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CHEREDNIK ALGEBRAS AND CALOGERO-MOSER CELLS

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The second author was partially supported by the NSF (grant DMS-1161999) and by a grant from the Simons Foundation (#376202).

October 26, 2016

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CONTENTS

PART I

REFLECTION GROUPS

CHAPTER 1

NOTATIONS

1.1. Integers

We put $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$.

1.2. Gradings

1.2.A. Let $\mathbf{k}$ be a ring and X a set. We denote by $\mathbf{k}X = \mathbf{k}^{(X)}$ the free $\mathbf{k}$-module with basis X. We sometimes denote elements of $\mathbf{k}^X$ as formal sums: $\sum_{x\in X} \alpha_x x$, where $\alpha_x \in \mathbf{k}$.

1.2.B. Let Γ be a monoid. We denote by $\mathbf{k}\Gamma$ (or $\mathbf{k}[\Gamma]$) the monoid algebra of Γ over $\mathbf{k}$. Its basis of elements of Γ is denoted by $\{t^\gamma\}_{\gamma\in\Gamma}$.

A Γ-*graded* $\mathbf{k}$-*module* is a $\mathbf{k}$-module L with a decomposition $L = \bigoplus_{\gamma\in\Gamma} L_\gamma$ (that is the same as a comodule over the coalgebra $\mathbf{k}\Gamma$). Given $\gamma_0 \in \Gamma$, we denote by $L\langle\gamma_0\rangle$ the Γ-graded $\mathbf{k}$-module given by $(L\langle\gamma_0\rangle)_\gamma = L_{\gamma\gamma_0}$. We denote by $\mathbf{k}\text{-free}^\Gamma$ the additive category of Γ-graded $\mathbf{k}$-modules L such that L_γ is a free $\mathbf{k}$-module of finite rank for all $\gamma \in \Gamma$. Given $L \in \mathbf{k}\text{-free}^\Gamma$, we put

$$\dim_{\mathbf{k}}^\Gamma(L) = \sum_{\gamma\in\Gamma} \operatorname{rank}_{\mathbf{k}}(L_\gamma) t^\gamma \in \mathbb{Z}^\Gamma.$$

We have defined an isomorphism of abelian groups $\dim_{\mathbf{k}}^\Gamma : K_0(\mathbf{k}\text{-free}^\Gamma) \xrightarrow{\sim} \mathbb{Z}^\Gamma$. This construction provides a bijection from the set of isomorphism classes of objects of $\mathbf{k}\text{-free}^\Gamma$ to $\mathbb{N}^\Gamma$. Given $P = \sum_{\gamma\in\Gamma} p_\gamma t^\gamma$ with $p_\gamma \in \mathbb{N}$, we define the Γ-graded $\mathbf{k}$-module $\mathbf{k}^P$ by $(\mathbf{k}^P)_\gamma = \mathbf{k}^{p_\gamma}$. We have $\dim_{\mathbf{k}}^\Gamma(\mathbf{k}^P) = P$.

We say that a subset E of a Γ-graded module L is *homogeneous* if every element of E is a sum of elements in $E \cap L_\gamma$ for various elements $\gamma \in \Gamma$.

1.2.C. A *graded* $\mathbf{k}$-*module* L is a $\mathbb{Z}$-graded $\mathbf{k}$-module. We put $L_+ = \bigoplus_{i>0} L_i$. If $L_i = 0$ for $i \ll 0$ (for example, if L is $\mathbb{N}$-graded), then $\dim_{\mathbf{k}}^{\mathbb{Z}}(L)$ is an element of the ring of

Laurent power series $\mathbb{Z}((\mathbf{t}))$: this is the Hilbert series of L. Similarly, if $L_i = 0$ for $i \gg 0$, then $\dim_{\mathbf{k}}^{\mathbb{Z}}(L) \in \mathbb{Z}((\mathbf{t}^{-1}))$.

When L has finite rank over $\mathbf{k}$, we define the *weight sequence* of L as the unique sequence of integers $r_1 \leqslant \cdots \leqslant r_m$ such that $\dim_k^{\mathbb{Z}}(L) = t^{r_1} + \cdots + t^{r_m}$.

A *bigraded* $\mathbf{k}$*-module* L is a $(\mathbb{Z}\times\mathbb{Z})$-graded $\mathbf{k}$-module. We put $\mathbf{t} = t^{(1,0)}$ and $\mathbf{u} = t^{(0,1)}$, so that $\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(L) = \sum_{i,j} \dim_{\mathbf{k}}(L_{i,j})\mathbf{t}^i\mathbf{u}^j$ for $L \in \mathbf{k}\text{-free}^{\mathbb{Z}\times\mathbb{Z}}$. When L is $(\mathbb{N}\times\mathbb{N})$-graded, we have $\dim_{\mathbf{k}}^{\mathbb{N}\times\mathbb{N}}(L) \in \mathbb{Z}[[\mathbf{t},\mathbf{u}]]$.

1.2.D. Assume $\mathbf{k}$ is a commutative ring. There is a tensor product of Γ-graded $\mathbf{k}$-modules given by $(L \otimes_{\mathbf{k}} L')_\gamma = \bigoplus_{\gamma'\gamma''=\gamma} L_{\gamma'} \otimes_{\mathbf{k}} L_{\gamma''}$. When the fibers of the multiplication map $\Gamma\times\Gamma \to \Gamma$ are finite, the multiplication in Γ provides $\mathbb{Z}^\Gamma$ with a ring structure, the tensor product preserves $\mathbf{k}\text{-free}^\Gamma$, and $\dim_{\mathbf{k}}^\Gamma(L \otimes_{\mathbf{k}} L') = \dim_{\mathbf{k}}^\Gamma(L)\dim_{\mathbf{k}}^\Gamma(L')$.

A Γ*-graded* $\mathbf{k}$*-algebra* is a $\mathbf{k}$-algebra A with a Γ-grading such that $A_\gamma \cdot A_{\gamma'} \subset A_{\gamma\gamma'}$.

1.3. Modules

Let A be a ring. Given L a subset of A, we denote by $< L >$ the two-sided ideal of A generated by L. Given M an A-module, we denote by $\mathrm{Rad}(M)$ the intersection of the maximal proper A-submodules of M. We denote by A-mod the category of finitely generated A-modules and we put $G_0(A) = K_0(A\text{-mod})$, where $K_0(\mathscr{C})$ denotes the Grothendieck group of an exact category $\mathscr{C}$.

Given $M \in A$-mod, we denote by $[M]_A$ (or simply $[M]$) its class in $G_0(A)$. When A is a graded ring and M is a finitely generated graded A-module, we denote by $[M]_A^{\mathrm{gr}}$ (or simply $[M]^{\mathrm{gr}}$) its class in the Grothendieck group of the category A-modgr of finitely generated graded A-modules. Note that $K_0(A\text{-modgr})$ is a $\mathbb{Z}[\mathbf{t}^{\pm 1}]$-module, with $\mathbf{t}[M]^{\mathrm{gr}} = [M\langle -1\rangle]^{\mathrm{gr}}$.

We denote by $\mathrm{Irr}(A)$ the set of isomorphism classes of simple A-modules. Assume A is a finite-dimensional algebra over the field $\mathbf{k}$. We have an isomorphism $\mathbb{Z}\,\mathrm{Irr}(A) \xrightarrow{\sim} G_0(A),\ M \mapsto [M]$. If A is semisimple, we have a bilinear form $\langle -,-\rangle_A$ on $G_0(A)$ given by $\langle [M],[N]\rangle = \dim_{\mathbf{k}} \mathrm{Hom}_A(M,N)$. When A is split semisimple, $\mathrm{Irr}(A)$ provides an orthonormal basis.

Let W be a finite group and assume $\mathbf{k}$ is a field. We denote by $\mathrm{Irr}_{\mathbf{k}}(W)$ (or simply by $\mathrm{Irr}(W)$) the set of irreducible characters of W over $\mathbf{k}$. When $|W| \in \mathbf{k}^\times$, there is a bijection $\mathrm{Irr}_{\mathbf{k}}(W) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{k}W),\ \chi \mapsto E_\chi$. The group $\mathrm{Hom}(W,\mathbf{k}^\times)$ of linear characters of W with values in $\mathbf{k}$ is denoted by $W^{\wedge_{\mathbf{k}}}$ (or $W^\wedge$) . We have an embedding $W^\wedge \subset \mathrm{Irr}(W)$, and equality holds if and only if W is abelian and $\mathbf{k}$ contains all e-th roots of unity, where e is the exponent of W.

CHAPTER 2

REFLECTION GROUPS

All along this book, we consider a fixed characteristic 0 field $\mathbf{k}$, a finite-dimensional $\mathbf{k}$-vector space V of dimension n and a finite subgroup W of $\mathrm{GL}_{\mathbf{k}}(V)$. We will write $\otimes$ for $\otimes_{\mathbf{k}}$. We denote by

$$\mathrm{Ref}(W) = \{s \in W \mid \dim_{\mathbf{k}} \mathrm{Im}(s - \mathrm{Id}_V) = 1\}$$

the set of reflections of W. We assume that W is generated by $\mathrm{Ref}(W)$.

2.1. Determinant, roots, coroots

We denote by ε the determinant representation of W

$$\begin{array}{rccc} \varepsilon: & W & \longrightarrow & \mathbf{k}^\times \\ & w & \longmapsto & \det_V(w). \end{array}$$

We have a perfect pairing between V and its dual V^*

$$\langle , \rangle : V \times V^* \longrightarrow \mathbf{k}.$$

Given $s \in \mathrm{Ref}(W)$, we choose $\alpha_s \in V^*$ and $\alpha_s^\vee \in V$ such that

$$\mathrm{Ker}(s - \mathrm{Id}_V) = \mathrm{Ker}\,\alpha_s \quad \text{and} \quad \mathrm{Im}(s - \mathrm{Id}_V) = \mathbf{k}\alpha_s^\vee$$

or equivalently

$$\mathrm{Ker}(s - \mathrm{Id}_{V^*}) = \mathrm{Ker}\,\alpha_s^\vee \quad \text{and} \quad \mathrm{Im}(s - \mathrm{Id}_{V^*}) = \mathbf{k}\alpha_s.$$

Note that, since $\mathbf{k}$ has characteristic 0, all elements of $\mathrm{Ref}(W)$ are diagonalizable, hence

$$\langle \alpha_s^\vee, \alpha_s \rangle \neq 0. \tag{2.1.1}$$

Given $x \in V^*$ and $y \in V$ we have

$$s(y) = y - (1 - \varepsilon(s)) \frac{\langle y, \alpha_s \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} \alpha_s^\vee \tag{2.1.2}$$

and

$$s(x) = x - (1 - \varepsilon(s)^{-1}) \frac{\langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} \alpha_s. \tag{2.1.3}$$

2.2. Invariants

We denote by $\mathbf{k}[V] = S(V^*)$ (respectively $\mathbf{k}[V^*] = S(V)$) the symmetric algebra of V^* (respectively V). We identify it with the algebra of polynomial functions on V (respectively V^*). The action of W on V induces an action by algebra automorphisms on $\mathbf{k}[V]$ and $\mathbf{k}[V^*]$ and we will consider the graded subalgebras of invariants $\mathbf{k}[V]^W$ and $\mathbf{k}[V^*]^W$. The *coinvariant algebras* $\mathbf{k}[V]^{\mathrm{co}(W)}$ and $\mathbf{k}[V^*]^{\mathrm{co}(W)}$ are the graded finite-dimensional $\mathbf{k}$-algebras

$$\mathbf{k}[V]^{\mathrm{co}(W)} = \mathbf{k}[V]/<\mathbf{k}[V]_+^W> \quad \text{and} \quad \mathbf{k}[V^*]^{\mathrm{co}(W)} = \mathbf{k}[V^*]/<\mathbf{k}[V^*]_+^W>.$$

Shephard-Todd-Chevalley's Theorem asserts that the property of W to be generated by reflections is equivalent to structural properties of $\mathbf{k}[V]^W$. We provide here a version augmented with quantitative properties (see for example [**Bro2**, Theorem 4.1]). We state a version with $\mathbf{k}[V]$, while the same statements hold with V replaced by V^*.

Let us define the sequence $d_1 \leqslant \cdots \leqslant d_n$ of *degrees of* W as the weight sequence of $<\mathbf{k}[V]_+^W>/<\mathbf{k}[V]_+^W>^2$ (cf §1.2.C).

***Theorem* 2.2.1 (Shephard-Todd, Chevalley).** — (a) *The algebra $k[V]^W$ is a polynomial algebra generated by homogeneous elements of degrees $d_1, \ldots, d_n$. We have*

$$|W| = d_1 \cdots d_n \quad \textit{and} \quad |\mathrm{Ref}(W)| = \sum_{i=1}^{n} (d_i - 1).$$

(b) *The* $(\mathbf{k}[V]^W[W])$*-module* $\mathbf{k}[V]$ *is free of rank* 1.
(c) *The* $\mathbf{k}W$*-module* $\mathbf{k}[V]^{\mathrm{co}(W)}$ *is free of rank* 1. *So,* $\dim_{\mathbf{k}} \mathbf{k}[V]^{\mathrm{co}(W)} = |W|$.

***Remark* 2.2.2.** — Note that when $\mathbf{k} = \mathbb{C}$, there is a skew-linear isomorphism between the representations V and V^* of W, hence the sequence of degrees for the action of W on V is the same as the one for the action of W on V^*. In general, note that the representation V of W can be defined over a finite extension of $\mathbb{Q}$, which can be embedded in $\mathbb{C}$: so, the equality of degrees for the actions on V and V^* holds for any $\mathbf{k}$.

This equality can also be deduced from Molien's formula [**Bro2**, Lemma 3.28]. ■

Let $N = |\mathrm{Ref}(W)|$. Since $\dim_{\mathbf{k}}^{\mathbb{Z}}(\mathbf{k}[V]^{\mathrm{co}(W)}) = \prod_{i=1}^{n} \frac{1-\mathbf{t}^{d_i}}{1-\mathbf{t}}$, we deduce that $\dim_{\mathbf{k}} \mathbf{k}[V]_N^{\mathrm{co}(W)} = 1$. A generator is given by the image of $\prod_{s\in \mathrm{Ref}(W)} \alpha_s$: this provides an isomorphism $h : \mathbf{k}[V]_N^{\mathrm{co}(W)} \xrightarrow{\sim} \mathbf{k}$.

The composition

$$\mathbf{k}[V]_N \otimes \mathbf{k}[V]^W \xrightarrow{\text{mult}} \mathbf{k}[V] \xrightarrow{\text{can}} \mathbf{k}[V]/(\mathbf{k}[V]^W \mathbf{k}[V]_{<N})$$

factors through an isomorphism $g : \mathbf{k}[V]_N^{\mathrm{co}(W)} \otimes \mathbf{k}[V]^W \xrightarrow{\sim} \mathbf{k}[V]/(\mathbf{k}[V]^W \mathbf{k}[V]_{<N})$. We denote by p_N the composition

$$p_N : \mathbf{k}[V] \xrightarrow{\text{can}} \mathbf{k}[V]/(\mathbf{k}[V]^W \mathbf{k}[V]_{<N}) \xrightarrow{g^{-1}} \mathbf{k}[V]_N^{\mathrm{co}(W)} \otimes \mathbf{k}[V]^W \xrightarrow[\sim]{h\otimes \mathrm{Id}} \mathbf{k}[V]^W.$$

We refer to §A.4 for basic facts on symmetric algebras.

Proposition 2.2.3. — *p_N is a symmetrizing form for the $\mathbf{k}[V]^W$-algebra $\mathbf{k}[V]$.*

Proof. — We need to show that the morphism of graded $\mathbf{k}[V]^W$-modules

$$\hat{p}_N : \mathbf{k}[V] \to \mathrm{Hom}_{\mathbf{k}[V]^W}(\mathbf{k}[V], \mathbf{k}[V]^W),\ a \mapsto (b \mapsto p_N(ab))$$

is an isomorphism. By the graded Nakayama lemma, it is enough to do so after applying $-\otimes_{\mathbf{k}[V]^W} \mathbf{k}$. We have $\hat{p}_N \otimes_{\mathbf{k}[V]^W} \mathbf{k} = \hat{\bar{p}}_N$, where $\bar{p}_N : \mathbf{k}[V]^{\mathrm{co}(W)} \to \mathbf{k}[V]_N^{\mathrm{co}(W)} \xrightarrow[\sim]{h} \mathbf{k}$ is the projection onto the homogeneous component of degree N. This is a symmetrizing form for $\mathbf{k}[V]^{\mathrm{co}(W)}$ [**Bro2**, Theorem 4.25], hence $\hat{\bar{p}}_N$ is an isomorphism. □

Note that the same statements hold for V replaced by V^*.

2.3. Hyperplanes and parabolic subgroups

Notation. *We fix an embedding of the group of roots of unity of* $\mathbf{k}$ *in* $\mathbb{Q}/\mathbb{Z}$. *When the class of* $\frac{1}{e}$ *is in the image of this embedding, we denote by* ζ_e *the corresponding element of* $\mathbf{k}$.

We denote by $\mathscr{A}$ the set of reflecting hyperplanes of W:

$$\mathscr{A} = \{\mathrm{Ker}(s - \mathrm{Id}_V) \mid s \in \mathrm{Ref}(W)\}.$$

There is a surjective W-equivariant map $\mathrm{Ref}(W) \to \mathscr{A},\ s \mapsto \mathrm{Ker}(s - \mathrm{Id}_V)$. Given X a subset of V, we denote by W_X the pointwise stabilizer of X:

$$W_X = \{w \in W \mid \forall\ x \in X,\ w(x) = x\}.$$

Given $H \in \mathscr{A}$, we denote by e_H the order of the cyclic subgroup W_H of W. We denote by s_H the generator of W_H with determinant ζ_{e_H}. This is a reflection with hyperplane H. We have

$$\mathrm{Ref}(W) = \{s_H^j \mid H \in \mathscr{A} \text{ and } 1 \leqslant j \leqslant e_H - 1\}.$$

The following lemma is clear.

***Lemma* 2.3.1.** — *s_H^j and $s_{H'}^{j'}$ are conjugate in W if and only if H and H' are in the same W-orbit and $j = j'$.*

Given Ω a W-orbit of hyperplanes of $\mathscr{A}$, we denote by e_Ω the common value of the e_H for $H \in \Omega$. Lemma 2.3.1 provides a bijection from $\mathrm{Ref}(W)/W$ to the set $\mathbf{\Omega}_W$ of pairs (Ω, j) where $\Omega \in \mathscr{A}/W$ and $1 \leqslant j \leqslant e_\Omega - 1$.

We denote by $\mathbf{\Omega}_W^\circ$ the set of pairs (Ω, j) with $\Omega \in \mathscr{A}/W$ and $0 \leqslant j \leqslant e_\Omega - 1$.

Let $V^{\mathrm{reg}} = \{v \in V \mid \mathrm{Stab}_W(v) = 1\}$. Define the discriminant $\delta = \prod_{H \in \mathscr{A}} \alpha_H^{e_H} \in \mathbf{k}[V]^W$. The following result shows that points outside reflecting hyperplanes have trivial stabilizers [**Bro2**, Theorem 4.7].

***Theorem* 2.3.2 (Steinberg).** — *Given $X \subset V$, the group W_X is generated by its reflections. As a consequence, $V^{\mathrm{reg}} = V \setminus \bigcup_{H \in \mathscr{A}} H$ and $\mathbf{k}[V^{\mathrm{reg}}] = k[V][\delta^{-1}]$.*

2.4. Irreducible characters

The rationality property of the reflection representation of W is classical.

***Proposition* 2.4.1.** — *Let $\mathbf{k}'$ be a subfield of $\mathbf{k}$ containing the traces of the elements of W acting on V. Then there exists a $\mathbf{k}'W$-submodule V' of V such that $V = \mathbf{k} \otimes_{\mathbf{k}'} V'$.*

Proof. — Assume first V is irreducible. Let V'' be a simple $\mathbf{k}'W$-module such that $\mathbf{k} \otimes_{\mathbf{k}'} V'' \simeq V^{\oplus m}$ for some integer $m \geqslant 1$. Let $s \in \mathrm{Ref}(W)$. Since s has only one non-trivial eigenvalue on V, it also has only one non-trivial eigenvalue on V''. Let L be the eigenspace of s acting on V'' for the non-trivial eigenvalue. This is an m-dimensional $\mathbf{k}'$-subspace of V'', stable under the action of the division algebra $\mathrm{End}_{\mathbf{k}'W}(V'')$. Since that division algebra has dimension m^2 over $\mathbf{k}'$ and has a module L that has dimension m over $\mathbf{k}'$, we deduce that $m = 1$. The proposition follows by taking for V' the image of V'' by an isomorphism $\mathbf{k} \otimes_{\mathbf{k}'} V'' \xrightarrow{\sim} V$.

Assume now V is arbitrary. Let $V = V^W \oplus \bigoplus_{i=1}^l V_i$ be a decomposition of the $\mathbf{k}W$-module V, where V_i is irreducible for $1 \leqslant i \leqslant l$. Let W_j be the subgroup of W of elements acting trivially on $\bigoplus_{i \neq j} V_i$. The group W_j is a reflection group on V_j. The discussion above shows there is a $\mathbf{k}'W_j$-submodule V_j' of V_j such that $V_j = \mathbf{k} \otimes_{\mathbf{k}'} V_j'$. Let V'' be a $\mathbf{k}'$-submodule of V^W such that $V^W = \mathbf{k} \otimes_{\mathbf{k}'} V''$. Let $V' = V'' \oplus \bigoplus_{j=1}^{l} V_j'$. We have $W = \prod_{i=1}^{l} W_j$ and $V = \mathbf{k} \otimes_{\mathbf{k}'} V'$: this proves the proposition. □

The following rationality property of all representations of complex reflection groups is proven using the classification of those groups [**Ben, Bes**].

***Theorem 2.4.2* (Benard, Bessis).** — *Let $\mathbf{k}'$ be a subfield of $\mathbf{k}$ containing the traces of the elements of W acting on V. Then the algebra $\mathbf{k}'W$ is split semisimple. In particular, $\mathbf{k}W$ is split semisimple.*

2.5. Hilbert series

2.5.A. Invariants. — The algebra $\mathbf{k}[V \times V^*] = \mathbf{k}[V] \otimes \mathbf{k}[V^*]$ admits a standard bi-grading, by giving to the elements of $V^* \subset \mathbf{k}[V]$ the bi-degree $(0,1)$ and to those of $V \subset \mathbf{k}[V^*]$ the bi-degree $(1,0)$. We clearly have

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V \times V^*]) = \frac{1}{(1-\mathbf{t})^n(1-\mathbf{u})^n}. \tag{2.5.1}$$

Using the notation of Theorem 2.2.1(a), we get also easily that

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V \times V^*]^{W\times W}) = \prod_{i=1}^{n} \frac{1}{(1-\mathbf{t}^{d_i})(1-\mathbf{u}^{d_i})}. \tag{2.5.2}$$

On the other hand, the bigraded Hilbert series of the diagonal invariant algebra $\mathbf{k}[V \times V^*]^{\Delta W}$ is given by a formula *à la Molien*

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V \times V^*]^{\Delta W}) = \frac{1}{|W|} \sum_{w\in W} \frac{1}{\det(1-w\mathbf{t})\ \det(1-w^{-1}\mathbf{u})}, \tag{2.5.3}$$

whose proof is obtained word by word from the proof of the usual Molien formula.

2.5.B. Fake degrees. — We identify $K_0(\mathbf{k}W\text{-modgr})$ with $G_0(\mathbf{k}W)[\mathbf{t},\mathbf{t}^{-1}]$: given $M = \bigoplus_{i\in\mathbb{Z}} M_i$ a finite dimensional $\mathbb{Z}$-graded $\mathbf{k}W$-module, we make the identification

$$[M]_{\mathbf{k}W}^{\mathrm{gr}} = \sum_{i\in\mathbb{Z}} [M_i]_{\mathbf{k}W}\ \mathbf{t}^i.$$

It is clear that $[M]_{\mathbf{k}W}$ is the evaluation at 1 of $[M]_{\mathbf{k}W}^{\mathrm{gr}}$ and that $[M\langle n\rangle]_{\mathbf{k}W}^{\mathrm{gr}} = \mathbf{t}^{-n}\ [M]_{\mathbf{k}W}^{\mathrm{gr}}$. If M is a bigraded $\mathbf{k}W$-module, we define similarly $[M]_{\mathbf{k}W}^{\mathbb{Z}\times\mathbb{Z}}$: it is an element of $K_0(\mathbf{k}W)[\mathbf{t},\mathbf{u},\mathbf{t}^{-1},\mathbf{u}^{-1}]$.

Let $(f_\chi(\mathbf{t}))_{\chi\in\mathrm{Irr}(W)}$ denote the unique family of elements of $\mathbb{N}[\mathbf{t}]$ such that

(2.5.4)
$$[k[V^*]^{\mathrm{co}(W)}]^{\mathbb{Z}\times\mathbb{Z}}_{\mathbf{k}W} = \sum_{\chi\in\mathrm{Irr}(W)} f_\chi(\mathbf{t})\,\chi.$$

Definition 2.5.5. — *The polynomial $f_\chi(\mathbf{t})$ is called the **fake degree** of χ. Its $\mathbf{t}$-valuation is denoted by $\mathbf{b}_\chi$ and is called the **b-invariant** of χ.*

The fake degree of χ satisfies

(2.5.6)
$$f_\chi(1)=\chi(1).$$

Note that

(2.5.7)
$$[\mathbf{k}[V]^{\mathrm{co}(W)}]^{\mathbb{Z}\times\mathbb{Z}}_{\mathbf{k}W} = \sum_{\chi\in\mathrm{Irr}(W)} f_\chi(\mathbf{u})\,\chi^*,$$

(here, χ^* denotes the dual character of χ, that is, $\chi^*(w)=\chi(w^{-1})$). Note also that, if $\mathbf{1}_W$ denotes the trivial character of W, then

$$[\mathbf{k}[V^*]^{\mathrm{co}(W)}]^{\mathbb{Z}\times\mathbb{Z}}_{\mathbf{k}W} \equiv \mathbf{1}_W \mod \mathbf{t}K_0(\mathbf{k}W)[\mathbf{t}]$$

and
$$[\mathbf{k}[V]^{\mathrm{co}(W)}]^{\mathbb{Z}\times\mathbb{Z}}_{\mathbf{k}W} \equiv \mathbf{1}_W \mod \mathbf{u}K_0(\mathbf{k}W)[\mathbf{u}].$$

We deduce:

Lemma 2.5.8. — *The elements $[\mathbf{k}[V]^{\mathrm{co}(W)}]^{\mathbb{Z}\times\mathbb{Z}}_{\mathbf{k}W}$ and $[\mathbf{k}[V^*]^{\mathrm{co}(W)}]^{\mathbb{Z}\times\mathbb{Z}}_{\mathbf{k}W}$ are not zero divisors in $K_0(\mathbf{k}W)[\mathbf{t},\mathbf{u},\mathbf{t}^{-1},\mathbf{u}^{-1}]$.*

Remark 2.5.9. — Note that

$$[\mathbf{k}[V]^{\mathrm{co}(W)}]_{\mathbf{k}W} = [\mathbf{k}[V^*]^{\mathrm{co}(W)}]_{\mathbf{k}W} = [\mathbf{k}W]_{\mathbf{k}W} = \sum_{\chi\in\mathrm{Irr}(W)} \chi(1)\chi$$

is a zero divisor in $K_0(\mathbf{k}W)$ (as soon as $W\neq 1$). ■

We can now give another formula for the Hilbert series $\dim^{\mathbb{Z}\times\mathbb{Z}}_{\mathbf{k}}(\mathbf{k}[V\times V^*]^{\Delta W})$:

Proposition 2.5.10. — $\dim^{\mathbb{Z}\times\mathbb{Z}}_{\mathbf{k}}(\mathbf{k}[V\times V^*]^W) = \dfrac{1}{\prod_{i=1}^n(1-\mathbf{t}^{d_i})(1-\mathbf{u}^{d_i})} \displaystyle\sum_{\chi\in\mathrm{Irr}(W)} f_\chi(\mathbf{t})\, f_\chi(\mathbf{u}).$

Proof. — Let $\mathscr{H}$ be a W-stable graded complement to $<\mathbf{k}[V]_+^W>$ in $\mathbf{k}[V]$. Since $\mathbf{k}[V]$ is a free $\mathbf{k}[V]^W$-module, we have isomorphisms of graded $\mathbf{k}[W]$-modules

$$\mathbf{k}[V] \simeq \mathbf{k}[V]^W \otimes \mathscr{H} \qquad \text{and} \qquad \mathbf{k}[V]^{\mathrm{co}(W)} \simeq \mathscr{H}.$$

Similarly, if $\mathscr{H}'$ is a W-stable graded complement of $<\mathbf{k}[V^*]_+^W>$ in $\mathbf{k}[V^*]$, then we have isomorphisms of graded $\mathbf{k}[W]$-modules

$$\mathbf{k}[V^*] \simeq \mathbf{k}[V^*]^W \otimes \mathscr{H}' \qquad \text{and} \qquad \mathbf{k}[V^*]^{\mathrm{co}(W)} \simeq \mathscr{H}'.$$

In other words, we have isomorphisms of graded $\mathbf{k}[W]$-modules

$$\mathbf{k}[V] \simeq \mathbf{k}[V]^W \otimes \mathbf{k}[V]^{\mathrm{co}(W)} \qquad \text{and} \qquad \mathbf{k}[V^*] \simeq \mathbf{k}[V^*]^W \otimes \mathbf{k}[V^*]^{\mathrm{co}(W)}.$$

We deduce an isomorphism of bigraded $\mathbf{k}$-vector spaces

$$(\mathbf{k}[V] \otimes \mathbf{k}[V^*])^{\Delta W} \simeq (\mathbf{k}[V]^W \otimes \mathbf{k}[V^*]^W) \otimes (\mathbf{k}[V]^{\mathrm{co}(W)} \otimes \mathbf{k}[V^*]^{\mathrm{co}(W)})^{\Delta W}.$$

By (2.5.4) and (2.5.7), we have

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V]^{\mathrm{co}(W)} \otimes \mathbf{k}[V^*]^{\mathrm{co}(W)})^{\Delta W} = \sum_{\chi,\psi \in \mathrm{Irr}(W)} f_\chi(\mathbf{t}) f_\psi(\mathbf{u}) \langle \chi\psi^*, \mathbf{1}_W \rangle_W.$$

So the formula follows from the fact that $\langle \chi\psi^*, \mathbf{1}_W \rangle = \langle \chi, \psi \rangle_W$. □

To conclude this section, we gather in a same formula Molien's Formula (2.5.3) and Proposition 2.5.10:

$$\begin{aligned}
\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V \times V^*]^{\Delta W}) &= \frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1-w\mathbf{t})\ \det(1-w^{-1}\mathbf{u})} \\
&= \frac{1}{\prod_{i=1}^n (1-\mathbf{t}^{d_i})(1-\mathbf{u}^{d_i})} \sum_{\chi \in \mathrm{Irr}(W)} f_\chi(\mathbf{t})\ f_\chi(\mathbf{u}).
\end{aligned}$$

2.6. Coxeter groups

Let us recall the following classical equivalences:

Proposition 2.6.1. — *The following assertions are equivalent:*

(1) *There exists a subset S of* $\mathrm{Ref}(W)$ *such that (W,S) is a Coxeter system.*
(2) *$V \simeq V^*$ as $\mathbf{k}W$-modules.*
(3) *There exists a W-invariant non-degenerate symmetric bilinear form $V \times V \to \mathbf{k}$.*
(4) *There exists a subfield $\mathbf{k}_{\mathbb{R}}$ of $\mathbf{k}$ and a W-stable $\mathbf{k}_{\mathbb{R}}$-vector subspace $V_{\mathbf{k}_{\mathbb{R}}}$ of V such that $V = \mathbf{k} \otimes_{\mathbf{k}_{\mathbb{R}}} V_{\mathbf{k}_{\mathbb{R}}}$ and $\mathbf{k}_{\mathbb{R}}$ embedds as a subfield of $\mathbb{R}$.*

Whenever one (or all the) assertion(s) of Proposition 2.6.1 is (are) satisfied, we say that *W is a Coxeter group*. In this case, the text will be followed by a gray line on the left, as below.

> **Assumption, choice.** *From now on, and until the end of §2.6, we assume that W is a Coxeter group. We fix a subfield $\mathbf{k}_{\mathbb{R}}$ of $\mathbf{k}$ that embedds as a subfield of $\mathbb{R}$ and a W-stable $\mathbf{k}_{\mathbb{R}}$-vector subspace $V_{\mathbf{k}_{\mathbb{R}}}$ of V such that $V = \mathbf{k} \otimes_{\mathbf{k}_{\mathbb{R}}} V_{\mathbf{k}_{\mathbb{R}}}$. We also fix a connected component $C_{\mathbb{R}}$ of $\{v \in \mathbb{R} \otimes_{\mathbf{k}_{\mathbb{R}}} V_{\mathbf{k}_{\mathbb{R}}} \mid \mathrm{Stab}_W(v) = 1\}$. We denote by S the set of $s \in \mathrm{Ref}(W)$ such that $\overline{C_{\mathbb{R}}} \cap \ker_{\mathbb{R} \otimes_{\mathbf{k}_{\mathbb{R}}} V_{\mathbf{k}_{\mathbb{R}}}}(s-1)$ has real codimension 1 in $\overline{C_{\mathbb{R}}}$. So, (W,S) is a Coxeter system. This notation will be used all along this book, provided that W is a Coxeter group.*

The following is a particular case of Theorem 2.4.2.

Theorem 2.6.2. — *The $\mathbf{k}_{\mathbb{R}}$-algebra $\mathbf{k}_{\mathbb{R}} W$ is split. In particular, the characters of W are real valued, that is, $\chi = \chi^*$ for all character χ of W.*

Recall also the following.

Lemma 2.6.3. — *If $s \in \mathrm{Ref}(W)$, then s has order 2 and $\varepsilon(s) = -1$.*

Corollary 2.6.4. — *The map $\mathrm{Ref}(W) \to \mathscr{A}$, $s \mapsto \mathrm{Ker}(s - \mathrm{Id}_V)$ is bijective and W-equivariant. In particular, $|\mathscr{A}| = |\mathrm{Ref}(W)| = \sum_{i=1}^n (d_i - 1)$ and $|\mathscr{A}/W| = |\mathrm{Ref}(W)/W|$.*

Let $\ell : W \to \mathbb{N}$ denote the length function with respect to S: given $w \in W$, the integer $\ell(w)$ is minimal such that w is a product of $\ell(w)$ elements of S. When $w = s_1 s_2 \cdots s_l$ with $s_i \in S$ and $l = \ell(w)$, we say that $w = s_1 s_2 \cdots s_l$ is a ***reduced decomposition*** of w. We denote by w_0 the longest element of W: we have $\ell(w_0) = |\mathrm{Ref}(W)| = |\mathscr{A}|$.

Remark 2.6.5. — If $-\mathrm{Id}_V \in W$, then $w_0 = -\mathrm{Id}_V$. Conversely, if w_0 is central and $V^W = 0$, then $w_0 = -\mathrm{Id}_V$. ■

PART II

CHEREDNIK ALGEBRAS

CHAPTER 3

GENERIC CHEREDNIK ALGEBRA

Let $\mathscr{C}$ be the $\mathbf{k}$-vector space of maps $c : \mathrm{Ref}(W) \to \mathbf{k}$, $s \mapsto c_s$ that are constant on conjugacy classes: this is the *space of parameters*, which we identify with the space of maps $\mathrm{Ref}(W)/W \to \mathbf{k}$.

Given $s \in \mathrm{Ref}(W)$ (or $s \in \mathrm{Ref}(W)/W$), we denote by C_s the linear form on $\mathscr{C}$ given by evaluation at s. The algebra $\mathbf{k}[\mathscr{C}]$ of polynomial functions on $\mathscr{C}$ is the algebra of polynomials on the set of indeterminates $(C_s)_{s\in\mathrm{Ref}(W)/W}$:

$$\mathbf{k}[\mathscr{C}] = \mathbf{k}[(C_s)_{s\in\mathrm{Ref}(W)/W}].$$

We denote by $\widetilde{\mathscr{C}}$ the $\mathbf{k}$-vector space $\mathbf{k}\times\mathscr{C}$ and we introduce $T : \widetilde{\mathscr{C}} \to \mathbf{k}$, $(t, c) \mapsto t$. We have $T \in \widetilde{\mathscr{C}}^*$ and

$$\mathbf{k}[\widetilde{\mathscr{C}}] = \mathbf{k}[T, (C_s)_{s\in\mathrm{Ref}(W)/W}].$$

We will use in this chapter results from Appendices A and B.

3.1. Structure

3.1.A. Symplectic action. — We consider here the action of W on $V \oplus V^*$.

Lemma B.1.1 and Proposition B.2.2 give the following result.

Proposition 3.1.1. — *We have* $Z(\mathbf{k}[V\oplus V^*]\rtimes W) = \mathbf{k}[V\oplus V^*]^W = \mathbf{k}[(V\oplus V^*)/W]$ *and there is an isomorphism*

$$Z(\mathbf{k}[V\oplus V^*]\rtimes W) \xrightarrow{\sim} e(\mathbf{k}[V\oplus V^*]\rtimes W)e,\ z\mapsto ze.$$

The action by left multiplication gives an isomorphism

$$\mathbf{k}[V\oplus V^*]\rtimes W \xrightarrow{\sim} \mathrm{End}_{\mathbf{k}[(V\oplus V^*)/W]^{\mathrm{opp}}}\big((\mathbf{k}[V\oplus V^*]\rtimes W)e\big)^{\mathrm{opp}}.$$

3.1.B. Definition. — The *generic rational Cherednik algebra* (or simply the *generic Cherednik algebra*) is the $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebra $\widetilde{\mathbf{H}}$ defined as the quotient of $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes (\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W)$ by the following relations (here, $\mathrm{T}_{\mathbf{k}}(V \oplus V^*)$ is the tensor algebra of $V \oplus V^*$):

$$\begin{cases} [x, x'] = [y, y'] = 0, \\ \\ [y, x] = T\langle y, x\rangle + \displaystyle\sum_{s \in \mathrm{Ref}(W)} (\varepsilon(s) - 1)\ C_s\ \frac{\langle y, \alpha_s\rangle \cdot \langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle}\ s, \end{cases} \tag{3.1.2}$$

for x, $x' \in V^*$ and y, $y' \in V$.

Remark 3.1.3. — Thanks to (2.1.2), the second relation is equivalent to

$$[y, x] = T\langle y, x\rangle + \sum_{s \in \mathrm{Ref}(W)} C_s \langle s(y) - y, x\rangle\ s \tag{3.1.4}$$

and to

$$[y, x] = T\langle y, x\rangle + \sum_{s \in \mathrm{Ref}(W)} C_s \langle y, s^{-1}(x) - x\rangle.\ s$$

This avoids the use of α_s and $\alpha_s^\vee$. ■

3.1.C. PBW Decomposition. — Given the relations (3.1.2), the following assertions are clear:

- There is a unique morphism of $\mathbf{k}$-algebras $\mathbf{k}[V] \to \widetilde{\mathbf{H}}$ sending $y \in V^*$ to the class of $y \in \mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$ in $\widetilde{\mathbf{H}}$.
- There is a unique morphism of $\mathbf{k}$-algebras $\mathbf{k}[V^*] \to \widetilde{\mathbf{H}}$ sending $x \in V$ to the class of $x \in \mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$ in $\widetilde{\mathbf{H}}$.
- There is a unique morphism of $\mathbf{k}$-algebras $\mathbf{k}W \to \widetilde{\mathbf{H}}$ sending $w \in W$ to the class of $w \in \mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$ in $\widetilde{\mathbf{H}}$.
- The $\mathbf{k}$-linear map $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*] \longrightarrow \widetilde{\mathbf{H}}$ induced by the three morphisms defined above and the multiplication map is surjective. Note that it is $\mathbf{k}[\widetilde{\mathscr{C}}]$-linear.

The last statement is strenghtened by the following fundamental result [**EtGi**, Theorem 1.3], for which we will provide a proof in Theorem 3.1.11.

***Theorem 3.1.5* (Etingof-Ginzburg)**. — *The multiplication map* $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*] \longrightarrow \widetilde{\mathbf{H}}$ *is an isomorphism of* $\mathbf{k}[\widetilde{\mathscr{C}}]$*-modules.*

3.1.D. Specialization. — Given $(t,c)\in\widetilde{\mathscr{C}}$, we denote by $\widetilde{\mathfrak{C}}_{t,c}$ the maximal ideal of $\mathbf{k}[\widetilde{\mathscr{C}}]$ given by $\widetilde{\mathfrak{C}}_{t,c}=\{f\in\mathbf{k}[\widetilde{\mathscr{C}}]\mid f(t,c)=0\}$: this is the ideal generated by $T-t$ and $(C_s-c_s)_{s\in\mathrm{Ref}(W)/W}$. We put

$$\widetilde{\mathbf{H}}_{t,c}=\mathbf{k}[\widetilde{\mathscr{C}}]/\widetilde{\mathfrak{C}}_{t,c}\otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]}\widetilde{\mathbf{H}}=\widetilde{\mathbf{H}}/\widetilde{\mathfrak{C}}_{t,c}\widetilde{\mathbf{H}}.$$

The $\mathbf{k}$-algebra $\widetilde{\mathbf{H}}_{t,c}$ is the quotient of $\mathrm{T}_{\mathbf{k}}(V\oplus V^*)\rtimes W$ by the ideal generated by the following relations:

$$\text{(3.1.6)}\qquad \begin{cases}[x,x']=[y,y']=0,\\[2ex] [y,x]=t\langle y,x\rangle+\displaystyle\sum_{s\in\mathrm{Ref}(W)}(\varepsilon(s)-1)\,c_s\,\frac{\langle y,\alpha_s\rangle\cdot\langle\alpha_s^\vee,x\rangle}{\langle\alpha_s^\vee,\alpha_s\rangle}\,s,\end{cases}$$

for x, $x'\in V^*$ and y, $y'\in V$.

Example 3.1.7. — We have $\widetilde{\mathbf{H}}_{0,0}=\mathbf{k}[V\oplus V^*]\rtimes W$ and $\widetilde{\mathbf{H}}_{T,0}=\mathscr{D}_T(V)\rtimes W$ (see §A.5). ■

More generally, given $\widetilde{\mathfrak{C}}$ a prime ideal of $\mathbf{k}[\widetilde{\mathscr{C}}]$, we put $\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})=\widetilde{\mathbf{H}}/\widetilde{\mathfrak{C}}\widetilde{\mathbf{H}}$.

3.1.E. Filtration. — We endow the $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebra $\widetilde{\mathbf{H}}$ with the filtration defined as follows:

- $\widetilde{\mathbf{H}}^{\leqslant -1}=0$
- $\widetilde{\mathbf{H}}^{\leqslant 0}$ is the $\mathbf{k}[\widetilde{\mathscr{C}}]$-subalgebra generated by V^* and W
- $\widetilde{\mathbf{H}}^{\leqslant 1}=\widetilde{\mathbf{H}}^{\leqslant 0}V+\widetilde{\mathbf{H}}^{\leqslant 0}$.
- $\widetilde{\mathbf{H}}^{\leqslant i}=(\widetilde{\mathbf{H}}^{\leqslant 1})^i$ for $i\geqslant 2$.

Specializing at $(t,c)\in\widetilde{\mathscr{C}}$, we have an induced filtration of $\widetilde{\mathbf{H}}_{t,c}$.

The canonical maps $\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V]\rtimes W\to(\mathrm{gr}\,\widetilde{\mathbf{H}})^0$ and $V\to(\mathrm{gr}\,\widetilde{\mathbf{H}})^1$ induce a surjective morphism of algebras $\rho:\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V\oplus V^*]\rtimes W\twoheadrightarrow\mathrm{gr}\,\widetilde{\mathbf{H}}$.

3.1.F. Localization at V^{reg}. — Recall that

$$V^{\mathrm{reg}}=V\setminus\bigcup_{H\in\mathscr{A}}H=\{v\in V\mid \mathrm{Stab}_G(v)=1\}\quad\text{and}\quad\mathbf{k}[V^{\mathrm{reg}}]=k[V][\delta^{-1}].$$

We put $\widetilde{\mathbf{H}}^{\mathrm{reg}}=\widetilde{\mathbf{H}}[\delta^{-1}]$, the non-commutative localization of $\widetilde{\mathbf{H}}$ obtained by adding a two-sided inverse to the image of δ. Note that the filtration of $\widetilde{\mathbf{H}}$ induces a filtration of $\widetilde{\mathbf{H}}^{\mathrm{reg}}$, with $(\widetilde{\mathbf{H}}^{\mathrm{reg}})^{\leqslant i}=\widetilde{\mathbf{H}}^{\leqslant i}[\delta^{-1}]$.

Note that multiplication induces an isomorphism of $\mathbf{k}$-vector spaces $\mathbf{k}[V^{\mathrm{reg}}]\otimes\mathbf{k}[V^*]\xrightarrow{\sim}\mathscr{D}(V^{\mathrm{reg}})=\mathscr{D}(V)[\delta^{-1}]$ (cf Appendix §A.5).

Lemma 3.1.8. — *We have:*

(a) *There is a Morita equivalence between* $\mathbf{k}[V^{\mathrm{reg}} \times V^*] \rtimes W$ *and* $\mathbf{k}[V^{\mathrm{reg}} \times V^*]^W$ *given by the bimodule* $\mathbf{k}[V^{\mathrm{reg}} \times V^*]$.

(b) *There is a Morita equivalence between* $\mathscr{D}(V^{\mathrm{reg}}) \rtimes W$ *and* $\mathscr{D}(V^{\mathrm{reg}})^W = \mathscr{D}(V^{\mathrm{reg}}/W)$ *given by the bimodule* $\mathscr{D}(V^{\mathrm{reg}})$.

(c) *The action of* $\mathscr{D}(V^{\mathrm{reg}}) \rtimes W$ *on* $\mathbf{k}[V^{\mathrm{reg}}]$ *is faithful.*

Proof. — (a) follows from Corollary B.2.1.

(b) becomes (a) after taking associated graded, hence (b) follows from Lemmas A.3.4 and B.1.1.

(c) It follows from (b) that every two-sided ideal of $\mathscr{D}(V^{\mathrm{reg}}) \rtimes W$ is generated by its intersection with $\mathscr{D}(V^{\mathrm{reg}})^W$. Since $\mathscr{D}(V^{\mathrm{reg}})$ acts faithfully on $\mathbf{k}[V^{\mathrm{reg}}]$ (cf §A.5), we deduce that the kernel of the action of $\mathscr{D}(V^{\mathrm{reg}}) \rtimes W$ vanishes. □

3.1.G. Polynomial representation and Dunkl operators. — Given $y \in V$, we define D_y, a $\mathbf{k}[\widetilde{\mathscr{C}}]$-linear endomorphism of $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V^{\mathrm{reg}}]$ by

$$D_y = T\partial_y - \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) C_s \langle y, \alpha_s \rangle \alpha_s^{-1} s.$$

Note that $D_y \in \mathbf{k}[\mathscr{C}] \otimes \mathscr{D}_T(V^{\mathrm{reg}}) \rtimes W \subset \mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathscr{D}(V^{\mathrm{reg}}) \rtimes W$.

Remark 3.1.9. — The Dunkl operators are traditionally defined as

$$T\partial_y - \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) C_s \langle y, \alpha_s \rangle \alpha_s^{-1}(s-1).$$

With this definition they preserve $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V]$. The results and proofs in this section apply also to those operators.

Proposition 3.1.10. — *There is a unique structure of* $\widetilde{\mathbf{H}}$*-module on* $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V^{\mathrm{reg}}]$ *where* $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V^{\mathrm{reg}}]$ *acts by multiplication,* W *acts through its natural action on* V *and* $y \in V$ *acts by* D_y.

Proof. — The following argument is due to Etingof. Let $y \in V$ and $x \in V^*$. We have

$$[\alpha_s^{-1} s, x] = (\varepsilon(s)^{-1} - 1) \frac{\langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s,$$

hence

$$[D_y, x] = T\langle y, x \rangle + \sum_s (\varepsilon(s) - 1) C_s \frac{\langle y, \alpha_s \rangle \cdot \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s.$$

Given $w \in W$, we have $w D_y w^{-1} = D_{w(y)}$.

Consider $y' \in V$. We have

$$[[D_y, D_{y'}], x] = [[D_y, x], D_{y'}] - [[D_{y'}, x], D_y]$$

and

$$\begin{aligned}[[D_y,x],D_{y'}] &= \sum_s (\varepsilon(s)-1)C_s \frac{\langle y,\alpha_s\rangle\cdot\langle\alpha_s^\vee,x\rangle}{\langle\alpha_s^\vee,\alpha_s\rangle}[s,D_{y'}] \\ &= \sum_s (\varepsilon(s)-1)^2 C_s \frac{\langle y,\alpha_s\rangle\cdot\langle y',\alpha_s\rangle\cdot\langle\alpha_s^\vee,x\rangle}{\langle\alpha_s^\vee,\alpha_s\rangle^2} D_{\alpha_s^\vee} s \\ &= [[D_{y'},x],D_y].\end{aligned}$$

We deduce that $[[D_y,D_{y'}],x]=0$ for all $x\in V^*$. On the other hand, $[D_y,D_{y'}]$ acts by zero on $1\in\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V^{\mathrm{reg}}]$, hence $[D_y,D_{y'}]$ acts by zero on $\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V^{\mathrm{reg}}]$. It follows from Lemma 3.1.8(c) that $[D_y,D_{y'}]=0$. The proposition follows. □

Proposition 3.1.10 provides a morphism of $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebras $\Theta:\widetilde{\mathbf{H}}\to\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})$. We denote by $\Theta^{\mathrm{reg}}:\widetilde{\mathbf{H}}^{\mathrm{reg}}\to\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})$ its extension to $\widetilde{\mathbf{H}}^{\mathrm{reg}}$.

Theorem 3.1.11. — *We have the following statements:*

(a) *The morphism Θ is injective, hence the polynomial representation of $\widetilde{\mathbf{H}}$ is faithful.*
(b) *The multiplication map is an isomorphism $\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W\xrightarrow{\sim}\widetilde{\mathbf{H}}$.*
(c) *We have an isomorphism of algebras $\rho:\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V\oplus V^*]\rtimes W\xrightarrow{\sim}\mathrm{gr}\,\widetilde{\mathbf{H}}$.*
(d) *The morphism Θ^{reg} is an isomorphism $\widetilde{\mathbf{H}}^{\mathrm{reg}}\xrightarrow{\sim}\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W$.*
(e) *Given $\mathfrak{c}$ a prime ideal of $\mathbf{k}[\widetilde{\mathscr{C}}]$, the morphism $(\mathbf{k}[\widetilde{\mathscr{C}}]/\mathfrak{c})\otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]}\Theta$ is injective. If $T\notin\mathfrak{c}$, then the polynomial representation of $\widetilde{\mathbf{H}}(\mathfrak{c})$ is faithful and $Z(\widetilde{\mathbf{H}}(\mathfrak{c}))=\mathbf{k}[\widetilde{\mathscr{C}}]/\mathfrak{c}$.*

Proof. — Let η be the composition

$$\eta:\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V^{\mathrm{reg}}]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W\xrightarrow{\text{mult}}\widetilde{\mathbf{H}}^{\mathrm{reg}}\xrightarrow{\Theta^{\mathrm{reg}}}\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W.$$

Note that gr η is an isomorphism, since it is equal to the graded map associated to the multiplication isomorphism

$$\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V^{\mathrm{reg}}]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W\xrightarrow{\sim}\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W.$$

We deduce that η is an isomorphism (Lemma A.3.1). Since the multiplication map is surjective, it follows that it is an isomorphism and Θ^{reg} is an isomorphism as well. We deduce also that ρ is injective, hence it is an isomorphism.

Since $\mathbf{k}[T]\otimes\mathbf{k}[V^{\mathrm{reg}}]$ is a faithful representation of $\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W$ (Lemma 3.1.8), we deduce that the polynomial representation induces an injective map

$$\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W\hookrightarrow\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathrm{End}_{\mathbf{k}}(\mathbf{k}[V^{\mathrm{reg}}]).$$

There is a commutative diagram

$$\begin{tikzcd}
\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W \arrow[r, two heads, "\mathrm{mult}"] \arrow[d, hook] & \widetilde{\mathbf{H}} \arrow[r, "\text{pol. rep.}"] \arrow[d, "\mathrm{can}"'] & \mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathrm{End}_{\mathbf{k}}(\mathbf{k}[V^{\mathrm{reg}}]) \\
\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V^{\mathrm{reg}}]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W \arrow[r, two heads, "\mathrm{mult}"'] \arrow[rr, bend right, "\eta", "\sim"'] & \widetilde{\mathbf{H}}^{\mathrm{reg}} \arrow[r, "\Theta^{\mathrm{reg}}"'] & \mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}})\rtimes W \arrow[u, "\text{pol. rep.}"']
\end{tikzcd}$$

It follows that the multiplication

$$\mathbf{k}[\widetilde{\mathscr{C}}]\otimes\mathbf{k}[V]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W\to\widetilde{\mathbf{H}}$$

is an isomorphism and the polynomial representation of $\widetilde{\mathbf{H}}$ is faithful.

Consider now $\mathfrak{c}$ a prime ideal of $\widetilde{\mathscr{C}}$ and let $A=\mathbf{k}[\widetilde{\mathscr{C}}]/\mathfrak{c}$. There is a commutative diagram

$$\begin{tikzcd}
A\otimes\mathbf{k}[V]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W \arrow[r, two heads, "\mathrm{mult}"] \arrow[d, hook] & \widetilde{\mathbf{H}}(\mathfrak{c}) \arrow[r, "\text{pol. rep.}"] \arrow[d, "\mathrm{can}"'] & A\otimes\mathrm{End}_{\mathbf{k}}(\mathbf{k}[V^{\mathrm{reg}}]) \\
A\otimes\mathbf{k}[V^{\mathrm{reg}}]\otimes\mathbf{k}[V^*]\otimes\mathbf{k}W \arrow[r, "\sim", "\mathrm{mult}"'] \arrow[rr, bend right, "\eta", "\sim"'] & \widetilde{\mathbf{H}}^{\mathrm{reg}}(\mathfrak{c}) \arrow[r, "\sim", "\Theta^{\mathrm{reg}}"'] & A\otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]}(\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}}))\rtimes W \arrow[u, "\text{pol. rep.}"']
\end{tikzcd}$$

We deduce as above that $(\mathbf{k}[\widetilde{\mathscr{C}}]/\mathfrak{c})\otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]}\Theta$ is injective. Assume now $T\notin\mathfrak{c}$. Then the polynomial representation of $A\otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]}(\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}}))\rtimes W$ is faithful, hence the polynomial representation of $\widetilde{\mathbf{H}}(\mathfrak{c})$ is faithful as well. Since $Z(\mathscr{D}(V^{\mathrm{reg}}))=\mathbf{k}$, we deduce that $Z(A\otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]}(\mathbf{k}[\mathscr{C}]\otimes\mathscr{D}_T(V^{\mathrm{reg}}))\rtimes W)=A$, hence $Z(\widetilde{\mathbf{H}}(\mathfrak{c}))=\mathbf{k}[\widetilde{\mathscr{C}}]/\mathfrak{c}$. ☐

Corollary 3.1.12. — *Given $f\in\mathbf{k}[V]$, we have*

$$[y,f]=T\partial_y(f)-\sum_{s\in\mathrm{Ref}(W)}\varepsilon(s)C_s\langle y,\alpha_s\rangle\frac{s(f)-f}{\alpha_s}s.$$

Proof. — The result follows from Proposition 3.1.10 and Theorem 3.1.11. Note that the corollary can also be proven directly by induction on the degree of f. ☐

3.1.H. Hyperplanes and parameters. — Recall that $(C_s)_{s\in\mathrm{Ref}(W)/W}$ is a $\mathbf{k}$-basis of $\mathscr{C}^*$. Let us construct a new basis of $\mathscr{C}^*$. We put $C_1=0\in\mathscr{C}^*$.

Denote by $(K_{\Omega,j})_{(\Omega,j)\in\mathbf{\Omega}_W^\circ}$ the unique family of elements of $\mathscr{C}^*$ such that, for all $H\in\mathscr{A}$ and all $i\in\{0,1,\dots,e_H-1\}$, we have

$$C_{s_H^i}=\sum_{j=0}^{e_H-1}\zeta_{e_H}^{i(j-1)}K_{H,j}.$$

Here, $K_{H,j} = K_{\Omega,j}$, where Ω is the W-orbit of H. The existence and unicity of the family $(K_{\Omega,j})_{(\Omega,j)\in\boldsymbol{\Omega}_W}$ is a consequence of the invertibility of the Vandermonde determinant. By restricting to elements of $\boldsymbol{\Omega}_W \subset \boldsymbol{\Omega}_W^\circ$, we deduce that

(3.1.13) $$(K_{\Omega,j})_{(\Omega,j)\in\boldsymbol{\Omega}_W} \textit{ is a } \mathbf{k}\textit{-basis of } \mathscr{C}^*.$$

Note that $K_{\Omega,0}$ is determined by the equation $K_{\Omega,0}+K_{\Omega,1}+\cdots+K_{\Omega,e_\Omega-1}=C_1=0$. Finally, note that

$$\sum_{w\in W_H} \varepsilon(w) C_w w = e_H \sum_{j=0}^{e_H-1} \varepsilon_{H,j} K_{H,j}$$

where $\varepsilon_{H,i} = e_H^{-1}\sum_{w\in W_H}\varepsilon(w)^i w$ and

(3.1.14) $$\sum_{s\in\mathrm{Ref}(W)} \varepsilon(s)\, C_s = \sum_{H\in\mathscr{A}} e_H K_{H,0} = -\sum_{H\in\mathscr{A}}\sum_{i=1}^{e_H-1} e_H K_{H,i}.$$

Given $H\in\mathscr{A}$, denote by $\alpha_H\in V^*$ a linear form such that $H=\mathrm{Ker}(\alpha_H)$ and let $\alpha_H^\vee\in V$ such that $V=H\oplus\mathbf{k}\alpha_H^\vee$ and $\mathbf{k}\alpha_H^\vee$ is stable under W_H. The second relation in (3.1.2) becomes

(3.1.15) $$[y,x]=T\langle y,x\rangle+\sum_{H\in\mathscr{A}}\sum_{i=0}^{e_H-1} e_H(K_{H,i}-K_{H,i+1})\frac{\langle y,\alpha_H\rangle\cdot\langle\alpha_H^\vee,x\rangle}{\langle\alpha_H^\vee,\alpha_H\rangle}\varepsilon_{H,i}$$

for $x\in V^*$ and $y\in V$, where $K_{H,e_H}=K_{H,0}$.

Given $y\in V$, we have

$$\Theta(y)=\partial_y-\sum_{H\in\mathscr{A}}\sum_{i=0}^{e_H-1}\frac{\langle y,\alpha_H\rangle}{\alpha_H}e_H K_{H,i}\varepsilon_{H,i}.$$

COMMENT - Our convention for the definition of Cherednik algebras differs from that of [**GGOR**, §3.1]: we have added a coefficient $\varepsilon(s)-1$ in front of the term C_s. On the other hand, our convention is the same as [**EtGi**, §1.15], with $c_s=c_{\alpha_s}$ (when W is a Coxeter group). Note that the $\mathrm{k}_{H,i}$'s from [**GGOR**] are related to the $K_{H,i}$'s above by the relation $\mathrm{k}_{H,i}=K_{H,0}-K_{H,i}$. ■

3.2. Gradings

The algebra $\tilde{\mathbf{H}}$ admits a natural $(\mathbb{N}\times\mathbb{N})$-grading, thanks to which we can associate, to each morphism of monoids $\mathbb{N}\times\mathbb{N}\to\mathbb{Z}$ (or $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$), a $\mathbb{Z}$-grading (or an $\mathbb{N}$-grading).

We endow the extended tensor algebra $\mathbf{k}[\tilde{\mathscr{C}}]\otimes(\mathrm{T}_{\mathbf{k}}(V\oplus V^*)\rtimes W)$ with an $(\mathbb{N}\times\mathbb{N})$-grading by giving to the elements of V the bi-degree $(1,0)$, to the elements of V^* the bi-degree $(0,1)$, to the elements of $\tilde{\mathscr{C}}^*$ the bi-degree $(1,1)$ and to those of W the bi-degree $(0,0)$. The relations (3.1.2) are homogeneous. Hence, $\widetilde{\mathbf{H}}$ inherits an $(\mathbb{N}\times\mathbb{N})$-grading whose homogeneous component of bi-degree (i,j) will be denoted by $\widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[i,j]$. We have

$$\widetilde{\mathbf{H}} = \bigoplus_{(i,j)\in\mathbb{N}\times\mathbb{N}} \widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[i,j] \quad \text{and} \quad \widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[0,0] = \mathbf{k}W.$$

Note that all homogeneous components have finite dimension over $\mathbf{k}$.

If $\varphi : \mathbb{N}\times\mathbb{N} \to \mathbb{Z}$ is a morphism of monoids, then $\widetilde{\mathbf{H}}$ inherits a $\mathbb{Z}$-grading whose homogenous component of degree i will be denoted by $\widetilde{\mathbf{H}}^{\varphi}[i]$:

$$\widetilde{\mathbf{H}}^{\varphi}[i] = \bigoplus_{\varphi(a,b)=i} \widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[a,b].$$

In this grading, the elements of V have degree $\varphi(1,0)$, the elements of V^* have degree $\varphi(0,1)$, the elements of $\widetilde{\mathscr{C}}^*$ have degree $\varphi(1,1)$ and those of W have degree 0.

***Example* 3.2.1 (ℤ-grading).** — The morphism of monoids $\mathbb{N}\times\mathbb{N}\to\mathbb{Z}$, $(i,j)\mapsto j-i$ induces a $\mathbb{Z}$-grading on $\widetilde{\mathbf{H}}$ for which the elements of V have degree -1, the elements of V^* have degree 1 and the elements of $\widetilde{\mathscr{C}}^*$ and W have degree 0. We denote by $\widetilde{\mathbf{H}}^{\mathbb{Z}}[i]$ the homogeneous component of degree i. Then

$$\widetilde{\mathbf{H}} = \bigoplus_{i\in\mathbb{Z}} \widetilde{\mathbf{H}}^{\mathbb{Z}}[i].$$

By specialization at $(t,c)\in\widetilde{\mathscr{C}}$, the algebra $\widetilde{\mathbf{H}}_{t,c}$ inherits a $\mathbb{Z}$-grading whose homogeneous component of degree i will be denoted by $\widetilde{\mathbf{H}}^{\mathbb{Z}}_{t,c}[i]$. ■

***Example* 3.2.2 (ℕ-grading).** — The morphism of monoids $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, $(i,j)\mapsto i+j$ induces an $\mathbb{N}$-grading on $\widetilde{\mathbf{H}}$ for which the elements of V or V^* have degree 1, the elements of $\widetilde{\mathscr{C}}^*$ have degree 2 and the elements of W have degree 0. We denote by $\widetilde{\mathbf{H}}^{\mathbb{N}}[i]$ the homogeneous component of degree i. Then

$$\widetilde{\mathbf{H}} = \bigoplus_{i\in\mathbb{N}} \widetilde{\mathbf{H}}^{\mathbb{N}}[i] \quad \text{and} \quad \widetilde{\mathbf{H}}^{\mathbb{N}}[0] = \mathbf{k}W.$$

Note that $\dim_{\mathbf{k}} \widetilde{\mathbf{H}}^{\mathbb{N}}[i] < \infty$ for all i. This grading is not inherited after specialization at $(t,c)\in\widetilde{\mathscr{C}}$, except whenever $(t,c)=(0,0)$: we retrieve the usual $\mathbb{N}$-grading on $\widetilde{\mathbf{H}}_{0,0} = \mathbf{k}[V\times V^*]\rtimes W$ (see Example 3.1.7). ■

3.3. Euler element

Let $(x_1,\dots,x_n)$ be a **k**-basis of V^* and let $(y_1,\dots,y_n)$ be its dual basis. We define the *generic Euler element* of $\widetilde{\mathbf{H}}$

$$\widetilde{\mathbf{eu}} = -nT + \sum_{i=1}^{n} y_i x_i + \sum_{s\in\mathrm{Ref}(W)} C_s s \in \widetilde{\mathbf{H}}.$$

Note that

$$\widetilde{\mathbf{eu}} = \sum_{i=1}^{n} x_i y_i + \sum_{s\in\mathrm{Ref}(W)} \varepsilon(s) C_s s = \sum_{i=1}^{n} x_i y_i + \sum_{H\in\mathcal{A}} \sum_{j=0}^{e_H-1} e_H\ K_{H,j}\varepsilon_{H,j}.$$

It is easy to check that $\widetilde{\mathbf{eu}}$ does not depend on the choice of the basis $(x_1,\dots,x_n)$ of V^*. Note that

$$\widetilde{\mathbf{eu}} \in \widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[1,1]. \tag{3.3.1}$$

We have

$$\Theta(\widetilde{\mathbf{eu}}) = T\sum_{i=1}^{n} y_i x_i$$

Thanks to Theorem 3.1.11, we deduce the following result [**GGOR**, §3.1(4)].

Proposition 3.3.2. — *If $x \in V^*$, $y \in V$ and $w \in W$, then*

$$[\widetilde{\mathbf{eu}}, x] = Tx, \qquad [\widetilde{\mathbf{eu}}, y] = -Ty \qquad \textit{and} \qquad [\widetilde{\mathbf{eu}}, w] = 0.$$

In [**GGOR**], the Euler element plays a fundamental role in the study of the category $\mathcal{O}$ associated with $\widetilde{\mathbf{H}}_{1,c}$. We will see in this book the role it plays in the theory of Calogero-Moser cells.

Proposition 3.3.3. — *If $h \in \widetilde{\mathbf{H}}^{\mathbb{Z}}[i]$, then $[\widetilde{\mathbf{eu}}, h] = iTh$.*

3.4. Spherical algebra

Notation. *All along this book, we denote by e the primitive central idempotent of* $\mathbf{k}W$ *defined by*

$$e = \frac{1}{|W|}\sum_{w\in W} w.$$

The $\mathbf{k}[\mathscr{C}]$*-algebra $e\widetilde{\mathbf{H}}e$ will be called the* ***generic spherical algebra***.

By specializing at (t,c), and since $e\widetilde{\mathbf{H}}e$ is a direct summand of the $\mathbf{k}[\widetilde{\mathscr{C}}]$-module $\widetilde{\mathbf{H}}$, we get

$$e\widetilde{\mathbf{H}}_{t,c}e = (\mathbf{k}[\widetilde{\mathscr{C}}]/\widetilde{\mathfrak{C}}_{t,c}) \otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]} e\widetilde{\mathbf{H}}e. \tag{3.4.1}$$

Since e has degree 0, the filtration of $\widetilde{\mathbf{H}}$ induces a filtration of the generic spherical algebra given by $(e\widetilde{\mathbf{H}}e)^{\leqslant i} = e(\widetilde{\mathbf{H}}^{\leqslant i})e$. It follows from Theorem 3.1.11 that

$$\mathrm{gr}(e\widetilde{\mathbf{H}}e) = e\,\mathrm{gr}(\widetilde{\mathbf{H}})e \simeq \mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V \times V^*]^{\Delta W}. \tag{3.4.2}$$

***Theorem 3.4.3* (Etingof-Ginzburg).** — *Let $\widetilde{\mathfrak{C}}$ be a prime ideal of $\mathbf{k}[\widetilde{\mathscr{C}}]$.*

(a) *The algebra $e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$ is a finitely generated $\mathbf{k}$-algebra without zero divisors.*
(b) *$\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$ is a finitely generated right $e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$-module.*
(c) *Left multiplication of $\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})$ on the projective module $\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$ induces an isomorphism $\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}}) \xrightarrow{\sim} \mathrm{End}_{(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}$.*
(d) *There is an isomorphism of algebras $Z(\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})) \xrightarrow{\sim} Z(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)$, $z \mapsto ze$.*

Proof. — The assertion (a) follows from Lemmas A.3.2 and A.2.1. The assertion (b) follows from Lemma A.3.2.

Let $\alpha : \widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}}) \to \mathrm{End}_{(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}$ be the morphism of the theorem. Lemma A.3.3 provides an injective morphism

$$\beta : \mathrm{gr}\,\mathrm{End}_{(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}} \hookrightarrow \mathrm{End}_{\mathrm{gr}(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\mathrm{gr}\,\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}.$$

The composition

$$\mathrm{gr}\,\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}}) \xrightarrow{\mathrm{gr}\,\alpha} \mathrm{gr}\,\mathrm{End}_{(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}} \xrightarrow{\beta} \mathrm{End}_{\mathrm{gr}(e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}}(\mathrm{gr}\,\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e)^{\mathrm{opp}}$$

is given by the left multiplication action. Via the isomorphism ρ (Theorem 3.1.11), it corresponds to the morphism given by left multiplication

$$\gamma : \mathbf{k}[\widetilde{\mathscr{C}}] \otimes \mathbf{k}[V \oplus V^*] \rtimes W \to \mathrm{End}_{\mathbf{k}[\widetilde{\mathscr{C}}] \otimes (e(\mathbf{k}[V\oplus V^*]\rtimes W)e)^{\mathrm{opp}}}(\mathbf{k}[\widetilde{\mathscr{C}}] \otimes (\mathbf{k}[V \oplus V^*] \rtimes W)e)^{\mathrm{opp}}.$$

Since the codimension of $(V \times (V^* \setminus (V^*)^{\mathrm{reg}})) \cup ((V \setminus V^{\mathrm{reg}}) \times V^*)$ in $V \times V^*$ is $\geqslant 2$, it follows from Proposition B.2.2 that γ is an isomorphism. So, $\mathrm{gr}\,\alpha$ is an isomorphism, hence α is an isomorphism by Lemma A.3.1.

The assertion (d) follows from (c) by Lemma B.1.4. □

Remark 3.4.4. — It can actually be shown **[EtGi**, Theorem 1.5] that if $\mathbf{k}[\widetilde{\mathscr{C}}]/\widetilde{\mathfrak{C}}$ is Gorenstein (respectively Cohen-Macaulay), then so is the algebra $e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$ as well as the right $e\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$-module $\widetilde{\mathbf{H}}(\widetilde{\mathfrak{C}})e$.

3.5. Some automorphisms of $\widetilde{\mathbf{H}}$

Let $\mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widetilde{\mathbf{H}})$ denote the group of automorphisms of the $\mathbf{k}$-algebra $\widetilde{\mathbf{H}}$.

3.5.A. Bigrading. — The bigrading on $\widetilde{\mathbf{H}}$ can be seen as an action of the algebraic group $\mathbf{k}^\times \times \mathbf{k}^\times$ on $\widetilde{\mathbf{H}}$. Indeed, if $(\xi,\xi') \in \mathbf{k}^\times \times \mathbf{k}^\times$, we define the automorphism $\mathrm{bigr}_{\xi,\xi'}$ of $\widetilde{\mathbf{H}}$ by the following formula:

$$\forall\,(i,j) \in \mathbb{N}\times\mathbb{N},\ \forall\ h \in \widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[i,j],\ \mathrm{bigr}_{\xi,\xi'}(h) = \xi^i \xi'^j h.$$

Then

$$\mathrm{bigr} : \mathbf{k}^\times \times \mathbf{k}^\times \longrightarrow \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widetilde{\mathbf{H}}) \tag{3.5.1}$$

is a morphism of groups. Concretely,

$$\begin{cases} \forall\ y \in V,\ \mathrm{bigr}_{\xi,\xi'}(y) = \xi y, \\ \forall\ x \in V^*,\ \mathrm{bigr}_{\xi,\xi'}(x) = \xi' x, \\ \forall\ C \in \widetilde{\mathscr{C}}^*,\ \mathrm{bigr}_{\xi,\xi'}(C) = \xi\xi' C, \\ \forall\ w \in W,\ \mathrm{bigr}_{\xi,\xi'}(w) = w. \end{cases}$$

After specialization, for all $\xi \in \mathbf{k}^\times$ and all $(t,c) \in \widetilde{\mathscr{C}}$, the action of $(\xi,1)$ induces an isomorphism of $\mathbf{k}$-algebras

$$\widetilde{\mathbf{H}}_{t,c} \xrightarrow{\sim} \widetilde{\mathbf{H}}_{\xi t,\xi c}. \tag{3.5.2}$$

3.5.B. Linear characters. — Let $\gamma : W \longrightarrow \mathbf{k}^\times$ be a linear character. It provides an automorphism of $\mathscr{C}$ by multiplication: given $c \in \mathscr{C}$, we define $\gamma \cdot c$ as the map $\mathrm{Ref}(W) \to \mathbf{k}$, $s \mapsto \gamma(s)c_s$. This induces an automorphism $\gamma_{\mathscr{C}} : \mathbf{k}[\mathscr{C}] \to \mathbf{k}[\mathscr{C}]$, $f \mapsto (c \mapsto f(\gamma^{-1}\cdot c))$ sending C_s on $\gamma(s)^{-1}C_s$. It extends to an automorphism $\gamma_{\widetilde{\mathscr{C}}}$ of $\mathbf{k}[\widetilde{\mathscr{C}}]$ by setting $\gamma_{\widetilde{\mathscr{C}}}(T) = T$.

On the other hand, γ induces also an automorphism of the group algebra $\mathbf{k}W$ given by $W \ni w \mapsto \gamma(w)w$. Hence, γ induces an automorphism of the $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebra $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \big(\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W\big)$ acting trivially on V et V^*: it will be denoted by γ_{T}. Of course,

$$(\gamma\gamma')_{\mathrm{T}} = \gamma_{\mathrm{T}}\gamma'_{\mathrm{T}}.$$

Since the relations (3.1.2) are stable by the action of γ_{T}, it follows that γ_{T} induces an automorphism γ_* of the $\mathbf{k}$-algebra $\widetilde{\mathbf{H}}$. The map

$$\begin{array}{ccc} W^\wedge & \longrightarrow & \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widetilde{\mathbf{H}}) \\ \gamma & \longmapsto & \gamma_* \end{array} \tag{3.5.3}$$

is an injective morphism of groups. Given $(t,c) \in \widetilde{\mathscr{C}}$ and $\gamma \in W^\wedge$, then γ_* induces an isomorphism of $\mathbf{k}$-algebras

$$\widetilde{\mathbf{H}}_{t,c} \xrightarrow{\sim} \widetilde{\mathbf{H}}_{t,\gamma\cdot c}. \tag{3.5.4}$$

3.5.C. Normalizer. — Let $\mathscr{N}$ denote the normalizer in $\mathrm{GL}_{\mathbf{k}}(V)$ of W. Then:

- $\mathscr{N}$ acts naturally on V and V^*;
- $\mathscr{N}$ acts on W by conjugacy;
- The action of $\mathscr{N}$ on W stabilizes $\mathrm{Ref}(W)$ and so $\mathscr{N}$ acts on $\mathscr{C}$: if $g \in \mathscr{N}$ and $c \in \mathscr{C}$, then ${}^g c : \mathrm{Ref}(W) \to \mathbf{k}$, $s \mapsto c_{g^{-1}sg}$.
- The action of $\mathscr{N}$ on $\mathscr{C}$ induces an action of $\mathscr{N}$ on $\mathscr{C}^*$ (and so on $\mathbf{k}[\mathscr{C}]$) such that, if $g \in \mathscr{N}$ and $s \in \mathrm{Ref}(W)$, then ${}^g C_s = C_{gsg^{-1}}$.
- $\mathscr{N}$ acts trivially on T.

Consequently, $\mathscr{N}$ acts on the $\mathbf{k}[\widetilde{\mathscr{C}}]$-algebra $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes \big(\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W\big)$ and it is easily checked, thanks to the relations (3.1.2), that this action induces an action on $\widetilde{\mathbf{H}}$: if $g \in \mathscr{N}$ and $h \in \widetilde{\mathbf{H}}$, we denote by ${}^g h$ the image of h under the action of g. By specialization at $(t,c) \in \widetilde{\mathscr{C}}$, an element $g \in \mathscr{N}$ induces an isomorphism of $\mathbf{k}$-algebras

$$\widetilde{\mathbf{H}}_{t,c} \xrightarrow{\sim} \widetilde{\mathbf{H}}_{t,{}^g c}. \tag{3.5.5}$$

Example 3.5.6. — If $\xi \in \mathbf{k}^\times$, then we can see ξ as an automorphism of V (by scalar multiplication) normalizing (and even centralizing) W. We then recover the automorphism of $\widetilde{\mathbf{H}}$ inducing the $\mathbb{Z}$-grading (up to a sign): if $h \in \widetilde{\mathbf{H}}$, then ${}^\xi h = \mathrm{bigr}_{\xi,\xi^{-1}}(h)$. ■

3.5.D. Compatibilities. — The automorphisms induced by $\mathbf{k}^\times \times \mathbf{k}^\times$ and those induced by $W^\wedge$ commute. On the other hand, the group $\mathscr{N}$ acts on the group $W^\wedge$ and on the $\mathbf{k}$-algebra $\widetilde{\mathbf{H}}$. This induces an action of $W^\wedge \rtimes \mathscr{N}$ on $\widetilde{\mathbf{H}}$ preserving the bigrading, that is, commuting with the action of $\mathbf{k}^\times \times \mathbf{k}^\times$. Given $\gamma \in W^\wedge$ and $g \in \mathscr{N}$, we will denote by $\gamma \rtimes g$ the corresponding element of $W^\wedge \rtimes \mathscr{N}$. We have a morphism of groups

$$\begin{array}{rcl} \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N}) & \longrightarrow & \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widetilde{\mathbf{H}}) \\ (\xi,\xi',\gamma \rtimes g) & \longmapsto & (h \mapsto \mathrm{bigr}_{\xi,\xi'} \circ \gamma_*({}^g h)). \end{array}$$

Given $\tau = (\xi,\xi',\gamma \rtimes g) \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ and $h \in \widetilde{\mathbf{H}}$, we set

$${}^\tau h = \mathrm{bigr}_{\xi,\xi'}\big(\gamma_*({}^g h)\big).$$

The following lemma is immediate.

Lemma 3.5.7. — *Let* $\tau = (\xi,\xi',\gamma \rtimes g) \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$. *Then:*

(a) τ *stabilizes the subalgebras* $\mathbf{k}[\widetilde{\mathscr{C}}]$, $\mathbf{k}[V]$, $\mathbf{k}[V^*]$ *and* $\mathbf{k}W$.
(b) τ *preserves the bigrading.*
(c) ${}^\tau \widetilde{\mathbf{eu}} = \xi\xi' \widetilde{\mathbf{eu}}$.
(d) ${}^\tau e = e$ *if and only if* $\gamma = 1$.

3.6. Special features of Coxeter groups

Assumption. *In this section 3.6, we assume that W is a Coxeter group, and we use the notation of §2.6.*

By Proposition 2.6.1, there exists a non-degenerate symmetric bilinear W-invariant form $\boldsymbol{\beta} : V \times V \to \mathbf{k}$. We denote by $\sigma : V \xrightarrow{\sim} V^*$ the isomorphism induced by $\boldsymbol{\beta}$: if $y, y' \in V$, then

$$\langle y, \sigma(y')\rangle = \boldsymbol{\beta}(y, y').$$

The W-invariance of $\boldsymbol{\beta}$ implies that σ is an isomorphism of $\mathbf{k}W$-modules and the symmetry of $\boldsymbol{\beta}$ implies that

$$\langle y, x\rangle = \langle \sigma^{-1}(x), \sigma(y)\rangle \tag{3.6.1}$$

for all $x \in V^*$ and $y \in V$. We denote by $\sigma_{\mathrm{T}} : \mathrm{T}_{\mathbf{k}}(V \oplus V^*) \to \mathrm{T}_{\mathbf{k}}(V \oplus V^*)$ the automorphism of algebras induced by the automorphism of the vector space $V \oplus V^*$ defined by $(y, x) \mapsto (-\sigma^{-1}(x), \sigma(y))$. It is W-invariant, hence extends to an automorphism of $\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$, with trivial action on W. By extension of scalars, we get another automorphism, still denoted by σ_{T}, of $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes (\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W)$. It is easy to check that σ_{T} induces an automorphism $\sigma_{\widetilde{\mathbf{H}}}$ of $\widetilde{\mathbf{H}}$. We have proven the following proposition.

***Proposition* 3.6.2.** — *There exists a unique automorphism $\sigma_{\widetilde{\mathbf{H}}}$ of $\widetilde{\mathbf{H}}$ such that*

$$\begin{cases} \sigma_{\widetilde{\mathbf{H}}}(y) = \sigma(y) & \text{if } y \in V, \\ \sigma_{\widetilde{\mathbf{H}}}(x) = -\sigma^{-1}(x) & \text{if } x \in V^*, \\ \sigma_{\widetilde{\mathbf{H}}}(w) = w & \text{if } w \in W, \\ \sigma_{\widetilde{\mathbf{H}}}(C) = C & \text{if } C \in \widetilde{\mathscr{C}}^*. \end{cases}$$

***Proposition* 3.6.3.** — *The following hold:*

(a) *$\sigma_{\widetilde{\mathbf{H}}}$ stabilizes the subalgebras $\mathbf{k}[\widetilde{\mathscr{C}}]$ and $\mathbf{k}W$ and exchanges the subalgebras $\mathbf{k}[V]$ and $\mathbf{k}[V^*]$.*

(b) *If $h \in \widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[i, j]$, then $\sigma_{\widetilde{\mathbf{H}}}(h) \in \widetilde{\mathbf{H}}^{\mathbb{N}\times\mathbb{N}}[j, i]$.*

(c) *If $h \in \widetilde{\mathbf{H}}^{\mathbb{N}}[i]$ (respectively $h \in \widetilde{\mathbf{H}}^{\mathbb{Z}}[i]$), then $\sigma_{\widetilde{\mathbf{H}}}(h) \in \widetilde{\mathbf{H}}^{\mathbb{N}}[i]$ (respectively $\sigma_{\widetilde{\mathbf{H}}}(h) \in \widetilde{\mathbf{H}}^{\mathbb{Z}}[-i]$).*

(d) *$\sigma_{\widetilde{\mathbf{H}}}$ commutes with the action of $W^\wedge$ on $\widetilde{\mathbf{H}}$.*

(e) *If $(t, c) \in \widetilde{\mathscr{C}}$, then $\sigma_{\widetilde{\mathbf{H}}}$ induces an automorphism of $\widetilde{\mathbf{H}}_{t,c}$, still denoted by $\sigma_{\widetilde{\mathbf{H}}}$ (or $\sigma_{\widetilde{\mathbf{H}}_{t,c}}$ if necessary).*

(f) $\sigma_{\widetilde{\mathbf{H}}}(\widetilde{\mathbf{eu}}) = -nT - \widetilde{\mathbf{eu}}$.

Remark 3.6.4 **(Action of $\mathbf{GL}_2(\mathbf{k})$).** — Let $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbf{k})$. The $\mathbf{k}$-linear map

$$\begin{array}{rcc} V \oplus V^* & \longrightarrow & V \oplus V^* \\ y \oplus x & \longmapsto & ay + b\sigma^{-1}(x) \oplus c\sigma(y) + dx \end{array}$$

is an automorphism of the $\mathbf{k}W$-module $V \oplus V^*$. It extends to an automorphism of the $\mathbf{k}$-algebra $\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$ and to an automorphism ρ_{T} of $\mathbf{k}[\widetilde{\mathscr{C}}] \otimes (\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W)$ by $\rho_{\mathrm{T}}(C) = \det(\rho) C$ for $C \in \widetilde{\mathscr{C}}^*$.

It is easy to check that ρ_{T} induces an automorphism $\rho_{\widetilde{\mathbf{H}}}$ of $\widetilde{\mathbf{H}}$. Moreover, $(\rho\rho')_{\widetilde{\mathbf{H}}} = \rho_{\widetilde{\mathbf{H}}} \circ \rho'_{\widetilde{\mathbf{H}}}$ for all ρ, $\rho' \in \mathbf{GL}_2(\mathbf{k})$. So, we obtain an action of $\mathbf{GL}_2(\mathbf{k})$ on $\widetilde{\mathbf{H}}$. This action preserves the $\mathbb{N}$-grading $\widetilde{\mathbf{H}}^{\mathbb{N}}$.

Finally, note that, for $\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have $\rho_{\widetilde{\mathbf{H}}} = \sigma_{\widetilde{\mathbf{H}}}$ and, if $\rho = \begin{pmatrix} \xi & 0 \\ 0 & \xi' \end{pmatrix}$, then $\rho_{\widetilde{\mathbf{H}}} = \mathrm{bigr}_{\xi,\xi'}$. Hence we have extended the action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$ to an action of $\mathbf{GL}_2(\mathbf{k}) \times (W^\wedge \rtimes \mathscr{N})$. ■

CHAPTER 4

CHEREDNIK ALGEBRAS AT $t = 0$

Notation. *We put* $\mathbf{H} = \widetilde{\mathbf{H}}/T\widetilde{\mathbf{H}}$. *The* $\mathbf{k}$*-algebra* $\mathbf{H}$ *is called the* ***Cherednik algebra at*** $t = 0$.

4.1. Generalities

We gather here those properties that are immediate consequences of results discussed in Chapter 3. We also introduce some notations.

Let us rewrite the defining relations (3.1.2). The algebra $\mathbf{H}$ is the $\mathbf{k}[\mathscr{C}]$-algebra quotient of $\mathbf{k}[\mathscr{C}] \otimes \big(\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W\big)$ by the ideal generated by the following relations:

$$\tag{4.1.1} \begin{cases} [x, x'] = [y, y'] = 0, \\ \\ [y, x] = \displaystyle\sum_{s \in \mathrm{Ref}(W)} (\varepsilon(s) - 1)\; C_s \; \frac{\langle y, \alpha_s \rangle \cdot \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} \; s, \end{cases}$$

for x, $x' \in V^*$ and y, $y' \in V$.

The PBW-decomposition (Theorem 3.1.5) takes the following form.

***Theorem 4.1.2* (Etingof-Ginzburg).** — *The multiplication map gives an isomorphism of* $\mathbf{k}[\mathscr{C}]$*-modules*

$$\mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*] \xrightarrow{\sim} \mathbf{H}.$$

Given $c \in \mathscr{C}$, we denote by $\mathfrak{C}_c$ the maximal ideal of $\mathbf{k}[\mathscr{C}]$ defined by $\mathfrak{C}_c = \{f \in \mathbf{k}[\mathscr{C}] \mid f(c) = 0\}$: it is the ideal generated by $(C_s - c_s)_{s \in \mathrm{Ref}(W)/W}$. We set

$$\mathbf{H}_c = (\mathbf{k}[\mathscr{C}]/\mathfrak{C}_c) \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{H} = \mathbf{H}/\mathfrak{C}_c\mathbf{H} = \widetilde{\mathbf{H}}_{0,c}.$$

The $\mathbf{k}$-algebra $\mathbf{H}_c$ is the quotient of the $\mathbf{k}$-algebra $\mathrm{T}_{\mathbf{k}}(V \oplus V^*) \rtimes W$ by the ideal generated by the following relations:

$$\begin{cases} [x,x'] = [y,y'] = 0, \\ \\ [y,x] = \displaystyle\sum_{s \in \mathrm{Ref}(W)} (\varepsilon(s)-1)\, c_s \,\frac{\langle y,\alpha_s\rangle \cdot \langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee,\alpha_s\rangle}\, s, \end{cases} \tag{4.1.3}$$

for $x, x' \in V^*$ and $y, y' \in V$.

Since T is bi-homogeneous, the $\mathbf{k}$-algebra $\mathbf{H}$ inherits all the gradings, filtrations of the algebra $\widetilde{\mathbf{H}}$: we will use the obvious notation $\mathbf{H}^{\mathbb{N}\times\mathbb{N}}[i,j]$, $\mathbf{H}^{\mathbb{N}}[i]$, $\mathbf{H}^{\mathbb{Z}}[i]$ and $\mathbf{H}^{\leqslant i}$ for the constructions obtained by quotient from $\widetilde{\mathbf{H}}$. We will denote by $\mathbf{eu}$ the image of $\widetilde{\mathbf{eu}}$ in $\mathbf{H}$. This is the *generic Euler element* of $\mathbf{H}$. Note that

$$\mathbf{eu} \in \mathbf{H}^{\mathbb{N}\times\mathbb{N}}[1,1] \subset \mathbf{H}^{\mathbb{Z}}[0] \tag{4.1.4}$$

The ideal generated by T is also stable by the action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathcal{N})$, so $\mathbf{H}$ inherits an action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathcal{N})$. The action of $\tau \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathcal{N})$ on $h \in \mathbf{H}$ is still denoted by ${}^\tau h$. The following lemma is immediate from Lemma 3.5.7:

Lemma 4.1.5. — *Let $\tau = (\xi, \xi', \gamma \rtimes g) \in \mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathcal{N})$. Then:*

(a) *τ stabilizes the subalgebras $\mathbf{k}[\mathscr{C}]$, $\mathbf{k}[V]$, $\mathbf{k}[V^*]$ and $\mathbf{k}W$.*
(b) *τ stabilizes the bigrading.*
(c) ${}^\tau\mathbf{eu} = \xi\xi'\,\mathbf{eu}$.

Theorem 3.4.3 implies the following result on the spherical algebra.

***Theorem 4.1.6* (Etingof-Ginzburg)**. — *Let $\mathfrak{C}$ be a prime ideal of $\mathbf{k}[\mathscr{C}]$ and let $\mathbf{H}(\mathfrak{C}) = \mathbf{H}/\mathfrak{C}\mathbf{H}$. Then:*

(a) *The algebra $e\mathbf{H}(\mathfrak{C})e$ is a finitely generated $\mathbf{k}$-algebra without zero divisors.*
(b) *Left multiplication of $\mathbf{H}(\mathfrak{C})$ on the projective module $\mathbf{H}(\mathfrak{C})e$ induces an isomorphism $\mathbf{H}(\mathfrak{C}) \xrightarrow{\sim} \mathrm{End}_{(e\mathbf{H}(\mathfrak{C})e)^{\mathrm{opp}}}(\mathbf{H}(\mathfrak{C})e)^{\mathrm{opp}}$.*

Let $\mathbf{H}^{\mathrm{reg}} = \mathbf{k}[\mathscr{C}] \otimes_{\mathbf{k}[\widetilde{\mathscr{C}}]} \widetilde{\mathbf{H}}^{\mathrm{reg}}$. Theorem 3.1.11 becomes the following result.

***Theorem 4.1.7* (Etingof-Ginzburg)**. — *There exists a unique isomorphism of $\mathbf{k}[\mathscr{C}]$-algebras*

$$\Theta : \mathbf{H}^{\mathrm{reg}} \xrightarrow{\sim} \mathbf{k}[\mathscr{C}] \otimes (\mathbf{k}[V^{\mathrm{reg}} \times V^*] \rtimes W)$$

such that

$$\begin{cases} \Theta(w) = w & \textit{for } w \in W, \\ \Theta(y) = y - \displaystyle\sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) C_s \frac{\langle y, \alpha_s \rangle}{\alpha_s} s & \textit{for } y \in V, \\ \Theta(x) = x & \textit{for } x \in V^*. \end{cases}$$

Given $\mathfrak{C}$ a prime ideal of $\mathbf{k}[\mathscr{C}]$, the restriction of $(\mathbf{k}[\mathscr{C}]/\mathfrak{C}) \otimes_{\mathbf{k}[\mathscr{C}]} \Theta$ to $(\mathbf{k}[\mathscr{C}]/\mathfrak{C}) \otimes_{\mathbf{k}[\mathscr{C}]} \mathbf{H}$ is injective.

4.2. Center

Notation. *All along this book, we denote by $Z = \mathrm{Z}(\mathbf{H})$ the center of $\mathbf{H}$. Given $c \in \mathscr{C}$, we set $Z_c = Z/\mathfrak{C}_c Z$. Let P denote the $\mathbf{k}[\mathscr{C}]$-algebra obtained by tensor product of algebras $P = \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V]^W \otimes \mathbf{k}[V^*]^W$. We identify P with a $\mathbf{k}[\mathscr{C}]$-submodule of $\mathbf{H}$ via Theorem 4.1.2.*

4.2.A. A subalgebra of Z. — The first fundamental result about the center Z of $\mathbf{H}$ is the next one [**EtGi**, Proposition 4.15] (we follow [**Gor1**, Proposition 3.6] for the proof).

Lemma 4.2.1. — *P is a subalgebra of Z stable under the action of $\mathbf{k}^\times \times \mathbf{k}^\times \times (W^\wedge \rtimes \mathscr{N})$. In particular, it is $(\mathbb{N} \times \mathbb{N})$-graded.*

Proof. — The subalgebra $\mathbf{k}[V]^W$ is central in $\mathbf{H}$ by Corollary 3.1.12. Dually, $\mathbf{k}[V^*]^W$ is central as well. The stability property is clear. □

Corollary 4.2.2. — *The PBW-decomposition is an isomorphism of P-modules. In particular, we have isomorphisms of P-modules:*

(a) $\mathbf{H} \simeq \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*]$.
(b) $\mathbf{H}e \simeq \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V] \otimes \mathbf{k}[V^*]$.
(c) $e\mathbf{H}e \simeq \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V \times V^*]^{\Delta W}$.

Hence, $\mathbf{H}$ (respectively $\mathbf{H}e$, respectively $e\mathbf{H}e$) is a free P-module of rank $|W|^3$ (respectively $|W|^2$, respectively $|W|$).

The principal theme of this book is to study the algebra $\mathbf{H}$, viewing it as a P-algebra: given $\mathfrak{p}$ is a prime ideal of P, we will be interested in the finite dimensional $k_P(\mathfrak{p})$-algebra $k_P(\mathfrak{p}) \otimes_P \mathbf{H}$ (splitting, simple modules, blocks, standard modules, decomposition matrix...). Here, $k_P(\mathfrak{p})$ is the fraction field of $P/\mathfrak{p}$, cf Appendix **??**.

Remark 4.2.3. — Let $(b_i)_{1 \leqslant i \leqslant |W|}$ be a $\mathbf{k}[V]^W$-basis of $\mathbf{k}[V]$ and let $(b_i^*)_{1 \leqslant i \leqslant |W|}$ be a $\mathbf{k}[V^*]^W$-basis of $\mathbf{k}[V^*]$. Corollary 4.2.2 shows that $(b_i w b_j^*)_{\substack{1 \leqslant i,j \leqslant |W| \\ w \in W}}$ is a P-basis of $\mathbf{H}$ and that $(b_i b_j^* e)_{1 \leqslant i,j \leqslant |W|}$ is a P-basis of $\mathbf{H}e$. ■

Set

$$P_\bullet = \mathbf{k}[V]^W \otimes \mathbf{k}[V^*]^W.$$

If $c \in \mathscr{C}$, then

$$P_\bullet \simeq \mathbf{k}[\mathscr{C}]/\mathfrak{C}_c \otimes_{\mathbf{k}[\mathscr{C}]} P = P/\mathfrak{C}_c P.$$

We deduce from Corollary 4.2.2 the next result:

Corollary 4.2.4. — *We have isomorphisms of $P_\bullet$-modules:*

(a) $\mathbf{H}_c \simeq \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*]$.
(b) $\mathbf{H}_c e \simeq \mathbf{k}[V] \otimes \mathbf{k}[V^*]$.
(c) $e\mathbf{H}_c e \simeq \mathbf{k}[V \times V^*]^{\Delta W}$.

In particular, $\mathbf{H}_c$ (respectively $\mathbf{H}_c e$, respectively $e\mathbf{H}_c e$) is a free $P_\bullet$-module of rank $|W|^3$ (respectively $|W|^2$, respectively $|W|$).

4.2.B. Satake isomorphism. — It follows from Proposition 3.3.2 that

$$\mathbf{eu} \in Z. \tag{4.2.5}$$

Given $c \in \mathscr{C}$, we denote by $\mathbf{eu}_c$ the image of $\mathbf{eu}$ in $\mathbf{H}_c$.

The next structural theorem is a cornerstone of the representation theory of $\mathbf{H}$.

***Theorem 4.2.6* (Etingof-Ginzburg).** — *The morphism of algebras $Z \longrightarrow e\mathbf{H}e$, $z \mapsto ze$ is an isomorphism of $(\mathbb{N} \times \mathbb{N})$-graded algebras. In particular, $e\mathbf{H}e$ is commutative.*

Proof. — Recall (Theorem 3.4.3) that the map $\pi_e : \mathrm{Z}(\mathbf{H}) \to \mathrm{Z}(e\mathbf{H}e)$, $z \mapsto ze$ is an isomorphism of algebras. Theorem 4.1.7 shows that $\Theta(e\mathbf{H}e) = \mathbf{k}[\mathscr{C}] \otimes (e\mathbf{k}[V^{\mathrm{reg}} \times V^*]^W)$ and Θ is injective, hence $e\mathbf{H}e$ is commutative. The theorem follows. □

Corollary 4.2.7. — *Let $\mathfrak{C}$ be a prime ideal of $\mathbf{k}[\mathscr{C}]$. Let $Z(\mathfrak{C}) = Z/\mathfrak{C}Z$ and $P(\mathfrak{C}) = P/\mathfrak{C}P$. We have:*

(a) $Z(\mathfrak{C}) = \mathrm{Z}(\mathbf{H}(\mathfrak{C}))$.
(b) *The map $Z(\mathfrak{C}) \to e\mathbf{H}(\mathfrak{C})e$, $z \mapsto ze$ is an isomorphism.*
(c) $\mathrm{End}_{\mathbf{H}(\mathfrak{C})}(\mathbf{H}(\mathfrak{C})e) = Z(\mathfrak{C})$ *and* $\mathrm{End}_{Z(\mathfrak{C})}(\mathbf{H}(\mathfrak{C})e) = \mathbf{H}(\mathfrak{C})$.

(d) $\mathbf{H}(\mathfrak{C}) = Z(\mathfrak{C}) \oplus e\mathbf{H}(\mathfrak{C})(1-e) \oplus (1-e)\mathbf{H}(\mathfrak{C})e \oplus (1-e)\mathbf{H}(\mathfrak{C})(1-e)$. *In particular, $Z(\mathfrak{C})$ is a direct summand of the $Z(\mathfrak{C})$-module $\mathbf{H}(\mathfrak{C})$.*

(e) *$Z(\mathfrak{C})$ is a free $P(\mathfrak{C})$-module of rank $|W|$.*

(f) *If $\mathbf{k}[\mathscr{C}]/\mathfrak{C}$ is integrally closed, then $Z(\mathfrak{C})$ is an integrally closed domain.*

Proof. — Assertion (b) follows from Theorem 4.2.6. We deduce now (c) from Theorem 4.1.6 and (e) from Corollary 4.2.2. We deduce also that $Z(\mathfrak{C})(1-e) \cap Z(\mathfrak{C}) = 0$. It follows also that $e\mathbf{H}(\mathfrak{C})e = Z(\mathfrak{C})e$, hence $e\mathbf{H}(\mathfrak{C})e \subset Z(\mathfrak{C}) + \mathbf{H}(\mathfrak{C})(1-e)$. The decomposition $\mathbf{H}(\mathfrak{C}) = e\mathbf{H}(\mathfrak{C})e \oplus e\mathbf{H}(\mathfrak{C})(1-e) \oplus (1-e)\mathbf{H}(\mathfrak{C})e \oplus (1-e)\mathbf{H}(\mathfrak{C})(1-e)$ implies (d).

The canonical map $Z(\mathbf{H}(\mathfrak{C})) \to e\mathbf{H}(\mathfrak{C})e,\ z \mapsto ze$ is injective since $\mathbf{H}(\mathfrak{C})$ acts faithfully on $\mathbf{H}(\mathfrak{C})e$ by (c). Since $Z(\mathfrak{C})$ is a direct summand of $\mathbf{H}(\mathfrak{C})$ contained in $Z(\mathbf{H}(\mathfrak{C}))$, the assertion (a) follows from (b).

The fact that $Z(\mathfrak{C}) \simeq e\mathbf{H}(\mathfrak{C})e$ is an integrally domain closed follows from the fact that $\mathrm{gr}(e\mathbf{H}(\mathfrak{C})e) \simeq (\mathbf{k}[\mathscr{C}]/\mathfrak{C}) \otimes \mathbf{k}[V \times V^*]^{\Delta W}$ is an integrally closed domain (Lemma A.2.2). □

4.3. Localization

4.3.A. Localization on V^{reg}. — Recall that

$$V^{\mathrm{reg}} = V \setminus \bigcup_{H \in \mathscr{A}} H = \{v \in V \mid \mathrm{Stab}_W(v) = 1\}.$$

Set $P^{\mathrm{reg}} = \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V^{\mathrm{reg}}]^W \times \mathbf{k}[V^*]^W$ and $Z^{\mathrm{reg}} = P^{\mathrm{reg}} \otimes_P Z$, so that $\mathbf{H}^{\mathrm{reg}} = P^{\mathrm{reg}} \otimes_P \mathbf{H} = Z^{\mathrm{reg}} \otimes_Z \mathbf{H}$. Given $s \in \mathrm{Ref}(W)$, let $\alpha_s^W = \prod_{w \in W} w(\alpha_s) \in P$, . The algebra P^{reg} (respectively Z^{reg}) is the localization of P (respectively Z) at the multiplicative subset $(\alpha_s^W)_{s \in \mathrm{Ref}(W)}$. As a consequence,

(4.3.1) $$\alpha_s \text{ is invertible in } \mathbf{H}^{\mathrm{reg}}.$$

Corollary 4.3.2. — *Θ restricts to an isomorphism of $\mathbf{k}[\mathscr{C}]$-algebras $Z^{\mathrm{reg}} \xrightarrow{\sim} \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V^{\mathrm{reg}} \times V^*]^W$. In particular, Z^{reg} is regular.*

Proof. — The first statement follows from the comparison between the centers of $\mathbf{H}^{\mathrm{reg}}$ and $\mathbf{k}[V^{\mathrm{reg}} \times V^*] \rtimes W$ (Theorem 4.1.7). The second statement follows from the fact that W acts freely on $V^{\mathrm{reg}} \times V^*$. □

Given $c \in \mathscr{C}$, let Z_c^{reg} denote the localization of Z_c at $P_\bullet^{\mathrm{reg}} = \mathbf{k}[V^{\mathrm{reg}}]^W \otimes \mathbf{k}[V^*]^W$. Corollary 4.3.2 shows that

(4.3.3) $$Z_c^{\mathrm{reg}} \simeq \mathbf{k}[V^{\mathrm{reg}} \times V^*]^W \text{ is a regular ring.}$$

4.3.B. Morita equivalences. — While Z and $\mathbf{H}$ are only related by a double endomorphism theorem, after restricting to a smooth open subset of Z, they become Morita equivalent.

Proposition 4.3.4. — *Let U be a multiplicative subset of Z such that $Z[U^{-1}]$ is regular. Then $\mathbf{H}[U^{-1}]e$ induces a Morita equivalence between the algebras $\mathbf{H}[U^{-1}]$ and $Z[U^{-1}]$.*

Proof. — Let $\mathfrak{m}$ be a maximal ideal of Z such that $Z_\mathfrak{m}$ is regular. Let i be maximal such that $\mathrm{Tor}_i^Z(\mathbf{H}e, Z/\mathfrak{m}) \neq 0$. Given any finite length Z-module L with support $\mathfrak{m}$, we have $\mathrm{Tor}_i^Z(\mathbf{H}e, L) \neq 0$.

Let $\mathfrak{n} = P \cap \mathfrak{m}$. We have $\mathrm{Tor}_*^Z(\mathbf{H}e, Z/\mathfrak{n}Z) \simeq \mathrm{Tor}_*^Z(\mathbf{H}e, Z \otimes_P P/\mathfrak{n}) \simeq \mathrm{Tor}_*^P(\mathbf{H}e, P/\mathfrak{n})$ since Z is a free P-module (Corollary 4.2.7). Since $\mathbf{H}e$ is a free P-module (Corollary 4.2.2), it follows that $\mathrm{Tor}_{>0}^Z(\mathbf{H}e, Z/\mathfrak{n}Z) = 0$, hence $\mathrm{Tor}_{>0}^Z(\mathbf{H}e, (Z/\mathfrak{n}Z)_\mathfrak{m}) = 0$. We deduce that $i = 0$, hence $(\mathbf{H}e)_\mathfrak{m}$ is a free $Z_\mathfrak{m}$-module.

We have shown that $\mathbf{H}[U^{-1}]e$ is a projective $Z[U^{-1}]$-module. The Morita equivalence follows from Corollary 4.2.7. □

Corollary 4.3.5. — *The $(\mathbf{H}^{\mathrm{reg}}, Z^{\mathrm{reg}})$-bimodule $\mathbf{H}^{\mathrm{reg}}e$ induces a Morita equivalence between $\mathbf{H}^{\mathrm{reg}}$ and Z^{reg}.*

Proof. — This follows from Proposition 4.3.4 and Corollary 4.3.2. □

4.3.C. Fraction field. — Let $\mathbf{K}$ denote the fraction field of P and let $\mathbf{K}Z = \mathbf{K} \otimes_P Z$. Since Z is a domain and is integral over P, it follows that

$$\mathbf{K}Z \textit{ is the fraction field of } Z. \tag{4.3.6}$$

In particular, $\mathbf{K}Z$ is a regular ring.

Theorem 4.3.7. — *The $\mathbf{K}$-algebras $\mathbf{KH}$ and $\mathbf{K}Z$ are Morita equivalent, the Morita equivalence being induced by $\mathbf{KH}e$. More precisely,*

$$\mathbf{KH} \simeq \mathrm{Mat}_{|W|}(\mathbf{K}Z).$$

Proof. — Proposition 4.3.4 shows the Morita equivalence. Recall that $\mathbf{H}e$ is a free P-module of rank $|W|^2$ and Z is a free P-module of rank $|W|$ (Corollary 4.2.2). It follows that $\mathbf{KH}e$ is a $\mathbf{K}Z$-vector space of dimension $|W|$, whence the result. □

4.4. Complements

4.4.A. Poisson structure. — The PBW-decomposition induces an isomorphism of $\mathbf{k}$-*vector spaces* $\mathbf{k}[T]\otimes\mathbf{H}\xrightarrow{\sim}\widetilde{\mathbf{H}}$. Given $h\in\mathbf{H}$, let $\tilde{h}$ denote the image of $1\otimes h\in\mathbf{k}[T]\otimes\mathbf{H}$ in $\widetilde{\mathbf{H}}$ through this isomorphism. If $z,\ z'\in Z$, then $[z,z']=0$, hence $[\tilde{z},\tilde{z}']\in T\widetilde{\mathbf{H}}$. We denote by $\{z,z'\}$ the image of $[\tilde{z},\tilde{z}']/T\in\widetilde{\mathbf{H}}$ in $\mathbf{H}=\widetilde{\mathbf{H}}/T\widetilde{\mathbf{H}}$. It is easily checked that $\{z,z'\}\in Z$ and that

$$\{-,-\}:Z\times Z\longrightarrow Z \tag{4.4.1}$$

is a $\mathbf{k}[\mathscr{C}]$-linear *Poisson bracket*. Given $c\in\mathscr{C}$, it induces a Poisson bracket

$$\{-,-\}:Z_c\times Z_c\longrightarrow Z_c. \tag{4.4.2}$$

4.4.B. Additional filtrations. — Define a P-algebra filtration of $\mathbf{H}$ by

$$\mathbf{H}^{\preccurlyeq -1}=0,\ \mathbf{H}^{\preccurlyeq 0}=P[W],\ \mathbf{H}^{\preccurlyeq 1}=\mathbf{H}^{\preccurlyeq 0}+\mathbf{H}^{\preccurlyeq 0}V+\mathbf{H}^{\preccurlyeq 0}V^* \text{ and } \mathbf{H}^{\preccurlyeq i}=\mathbf{H}^{\preccurlyeq 1}\mathbf{H}^{\preccurlyeq i-1} \text{ for } i\geqslant 2.$$

Note that $\mathbf{H}^{\preccurlyeq 2N-1}\neq\mathbf{H}$ and $\mathbf{H}^{\preccurlyeq 2N}=\mathbf{H}$.

Let V' be the $\mathbf{k}W$-stable complement to V^W in V. We have an injection of P-modules $P\otimes(V'^*\oplus V')\otimes\mathbf{k}[W]\hookrightarrow\mathbf{H}^{\preccurlyeq 1}$. It extends to a morphism of graded P-algebras

$$f:P\otimes(\mathbf{k}[V']^{\mathrm{co}(W)}\otimes\mathbf{k}[V'^*]^{\mathrm{co}(W)})\rtimes W\to\mathrm{gr}^{\preccurlyeq}\mathbf{H}$$

where P and W are in degree 0 and V'^* and V' are in degree 1.

Proposition 4.4.3. — *The morphism f is an isomorphism of graded P-algebras.*

Proof. — This follows from the PBW decomposition (Corollary 4.2.2). □

Let us define $\dot{\mathbf{H}}=\widetilde{\mathbf{H}}\otimes_{\mathbf{k}[T]}\big(\mathbf{k}[T]/(T-1)\big)$, an algebra over $\mathbf{k}[\widetilde{\mathscr{C}}]/(T-1)$ (we identify that ring with $\mathbf{k}[\mathscr{C}]$). We define a $\mathbf{k}$-algebra filtration of $\dot{\mathbf{H}}$

$$\dot{\mathbf{H}}^{\trianglelefteq -1}=0,\ \dot{\mathbf{H}}^{\trianglelefteq 0}=\mathbf{k}[V]\rtimes W,\ \dot{\mathbf{H}}^{\trianglelefteq 1}=\dot{\mathbf{H}}^{\trianglelefteq 0}V+\dot{\mathbf{H}}^{\trianglelefteq 0}\mathscr{C}^*+\dot{\mathbf{H}}^{\trianglelefteq 0}$$

$$\text{and } \dot{\mathbf{H}}^{\trianglelefteq i}=(\dot{\mathbf{H}}^{\trianglelefteq 1})^i \text{ for } i\geqslant 2.$$

The canonical maps $\mathbf{k}[V]\rtimes W\to(\mathrm{gr}^{\trianglelefteq}\dot{\mathbf{H}})^0$ and $V\oplus\mathscr{C}^*\to(\mathrm{gr}^{\trianglelefteq}\dot{\mathbf{H}})^1$ induce a morphism of $\mathbb{N}$-graded algebras $g:\mathbf{H}\to\mathrm{gr}^{\trianglelefteq}\dot{\mathbf{H}}$

The PBW decomposition (Theorem 3.1.5) shows the following result.

Proposition 4.4.4. — *The morphism g is an isomorphism.*

Note that this proposition shows that $\widetilde{\mathbf{H}}$ is the Rees algebra of $\dot{\mathbf{H}}$ for this filtration.

4.4.C. Symmetrizing form. — Recall (Proposition 2.2.3) that we have symmetrizing forms $p_N : \mathbf{k}[V] \to \mathbf{k}[V]^W$ and $p_N^* : \mathbf{k}[V^*] \to \mathbf{k}[V^*]^W$.

We define a P-linear map

$$\begin{aligned} \boldsymbol{\tau}_{\mathbf{H}} : \mathbf{H} = \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*] &\longrightarrow P \\ a \otimes b \otimes w \otimes c &\longmapsto a\delta_{1w}\, p_N(b) p_N^*(c). \end{aligned}$$

Theorem 4.4.5 **([BrGoSt**, Theorem 4.4**]).** — *The form* $\boldsymbol{\tau}_{\mathbf{H}}$ *is symmetrizing for the* P*-algebra* $\mathbf{H}$.

Proof. — We have an isomorphism

$$\begin{aligned} (\mathrm{gr}^{\preccurlyeq}\mathbf{H})^{2N} &\xrightarrow{\sim} P \\ \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V] \otimes \mathbf{k}W \otimes \mathbf{k}[V^*] \ni a \otimes b \otimes w \otimes c &\longmapsto p_N(b) \otimes aw \otimes p_N^*(c) \end{aligned}$$

Via the isomorphism of Proposition 4.4.4, the P-linear form on $\mathrm{gr}^{\preccurlyeq}\mathbf{H}$ induced by $\boldsymbol{\tau}_{\mathbf{H}}$ is given by

$$P \otimes (\mathbf{k}[V]_W \otimes \mathbf{k}[V^*]_W) \rtimes W \ni a \otimes (b \otimes c) \otimes w \mapsto a\delta_{1w} \langle p_N(b), p_N(c) \rangle.$$

It follows from Lemma A.4.1 that this is a symmetrizing form.

Let $L = V' \oplus V'^*$. We have $S^{N+1}(V') \subset S(V')^{\mathrm{co}(W)}_{>0} \cdot S^{\leqslant N-1}(V')$ (and similarly with V'^*), hence

$$L^{2N+1} \subset (S^{N+1}(V') \otimes S^N(V'^*)) + (S^N(V') \otimes S^{N+1}(V'^*)) + \mathbf{H}^{\preccurlyeq 2N-1} \subset \mathbf{H}^{\preccurlyeq 2N-1}.$$

It follows from Lemma A.4.2 that $\tau_{\mathbf{H}}$ is a trace. We deduce from Proposition A.4.3 that $\boldsymbol{\tau}_{\mathbf{H}}$ is symmetrizing. □

Remark 4.4.6. — Note that while the identification $\mathbf{k}[V]_N^{\mathrm{co}(W)} \xrightarrow{\sim} \mathbf{k}$ is not canonical, there is a canonical choice of isomorphism $\mathbf{k}[V]_N^{\mathrm{co}(W)} \otimes_{\mathbf{k}} \mathbf{k}[V^*]_N^{\mathrm{co}(W)} \xrightarrow{\sim} \mathbf{k}$ obtained by requiring $\langle \alpha_s^\vee, \alpha_s \rangle = 1$ for all $s \in \mathrm{Ref}(W)$. This provides a canonical choice for $\boldsymbol{\tau}_{\mathbf{H}}$. ■

We denote by $\mathrm{cas}_{\mathbf{H}} \in Z$ the central Casimir element of $\mathbf{H}$ (cf §A.4).

4.4.D. Hilbert series. — We compute here the bigraded Hilbert series of $\mathbf{H}$, P, Z and $e\mathbf{H}e$. First of all, note that

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[\mathscr{C}]) = \frac{1}{(1-\mathbf{tu})^{|\mathrm{Ref}(W)/W|}},$$

so that it becomes easy to deduce the Hilbert series for $\mathbf{H}$, using the PBW-decomposition:

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{H}) = \frac{|W|}{(1-\mathbf{t})^n\,(1-\mathbf{u})^n\,(1-\mathbf{tu})^{|\mathrm{Ref}(W)/W|}}. \tag{4.4.7}$$

On the other hand, using the notation of Theorem 2.2.1, we get, thanks to (2.5.2),

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(P) = \frac{1}{(1-\mathbf{tu})^{|\mathrm{Ref}(W)/W|} \prod_{i=1}^{n} (1-\mathbf{t}^{d_i})(1-\mathbf{u}^{d_i})}. \tag{4.4.8}$$

Finally, note that the PBW-decomposition is a W-equivariant isomorphism of bigraded $\mathbf{k}[\mathscr{C}]$-modules, from which we deduce that $\mathbf{H}e \simeq \mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V] \otimes \mathbf{k}[V^*]$ as bigraded $\mathbf{k}W$-modules. So

(4.4.9) *the bigraded* $\mathbf{k}$*-vector spaces* Z *and* $\mathbf{k}[\mathscr{C}] \otimes \mathbf{k}[V \times V^*]^{\Delta W}$ *are isomorphic.*

We deduce that $\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z) = \dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[\mathscr{C}]) \cdot \dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(\mathbf{k}[V\times V^*]^{\Delta W})$. By (2.5.3) and Proposition 2.5.10, we get

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z) = \frac{1}{|W|\,(1-\mathbf{tu})^{|\mathrm{Ref}(W)/W|}} \sum_{w\in W} \frac{1}{\det(1-w\mathbf{t})\,\det(1-w^{-1}\mathbf{u})} \tag{4.4.10}$$

and

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z) = \frac{\displaystyle\sum_{\chi\in\mathrm{Irr}(W)} f_\chi(\mathbf{t})\, f_\chi(\mathbf{u})}{(1-\mathbf{tu})^{|\mathrm{Ref}(W)/W|} \prod_{i=1}^{n} (1-\mathbf{t}^{d_i})(1-\mathbf{u}^{d_i})}. \tag{4.4.11}$$

Example 4.4.12. — Assume here that $n = \dim_{\mathbf{k}}(V) = 1$ and let $d = |W|$. Let $y \in V \setminus \{0\}$ and $x \in V^*$ with $\langle y, x\rangle = 1$. Then $P_\bullet = \mathbf{k}[x^d, y^d]$, $\mathbf{eu}_0 = xy$ and it is easily checked that $Z_0 = \mathbf{k}[x^d, y^d, xy]$, that is, $Z_0 = P_\bullet[\mathbf{eu}_0]$. We will prove here that

$$Z = P[\mathbf{eu}].$$

Indeed, $\mathrm{Irr}(W) = \{\varepsilon^i \mid 0 \leqslant i \leqslant d-1\}$ and $f_{\varepsilon^i}(\mathbf{t}) = \mathbf{t}^i$ for $0 \leqslant i \leqslant d-1$. Consequently, (4.4.11) implies that

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z) = \frac{1 + (\mathbf{tu}) + \cdots + (\mathbf{tu})^{d-1}}{(1-\mathbf{tu})^{d-1}\,(1-\mathbf{t}^d)\,(1-\mathbf{u}^d)}$$

whereas, since $P[\mathbf{eu}] = P \oplus P\mathbf{eu} \oplus \cdots \oplus P\mathbf{eu}^{d-1}$ by Proposition **??**, we have

$$\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(P[\mathbf{eu}]) = \frac{1 + (\mathbf{tu}) + \cdots + (\mathbf{tu})^{d-1}}{(1-\mathbf{tu})^{d-1}\,(1-\mathbf{t}^d)\,(1-\mathbf{u}^d)}.$$

Hence, $\dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(P[\mathbf{eu}]) = \dim_{\mathbf{k}}^{\mathbb{Z}\times\mathbb{Z}}(Z)$, so $Z = P[\mathbf{eu}]$. ■

In fact, it almost never happens that $Z = P[\mathbf{eu}]$, cf Proposition **??**.

4.5. Special features of Coxeter groups

> **Assumption.** *In this section §4.5, we assume that W is a Coxeter group, and we use the notation of §2.6.*

In relation with the aspects studied in this chapter, one of the features of the situation is that the algebra $\mathbf{H}$ admits another automorphism $\sigma_{\mathbf{H}}$, induced by the isomorphism of W-modules $\sigma : V \xrightarrow{\sim} V^*$. It is the reduction modulo T of the automorphism $\sigma_{\widetilde{\mathbf{H}}}$ de $\widetilde{\mathbf{H}}$ defined in §3.6. Propositions 3.6.2 and 3.6.3 now become:

Proposition 4.5.1. — *There exists a unique automorphism $\sigma_{\mathbf{H}}$ of $\mathbf{H}$ such that*

$$\begin{cases} \sigma_{\mathbf{H}}(y) = \sigma(y) & \textit{if } y \in V, \\ \sigma_{\mathbf{H}}(x) = -\sigma^{-1}(x) & \textit{if } x \in V^*, \\ \sigma_{\mathbf{H}}(w) = w & \textit{if } w \in W, \\ \sigma_{\mathbf{H}}(C) = C & \textit{if } C \in \mathscr{C}^*. \end{cases}$$

Proposition 4.5.2. — *We have the following statements:*

(a) $\sigma_{\mathbf{H}}$ *stabilizes the subalgebras* $\mathbf{k}[\mathscr{C}]$ *and* $\mathbf{k}W$ *and exchanges the subalgebras* $\mathbf{k}[V]$ *and* $\mathbf{k}[V^*]$.

(b) *Given* $h \in \mathbf{H}^{\mathbb{N}\times\mathbb{N}}[i,j]$, *we have* $\sigma_{\mathbf{H}}(h) \in \mathbf{H}^{\mathbb{N}\times\mathbb{N}}[j,i]$.

(c) *Given* $h \in \mathbf{H}^{\mathbb{N}}[i]$ *(respectively* $h \in \mathbf{H}^{\mathbb{Z}}[i]$*), we have* $\sigma_{\mathbf{H}}(h) \in \mathbf{H}^{\mathbb{N}}[i]$ *(respectively* $h \in \mathbf{H}^{\mathbb{Z}}[-i]$*).*

(d) $\sigma_{\mathbf{H}}$ *commutes with the action of* $W^\wedge$ *on* $\mathbf{H}$.

(e) *Given* $c \in \mathscr{C}$, *then* $\sigma_{\mathbf{H}}$ *induces an automorphism of* $\mathbf{H}_c$, *still denoted by* $\sigma_{\mathbf{H}}$ *(or* $\sigma_{\mathbf{H}_c}$ *if necessary).*

(f) $\sigma_{\mathbf{H}}(\mathbf{eu}) = -\mathbf{eu}$.

Similarly, there exists an action of $\mathbf{GL}_2(\mathbf{k})$ on $\mathbf{H}$, which is obtained by reduction modulo T of the action on $\widetilde{\mathbf{H}}$ defined in Remark 3.6.4.

APPENDICES

APPENDIX A

FILTRATIONS

A.1. Filtered modules

Let R be a commutative ring. A *filtered R-module* is an R-module M together with R-submodules $M^{\leqslant i}$ for $i \in \mathbb{Z}$ such that

$$M^{\leqslant i} \subset M^{\leqslant i+1} \text{ for } i \in \mathbb{Z},\ M^{\leqslant i} = 0 \text{ for } i \ll 0 \text{ and } M = \bigcup_{i \in \mathbb{Z}} M^{\leqslant i}.$$

Given M a filtered R-module, the *associated $\mathbb{Z}$-graded* R-module $\operatorname{gr} M$ is given by

$$(\operatorname{gr} M)_i = M^{\leqslant i}/M^{\leqslant i-1}.$$

The *principal symbol map* $\xi : M \to \operatorname{gr} M$ is defined by $\xi(m) = m \mod M^{\leqslant i-1} \in (\operatorname{gr} M)_i$, where i is minimal such that $m \in M^{\leqslant i}$. The principal symbol map is injective but not additive.

The *Rees* module associated with M is the $R[T]$-submodule $\operatorname{Rees}(M) = \sum_{i \in \mathbb{Z}} T^i M^{\leqslant i}$ of $R[T, T^{-1}] \otimes_R M$. We have $R[T, T^{-1}] \otimes_{R[T]} \operatorname{Rees}(M) = R[T, T^{-1}] \otimes_R M$. In particular, given $t \in R^\times$, we have an isomorphism of R-modules

$$R[T]/\langle T - t\rangle \otimes_{R[T]} \operatorname{Rees}(M) \xrightarrow{\sim} M,\ T^i m \mapsto t^i m.$$

There is an isomorphism of R-modules

$$R[T]/\langle T\rangle \otimes_{R[T]} \operatorname{Rees}(M) \xrightarrow{\sim} \operatorname{gr} M,\ T^i m \mapsto \begin{cases} 0 & \text{if } m \in M^{<i} \\ \xi(m) & \text{otherwise.} \end{cases}$$

A.2. Filtered algebras

Let A be an R-algebra. A (bounded below) *filtration* on A is the data of a filtered Rb-module structure on A such that

$$1 \in A^{\leqslant 0} \setminus A^{\leqslant -1} \text{ and } A^{\leqslant i} \cdot A^{\leqslant j} \subset A^{\leqslant i+j} \text{ for all } i, j \in \mathbb{Z}.$$

The associated graded R-module $\mathrm{gr}A$ is a graded R-algebra. The Rees module associated with A is a $R[T]$-algebra. An immediate consequence is the following lemma.

Lemma A.2.1. — *If* $\mathrm{gr}A$ *has no* 0 *divisors, then the principal symbol map* $\xi : A \to \mathrm{gr}A$ *is multiplicative and* A *has no* 0 *divisors.*

Proof. — Let $a, b \in A$ be two non-zero elements and let i, j minimal such that $a \in A^{\leqslant i}$ and $b \in A^{\leqslant j}$. Since $\mathrm{gr}A$ has no 0 divisors, it follows that $\xi(a)\xi(b) \neq 0$, hence $ab \notin A^{<i+j}$. This shows that $\xi(ab) = \xi(a)\xi(b)$, and that $ab \neq 0$. □

Let us recall some facts of commutative algebra (cf [**Mat**, Exercices 9.4-9.5]). Let R be a commutative domain with field of fractions K. An element $x \in K$ is said to be *almost integral* over R if there exists $a \in R$, $a \neq 0$, such that for all $n \geqslant 0$, we have $ax^n \in R$. If x is integral over R, then x is almost integral over R, and the converse holds if R is noetherian.

We say that R is *completely integrally closed* if the elements of K that are almost integral over R are in R.

Lemma A.2.2. — *Assume* A *is a commutative ring. If* $\mathrm{gr}A$ *is a completely integrally closed domain, then* A *is a completely integrally closed domain.*

Proof. — Lemma A.2.1 shows that A is a domain. Let K be its field of fractions. Let $x \in K$ be almost integral over R. Let $c, d \in A$ such that $x = c/d$. Let i (resp. j) be minimal such that $c \in A^{\leqslant i}$ (resp. $d \in A^{\leqslant j}$). We show by induction on i that $x \in A$.

Let $a \in A$, $a \neq 0$, such that $ax^n \in A$ for all $n \geqslant 0$. Let $\alpha_n = ax^n$. We have $d^n\alpha_n = ac^n$, hence $\xi(d)^n\xi(\alpha_n) = \xi(a)\xi(c)^n$ (cf Lemma A.2.1). It follows that $\frac{\xi(c)}{\xi(d)}$ is an element of the field of fractions of $\mathrm{gr}A$ that is almost integral over $\mathrm{gr}A$. Consequently, it is in $\mathrm{gr}A$. Since $\mathrm{gr}A$ has no zero divisors, it follows that it is homogeneous of degree $i - j$. Let $u \in A$ with $\xi(u) = \frac{\xi(c)}{\xi(d)}$. Let $x' = x - u = \frac{c-ud}{d}$. We have $c - ud \in A^{\leqslant i-1}$ and x' is almost integral over A. It follows by induction that $x' \in A$, hence $x \in A$. □

A.3. Filtered modules over filtered algebras

A *filtered A-module* is an A-module M together with a structure of filtered R-module such that

$$A^{\leqslant i} \cdot M^{\leqslant j} \subset M^{\leqslant i+j} \text{ for all } i, j \in \mathbb{Z}.$$

A *filtered morphism* of A-modules is a morphism $f : M \to N$, where M and N are filtered A-modules, such that $f(M^{\leqslant i}) \subset N^{\leqslant i}$ for all $i \in \mathbb{Z}$.

Lemma A.3.1. — *Let $f : M \to N$ be a filtered morphism of A-modules. If* $\operatorname{gr} f$ *is surjective (resp. injective), then f is surjective (resp. injective).*

Proof. — Assume $\operatorname{gr} f$ is surjective. We have $N^{\leqslant i} = f(M^{\leqslant i}) + N^{\leqslant i-1}$. Since $N^{\leqslant i} = 0$ for $i \ll 0$, it follows by induction that $N^{\leqslant i} = f(M^{\leqslant i})$, hence f is surjective.

Assume $\operatorname{gr} f$ is injective. Let $m \in M - \{0\}$ and let i be minimal such $m \in M^{\leqslant i}$. We have $f(m) \notin N^{\leqslant i-1}$, hence $f(m) \neq 0$. □

Lemma A.3.2. — *Let M be a filtered A-module and E a subset of M. If $\xi(E)$ generates* $\operatorname{gr} M$ *as an A-module, then E generates M as an A-module.*

Let F be a subset of A. If $\xi(F)$ generates $\operatorname{gr} A$ *as an R-algebra, then F generates A as an R-algebra.*

Proof. — We have a canonical morphism of filtered A-modules $f : A^{(E)} \to M$. By assumption, $\operatorname{gr} f$ is surjective, hence f is surjective by lemma A.3.1.

The second assertion follows from the first one by taking $A = R$, $M = A$ and E the set of elements of A that are products of elements of F. □

Let M and N be two finitely generated filtered A-modules. We endow the R-module $\operatorname{Hom}_A(M, N)$ with the filtration given by

$$\operatorname{Hom}_A(M, N)^{\leqslant i} = \{f \in \operatorname{End}_A(M) | f(M^{\leqslant j}) \subset N^{\leqslant i+j} \ \forall j \in \mathbb{Z}\}.$$

A map $f \in \operatorname{Hom}_A(M, N)^{\leqslant i}$ induces a morphism of $\operatorname{gr} A$-modules $\operatorname{gr} M \to \operatorname{gr} N$, homogeneous of degree i, that vanishes if $f \in \operatorname{Hom}_A(M, N)^{\leqslant i-1}$.

Lemma A.3.3. — *The construction above provides an injective morphism of graded R-modules*

$$\operatorname{gr} \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\operatorname{gr} A}(\operatorname{gr} M, \operatorname{gr} N).$$

Lemma A.3.4. — *Let e be an idempotent of $A^{\leqslant 0} \setminus A^{\leqslant -1}$ Then* $\operatorname{gr} A \cdot \xi(e)$ *is a progenerator for* $\operatorname{gr} A$ *if and only if Ae is a progenerator for A.*

Proof. — Note that $\xi(e)$ is an idempotent of $\operatorname{gr} A$. The A-module Ae is a progenerator if and only if e generates A as an (A, A)-bimodule. The lemma follows from Lemma A.3.2. □

A.4. Symmetric algebras

Let us recall some basic facts about symmetric algebras (cf for example [**Bro1**, §2, 3]). A *symmetric R-algebra* is an R-algebra A, finitely generated and projective as an R-module, and endowed with an R-linear map $\tau_A : A \to R$ such that

- $\tau_A(ab) = \tau_A(ba)$ for all $a, b \in A$ (i.e., τ_A is a trace) and
- the morphism of (A,A)-bimodules
$$\hat{\tau}_A : A \to \mathrm{Hom}_R(A,R),\ a \mapsto (b \mapsto \tau(ab))$$
is an isomorphism.

Such a form τ_A is called a *symmetrizing form* for A.

Consider the sequence of isomorphisms
$$A \otimes_R A \xrightarrow[\sim]{\hat{\tau}\otimes \mathrm{id}} \mathrm{Hom}_R(A,R) \otimes_R A \xrightarrow[\sim]{f\otimes a \mapsto (b \mapsto f(b)a)} \mathrm{End}_R(A).$$
The *Casimir element* is the inverse image of id_A through the composition of maps above. The *central Casimir element* cas_A is its image in A by the multiplication map $A \otimes_R A \to A$. It is an element of $Z(A)$.

Assume A is free over R, with R-basis $\mathscr{B}$ and dual basis $(b^\vee)_{b\in\mathscr{B}}$ for the bilinear form $A \otimes_R A \to R,\ a \otimes a' \mapsto \tau(aa')$. We have $\mathrm{cas}_A = \sum_{b\in\mathscr{B}} bb^\vee$.

Lemma A.4.1. — *Let (A, τ_A) be a symmetric R-algebra and G a finite group acting on the R-algebra A and such that $\tau_A(g(a)) = \tau_A(a)$ for all $g \in G$ and $a \in A$.*

Let $B = A \rtimes G$ and define an R-linear form $\tau_B : B \to R$ by $\tau_B(a \otimes g) = \tau_A(a)\delta_{1,g}$ for $a \in A$ and $g \in G$.

The form τ_B is symmetrizing for B.

Proof. — We have
$$\tau_B((a\otimes g)(a'\otimes g')) = \tau_A(ag(a'))\delta_{g^{-1},g'} = \tau_A(a'g^{-1}(a))\delta_{g^{-1},g'} = \tau_B((a'\otimes g')(a\otimes g)).$$

Given $g \in G$, let $B_g = A \otimes g$ and $C_g = \mathrm{Hom}_R(B_g, R)$. We have $B = \bigoplus_{g\in G} B_g$ and $\mathrm{Hom}_R(B,R) = \bigoplus_g C_g$. Given $g \in G$ we have $\hat{\tau}_B(B_g) \subset C_{g^{-1}}$ and $\hat{\tau}_B(a\otimes g)(a' \otimes g^{-1}) = \tau_A(ag(a'))$. It follows that the restriction of τ_B to B_g is an isomorphism $B_g \xrightarrow{\sim} C_{g^{-1}}$. □

Let now A be a filtered R-algebra, with $A^{\leqslant -1} = 0$, $A^{\leqslant d-1} \neq A$ and $A^{\leqslant d} = A$ for some $d \geqslant 0$. Let $\bar{\tau} : (\mathrm{gr}A)^d \to R$ be an R-linear form. We extend it to an R-linear form on $\mathrm{gr}A$ by setting it to 0 on $(\mathrm{gr}A)^i$ for $i < d$. We define an R-linear form τ on A as the composition
$$\tau : A \xrightarrow{\mathrm{can}} (\mathrm{gr}A)^d \xrightarrow{\bar{\tau}} R.$$

Denote by $p_i : A^{\leqslant i} \to (\mathrm{gr} A)^i$ the canonical projection. Let $x \in A^{\leqslant i}$ and $y \in A^{\leqslant j}$. We have $\tau(xy) = \bar{\tau}(p_d(xy))$. We have $p_d(xy) = 0$ if $i + j < d$. If $i + j = d$, we have $p_d(xy) = p_i(x)p_j(y)$, hence $\tau(xy) = \bar{\tau}(p_i(x)p_j(y))$. The R-module $\mathrm{Hom}_R(A, R)$ is filtered with $\mathrm{Hom}_R(A, R)^{\leqslant i} = \mathrm{Hom}_R(A/A^{\leqslant d-i-1}, R)$ and $\hat{\tau}$ is a morphism of filtered R-modules with $\mathrm{gr}(\hat{\tau}) = \hat{\bar{\tau}}$.

Lemma A.4.2. — *Let L be an R-submodule of $A^{\leqslant 1}$ such that $A = A^{\leqslant 0}(R + L)^d$, $L^{d+1} \subset A^{<d}$ and $A^{\leqslant 0}L = LA^{\leqslant 0}$.*

If $\bar{\tau}$ is a trace, then τ is a trace.

Proof. — Note that $A = A^{\leqslant 0}L^d + A^{<d}$. We have $p_d(A^{\leqslant 0}L^d L) \subset p_d(A^{<d}) = 0$ and $p_d(LA^{\leqslant 0}L^d) = p_d(A^{\leqslant 0}LL^d) = 0$. It follows that $\tau(al) = \tau(la)$ for $a \in A^{\leqslant 0}L^d$ and $l \in L$. The considerations above show that $\tau(al) = \tau(la)$ for $a \in A^{<d}$ and $l \in L$ and $\tau(ba) = \tau(ab)$ for $b \in A$ and $a \in A^{\leqslant 0}$. □

The next proposition is inspired by a result of Brundan and Kleshchev on degenerate cyclotomic Hecke algebras [**BrKl**, Theorem A.2].

Proposition A.4.3. — *Assume* $\mathrm{gr} A$ *is projective and finitely generated as an R-module and assume τ and $\bar{\tau}$ are traces.*

If $\bar{\tau}$ is a symmetrizing form for $\mathrm{gr} A$*, then τ is a symmetrizing form for A.*

Proof. — Note that A is a finitely generated projective R-module. Since $\hat{\bar{\tau}}$ is an isomorphism, it follows that $\hat{\tau}$ is an isomorphism (Lemma A.3.1). □

A.5. Weyl algebras

Let V be a finite dimensional vector space over a field $\mathbf{k}$ of characteristic 0. Let $\mathscr{D}(V) = \widetilde{\mathbf{H}}_{1,0}$ be the Weyl algebra of V. This is the quotient of the tensor algebra $\mathrm{T}_{\mathbf{k}}(V \oplus V^*)$ by the relations

$$[x, x'] = [y, y'] = 0,\ [y, x] = \langle y, x\rangle \text{ for } x, x' \in V^* \text{ and } y, y' \in V.$$

There is an isomorphism of $\mathbf{k}$-modules given by multiplication: $\mathbf{k}[V] \otimes \mathbf{k}[V^*] \xrightarrow{\sim} \mathscr{D}(V)$.

The $\mathbf{k}$-algebra $\mathscr{D}(V)$ is filtered, with $\mathscr{D}(V)^{\leqslant -1} = 0$, $\mathscr{D}(V)^{\leqslant 0} = \mathbf{k}[V]$, $\mathscr{D}(V)^{\leqslant 1} = \mathbf{k}[V] \oplus \mathbf{k}[V] \otimes V$ and $\mathscr{D}^{\leqslant i} = (\mathscr{D}^{\leqslant 1})^i$ for $i \geqslant 2$. The associated graded algebra $\mathrm{gr}\,\mathscr{D}(V)$ is $\mathbf{k}[V \oplus V^*]$. The associated Rees algebra $\mathscr{D}_T(V)$ is the quotient of $\mathbf{k}[T] \otimes \mathrm{T}_{\mathbf{k}}(V \oplus V^*)$ by the relations

$$[x, x'] = [y, y'] = 0,\ [y, x] = T\langle y, x\rangle \text{ for } x, x' \in V^* \text{ and } y, y' \in V.$$

Consider the induced $\mathscr{D}(V)$-module $\mathscr{D}(V)\otimes_{\mathbf{k}[V^*]}\mathbf{k}$, where $\mathbf{k}[V^*]$ acts on $\mathbf{k}$ by evaluation at 0. Via the canonical isomorphism $\mathbf{k}[V] \xrightarrow{\sim} \mathscr{D}(V)\otimes_{\mathbf{k}[V^*]}\mathbf{k}$, $a \mapsto a\otimes 1$, we obtain the faithful action of $\mathscr{D}(V)$ by polynomial differential operators on $\mathbf{k}[V]$: an element $x \in V^*$ acts by multiplication, while $y \in V$ acts by $\partial_y = \frac{\partial}{\partial y}$. As a consequence of the faithfulness of the action, the centralizer of $\mathbf{k}[V]$ in $\mathscr{D}(V)$ is $\mathbf{k}[V]$.

Note that there is an injective morphism of $\mathbf{k}[T]$-algebras

$$\mathscr{D}_T(V) \hookrightarrow \mathbf{k}[T]\otimes\mathscr{D}(V),\ V^*\ni x\mapsto x,\ V\in y\mapsto Ty.$$

This provides by restriction $\mathbf{k}[T]\otimes\mathbf{k}[V]$ with the structure of a faithful representation of $\mathscr{D}_T(V)$.

APPENDIX B

INVARIANT RINGS

Let $\mathbf{k}$ be a field. Let A be a $\mathbf{k}$-algebra acted on by a finite group G whose order is invertible in $\mathbf{k}$. Let $e = \frac{1}{|G|}\sum_{g\in G} g$, a central idempotent of $\mathbf{k}[G]$. Let $R = A \rtimes G$. The aim of this appendix is to relate the representation theory of R and that of A^G. We are mainly interested in the case where A is the algebra of regular functions on an affine scheme.

B.1. Morita equivalence

The following lemma is clear.

Lemma B.1.1. — *There is an isomorphism of R-modules $A \xrightarrow{\sim} Re$, $a \mapsto ae$ that restricts to an isomorphism of $\mathbf{k}$-algebras $A^G \xrightarrow{\sim} eRe$.*

Let M be an A-module whose isomorphism class is stable under the action of a subgroup H of G. There are isomorphisms of A-modules $\phi_h : h^*(M) \xrightarrow{\sim} M$ for $h \in H$, unique up to left multiplication by $\mathrm{Aut}_A(M)$. Consequently, the elements $\phi_h \in N_{\mathrm{Aut}_{A^G}(M)}(\mathrm{Aut}_A(M))$ define a morphism of groups $H \to \mathrm{Aut}_{A^G}(M)/\mathrm{Aut}_A(M)$.

Proposition B.1.2. — *The following assertions are equivalent:*

(1) *Re is a progenerator for R*
(2) *Re induces a Morita equivalence between R and A^G*
(3) *$R = ReR$*
(4) *for every simple R-module S, we have $S^G \neq 0$.*
(5) *for every simple A-module T whose isomorphism class is stable under the action of a subgroup H of G and for every non-zero direct summand U of $\mathrm{Ind}_A^{A\rtimes H} T$, we have $U^H \neq 0$.*

Proof. — Note that Re is a direct summand of R, as a left R-module, hence Re is a finitely generated projective R-module. The equivalence between (1) and (2) follows from Lemma B.1.1. If Re is a progenerator, then R is isomorphic to a quotient of a multiple of Re. Since the image of a morphism $Re \to R$ is contained in ReR, we deduce that if (1) holds, then (3) holds. Conversely, assume (3). There are $r_1,\dots,r_n \in R$ such that $1 \in Rer_1 + \cdots + Rer_n$, hence the morphism $(Re)^n \to R$, $(a_1,\dots,a_n) \mapsto a_1 r_1 + \cdots + a_n r_n$ is surjective and (1) follows.

We have $R/ReR = 0$ if and only if R/ReR has no simple module, hence if and only if e does not act by 0 on any simple R-module. This shows the equivalence of (3) and (4).

Let S be a simple R-module. There is a simple A-module T such that S is a direct summand of $\mathrm{Ind}_A^R(T)$. Let H be the stabilizer of the isomorphism class of T. There is a simple $(A \rtimes H)$-module U such that S is a direct summand of $\mathrm{Ind}_{A\rtimes H}^R(U)$. We have $S^G \neq 0$ if and only if $U^H \neq 0$. This shows the equivalence of (4) and (5). □

Corollary B.1.3. — *If $ReR = R$, then $\mathrm{Z}(R) = \mathrm{Z}(A^G)$.*

Proof. — By Proposition B.1.2, the rings R and $eRe \simeq A^G$ are Morita equivalent thanks to the bimodule Re, so $\mathrm{Z}(R) \simeq \mathrm{Z}(A^G)$, the isomorphism being determined by the action on the bimodule Re (Lemma B.1.4 below). The result follows. □

Lemma B.1.4. — *Let A and B be two rings and M an (A,B)-bimodule such that the canonical maps give isomorphisms $B \xrightarrow{\sim} \mathrm{End}_A(M)$ and $A \xrightarrow{\sim} \mathrm{End}_{B^{\mathrm{opp}}}(M)^{\mathrm{opp}}$. Then we have an isomorphism $\mathrm{Z}(A) \xrightarrow{\sim} \mathrm{Z}(B)$.*

In particular, if e is an idempotent of a ring A and if left multiplication gives an isomorphism $A \xrightarrow{\sim} \mathrm{End}_{(eAe)^{\mathrm{opp}}}(Ae)^{\mathrm{opp}}$, then there is an isomorphism $\mathrm{Z}(A) \xrightarrow{\sim} \mathrm{Z}(eAe)$, $a \mapsto ae$.

Proof. — The left multiplication on M induces induces a ring morphism $\alpha : \mathrm{Z}(A) \to \mathrm{Z}(B)$ such that $zm = m\alpha(z)$ for all $z \in \mathrm{Z}(A)$ and $m \in M$. Similarly, the right multiplication induces a ring morphism $\beta : \mathrm{Z}(B) \to \mathrm{Z}(A)$ such that $mz = \beta(z)m$ for all $z \in \mathrm{Z}(B)$ and $m \in M$. Hence, if $z \in \mathrm{Z}(A)$ and $m \in M$, then $zm = \beta(\alpha(z))m$, and so $\beta \circ \alpha = \mathrm{Id}_{\mathrm{Z}(A)}$ since the action of A on M is faithful by assumption. Similarly $\alpha \circ \beta = \mathrm{Id}_{\mathrm{Z}(B)}$. □

B.2. Geometric setting

We assume now that $A = \mathbf{k}[X]$, where X is an affine scheme of finite type over $\mathbf{k}$, i.e., A is a finitely generated commutative $\mathbf{k}$-algebra. Then Proposition B.1.2 has the following consequence.

Corollary B.2.1. — *If G acts freely on X, then Re induces a Morita equivalence between R and A^G.*

Let $X^{\mathrm{reg}} = \{x \in X | \mathrm{Stab}_G(x) = 1\}$ and let $R^{\mathrm{reg}} = \mathbf{k}[X^{\mathrm{reg}}] \rtimes G$. We assume X^{reg} is dense in X, i.e., the pointwise stabilizer of an irreducible component of X is trivial. The following proposition gives a sufficient condition for a double centralizer theorem.

Proposition B.2.2. — *Assume that X is a normal variety, i.e., all localizations of A at prime ideals are integral and integrally closed.*

(1) *The canonical morphism of algebras $R \to \mathrm{End}_{(A^G)^{\mathrm{opp}}}(A)^{\mathrm{opp}}$ is injective.*
(2) *If the codimension of $X \setminus X^{\mathrm{reg}}$ is $\geqslant 2$ in each connected component of X, then the morphism above is an isomorphism and $Z(R) = Z(A^G)$.*

Proof. — It follows from Corollary B.2.1 that given $f \in A^G$ such that $D(f) \subset X^{\mathrm{reg}}$, then the canonical morphism $A[f^{-1}] \rtimes G \to \mathrm{End}_{A^G[f^{-1}]}(A[f^{-1}])^{\mathrm{opp}}$ is an isomorphism. In particular, the morphism of the proposition $R \to \mathrm{End}_{(A^G)^{\mathrm{opp}}}(A)^{\mathrm{opp}}$ is an injective morphism of A-modules, since X^{reg} is dense in X.

Let K be the cokernel of the canonical morphism $R \to \mathrm{End}_{(A^G)^{\mathrm{opp}}}(A)^{\mathrm{opp}}$. We have $K \otimes_A Ak[X^{\mathrm{reg}}] = 0$, hence the support of K has codimension $\geqslant 2$. Since A is normal, it has depth $\geqslant 2$, hence $\mathrm{Ext}^1_A(K, A) = 0$. We deduce that K is a direct summand of the torsion free A-module $\mathrm{End}_{(A^G)^{\mathrm{opp}}}(A)^{\mathrm{opp}}$, hence $K = 0$. The last statement follows from Lemma B.1.4. □

We conclude by a description of the simple R-modules whenever $R = ReR$. In this case, using the Morita equivalence between R and A^G induced by the bimodule Re, we obtain a bijective map

(B.2.3)
$$\begin{array}{ccc} \mathrm{Irr}(R) & \xrightarrow{\sim} & \mathrm{Irr}(A^G) \\ S & \longmapsto & eS. \end{array}$$

Since A is commutative, $\mathrm{Irr}(A)$ (respectively $\mathrm{Irr}(A^G)$) is in one-to-one correspondence with the maximal ideals of A (respectively of A^G), so we obtain a bijective map

(B.2.4)
$$\mathrm{Irr}(A)/G \xleftrightarrow{\sim} \mathrm{Irr}(A^G)$$

(see Propositions **??** and **??**). By composing the two previous bijective maps, we obtain a third bijective map

(B.2.5)
$$\mathrm{Irr}(A)/G \xleftrightarrow{\sim} \mathrm{Irr}(R)$$

We will describe more concretely this last map. In order to do that, let Ω be a G-orbit of (isomorphism classes of) simple A-modules. The A-module $S_\Omega = A/\cap_{T \in \Omega} \mathrm{Ann}_A(T)$ inherits an action of G, hence it becomes an R-module.

Proposition B.2.6. — *Assume that $R = ReR$ and that A is commutative and finitely generated. Then:*

(a) *If $\Omega \in \mathrm{Irr}(A)/G$, then S_Ω is a simple R-module.*

(b) *The map $\mathrm{Irr}(A)/G \longrightarrow \mathrm{Irr}(R)$, $\Omega \mapsto S_\Omega$ is bijective (and coincides with the bijective map B.2.4).*

(c) *If S is a simple A-module, then $\mathrm{Res}_A^R(S)$ is semisimple and multiplicity-free, and two simple A-modules occuring in $\mathrm{Res}_A^R(S)$ are in the same G-orbit.*

(d) *If S and S' are two simple R-modules, then $S \simeq S'$ if and only if $\mathrm{Res}_A^R(S)$ and $\mathrm{Res}_A^R(S')$ have a common irreducible submodule.*

Proof. — (a) By construction, we have a well-defined injective morphism of A-modules $S_\Omega \hookrightarrow \bigoplus_{T\in\Omega} T$ (here, we identify T and $A/\mathrm{Ann}_A(T)$. So, if S is a non-zero R-submodule of S_Ω, then it is a non-zero A-submodule of S_Ω. Therefore, S contains some submodule isomorphic to $T \in \Omega$. Since the action of G stabilizes S, it follows that $S = S_\Omega$ and that

$$(*) \qquad \mathrm{Res}_A^R(S_\Omega) = \bigoplus_{T\in\Omega} T.$$

This proves (a).

(b) It follows from $(*)$ that the map $\mathrm{Irr}(A)/G \longrightarrow \mathrm{Irr}(R)$, $\Omega \mapsto S_\Omega$ is injective. Now, let $\Omega \in \mathrm{Irr}(A)/G$, let $T \in \Omega$ and let $\mathfrak{m} = \mathrm{Ann}_A(T)$. We denote by H the stabilizer of $\mathfrak{m}$ in G (that is, the decomposition group of $\mathfrak{m}$). Then $eS = S_\Omega^G \simeq T^H = (A/\mathfrak{m})^H$. But, by Theorem **??**, $(A/\mathfrak{m})^H = A^G/(\mathfrak{m} \cap A^G)$. This proves that eS_Ω is the simple A^G-module associated with the maximal ideal $\mathfrak{m} \cap A^G$ of A^G or, in other words, is the simple A^G-module associated with Ω through the bijective map B.2.4. This completes the proof of (b).

(c) and (d) now follow from (a), (b) and $(*)$. $\square$

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Questions.

- je comprends pas le definition de $Hom_A(M,n)^{\leqslant i}$.
- notations homogenes pour ideaux engendres, pour ker (ou Ker)
- commentaires sur le calcul de B2 et la verification d'Ulrich
- Magma pour calcul de polynome