{Res}_B^{D(B)}"] \arrow[d, "\mathscr{F}"'] & B\text{-mod} \arrow[d, "\mathrm{can}_{\mathrm{st}}^{B}"] \\
\mathbf{Z}(B\text{-stab}) \arrow[r, "\mathbf{For}_{B\text{-stab}}"'] & B\text{-stab}\,.
\end{tikzcd}$$

Since any $D(B)$-module that is projective as a B-module is sent to the zero object of $\mathbf{Z}(B\text{-stab})$ through $\mathscr{F}$, the functor $\mathscr{F}$ factors through $D(B)\text{-proj}_{\mathrm{B}}$ and we obtain a commutative diagram of functors

$$\begin{tikzcd}
 & D(B)\text{-mod} \arrow[r, "\mathrm{Res}_B^{D(B)}"] \arrow[dl, "\mathrm{can}_{\mathrm{st_B}}^{D(B)}"'] \arrow[dd, "\mathscr{F}"'] & B\text{-mod} \arrow[dd, "\mathrm{can}_{\mathrm{st}}^{B}"] \\
D(B)\text{-stab}_{\mathrm{B}} \arrow[dr, "\overline{\mathscr{F}}"'] & & \\
 & \mathbf{Z}(B\text{-stab}) \arrow[r, "\mathbf{For}_{B\text{-stab}}"'] & B\text{-stab}\,.
\end{tikzcd}$$

Question. Is $\overline{\mathscr{F}} : D(B)\text{-stab}_{\mathrm{B}} \to \mathbf{Z}(B\text{-stab})$ an equivalence of categories?

6. Fusion datum

6.1. Quantum traces

The element $R \in D(B) \otimes D(B)$ defined in §1.6 is a universal R-matrix which endows $D(B)$ with a structure of braided Hopf algebra. The category $D(B)$-mod is braided as follows: given M and N two $D(B)$-modules, the braiding $c_{M,N} : M \otimes N \xrightarrow{\sim} N \otimes M$ is given by

$$c_{M,N}(m \otimes n) = \tau(R)(n \otimes m).$$

Recall that $\tau : D(B) \otimes D(B) \xrightarrow{\sim} D(B) \otimes D(B)$ is given by $\tau(a \otimes b) = b \otimes a$. In particular,

$$c_{N,M} c_{M,N} : M \otimes N \xrightarrow{\sim} M \otimes N \text{ is given by the action of } \tau(R)R. \tag{6.1}$$

Given $i \in \mathbb{Z}$, we have $S^2(b) = (z^{-i}K)b(z^{-i}K)^{-1}$ for all $b \in D(B)$ and $z^{-i}K$ is group-like, so the algebra $D(B)$ is *pivotal* with pivot $z^{-i}K$. This endows the tensor category $D(B)$-mod with a structure of pivotal category (see Appendix A) whose associated traces $\mathrm{Tr}_+^{(i)}$ and $\mathrm{Tr}_-^{(i)}$ are given as follows: given M a $D(B)$-module and $f \in \mathrm{End}_{D(B)}(M)$, we have

$$\mathrm{Tr}_+^{(i)}(f) = \mathbf{Tr}(z^{-i}Kf) \qquad \text{and} \qquad \mathrm{Tr}_-^{(i)}(f) = \mathbf{Tr}(fK^{-1}z^i).$$

Recall that $\mathbf{Tr}$ denotes the "classical" trace for endomorphisms of a finite dimensional vector space. So the pivotal structure depends on the choice of i (modulo d). The corresponding twist is $\theta_i = z^i\theta$, which endows $D(B)$-mod with a structure of balanced braided category (depending on i).

Hypothesis and notation. *From now on, and until the end of this paper, we assume that the Hopf algebra $D(B)$ is endowed with the pivotal structure whose pivot is $z^{-1}K$. The structure of balanced braided category is given by $\theta_1 = z\theta$ and the associated quantum traces $\mathrm{Tr}_\pm^{(1)}$ are denoted by $\mathrm{Tr}_\pm$.*

Given M a $D(B)$-module, we set $\dim_\pm(M) = \mathrm{Tr}_\pm(\mathrm{Id}_M)$.

We define

$$\dim(D(B)) = \sum_{M \in \mathrm{Irr}\, D(B)} \dim_-(M) \dim_+(M).$$

We have

$$\dim(D(B)) = \frac{2d^2}{(1-\zeta)(1-\zeta^{-1})}. \tag{6.2}$$

This follows easily from the fact that

$$\dim_+ M_{l,p} = \zeta^{1-l-p}(l)_\zeta \quad \text{and} \quad \dim_- M_{l,p} = \zeta^{p+l-1}(l)_{\zeta^{-1}} = \zeta^p(l)_\zeta. \tag{6.3}$$

6.2. Characters of $\mathrm{Gr}(D(B))$ *via the pivotal structure*

As in Appendix A, these structures (braiding, pivot) allow to define characters of $\mathrm{Gr}(D(B))$ associated with simple modules (or bricks). Given $\lambda \in \Lambda(d)$, we set

$$\begin{array}{rccc} s^+_{M_\lambda}: & \mathrm{Gr}(D(B)) & \longrightarrow & \mathbb{C} \\ & [M] & \longmapsto & (\mathrm{Id}_{M_\lambda} \otimes \mathrm{Tr}^M_+)(c_{M,M_\lambda} c_{M_\lambda,M}) \end{array}$$

and

$$\begin{array}{rccc} s^-_{M_\lambda}: & \mathrm{Gr}(D(B)) & \longrightarrow & \mathbb{C} \\ & [M] & \longmapsto & (\mathrm{Tr}^M_- \otimes \mathrm{Id}_{M_\lambda})(c_{M,M_\lambda} c_{M_\lambda,M}). \end{array}$$

These are morphisms of rings (see Proposition A.4). The main result of this section is the following.

Theorem 6.4. *Given* $\lambda \in \Lambda(d)$, *we have*

$$s^+_{M_\lambda} = \chi_{-\lambda} \qquad \textit{and} \qquad s^-_{M_{(l,p)}} = \chi_{(0,1)-\lambda}.$$

Proof. Write $\lambda = (l, p)$ and

$$\gamma_{i,j,k} = \frac{\zeta^{(i-k)(i+j)-i(i+1)/2}}{(i)!_\zeta}.$$

We have

$$\tau(R)R = \frac{1}{d^2} \sum_{i,i',j,j',k,k'=0}^{d-1} \gamma_{i,j,k}\gamma_{i',j',k'} \zeta^{i(k'-j')} (z^{-k'} F^{i'} E^i K^{k'+j}) \otimes (z^{-k} E^{i'} F^i K^{j'+k}).$$

We need to compute the endomorphism of $M_{l,p}$ given by

$$(\mathrm{Id}_{M_{l,p}} \otimes \mathrm{Tr}^M_+)(\tau(R)R|_{M_{l,p}\otimes M}).$$

Since $M_{l,p}$ is simple, this endomorphism is the multiplication by a scalar ϖ, and so it is sufficient to compute the action on $e_1^{(l,p)} \in M_{l,p}$. Therefore, all the terms (in the big sum giving $\tau(R)R$) corresponding to $i \neq 0$ disappear (because $Ee_1^{(l,p)} = 0$). Also, since we are only interested in the coefficient on $e_1^{(l,p)}$ of the result (because the coefficients on other vectors will be zero), all the terms corresponding to $i' \neq 0$ also disappear. Therefore,

$$\varpi = \frac{1}{d^2} \sum_{j,j',k,k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-kj-k'j'} (\zeta^{2p+l-1})^{-k'} (\zeta^{p+l-1})^{k'+j} \mathbf{Tr}(z^{-1} K z^{-k} K^{j'+k}|_M).$$

So it remains to compute the element

$$b = \frac{1}{d^2} \sum_{j,j',k,k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-kj-k'j'} (\zeta^{2p+l-1})^{-k'} (\zeta^{p+l-1})^{k'+j} z^{-k-1} K^{j'+k+1}$$

of $D(B)$. Since

$$b = \frac{1}{d^2} \sum_{j',k \in \mathbb{Z}/d\mathbb{Z}} \Bigl(\sum_{j,k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{l(p+l-1-k)} \zeta^{k'(-p-j')} \Bigr) z^{-k-1} K^{j'+k+1},$$

only the terms corresponding to $k = l + p - 1$ and $j' = -p$ remain, hence

$$b = z^{-l-p} K^l.$$

So $s^+_{M_\lambda} = \chi_{-\iota(\lambda)} = \chi_{-\lambda}$, as expected.

The other formula is obtained via a similar computation. $\square$

We denote by $\mathbb{S}^\pm = (\mathbb{S}^\pm_{\lambda,\lambda'})_{\lambda,\lambda' \in \Lambda(d)}$ the square matrix defined by

$$\mathbb{S}^\pm_{\lambda,\lambda'} = \mathrm{Tr}_\pm(c_{M_{\lambda'},M_\lambda} \circ c_{M_\lambda,M_{\lambda'}}).$$

Similarly, we define $\mathbb{T}^\pm$ to be the diagonal matrix (whose rows and columns are indexed by $\lambda \in \Lambda(d)$) and whose λ-entry is

$$\mathbb{T}^\pm_\lambda = \omega_\lambda(\theta_1^{\mp 1}).$$

Let us first give a formula for $\mathbb{S}^\pm_{\lambda,\lambda'}$ and $\mathbb{T}^\pm_\lambda$.

Corollary 6.5. *Let (l, p), $(l', p') \in \Lambda(d)$. We have*

$$\mathbb{S}^+_{(l,p),(l',p')} = \frac{\zeta}{1-\zeta} \zeta^{-ll'-lp'-pl'-2pp'} (1 - \zeta^{ll'}), \qquad \mathbb{T}^+_{(l,p)} = \zeta^{-p(p+l)},$$

$$\mathbb{S}^-_{(l,p),(l',p')} = \frac{\zeta^{2p+l+2p'+l'-1}}{1-\zeta} \zeta^{-ll'-lp'-pl'-2pp'} (1 - \zeta^{ll'}) \qquad \textit{and} \qquad \mathbb{T}^-_{(l,p)} = \zeta^{p(p+l)}.$$

Proof. This follows immediately from formulas (2.3), (4.4), (6.3) and Theorem 6.4. $\square$

6.3. Fusion datum associated with $D(B)$-$\mathrm{stab}_{\mathrm{B}}$

Let $\mathscr{E}$ denote a set of representatives of ι-orbits in $\{1, 2, \ldots, d-1\} \times \mathbb{Z}/d\mathbb{Z}$. We define

$$\dim^{\mathrm{st}_{\mathrm{B}}}(D(B)) = \sum_{(l,p)\in\mathscr{E}} \dim_-(M_{l,p}) \dim_+(M_{l,p}).$$

This can be understood in terms of super-categories, as explained recently by Lacabanne [8]. We have

$$\dim^{\mathrm{st}_{\mathrm{B}}}(D(B)) = \frac{1}{2}\dim(D(B)) = \frac{d^2}{(1-\zeta)(1-\zeta^{-1})}.$$

So $\dim^{\mathrm{st}_{\mathrm{B}}} D(B)$ is a positive real number and we denote by $\sqrt{\dim^{\mathrm{st}_{\mathrm{B}}} D(B)}$ its positive square root. Since $1-\zeta^{-1} = -\zeta^{-1}(1-\zeta)$, there exists a unique square root $\sqrt{-\zeta}$ of $-\zeta$ such that

$$\sqrt{\dim^{\mathrm{st}_{\mathrm{B}}}(D(B))} = \frac{d\sqrt{-\zeta}}{1-\zeta}.$$

We denote by $\mathbb{S}^{\mathrm{st}_{\mathrm{B}}} = (\mathbb{S}_{\lambda,\lambda'})_{\lambda,\lambda'\in\mathscr{E}}$ the square matrix defined by

$$\mathbb{S}^{\mathrm{st}_{\mathrm{B}}}_{\lambda,\lambda'} = \frac{\mathbb{S}^+_{\lambda,\lambda'}}{\sqrt{\dim^{\mathrm{st}_{\mathrm{B}}}(D(B))}}.$$

We denote by $\mathbb{T}^{\mathrm{st}_{\mathrm{B}}}$ the diagonal matrix whose λ-entry is $\mathbb{T}^+_\lambda$ (for $\lambda \in \mathscr{E}$). It follows from Corollary 6.5 that

$$\mathbb{S}^{\mathrm{st}_{\mathrm{B}}}_{(l,p),(l',p')} = \frac{\sqrt{-\zeta}}{d}\zeta^{-ll'-lp'-pl'-2pp'}(\zeta^{ll'}-1) \quad \text{and} \quad \mathbb{T}^{\mathrm{st}_{\mathrm{B}}}_{(l,p)} = \zeta^{-p(l+p)}. \tag{6.6}$$

The root of unity $\sqrt{-\zeta}$ appearing in this formula has been interpreted in terms of super-categories by Lacabanne [8]: it is due to the fact that our category is not spherical. Finally, note that

$$\mathbb{S}^{\mathrm{st}_{\mathrm{B}}}_{(l,p),(l',p')} = -\mathbb{S}^{\mathrm{st}_{\mathrm{B}}}_{\iota(l,p),(l',p')} = -\mathbb{S}^{\mathrm{st}_{\mathrm{B}}}_{(l,p),\iota(l',p')} = \mathbb{S}^{\mathrm{st}_{\mathrm{B}}}_{\iota(l,p),\iota(l',p')}. \tag{6.7}$$

7. Comparison with Malle $\mathbb{Z}$-fusion datum

We refer to [10] and [3] for most of the material of this section. We denote by $\mathscr{E}(d)$ the set of pairs (i, j) of integers with $0 \leqslant i < j \leqslant d-1$.

7.1. Set-up

Let $Y = \{0, 1, \dots, d\}$ and let $\pi : Y \to \{0, 1\}$ be the map defined by

$$\pi(i) = \begin{cases} 1 & \text{if } i \in \{0,1\}, \\ 0 & \text{if } i \geqslant 2. \end{cases}$$

We denote by $\Psi(Y, \pi)$ the set of maps $f : Y \to \{0, 1, \dots, d-1\}$ such that f is strictly increasing on $\pi^{-1}(0) = \{2, 3, \dots, d\}$ and strictly increasing on $\pi^{-1}(1) = \{0, 1\}$. Since f is injective on $\{2, 3, \dots, d\}$, there exists a unique element $\mathbf{k}(f) \in \{0, 1, \dots, d-1\}$ which does not belong to $f(\{2, 3, \dots, d\})$. Note that, since f is strictly increasing on $\{2, 3, \dots, d\}$, the element $\mathbf{k}(f)$ determines the restriction of f to $\{2, 3, \dots, d\}$. So the map

$$\begin{array}{ccc} \Psi(Y,\pi) & \longrightarrow & \mathscr{E}(d) \times \{0,1,\dots,d-1\} \\ f & \longmapsto & (f(0), f(1), \mathbf{k}(f)) \end{array} \tag{7.1}$$

is bijective. For $f \in \Psi(Y, \pi)$, we set

$$\varepsilon(f) = (-1)^{|\{(y,y') \in Y \times Y \ \mid \ y < y' \text{ and } f(y) < f(y')\}|}.$$

We put by $V = \bigoplus_{i=0}^{d-1} \mathbb{C} v_i$ and we denote by $\mathscr{S}$ the square matrix $(\zeta^{ij})_{0 \leqslant i,j \leqslant d-1}$, which will be viewed as an automorphism of V. Note that $\mathscr{S}$ is the character stable of the cyclic group $\boldsymbol{\mu}_d$. We set $\delta(d) = \det(\mathscr{S}) = \prod_{0 \leqslant i<j \leqslant d-1} (\zeta^j - \zeta^i)$. Recall that

$$\delta(d)^2 = (-1)^{(d-1)(d-2)/2} d^d.$$

Given $f \in \Psi(Y, \pi)$, let

$$\mathbf{v}_f = (v_{f(0)} \wedge v_{f(1)}) \otimes (v_{f(2)} \wedge v_{f(3)} \wedge \cdots \wedge v_{f(d)}) \in \Big(\bigwedge^2 V\Big) \otimes \Big(\bigwedge^{d-1} V\Big).$$

Note that $(\mathbf{v}_f)_{f \in \Psi(Y,\pi)}$ is a $\mathbb{C}$-basis of $\big(\bigwedge^2 V\big) \otimes \big(\bigwedge^{d-1} V\big)$. Given $f' \in \Psi(Y, \pi)$, we put

$$\Big(\big(\bigwedge^2 \mathscr{S}\big) \otimes \big(\bigwedge^{d-1} \mathscr{S}\big)\Big)(\mathbf{v}_{f'}) = \sum_{f \in \Psi(Y,\pi)} \mathbf{S}_{f,f'} \mathbf{v}_f.$$

In other words, $(\mathbf{S}_{f,f'})_{f,f' \in \Psi(Y,\pi)}$ is the matrix of the automorphism $\big(\bigwedge^2 \mathscr{S}\big) \otimes \big(\bigwedge^{d-1} \mathscr{S}\big)$ of $\big(\bigwedge^2 V\big) \otimes \big(\bigwedge^{d-1} V\big)$ in the basis $(\mathbf{v}_f)_{f \in \Psi(Y,\pi)}$.

Lemma 7.2. *Let f, $f' \in \Psi(Y, \pi)$. We define*

$$i = f(0), \quad j = f(1), \quad k = \mathbf{k}(f),$$
$$i' = f'(0), \quad j' = f'(1), \quad k' = \mathbf{k}(f').$$

We have

$$\mathbf{S}_{f,f'} = (-1)^{k+k'}\frac{\delta(d)}{d}\zeta^{-kk'}(\zeta^{ii'+jj'} - \zeta^{ij'+ji'}).$$

Proof. The computation of the action of $\bigwedge^2 \mathscr{S}$ is easy, and gives the term $\zeta^{ii'+jj'} - \zeta^{ij'+ji'}$. It remains to show that the determinant of the matrix $\mathscr{S}(k,k')$ obtained from $\mathscr{S}$ by removing the k-th row and the k'-th column is equal to $(-1)^{k+k'}\zeta^{-kk'}\delta(d)/d$. For this, let $\mathscr{S}'(k)$ denote the matrix whose k-th row is equal to $(1,t,t^2,\dots,t^{d-1})$ (where t is an indeterminate) and whose other rows coincide with those of $\mathscr{S}$. Then $(-1)^{k+k'}\det(\mathscr{S}(k,k'))$ is equal to the coefficient of $t^{k'}$ in the polynomial $\det(\mathscr{S}'(k))$. This is a Vandermonde determinant and

$$\begin{aligned}\det(\mathscr{S}'(k)) &= \prod_{\substack{0 \leqslant i<j \leqslant d-1\\ i\neq k, j\neq k}} (\zeta^j - \zeta^i)\cdot \prod_{i=0}^{k-1}(t-\zeta^i)\cdot \prod_{i=k+1}^{d-1}(\zeta^i - t)\\ &= \delta(d)\prod_{\substack{i=0\\ i\neq k}}^{d-1}\frac{(t-\zeta^i)}{(\zeta^k-\zeta^i)}.\end{aligned}$$

Since

$$\prod_{\substack{i=0\\ i\neq k}}^{d-1}(t-\zeta^i) = \frac{t^d-1}{t-\zeta^k} = \sum_{i=0}^{d-1} t^i\zeta^{(d-1-i)k},$$

we have

$$\det(\mathscr{S}(k,k')) = (-1)^{k+k'}\delta(d)\frac{\zeta^{(d-1-k')k}}{d\zeta^{(d-1)k}} = (-1)^{k+k'}\frac{\delta(d)}{d}\zeta^{-kk'},$$

as desired. □

7.2. Malle $\mathbb{Z}$-fusion datum

Let

$$\Psi^{\#}(Y,\pi) = \{f \in \Psi(Y,\pi) \mid \sum_{y\in Y} f(y) \equiv \frac{d(d-1)}{2} \mod d\}.$$

Given $f \in \Psi^{\#}(Y,\pi)$, we define

$$\mathrm{Fr}(f) = \zeta_*^{d(1-d^2)}\prod_{y\in Y}\zeta_*^{-6(f(y)^2+df(y))},$$

where ζ_* is a primitive $(12d)$-th root of unity such that $\zeta_*^{12} = \zeta$.

We denote by $\mathbb{T}$ diagonal matrix (whose rows and columns are indexed by $\Psi^{\#}(Y,\pi)$) equal to $\mathrm{diag}(\mathrm{Fr}(f))_{f\in\Psi^{\#}(Y,\pi)}$. We denote by $\mathbb{S} = (\mathbb{S}_{f,g})_{f,g\in\Psi^{\#}(Y,\pi)}$ the square matrix defined by

$$\mathbb{S}_{f,g} = \frac{(-1)^{d-1}}{\delta(d)}\varepsilon(f)\varepsilon(g)\ \overline{\mathbf{S}}_{f,g}.$$

Note that $\mathbb{S}_{f,f_{0,1}} \neq 0$ for all $f \in \Psi^{\#}(Y,\pi)$ (see Lemma 7.2).

Proposition 7.3 *(Malle [10], Cuntz [3]). With the previous notation, we have:*

(a) $\mathbb{S}^4 = (\mathbb{S}\mathbb{T})^3 = [\mathbb{S}^2, \mathbb{T}] = 1$.
(b) ${}^t\mathbb{S} = \mathbb{S}$ *and* ${}^t\overline{\mathbb{S}}\ \mathbb{S} = 1$.
(c) *For all f, g, $h \in \Psi^{\#}(Y,\pi)$, the number*

$$N_{f,g}^{h} = \sum_{i\in\Psi^{\#}(Y,\pi)} \frac{\mathbb{S}_{i,f}\mathbb{S}_{i,g}\overline{\mathbb{S}}_{i,h}}{\mathbb{S}_{i,f_{0,1}}}$$

belongs to $\mathbb{Z}$.

The pair $(\mathbb{S},\mathbb{T})$ is called the *Malle $\mathbb{Z}$-fusion datum.*

7.3. Comparison

We wish to compare the $\mathbb{Z}$-fusion datum $(\mathbb{S},\mathbb{T})$ with the ones obtained from the tensor categories $D(B)$-mod and $D(B)$-stab. For this, we will use the bijection (7.1) to characterize elements of $\Psi^{\#}(Y,\pi)$. Given $k \in \mathbb{Z}$, we denote by k^{res} the unique element in $\{0,1,\ldots,d-1\}$ such that $k \equiv k^{\mathrm{res}} \mod d$.

Lemma 7.4. *Let $f \in \Psi(Y,\pi)$. Then $f \in \Psi^{\#}(Y,\pi)$ if and only if $\mathbf{k}(f) = (f(0)+f(1))^{\mathrm{res}}$. Consequently, the map*

$$\begin{array}{ccc} \Psi^{\#}(Y,\pi) & \longrightarrow & \mathscr{E}(d) \\ f & \longmapsto & (f(0),f(1)) \end{array}$$

is bijective

Proof. We have

$$\sum_{y\in Y} f(y) = f(0)+f(1)+\frac{d(d-1)}{2} - \mathbf{k}(f)$$

and the result follows. □

Given $(i,j) \in \mathscr{E}(d)$, we denote by $f_{i,j}$ the unique element of $\Psi^{\#}(Y,\pi)$ such that $f_{i,j}(0) = i$ and $f_{i,j}(1) = j$. We have

$$\mathrm{Fr}(f_{i,j}) = \zeta^{ij} \tag{7.5}$$

and, if (i,j), $(i',j') \in \Lambda(d)$, then

$$\mathbb{S}_{f_{i,j},f_{i',j'}} = \frac{(-1)^{(i+j)^{\mathrm{res}}+(i'+j')^{\mathrm{res}}}}{d}\varepsilon(f_{i,j})\varepsilon(f_{i',j'})(\zeta^{ij'+ji'} - \zeta^{ii'+jj'}). \tag{7.6}$$

Proof. The second equality follows immediately from Lemmas 7.2 and 7.4. Let us prove the first one. By definition, $\mathrm{Fr}(f_{i,j}) = \zeta_*^{\alpha}$, where

$$\alpha = d(1-d^2) - 6\sum_{y\in Y}(f_{i,j}(y)^2 + df_{i,j}(y)).$$

The construction of $f_{i,j}$ shows that

$$\alpha = d(1-d^2) - 6(i^2+di) - 6(j^2+dj) - 6\sum_{k=0}^{d-1}(k^2+dk) + 6(((i+j)^{\mathrm{res}})^2 + d(i+j)^{\mathrm{res}}).$$

Write $i+j = (i+j)^{\mathrm{res}} + \eta d$, with $\eta \in \{0,1\}$. Then $\eta^2 = \eta$ and so

$$\begin{aligned}(i+j)^2 + d(i+j) &= ((i+j)^{\mathrm{res}})^2 + d(i+j)^{\mathrm{res}} + 2\eta d(i+j) + 2\eta d^2\\ &\equiv ((i+j)^{\mathrm{res}})^2 + d(i+j)^{\mathrm{res}} \mod 2d.\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha &\equiv 12ij + d(1-d^2) - 6\sum_{k=0}^{d-1}(k^2+dk) \mod 12d\\ &\equiv 12ij \mod 12d.\end{aligned}$$

So $\mathrm{Fr}(f_{i,j}) = \zeta_*^{12ij} = \zeta^{ij}$. $\square$

We define

$$\begin{aligned}\varphi : \quad \mathscr{E}(d) &\longrightarrow \quad \Lambda^{\#}(d)\\ (i,j) &\longmapsto (j-i,i).\end{aligned}$$

Note that $\varphi(\mathscr{E}(d))$ is a set of representatives of ι-orbits in $\Lambda^{\#}(d)$. We set

$$\tilde{\varphi}(i,j) = \begin{cases}\varphi(i,j) & \text{if } (-1)^{(i+j)^{\mathrm{res}}}\varepsilon(f_{i,j}) = 1,\\ \iota(\varphi(i,j)) & \text{if } (-1)^{(i+j)^{\mathrm{res}}}\varepsilon(f_{i,j}) = -1.\end{cases}$$

Then $\tilde{\varphi}(\mathscr{E}(d))$ is also a set of representatives of ι-orbits in $\Lambda^{\#}(d)$ and the pairs of matrices $(\mathbb{S}^{\mathrm{st_B}}, \mathbb{T}^{\mathrm{st_B}})$ and $(\mathbb{S}, \mathbb{T})$ are related by the following equality (which follows immediately from Corollary 6.5 and formulas (6.7), (7.5) and (7.6)):

$$\overline{\mathbb{S}}_{f_{i,j}, f_{i',j'}} = \sqrt{-\zeta}\; \mathbb{S}^{\mathrm{st_B}}_{\tilde{\varphi}(i,j), \tilde{\varphi}(i',j')} \qquad \text{and} \qquad \overline{\mathbb{T}}_{f_{i,j}} = \mathbb{T}^{\mathrm{st_B}}_{\tilde{\varphi}(i,j)}. \tag{7.7}$$

Therefore, up to the change of ζ into ζ^{-1}, we obtain our main result.

Theorem 7.8. *Malle $\mathbb{Z}$-fusion datum $(\mathbb{S}, \mathbb{T})$ can be categorified by the monoidal category $D(B)$-stab$_{\mathrm{B}}$, endowed with the pivotal structure induced by the pivot $z^{-1}K$ and the balanced structure induced by $z\theta$.*

Appendix A. Reminders on S-matrices

We follow closely [6, Chapters 4 and 8].

Let $\mathscr{C}$ be a *tensor category* over $\mathbb{C}$, as defined in [6, Definition 4.1.1]: $\mathscr{C}$ is a locally finite $\mathbb{C}$-linear rigid monoidal category (whose unit object is denoted by $\mathbf{1}$) such that the bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is $\mathbb{C}$-bilinear on morphisms and $\mathrm{End}_{\mathscr{C}}(\mathbf{1}) = \mathbb{C}$. If X is an object in $\mathscr{C}$, its left (respectively right) dual is denoted by X^* (respectively ${}^*X$) and we denote by

$$\mathrm{coev}_X : \mathbf{1} \longrightarrow X \otimes X^* \qquad \text{and} \qquad \mathrm{ev}_X : X^* \otimes X \longrightarrow \mathbf{1}$$

the *coevaluation* and *evaluation* morphisms respectively.

We assume that $\mathscr{C}$ is *braided*, namely that it is endowed with a bifunctorial family of isomorphisms $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ such that

$$c_{X,Y\otimes Y'} = (\mathrm{Id}_Y \otimes c_{X,Y'}) \circ (c_{X,Y} \otimes \mathrm{Id}_{Y'}) \tag{A.1}$$

and

$$c_{X\otimes X',Y} = (c_{X,Y} \otimes \mathrm{Id}_{X'}) \circ (\mathrm{Id}_X \otimes c_{X',Y}), \tag{A.2}$$

for all objects X, X', Y and Y' in $\mathscr{C}$ (we have omitted the associativity constraints).

Finally, we also assume that $\mathscr{C}$ is *pivotal* [6, Definition 4.7.8], i.e. that it is equipped with a family of functorial isomorphisms $a_X : X \to X^{**}$ (for X running over the objects of $\mathscr{C}$) such that $a_{X\otimes Y} = a_X \otimes a_Y$. Given $f \in \mathrm{End}_{\mathscr{C}}(X)$, the pivotal structure allows to define two *traces*:

$$\mathrm{Tr}_+(f) = \mathrm{ev}_{X^*} \circ (a_X f \otimes \mathrm{Id}_{X^*}) \circ \mathrm{coev}_X \in \mathrm{End}_{\mathscr{C}}(\mathbf{1}) = \mathbb{C}$$

and

$$\mathrm{Tr}_-(f) = \mathrm{ev}_X \circ (\mathrm{Id}_{X^*} \otimes f a_X^{-1}) \circ \mathrm{coev}_{X^*} \in \mathrm{End}_{\mathscr{C}}(\mathbf{1}) = \mathbb{C}.$$

We will sometimes write $\mathrm{Tr}_+^X(f)$ or $\mathrm{Tr}_-^X(f)$ for $\mathrm{Tr}_+(f)$ and $\mathrm{Tr}_-(f)$. We define two *dimensions*

$$\dim_+(X) = \mathrm{Tr}_+(\mathrm{Id}_X) \qquad \text{and} \qquad \dim_-(X) = \mathrm{Tr}_-(\mathrm{Id}_X).$$

To summarize, we will work under the following hypothesis:

Hypothesis and notation. *We fix in this section a braided pivotal tensor category $\mathscr{C}$ as above. We denote by $\mathrm{Gr}(\mathscr{C})$ its Grothendieck ring. Given X is an object in $\mathscr{C}$, we denote by $[X]$ its class in $\mathrm{Gr}(\mathscr{C})$. The set of isomorphism classes of simple objects in $\mathscr{C}$ will be denoted by $\mathrm{Irr}(\mathscr{C})$. If $X \in \mathrm{Irr}(\mathscr{C})$ and Y is any object in $\mathscr{C}$, we denote by $[Y : X]$ the multiplicity of X in a Jordan-Hölder series of Y.*

Given X, Y two objects in $\mathscr{C}$, we set

$$s_{X,Y}^+ = (\mathrm{Id}_X \otimes \mathrm{Tr}_+^Y)(c_{Y,X} c_{X,Y}) \in \mathrm{End}_{\mathscr{C}}(X),$$

and

$$s_{X,Y}^- = (\mathrm{Tr}_-^Y \otimes \mathrm{Id}_X)(c_{Y,X} c_{X,Y}) \in \mathrm{End}_{\mathscr{C}}(X).$$

These induce two morphisms of abelian groups

$$\begin{array}{rccc} s_X^+ : & \mathrm{Gr}(\mathscr{C}) & \longrightarrow & \mathrm{End}_{\mathscr{C}}(X) \\ & [Y] & \longmapsto & s_{X,Y}^+ \end{array} \qquad \text{and} \qquad \begin{array}{rccc} s_X^- : & \mathrm{Gr}(\mathscr{C}) & \longrightarrow & \mathrm{End}_{\mathscr{C}}(X) \\ & [Y] & \longmapsto & s_{X,Y}^-. \end{array}$$

Definition A.3. An object X in $\mathscr{C}$ is called a **brick** if $\mathrm{End}_{\mathscr{C}}(X) = \mathbb{C}$.

For instance, a simple object is a brick (and $\mathbf{1}$ is also a brick, but $\mathbf{1}$ is simple in a tensor category [6, Theorem 4.3.1]). Note also that a brick is indecomposable. So if $\mathscr{C}$ is moreover semisimple, then an object is a brick if and only if it is simple.

If X is a brick, then we will view $s_{X,Y}^+$ and $s_{X,Y}^-$ as elements of $\mathbb{C} = \mathrm{End}_{\mathscr{C}}(X)$.

Proposition A.4. *If X is a brick, then $s_X^+ : \mathrm{Gr}(\mathscr{C}) \to \mathbb{C}$ and $s_X^- : \mathrm{Gr}(\mathscr{C}) \to \mathbb{C}$ are morphisms of rings.*

Proof. Assume that X is a brick. We will only prove the result for s_X^+, which amounts to show that

$$s^+_{X,Y\otimes Y'} = s^+_{X,Y} s^+_{X,Y'}. \tag{$*$}$$

First, note that the following equality

$$c_{Y\otimes Y',X} c_{X,Y\otimes Y'} = (c_{Y,X} \otimes \mathrm{Id}_{Y'}) \circ (\mathrm{Id}_Y \otimes c_{Y',X} c_{X,Y'}) \circ (c_{X,Y} \otimes \mathrm{Id}_{Y'})$$

holds by (A.1) and (A.2). Taking $\mathrm{Id}_X \otimes \mathrm{Id}_Y \otimes \mathrm{Tr}_+^{Y'}$ on the right-hand side, one gets $s^+_{X,Y'} c_{Y,X} c_{X,Y} \in \mathrm{End}_{\mathscr{C}}(X \otimes Y)$ (because X is a brick). Applying now $\mathrm{Id}_X \otimes \mathrm{Tr}_+^Y$, one gets $s^+_{X,Y'} s^+_{X,Y} \mathrm{Id}_X$. Since

$$(\mathrm{Id}_X \otimes \mathrm{Tr}_+^Y) \circ (\mathrm{Id}_X \otimes \mathrm{Id}_Y \otimes \mathrm{Tr}_+^{Y'}) = \mathrm{Id}_X \otimes \mathrm{Tr}_+^{Y\otimes Y'},$$

this proves ($*$). □

Proposition A.5. *Let X be a brick and let X' be a subquotient of X which is also a brick. Then*

$$s^+_X = s^+_{X'} \qquad \textit{and} \qquad s^-_X = s^-_{X'}.$$

Proof. Indeed, the endomorphism $(\mathrm{Id}_X \otimes \mathrm{Tr}_+^Y)(c_{Y,X} c_{X,Y})$ of X is multiplication by a scalar, and this scalar can be computed on any non-trivial subquotient of X. □

Corollary A.6. *Let X and X' be two bricks in $\mathscr{C}$ belonging to the same block. Then*

$$s^+_X = s^+_{X'} \qquad \textit{and} \qquad s^-_X = s^-_{X'}.$$

Proof. By Proposition A.5, we may assume that X and X' are simple. We may also assume that X is not isomorphic to X' and that $\mathrm{Ext}^1_{\mathscr{C}}(X, X') = 0$. Let $\mathbf{X} \in \mathscr{C}$ such that there exists a non-split exact sequence

$$0 \longrightarrow X' \longrightarrow \mathbf{X} \longrightarrow X \longrightarrow 0.$$

Since $X \not\simeq X'$, we have $\mathrm{End}_{\mathscr{C}}(\mathbf{X}) = \mathbb{C}$, hence $\mathbf{X}$ is a brick. It follows from Proposition A.5 that

$$s^+_{\mathbf{X}} = s^+_X = s^+_{X'} \qquad \text{and} \qquad s^-_{\mathbf{X}} = s^-_X = s^-_{X'},$$

as desired. □

References

[1] M. Broué, J. Michel, G. Malle, Split Spetses for primitive reflection groups, Astérisque 359 (2014), vi+146 pp.

[2] H.-X. Chen, Irreducible representations of a class of quantum doubles, J. Algebra 225 (2000) 391–409.

[3] M. Cuntz, Fusion algebras for imprimitive complex reflection groups, J. Algebra 311 (2007) 251–267.

[4] K. Erdmann, E.L. Green, N. Snashall, R. Taillefer, Representation theory of the Drinfeld doubles of a family of Hopf algebras, J. Pure Appl. Algebra 204 (2006) 413–454.

[5] K. Erdmann, E.L. Green, N. Snashall, R. Taillefer, Stable Green ring of the Drinfeld doubles of the generalised Taft algebras (corrections and new results), Algebr. Represent. Theory 22 (2019) 757–783.

[6] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Tensor Categories, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, Providence, RI, 2015, xvi+343 pp.

[7] C. Kassel, Quantum Groups, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, 1995, xii+531 pp.

[8] A. Lacabanne, Slightly degenerate categories and $\mathbb{Z}$-modular data, preprint, arXiv:1807.00766, 2018, to appear in Int. Math. Res. Not.

[9] A. Lacabanne, Drinfeld double of quantum groups, tilting modules and $\mathbb{Z}$-modular data associated to complex reflection groups, preprint, arXiv:1807.00770, 2018.

[10] G. Malle, Unipotente Grade imprimitiver komplexer Spiegelungsgruppen, J. Algebra 177 (1995) 768–826.