A theorem and a remark with the purpose of comparing the role and origin of probabilities in non-collapse and in collapse quantum mechanics

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Abstract

It is proved that in non-collapse quantum mechanics the state of a closed system can always be expressed as a superposition of states all of which describe histories that conform to Born's probability rule. This theorem allows one to see the probabilities in non-collapse quantum mechanics as a prediction made by the theory, and renders non-collapse quantum mechanics with the same predictive power as standard quantum mechanics with collapse according to Born's rule. By adding the remark that collapse quantum mechanics is logically compatible with probabilities different from those given by Born's rule, it is argued that the fact that the experimental observations support Born's probability rule can be seen as evidence in support of the non-collapse interpretation of quantum mechanics, rather than as a problem for that interpretation. This remark should also be used to scrutinize derivations of Born's rule in the context of collapse and of non-collapse quantum mechanics.

Key words and phrases: Non-collapse quantum mechanics, Everett, Born's rule, origin of probability in quantum mechanics.

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1 Introduction

First a few words about terminology. We will use the expression "Born-rule collapse quantum mechanics" for the standard quantum mechanics theory, as presented in our textbooks, including the assumption that measurements lead to collapses of the state of the system according to Born's rule. "Collapse quantum mechanics" will be used for a broader set of theories, in which the collapses follow some probability distribution that may or not be the one given by Born's rule. And by "non-collapse quantum mechanics" we simply mean that we eliminate the assumption of collapse when measurements are performed. In non-collapse quantum mechanics, we do not include the words "measurement" or "observation" in the axioms of the theory, and use them only informally when applying the theory to explain and predict our experiences.

Readers who want an introduction to non-collapse quantum mechanics will benefit from the classic [6], where papers by those who first proposed and advertised it as a (better) alternative to collapse quantum mechanics are collected. The subject is not standard in textbooks geared to physicists, but is standard in texts concerned with the philosophy of quantum mechanics; see, e.g., [2], [3], [20] and [11]. For a positive appraisal of the theory, written for the general scientific public, see, e.g., [17]. For expositions for the general public, see, e.g., [5] and [18]. For a recent collection of mostly philosophical discussions see [14]. And for two among the many recent research papers on the subject, see, e.g., [1] and [16], which also provide extensive additional references.

Our concern here is with the origin of our perception of Born-rule probabilities in a theory, non-collapse quantum mechanics, in which everything is deterministic and, in particular, no probabilities are introduced in its axioms. A great deal has been written about this problem, e.g., in the references cited in the last paragraph and references therein, with opinions ranging from "the problem is solved" (sometimes by the authors themselves), to "the problem is hopeless and the proposed solutions all flawed". I hope nevertheless to convince the reader that this paper adds substantially to the solution of this puzzle and explains how our perception of probabilities, as given by Born's rule, emerges in non-collapse quantum mechanics. I will go as far as suggesting that the puzzle be turned around: If collapses do happen, why do they happen precisely with the same rule that comes out of quantum mechanics without collapse? Additionally, I hope to convince the skeptical reader that this paper will be helpful even to those who disagree with my conclusions, as the theorem that will be proved and the counterexamples that will be provided will help them pinpoint where their objections lie, and also provide some guidance on what to possibly do and what to avoid if one wants to propose empirical tests to disprove that we live in a universe ruled by non-collapse quantum mechanics.

In Section 2 we will state the theorem alluded to in the title, in a mathematically self-contained fashion, but without emphasizing the corresponding physics, which will then be discussed in the following two sections. To help the reader keep in mind what is planned, we include here a few words of introduction on how the mathematical setting in Section 2 is motivated by collapse quantum mechanics.

We will be working in the Heisenberg picture (operators evolve in time, rather than states), as applied to a closed system (possibly the whole universe). Associated to the system there is a Hilbert space $\mathcal{H}$ (not assumed in this paper to be necessarily separable). The state of the system is given at any time by a non-null vector in $\mathcal{H}$ (with non-null scalar multiples of a vector corresponding to the same state). This state does not change with time except when there is a collapse. Collapses are associated with measurements and with their corresponding self-adjoint operators (which in the Heisenberg picture are time dependent). In each collapse, the state immediately after the collapse is a projection of the state immediately before the collapse on a subspace (a subset of $\mathcal{H}$ closed linearly and topologically) chosen at random, according to a specified probability law (in the standard case, Born's rule), from among the eigenspaces of that operator, one eigenspace for each possible outcome of the experiment. (To avoid unnecessary mathematical complications, and on physical grounds, we are assuming that every experiment can only have a finite number of possible outcomes.) To each subspace of $\mathcal{H}$ there is associated a projection operator (self-adjoint idempotent operator on $\mathcal{H}$) that projects on that subspace. If initially the state was a vector $\psi \in \mathcal{H}$, then immediately after a collapse the state can be expressed as $\mathrm{Proj}\,\psi$, where Proj is the composition of the projections that took place after each collapse, up to and including this last one.

It is natural to represent all the possible ways in which the system can evolve using a rooted (oriented) tree. The root vertex of the tree will correspond to the beginning of times for the system under study, and the other vertices will either correspond to collapse events, or be terminal vertices (vertices of degree 1) that signal that no further experiment is performed along a branch of the tree. (In the interesting cases the tree will be infinite. One can think of terminal vertices as uncommon in the tree, possibly even absent.) The projections associated to the possible outcomes in the collapses, as described at the end of the last paragraph, will then be indexed by the edges of the tree. The tree does not have to be homogeneous, as, e.g., decisions on what experiments to perform in a lab may depend on the outcomes of previous experiments. More interesting and dramatic examples of non-homogeneity of the tree occur if one thinks of some major human decisions being made by use of "quantum coins", i.e., outcomes of experiments performed for this purpose (depending on these decisions the future of humanity may take quite different turns).

After stating the theorem in Section 2, we will discuss in Section 3 how it relates to quantum mechanics with and without collapse and its relevance. In Section 4 we will discuss the fact that Born's rule is not logically necessary in quantum mechanics with collapse, while the theorem stated in Section 2 makes it necessary (in a sense that will be made explicit in Section 3) in non-collapse quantum mechanics. In Section 5 we provide a proof of the theorem. Section 6 is an appendix, in which we briefly digress on the meaning of probabilities in collapse quantum mechanics.

2 The theorem

Let $(\mathbf{V}, \mathbf{E})$ be a tree with vertex set $\mathbf{V}$, including a singled out vertex called the root vertex, and edge set $\mathbf{E}$. We assume that the root vertex has a single edge incident to it and call it the root edge. Such a tree will be called an edge-rooted tree. We orient the root edge from the root vertex to its other end, and we give an orientation to every edge in the tree, so that each vertex other than the root vertex has exactly one edge oriented towards it. If e is the edge oriented towards vertex v and $e_1, ..., e_n$ are the edges incident to v and oriented away from it, we call $e_1, ..., e_n$ the children of e, and we refer to $\{e_1, ...e_n\}$ as a set of siblings and to e as their parent. (The advantage of using such "family" language, even if a bit funny, is that the terminology becomes easy to remember and easy to extend.) Childless edges will be called terminal edges, and the vertices to which terminal edges point will be called terminal vertices. Each edge belongs to a generation defined inductively by declaring the generation of the root edge as 1, and the generation of the children of the edges of generation i to be $i+1$. It will be convenient to declare that childless edges that belong to generation i also belong to generations $i+1$, $i+2$, ... A partial history is a finite sequence of edges $(e_1, e_2, ..., e_n)$, where each e_i is a child of e_{i-1}, $i = 2, ..., n$. A complete history (or just a history) is either a partial history in which e_1 is the root edge and the last edge is a terminal edge, or an infinite sequence of edges $(e_1, e_2, ...)$, where e_1 is the root edge and each e_i is a child of e_{i-1}, $i = 2, ...$

Definition 1 *A tree-structured set of projections on a Hilbert space $\mathcal{H}$ is a collection of such projections, $\mathcal{P} = \{\mathrm{Proj}_e : e \in \mathbf{E}\}$, where the index set $\mathbf{E}$ is the set of edges of an edge-rooted tree, and the following conditions are satisfied:*

1. *If e is the root edge, then Proj_e is the identity operator.*
2. *If $e_1, ..., e_n$ are the children of e, then $\sum_{i=1}^n \mathrm{Proj}_{e_i} = \mathrm{Proj}_e$.*

We write $\mathcal{H}_e = \mathrm{Proj}_e\mathcal{H}$ for the subspace associated to Proj_e. The first condition means that $\mathcal{H}_e = \mathcal{H}$ when e is the root edge, while the second one means that the subspaces $\mathcal{H}_{e_i}$ associated to a set of siblings $\{e_1, ..., e_n\}$ are orthogonal to each other and their linear span is the subspace $\mathcal{H}_e$ associated to the parent e.

Implicit in the definition of a tree-structured set of projections $\mathcal{P}$ is the associated edge-rooted tree $(\mathbf{V}, \mathbf{E})$. The set of histories on this tree, denoted Ω, is the sample space on which one defines Born's probabilities (and alternative ones) associated to $\mathcal{P}$. Recall that, informally speaking, an element $\omega \in \Omega$ is a sequence of edges starting from the root and having each of its elements succeeded by one of its children, either with no end, or ending at a terminal edge. Abusing notation, we will write $e \in \omega$ for the statement that the edge e is an element of the sequence ω. For each $e \in \mathbf{E}$, we define $\Omega_e = \{\omega : e \in \omega\}$, the set of histories that go through e. Unions of finitely many sets Ω_e define an algebra of sets (a class of sets that is closed with respect to complements, finite unions and finite intersections) that we denote by $\mathcal{A}$. (This statement requires a proof, which is easily obtained by noting that every set $A \in \mathcal{A}$ can be written as a union over sets Ω_e with all e in the same generation, and that A^c is then the union of the sets Ω_e over the other e belonging to this same generation. This shows closure under complements. Closure under unions is immediate and De Morgan's law then provides closure under intersections.) The smallest σ-algebra that contains $\mathcal{A}$ will be denoted by $\mathcal{B}$.

Born's probabilities are defined on the measure space $(\Omega, \mathcal{B})$ and, in addition to $\mathcal{P}$, depend on a vector $\psi \in \mathcal{H}\backslash\{0\}$. (In the theorem below, ψ is arbitrary, but in all our applications it will be the initial state of our system. In collapse quantum mechanics, ψ will be chosen as the state, in the Heisenberg picture, before collapses. In non-collapse quantum mechanics, ψ will be chosen as the unchanging state, in the Heisenberg picture.) Born's probability corresponding to ψ will be denoted by $I\!P_\psi$. It is described informally by imagining a walker that moves on the edges of the tree. The walker starts at the root vertex of the tree and then moves in the direction of the orientation, deciding at each vertex where to go in a probabilistic fashion, with edges chosen with probability proportional to norm-squared, i.e., when at a vertex that separates a parent e from its children, the walker chooses child e' with probability $||\text{Proj}_{e'}\psi||^2/||\text{Proj}_e\psi||^2$, independently of past choices. If ever at a terminal vertex, the walker stops. A simple inductive computation shows that this is equivalent to the statement

$$I\!P_\psi(\Omega_e) = \frac{||\text{Proj}_e\psi||^2}{||\psi||^2}, \quad \text{for each } e \in \mathbf{E}. \tag{1}$$

It is standard to show that (1) extends in a unique fashion to $\mathcal{A}$ and then to $\mathcal{B}$, defining in this way a unique probability measure on $(\Omega, \mathcal{B})$. Actually, for our purposes it will be important to observe that this standard procedure yields even more. For each $\psi \in \mathcal{H}$, the extension of the probability measure is to a larger measure space, $(\Omega, \mathcal{M}_\psi)$, where $\mathcal{M}_\psi \supset \mathcal{B}$, completes $\mathcal{B}$ with respect to the measure $I\!P_\psi$, meaning that if $A \in \mathcal{B}$, $I\!P_\psi(A) = 0$ and $B \subset A$, then also $B \in \mathcal{M}_\psi$ and $I\!P_\psi(B) = 0$. In preparing for the remark that will be made in Section 4, we should note that all that is needed to implement this extension is contained in two facts about the non-negative numbers $p_e = I\!P_\psi(\Omega_e)$, which are similar to conditions 1 and 2 in Definition 1: $p_e = 1$, when e is the root edge, and $\sum_{i=1,\ldots,n} p_{e_i} = p_e$, when $e_1, \ldots, e_n$ are the children of e. (In obtaining the extension of $I\!P_\psi$ to the algebra $\mathcal{A}$ as a premeasure, the only non-trivial claim that has to be checked is that if $A \in \mathcal{A}$ is described in two distinct ways as finite disjoint unions of sets Ω_e, then the sum of the p_e over these sets is the same for both descriptions. And this is not difficult, if one realizes that it is possible to compare both representations to a third one, in which all the sets Ω_e have all e in the same sufficiently large generation. The extension from a premeasure on $\mathcal{A}$ to a measure on $\mathcal{M}_\psi$ is an application of Carathéodory's Extension Theorem; see Sections 1 and 2 of Chapter 12 in [13], or Section 4 of Chapter 1 in [8].)

Before stating our theorem, we need to introduce a few more definitions, which will play a fundamental role in this paper. Given $\phi \in \mathcal{H}$ and $\omega \in \Omega$, we say that ϕ persists on ω if for each $e \in \omega$, $\text{Proj}_e\phi \neq 0$. Otherwise we say that ϕ terminates on ω. We set now

$$\Omega(\phi) = \{\omega \in \Omega : \phi \text{ persists on } \omega\}, \tag{2}$$

and

$$\Omega^c(\phi) = \Omega\backslash\Omega(\phi) = \{\omega \in \Omega : \phi \text{ terminates on } \omega\}. \tag{3}$$

Keep in mind that the choice of $\mathcal{P}$ is implicit in the definitions in the last two paragraphs. We omitted it from the notation, but should not forget that Ω, $I\!P_\psi$, $\Omega(\phi)$, etc, depend on this choice.

Theorem 1 *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{P}$ be a tree-structured set of projections on $\mathcal{H}$. For any $\psi \in \mathcal{H}\backslash\{0\}$ and $A \subset \Omega$, the following are equivalent.*

(1) $I\!P_\psi(A) = 0$.

(2.i) There exist $\phi_1, \phi_2, ...$ orthogonal to each other, such that $\psi = \sum \phi_i$ and $\Omega(\phi_i) \subset A^c$, for each i.

(2.ii) There exist $\zeta_1, \zeta_2, ...$ such that $\zeta_n \to \psi$ and $\Omega(\zeta_n) \subset A^c$, for each n.

Note that we are not, a priori, making any assumption of measurability on A. But if we assume that one of (2.i), (2.ii) is true, then we learn from the theorem that $A \in \mathcal{M}_\psi$ (and $I\!P_\psi(A) = 0$). On the other hand, assuming that (1) holds means assuming that $A \in \mathcal{M}_\psi$ (and $I\!P_\psi(A) = 0$).

The reader may choose to either first read the proof of Theorem 1, in Section 5, or continue reading the sections below in the order they are presented, with no loss of continuity. The first two propositions stated and proved in Section 5 will add mathematical structure to the content of Theorem 1, and allow it to be restated in a very compact form in display (10). These additional results from Section 5 are not used, or mentioned, in the discussion in the other sections of this paper. They are, nevertheless, being used in the developing project mentioned at the end of Section 3.

3 Relevance of the theorem

The theorem stated in the previous section holds for any Hilbert space $\mathcal{H}$ and any choice of tree-structured set of projections $\mathcal{P}$ on it. The arbitrariness of $\mathcal{P}$ should be kept in mind as our discussion returns to Physics. When we consider collapse quantum mechanics, there is a special choice of $\mathcal{P}$, namely the one described in the introduction: vertices (other than the root vertex and terminal ones) correspond to experiments and edges (other than the root edge) correspond to the possible outcomes in each experiment. In the case of collapse quantum mechanics, and with this choice of $\mathcal{P}$, Ω is the set of possible histories that could materialize from the collapses. And in the special case of Born-rule collapse quantum mechanics, statement (1) in Theorem 1 means that event A is (probabilistically) precluded from happening. In the case of non-collapse quantum mechanics there is in principle no special choice of $\mathcal{P}$. But, as the reader may have anticipated, for the purpose of comparing non-collapse to collapse quantum mechanics, via Theorem 1, it is natural to choose precisely the same $\mathcal{P}$. We will argue below that with this choice, the equivalence between statements (1) and (2.i) in the theorem implies (modulo plausible postulates on how the theories provide predictions) that quantum mechanics without collapse gives raise to the same predictions as Born-rule collapse quantum mechanics.

In collapse quantum mechanics only one history $\omega = (e_1, e_2, e_3, ...)$ materializes. In the Heisenberg picture that we are considering, the state of our system is initially some $\psi \in \mathcal{H}$, but it changes at each collapse, following the path $(\mathrm{Proj}_{e_1}\psi, \mathrm{Proj}_{e_2}\psi, \mathrm{Proj}_{e_3}\psi, ...) = (\psi, \mathrm{Proj}_{e_2}\psi, \mathrm{Proj}_{e_3}\psi, ...)$. In non-collapse quantum mechanics in the Heisenberg picture, ψ never changes. Everett [7] was the first to make the observation that this would still be compatible with our perception that collapses happen. As observers who are part of the system (otherwise we would not be able to interact with the experimental arrangement and observe it), the particles that form our bodies and in particular our brains must follow the same quantum mechanics that describes the rest of the system that we are observing. So that at the end of an experiment we can be described as being in a superposition of states, each one with a brain that encodes a different outcome for this experiment. All the possible outcomes materialize, and versions of the human observers, entangled to each possible experimental outcome, are included in this superposition.

The non-collapse view of quantum mechanics has the significant advantage of eliminating the mystery of collapse: How can systems behave differently when they are being "measured"? It yields a much simpler and consistent theory. One of the main hurdles that prevents its acceptance is probably psychological, as it affects substantially our sense of identity and of our reality. But other than this, probably the greatest obstacle to its acceptance is the issue addressed (once more) in this paper: even accepting Everett's observation that we will see collapses even if they do not happen, the question remains of why it is that we perceive them happening as if they were produced according to Born's probability rule. I will not discuss here the various previous approaches to this problem, and rather refer the reader to the recent papers [1], [16], references therein and papers in the collection [14] for background and recent ideas. Here I will argue briefly that Theorem 1 presents an answer.

A complete discussion of the relevance of Theorem 1 requires more knowledge of philosophy, and especially philosophy of probability, than I have. (See [9] for a nice introduction to the subject). Depending on the way in which one interprets probability theory, one will interpret collapse quantum mechanics differently. But I anticipate that even starting from different views on the meaning of the probabilities in collapse quantum mechanics, one will find value in Theorem 1. I will elaborate on this point twice later in this section.

I will focus my comments on what I derive from my way of understanding the role of probabilities in collapse quantum mechanics. Stating that collapses do happen according to Born's probabilities can only have meaning if we add some postulate telling us how this leads to predictions. I will argue based on the understanding that, in collapse quantum mechanics, the predictive power derived from the collapse axiom is fully contained in the following postulate.

Postulate 1 *In making predictions in collapse quantum mechanics, events of probability 0 can be deemed as sure not to happen.*

If it is accepted that this postulate covers the full predictive power of the probability axiom in collapse quantum mechanics, and in particular of Born's rule in Born-rule collapse quantum mechanics, then Theorem 1 tells us that non-collapse quantum mechanics will yield the same predictions as Born-rule collapse quantum mechanics, provided we accept the following postulate for non-collapse quantum mechanics.

Postulate 2 *In making predictions in non-collapse quantum mechanics, if the state of our system is a superposition of states all of which exclude a certain event, then this event can be deemed as sure not to happen.*

Before proceeding, we have to clarify exactly what we mean by "ϕ excludes A", implicit in Postulate 2. The definition $\Omega(\phi) \subset A^c$ (or equivalently, $A \subset \Omega^c(\phi)$) is, of course, what we have in mind. But $\Omega(\phi)$ has only been defined in terms of a given tree-structured set of projections, $\mathcal{P}$. If there are no collapses, what projections are we talking about? Any $\mathcal{P}$ gives raise to a definition of Ω and $\Omega(\phi)$ and hence gives meaning to $\Omega(\phi) \subset A^c$. In Postulate 2, $\mathcal{P}$ should be understood as arbitrary, i.e., Postulate 2 should be valid for any $\mathcal{P}$. But for (1), and hence (2.i), in Theorem 1 to be interesting, we should take $\mathcal{P}$ in a special way, which depends on our goals. (This is similar to the flexibility that one has, e.g., in taking different coordinate system to describe a classical mechanics problem; some are more relevant than others for a given problem. Theorem 1 is true for any $\mathcal{P}$, and it states that (1), (2.i) and (2.ii) are equivalent for each $\mathcal{P}$; all of them true or all of them false. To learn something interesting from this theorem, we must choose $\mathcal{P}$ in a judicious manner.) Our goal is to understand why in non-collapse quantum mechanics our experimental records show what we perceive as collapses being well modeled by Born's probability rule (and, in particular, not by some other probability rule, as we will discuss in the next section). For this purpose we should take for $\mathcal{P}$ precisely the one we described in the introduction and again in the first paragraph of this section: The vertices of the tree (other than the root vertex and the terminal vertices) in the definition of $\mathcal{P}$ should correspond to what we call experiments, and its edges (other than the root edge) should correspond to the possible outcomes in each experiment. Each edge $e \in \mathbf{E}$ is then associated to the subspace $\mathcal{H}_e$ (and corresponding projection Proj_e) in which there are versions of the observers with records of the various outcomes of experiments in the partial history that starts at the root edge and ends at e.

Suppose that we accept Everett's hypothesis that in our universe collapses do not happen, and that all the possible outcomes of each future experiment will materialize. And suppose also that we accept Postulate 2, as we make predictions about our possible futures (perhaps with the purpose of making decisions). Theorem 1, applied with the choice of $\mathcal{P}$ described above, implies that our predictions should be the same as if we assumed that we lived in a universe in which the collapses did happen in accordance with Born's rule, interpreted by Postulate 1. (In our non-collapse quantum mechanics universe, the probability measure P_ψ can be seen as a mathematical object that in principle has no physical significance. But the combination of Postulate 2 with Theorem 1 gives it physical meaning: the same physical meaning that it has in Born-rule collapse quantum mechanics, with Postulate 1 added to Kolmogorov's probability axioms as the physical interpretation of the word "probability" in the collapse axiom.) This means that for all practical purposes, we should continue to make predictions using the good old standard textbook Born-rule collapse quantum mechanics ... And the rest is philosophy, ... important, mind boggling and subject to being controversial. (Compute, ... but no need to shut up!)

I anticipate little objection to Postulate 2. After all, rejecting it would require an understanding of the role of superpositions of states in quantum mechanics that would be at odds with the notion that they cover all the possibilities (but see the last paragraph of this section). And I do not expect to see proposals suggesting that there is more to how predictions can be made in non-collapse quantum mechanics than contained in Postulate 2. But I do foresee that some will object to my understanding that collapse quantum mechanics derives its predictive power from Postulate 1. After all, the typical and interesting events of probability 0 require the performance of infinitely many experiments, something not accessible to us. This is a well known and well taken fundamental issue in philosophy of probability and statistics, often discussed and certainly contentious. I will provide, in the appendix, a brief discussion of this problem. This discussion will, by necessity, be based mostly on heuristic reasoning, rather than on solid logical/mathematical argumentation. Readers can either continue reading this section without loss of continuity, or first refer to the appendix.

As I said before, I anticipate that those who interpret the role of probabilities in collapse quantum mechanics differently will find alternative ways of applying Theorem 1, with the purpose of comparing non-collapse quantum mechanics with collapse quantum mechanics. For instance, if one could produce convincing arguments why Postulate 1 does not convey the full predictive power of collapse quantum mechanics, while Postulate 2 does convey the full predictive power of non-collapse quantum mechanics, then Theorem 1 would imply that, contrary to my thesis, collapse quantum mechanics has greater predictive power than non-collapse quantum mechanics: it makes additional predictions! A very interesting conclusion, that would enlighten our understanding of probabilities in a fundamental theory of physics, and should provide ways of distinguishing Born-rule collapse from non-collapse quantum mechanics empirically! A similar conclusion would follow if one could convincingly argue against Postulate 2.

It is, nevertheless, hard to imagine how one would be able to add predictions to collapse quantum mechanics, beyond what can be inferred (rigorously, or at least heuristically) from Postulate 1. This postulate predicts that events of probability 0 will not occur, and hence also that events of probability 1 will occur. But what prediction can be made about an event by knowing that it has a certain probability p strictly between 0 and 1? Saying that it will either occur or not occur is true, but not much of a prediction. It is true that we can predict that independent repetitions will produce a fraction converging to p of outcomes corresponding to that event, but this is a prediction made using the strong law of large numbers and Postulate 1. In the appendix, I present a heuristic argument using this last fact and symmetry, to argue that facing a one-shot outcome that has Born probability p of happening should, for all practical purposes, be seen as equivalent to facing another random experiment, in which a single ball is picked from a box containing a large number of balls, a fraction p of them being marked, and all balls being equally likely to be picked. This gives us heuristic intuition on what we are facing, and is of great value, e.g., in making decisions.

But it is not a prediction; and it is derived heuristically from Postulate 1, combined with a fundamental theorem in mathematical probability theory and symmetry considerations.

There are some additional considerations well worth pointing out to those who accept Postulates 1 and 2 (assumed to be true in this and the next two paragraphs), but perhaps not that the former exhausts the predictive power of Born's rule. Let's suppose that our universe is ruled by non-collapse quantum mechanics, and its state in the Heisenberg picture is ψ. Let's now consider a set S of hypothesis tests aimed at testing the hypothesis H that "the (illusory) collapses that we perceive are well described by Born's rule (as opposed to some other probability distribution, or no probability distribution at all)". And let's consider the set of histories, A, defined as those "along which we continue forever collecting data to test the hypothesis H by means of the tests in the set S, and the aggregate of our experimental results will never finally stop rejecting H. (This means that for any time t there is a later time $t' > t$ at which the tests applied to the data collected to that time, t', reject H. The technical expression for this is that H is rejected by the data applied to S infinitely often)". What assumptions should we impose on S so that it can be regarded as a well devised set of tests? One common assumption is consistency, in the sense that the probability of rejecting a true hypothesis should vanish as the amount of data goes to infinity. This is the case with typical tests that we currently use. But this condition implies (1) in Theorem 1: $I\!P_\psi(A) = 0$. And now the theorem tells us that (2.i) is also satisfied. Postulate 2 then implies that A will not happen in our universe! In conclusion, assuming that we live in a non-collapse quantum mechanics universe, statistical tests based on the outcomes of the experiments in which we perceive collapses, and aimed at showing that they are not described by Born's probabilities could only be successful if they violated the assumption of consistency (as defined above). And this means that if these tests were used in a universe ruled by Born-rule collapse quantum mechanics, there would be a positive probability that they would never finally stop rejecting the hypothesis that the (real) collapses that are perceived there are well described by Born's rule, despite experiments to test this hypothesis being continued forever!

Things get even more interesting if in the discussion above we require the set of tests S, regardless of consistency, to satisfy another reasonable assumption, that I will call the tail property. Informally this condition simply expresses the idea that any finite amount of data becomes irrelevant if we later collect a much larger amount of data. Formally we require that for any time the set of tests be equivalent to a set of tests that only uses the data collected after that time. (Recall that we are assuming in the definition of the event A that we keep collecting data, so that the data collected before any given time will be negligible in face of additional unlimited data.) Under this assumption, A is what is called a tail event, and the well known Kolmogorov 0-1 law applies, telling us that $I\!P_\psi(A) = 0$, or $I\!P_\psi(A) = 1$. (See, e.g., Theorem 3.12 of [4], and note that under the probability law $I\!P_\psi$ the outcomes of experiments are independent random variables.) Now Theorem 1 implies that there is a dichotomy: Either (1) and (2.i) of the theorem hold, or else their analogues hold for A^c. And if we accept Postulates 1 and 2, we must conclude that, if we continue collecting data forever, the set of tests S will never finally stop rejecting the hypothesis H in our non-collapse quantum mechanics universe if and only if it never finally stops rejecting H in a Born-rule collapse quantum mechanics universe. Equivalently, if we continue collecting data forever, the set of tests S will eventually stop rejecting the hypothesis H in our non-collapse quantum mechanics universe if and only if it eventually stops rejecting H in a Born-rule collapse quantum mechanics universe!

Combining the conclusions in the last two paragraphs (where we are assuming Postulates 1 and 2 true): In a non-collapse quantum mechanics universe, a set of tests that are consistent (in the sense defined above) and also have the tail property, will eventually stop (forever) rejecting the hypothesis H that the (illusory) collapses that we perceive are well described by Born's rule, assuming that data continues to be collected forever to be used in these tests!

The discussion so far has relied on the equivalence between (1) and (2.i) in Theorem 1. There is an additional comment that can be made based on the equivalence between (1) and (2.ii), and that may perhaps appeal especially to readers with a Bayesian view of probabilities. Let's suppose that our universe is ruled by non-collapse quantum mechanics. And suppose that, as observers inside the system, we want to infer an approximation $\widehat{\psi}$ for the actual state ψ (in the Heisenberg picture), from all our available data. Suppose also that A is an event that satisfies statement (1) of Theorem 1. Then the theorem tells us that (2.ii) is also true. And (2.ii) implies that, because we have limits in precision, we will not be able to distinguish any proposed $\widehat{\psi}$ from others that excludes A from happening. An interesting special choice of A to which this reasoning applies is the one discussed in the last three paragraphs.

There is a different philosophical issue that should not be avoided, but that is very slippery. How about "physical reality"? One possible reaction to the question is to regard it as irrelevant, as the discussion above can be taken in a purely operational fashion that avoids such metaphysical issues. And perhaps that is all that can be said and done on the issue. But I will add here a proposal, leaving it open for further analysis. In collapse quantum mechanics, reality is usually associated to the single history that materializes, while in non-collapse quantum mechanics one usually sees all the histories as equally real. But this may be a misinterpretation. Non-collapse quantum mechanics asserts that in each experiment all the possible outcomes materialize, and this should imply that every finite partial history should materialize. But it does not assert that every infinite history materializes. Can one build a theory of physical reality along these lines, in which all finite partial histories are real, but typically not the infinite ones, which is logically sound, philosophically satisfying and compatible with Postulate 2? In work in progress, I am aiming at answering this question in the affirmative.

4 A remark on the non-necessity of Born's rule in collapse quantum mechanics

In the last section we saw that in quantum mechanics without collapse, if we accept Postulate 2, then Theorem 1 tells us that we (internal observers in the system) should expect to observe experimental outcomes that are fully compatible with Born-rule collapse quantum mechanics, interpreted through Postulate 1. In other words, in non-collapse quantum mechanics, Born's rule, rather than "complete chaos", or some other probability distribution emerges from the theory. Or still more succinctly: Non-collapse quantum mechanics forces Born's rule.

This is not the case in collapse quantum mechanics, for a very simple reason that becomes apparent when we consider the tree-structured set of projections associated to the collapse events. The Born rule, introduced by (1), can be modified into a different one. For instance, it is natural to consider the class of probability distributions given informally by imagining again a walker that moves on our tree. As before, the walker starts at the root vertex of the tree and then moves in the direction of the orientation, deciding at each vertex where to go in a probabilistic fashion, with edges chosen with probability proportional to norm to some power a, i.e., when at a vertex that separates a parent e from its children, the walker chooses child e' with probability proportional to $||\mathrm{Proj}_{e'}\psi||^a$, if $||\mathrm{Proj}_{e'}\psi|| \neq 0$, and 0 otherwise, independently of past choices. If ever at a terminal vertex, the walker stops. As with (1), a simple inductive computation shows that this is equivalent to the statement

$$I\!P_{a,\psi}(\Omega_e) \;=\; p_{a,\psi,e}, \tag{4}$$

for appropriate non-negative numbers $p_{a,\psi,e}$ that satisfy

1. If e is the root edge, then $p_{a,\psi,e} = 1$.

2. If $e_1, ..., e_n$ are the children of e, then $\sum_{i=1}^{n} p_{a,\psi,e_i} = p_{a,\psi,e}$.

When $a \neq 2$, replacing (1) with (4) produces, for each $\psi \in \mathcal{H}$, a probability measure $I\!P_{a,\psi}$ on $(\Omega, \mathcal{B})$ that replaces $I\!P_\psi = I\!P_{2,\psi}$. (Technical observation: If we accept Postulate 1 as the source of predictions in collapse quantum mechanics, then $(I\!P_{a',\psi})$-rule collapse quantum mechanics will be a distinct theory from $(I\!P_{a'',\psi})$-rule collapse quantum mechanics if and only if there is some set $A \in \mathcal{B}$ such that of $I\!P_{a',\psi}(A)$ and $I\!P_{a'',\psi}(A)$ one equals 0 and the other does not. By the strong law of large numbers, this condition is certainly satisfied for any $a' \neq a''$, if our tree structured set of projections includes representations of infinite sequences of the same appropriate experiment.)

The concept of a universe in which reality corresponds to a single history chosen at random according to $I\!P_{a,\psi}$ (where as before, ψ is the initial state) is as mathematically sound for any value of a as it is for the especial value 2. Such a universe is ruled by the same laws of collapse quantum mechanics that we read in our textbooks, except for the different probabilities that replace Born's rule. For instance, in a $(I\!P_{0,\psi})$-rule collapse quantum mechanics universe, in each collapse, each of the possible outcomes is equally likely. If we prepare a spin 1/2 particle in a state in which the spin points in a direction that makes an angle $\theta \neq 0$ with the x-axis, and then measure the spin along the x-axis, the two possible outcomes will be equally likely, regardless of the value of θ. If we repeat this experiment many times, we should predict roughly half of the outcomes to be $(+)$ and roughly half to be $(-)$. In this universe, intelligent beings would still be puzzled by the collapses, and one of them could perhaps have also proposed, following the same rational as Everett, that collapses do not happen. But then the title of the current paper would have been "A theorem with the purpose of showing that non-collapse quantum mechanics is incompatible with our experimental records".

There are three conclusions that can be drawn from this discussion, as alluded to in the abstract.

First, while Born's rule has a special status in non-collapse quantum mechanics if we accept Postulate 2 (thanks to Theorem 1, as explained in the last section), this is not the case in collapse quantum mechanics. Suppose that in the future, as we accumulate data, we decide that in actuality there is reason for rejecting Born's rule, and that an empirical power $a = 2.1 \pm 0.05$ better describes the data. This would then be seen as evidence to reject the hypothesis that we live in a universe without quantum collapses (or alternatively, reject Postulate 2). In other words, as much as Born's rule is falsifiable (a delicate issue, certainly, as it is a probabilistic statement), Everett's non-collapse hypothesis is falsifiable. And the non-falsification of Born's rule in our current massive data can therefore be seen as giving some support to Everett's hypothesis. If collapses do not happen in our universe, Born's rule is explained, invoking Theorem 1 and Postulate 2. But if collapses happen in our universe, then why do we have them according to Born's rule and not, e.g., according to some other $I\!\!P_{a,\psi}$, as given by (4)?

Second, when proposing derivations of Born's rule from a set of assumptions, one should be sure that it is not the case that these assumptions hold in a collapse-quantum-mechanics universe with collapse probabilities distinct from Born's rule. I add this comment, as there are published papers that seem to provide invalid derivations of Born's rule (in some cases later pointed out in papers by others), while this simple test would have avoided the mistake. (I refrain from giving examples, as my list is probably incomplete and this is not a central issue in this paper.)

Third, also arguments presented to explain why in non-collapse quantum mechanics Born's rule emerges should be subject to such a scrutiny. By this I mean that one should be sure that the arguments given apply to a non-collapse quantum mechanics universe, but fail in a non-Born-rule collapse quantum mechanics universe.

5 Proof of the theorem

The orientation that was introduced on the tree $(\mathbf{V}, \mathbf{E})$ induces a partial order on the set of edges: for any two edges we write $e' \leq e''$ if there is a partial history that starts with e' and ends with e''. We write $e' < e''$ if $e' \leq e''$ and $e' \neq e''$. If neither $e' \leq e''$, nor $e'' \leq e'$, then we say that e' and e'' are not comparable.

Definition 1 has some simple consequences. If e'' is a child of e', then $\mathcal{H}_{e''} \subset \mathcal{H}_{e'}$. By induction along a partial history line, this extends to:

$$\text{If } e' \leq e'', \text{ then } \mathcal{H}_{e''} \subset \mathcal{H}_{e'}. \tag{5}$$

In contrast, if e' and e'' are siblings, then $\mathcal{H}_{e'} \perp \mathcal{H}_{e''}$. By induction along partial history lines, this extends to:

$$\text{If } e' \text{ and } e'' \text{ are not comparable, then } \mathcal{H}_{e'} \perp \mathcal{H}_{e''}. \tag{6}$$

Proposition 1 *For every $\phi_1, \phi_2 \in \mathcal{H}$, $\Omega(\phi_1+\phi_2) \subset \Omega(\phi_1) \cup \Omega(\phi_2)$, or equivalently $\Omega^c(\phi_1) \cap \Omega^c(\phi_2) \subset \Omega^c(\phi_1 + \phi_2)$.*

Proof: Suppose $\omega \in \Omega^c(\phi_1) \cap \Omega^c(\phi_2)$. Then there are $e_1, e_2 \in \omega$ such that $\mathrm{Proj}_{e_1}\phi_1 = \mathrm{Proj}_{e_2}\phi_2 = 0$. As ω is a history, e_1 and e_2 are comparable. Let e be the larger between e_1 and e_2. Also because ω is a history, for $i = 1, 2$ we have now, from (5), $\mathcal{H}_e \subset \mathcal{H}_{e_i}$ and hence $\mathrm{Proj}_e\phi_i = 0$. Therefore $\mathrm{Proj}_e(\phi_1 + \phi_2) = \mathrm{Proj}_e\phi_1 + \mathrm{Proj}_e\phi_2 = 0$, which means that $\omega \in \Omega^c(\phi_1 + \phi_2)$. $\square$

For each $A \subset \Omega$ we define the following two sets (T stands for "truth" and F for "falsehood"):

$$T(A) = \{\phi \in \mathcal{H} : \Omega(\phi) \subset A\}, \tag{7}$$

and

$$F(A) = T(A^c) = \{\phi \in \mathcal{H} : \Omega(\phi) \subset A^c\} = \{\phi \in \mathcal{H} : A \subset \Omega^c(\phi)\}. \tag{8}$$

Proposition 2 *For every $A \subset \Omega$, $T(A)$ and $F(A)$ are vector spaces.*

Proof: Since $F(A) = T(A^c)$, it suffices to prove the statement for $T(A)$. Suppose $\phi_1, \phi_2 \in T(A)$, a_1, a_2 scalars. Then, for $i = 1, 2$, $\Omega(a_i\phi_i) = \Omega(\phi_i)$, if $a_i \neq 0$, and $\Omega(a_i\phi_i) = \emptyset$, if $a_i = 0$. In any case $\Omega(a_i\phi_i) \subset \Omega(\phi_i) \subset A$. From Proposition 1 we obtain $\Omega(a_1\phi_1 + a_2\phi_2) \subset \Omega(a_1\phi_1) \cup \Omega(a_2\phi_2) \subset A$, which means $a_1\phi_1 + a_2\phi_2 \in T(A)$. $\square$

We can rephrase Statement (2.ii) in Theorem 1 as

$$\psi \in \overline{F(A)}, \tag{9}$$

where the bar denotes topological closure in the Hilbert space $\mathcal{H}$.

The equivalence of (2.ii) and the apparently stronger statement (2.i) in Theorem 1 can be obtained, in a standard fashion, by applying the Gram-Schmidt orthonormalization procedure (see p.46 of [12], or p.167 of [8]) to the vectors $\zeta_1, \zeta_2 - \zeta_1, \zeta_3 - \zeta_2, ...$ to produce an orthonormal system with the same span. Proposition 2 assures us that the orthonormal system will be contained in $F(A)$, since the ζ_i are. The vectors $\phi_1, \phi_2, \phi_3, ...$ are then obtained by expanding ψ in this orthonormal system.

The proof of Theorem 1 is now reduced to showing that for any $\psi \in \mathcal{H}\backslash\{0\}$ and $A \subset \Omega$,

$$I\!P_\psi(A) = 0 \iff \psi \in \overline{F(A)}. \tag{10}$$

The class $\mathcal{A}_\sigma$ of subsets of Ω obtained by countable unions of elements of $\mathcal{A}$ will play a major role in the proof of (10). Every $A \in \mathcal{A}_\sigma$ is a union of sets in the countable class $\{\Omega_e\colon e \in \mathbf{E}\}$. But since $\Omega_{e''} \subset \Omega_{e'}$, whenever $e' \le e''$, we will avoid redundancies in this union by writing it as

$$A = \bigcup_{e\in\mathbf{E}(A)} \Omega_e, \tag{11}$$

where

$$\mathbf{E}(A) = \{e \in \mathbf{E} : \Omega_e \subset A \text{ and there is no } e' \in \mathbf{E} \text{ such that } e' < e \text{ and } \Omega_{e'} \subset A\}. \tag{12}$$

Any two distinct elements of $\mathbf{E}(A)$ are not comparable. And since $\Omega_{e'} \cap \Omega_{e''} = \emptyset$, whenever e' and e'' are not comparable, (11) is a disjoint union. Moreover, using (6) we see that $\{\mathcal{H}_e : e \in \mathbf{E}(A)\}$ is a countable collection of orthogonal subspaces of $\mathcal{H}$. We will associate to A their direct sum (the topological closure of the linear span of vectors in these $\mathcal{H}_e$), which we denote by

$$\mathcal{H}(A) = \bigoplus_{e\in\mathbf{E}(A)} \mathcal{H}_e. \tag{13}$$

If $\mathcal{S}$ is a subspace of $\mathcal{H}$ and $\phi \in \mathcal{H}$, we will use the notation $\mathrm{Proj}(\phi|\mathcal{S})$ to denote the projection of ϕ on $\mathcal{S}$. For instance $\mathrm{Proj}(\phi|\mathcal{H}_e) = \mathrm{Proj}_e\phi$.

Lemma 1 *For any $\phi \in \mathcal{H}$ and $A \in \mathcal{A}_\sigma$,*

(i) For any $e \in \mathbf{E}$, $\mathit{Proj}_e\phi = 0 \iff \Omega_e \subset \Omega^c(\phi)$.

(ii) $\mathcal{H}^\perp(A) = F(A)$.

(iii) $\Omega^c(\phi) \in \mathcal{A}_\sigma$.

(iv) $\phi \in \mathcal{H}^\perp(\Omega^c(\phi))$.

(v) $||\mathit{Proj}(\phi|\mathcal{H}(A))||^2 = ||\phi||^2\, I\!P_\phi(A)$, if $\phi \neq 0$.

Proof: (i) The implication ($\Longrightarrow$) is clear. To prove ($\Longleftarrow$) suppose that $\mathrm{Proj}_e\phi \neq 0$. Then either e is a terminal edge, or it has a child e' with $\mathrm{Proj}_{e'}\phi \neq 0$. Repeating inductively this reasoning, we produce a history ω such that $e \in \omega$ and ϕ persists on ω. Hence $\Omega_e \not\subset \Omega^c(\phi)$.

(ii)

$$\mathcal{H}^\perp(A) = \bigcap_{e\in\mathbf{E}(A)} \mathcal{H}_e^\perp = \bigcap_{e\in\mathbf{E}(A)} \{\phi \in \mathcal{H} : \Omega_e \subset \Omega^c(\phi)\} = \{\phi \in \mathcal{H} : A \subset \Omega^c(\phi)\} = F(A),$$

where in the first equality we used the definition (13) of $\mathcal{H}(A)$, in the second equality we used part (i) of the lemma, in the third equality we used (11), and in the fourth equality we used (8)

(iii) $\Omega^c(\phi) = \cup\{\Omega_e : e \in \mathbf{E}, \text{Proj}_e\phi = 0\}$. And this set belongs to $\mathcal{A}_\sigma$, since this union is countable.

(iv) Thanks to part (iii) of the lemma, we can take $A = \Omega^c(\phi)$ in part (ii) of the lemma. Using then (8), we obtain

$$\mathcal{H}^\perp(\Omega^c(\phi)) = F(\Omega^c(\phi)) = \{\phi' \in \mathcal{H} : \Omega^c(\phi) \subset \Omega^c(\phi')\} \ni \phi.$$

(v)

$$\begin{aligned} ||\text{Proj}(\phi|\mathcal{H}(A))||^2 &= \sum_{e\in\mathbf{E}(A)} ||\text{Proj}_e(\phi)||^2 = \sum_{e\in\mathbf{E}(A)} ||\phi||^2 I\!P_\phi(\Omega_e) \\ &= ||\phi||^2 I\!P_\phi(\cup_{e\in\mathbf{E}(A)}\Omega_e) \;=\; ||\phi||^2 I\!P_\phi(A), \end{aligned}$$

where in the first equality we used the definition (13) of $\mathcal{H}(A)$, in the second equality we used (1), in the third equality we used the disjointness of the sets involved, and in the fourth equality we used (11). □

We will use some consequences of Carathéodory's theorem that extends the measure $I\!P_\psi$ from $\mathcal{A}$ to $\mathcal{M}_\psi$ (see Sections 1 and 2 of Chapter 12 in [13], or Section 4 of Chapter 1 in [8]). Given $\psi \in \mathcal{H}$, define the outer measure of any set $A \subset \Omega$ by

$$I\!P^*_\psi(A) = \inf\{I\!P_\psi(A') : A' \in \mathcal{A}_\sigma,\ A \subset A'\}. \tag{14}$$

Then it follows from Carathéodory's Extension Theorem that $I\!P^*_\psi(A) = I\!P_\psi(A)$ for every $A \in \mathcal{M}_\psi$, in particular for every $A \in \mathcal{B}$ and therefore for every $A \in \mathcal{A}_\sigma$. It also follows that $I\!P^*_\psi(A) = 0$ implies $A \in \mathcal{M}_\psi$ and is necessary and sufficient for $I\!P_\psi(A) = 0$.

The next two lemmas prove each one of the directions of the equivalence (10), completing the proof of Theorem 1.

Lemma 2 *For any $\psi \in \mathcal{H}\backslash\{0\}$ and $A \subset \Omega$,*

$$\psi \in \overline{F(A)} \implies I\!P_\psi(A) = 0.$$

Proof: If $\psi \in \overline{F(A)}$, there are $\zeta_n \in F(A)$ such that $\zeta_n \to \psi$. Set $B_n = \Omega^c(\zeta_n)$. From (8) and Lemma 1(iii) we have $A \subset B_n \in \mathcal{A}_\sigma$. Using (14) and Lemma 1(v), we obtain

$$0 \;\le\; I\!P^*_\psi(A) \;\le\; I\!P_\psi(B_n) \;=\; \frac{||\text{Proj}(\psi|\mathcal{H}(B_n))||^2}{||\psi||^2}.$$

But since Lemma 1(iv) tells us that $\zeta_n \in \mathcal{H}^\perp(B_n)$, we can write

$$\begin{aligned} ||\mathrm{Proj}(\psi|\mathcal{H}(B_n))||^2 &= ||\mathrm{Proj}(\psi - \zeta_n|\mathcal{H}(B_n)) + \mathrm{Proj}(\zeta_n|\mathcal{H}(B_n))||^2 \\ &= ||\mathrm{Proj}(\psi - \zeta_n|\mathcal{H}(B_n))||^2 \le ||\psi - \zeta_n||^2. \end{aligned}$$

Since n is arbitrary, the two displays combined give

$$0 \le I\!P^*_\psi(A) \le \lim_{n\to\infty} \frac{||\psi - \zeta_n||^2}{||\psi||^2} = 0,$$

proving that $I\!P^*_\psi(A) = 0$ and hence $A \in \mathcal{M}_\psi$ and $I\!P_\psi(A) = 0$. $\Box$

Lemma 3 *For any $\psi \in \mathcal{H}\backslash\{0\}$ and $A \subset \Omega$,*

$$I\!P_\psi(A) = 0 \implies \psi \in \overline{F(A)}.$$

Proof: If $I\!P_\psi(A) = 0$, (14) tells us that there are $A_n \in \mathcal{A}_\sigma$ such that $A \subset A_n$ and $I\!P_\psi(A_n) \to 0$. Set $\xi_n = \mathrm{Proj}(\psi|\mathcal{H}^\perp(A_n))$. Then $\xi_n \in \mathcal{H}^\perp(A_n) = F(A_n) \subset F(A)$, where the equality is Lemma 1(ii), and in the last step we are using (8). Therefore, using Lemma 1(v), we obtain

$$||\xi_n - \psi||^2 = ||\mathrm{Proj}(\psi|\mathcal{H}(A_n))||^2 = ||\psi||^2\, I\!P_\psi(A_n) \to 0,$$

as $n \to \infty$. This shows that (ξ_n) is a sequence in $F(A)$ that converges to ψ and therefore $\psi \in \overline{F(A)}$. $\Box$

6 Appendix: A digression on the meaning of probabilities in collapse quantum mechanics

This appendix deals briefly with the question whether Postulate 1 can convey the full predictive power of probabilities in collapse quantum mechanics. The well known issue, as was pointed out in Section 3, is that the interesting events of probability 0 require infinitely many repetitions of experiments.

A operational implementation of Postulate 1 is achieved by adopting some "approximation to 0", called, in some contexts, the "significance level". For instance, it is current practice in Physics to accept a major discovery, when the data implies an estimated probability of less than 10^{-5} for what was observed, if the proposed discovery were false. In making certain kinds of decisions, it is common practice to set a "level of safety" that depends on the gravity of the consequences, so that events that have probabilities below that level are in practice deemed as "almost-sure not to happen", and we can feel safe.

One way in which one can try to make sense of the use of such "approximations to 0", is by a symmetry argument. Suppose that we observed the outcome of an experiment (or a series of experiments), and let's suppose that what we observed had a very small but positive predicted probability p. How can we explain/justify that we are surprised, and possibly suspicious of the hypothesis that led to the computation of p? After all we are accepting as our single input from probability that events of probability 0 are excluded. Our observation is not in contradiction with this postulate! But now, if we imagine repeating this experiment (or series of experiments) again and again, forever, our postulate, combined with the strong law of large numbers, tells us that only a fraction p of these repetitions will result in what we saw. Symmetry suggests to us that there is nothing special about the first experiment in this imagined series (the only one we actually performed). And this seems to justify the surprise and the suspicion. We reduced our reasoning to a "classical probability" issue, meaning a situation in which there is symmetry among possible outcomes, and surprise/suspicion seems to be warranted if we observe an outcome that is rare in the population of possible outcomes. We have not violated our assertion that Postulate 1 conveys the full meaning of the probabilities in the collapse axiom of collapse quantum mechanics. The "classical probability" notion that we used (when there is symmetry among possible outcomes) is meaningful regardless of and beyond quantum mechanics, and is not the concept of probability that appears in the axioms of collapse quantum mechanics. What we did was to show that Postulate 1 combined with symmetry arguments and the strong law of large numbers makes our puzzle (why the surprise/suspicion?) amenable to a "classical probability" heuristic argumentation and solution. For instance, in deciding "how small is small" (as applied to the level of significance, or the safety level), we can use our feelings about the answer to this question as if the random experiment that we are facing were the one in which there is a box with a large number of balls, a fraction p of them marked, and one will be picked, under assumption of symmetry among all the balls as to which one will be picked. (There is a remaining issue of how to lump together possible outcomes when setting up hypothesis tests. But this is not the issue that concerns us here, as we are discussing only the power of Postulate 1 in giving the full meaning to probabilities in collapse quantum mechanics.)

The heuristics above can also be used when p is not necessarily small. It tells us that when facing a quantum experiment in which there are n possible outcomes, with Born probabilities $p_1, ..., p_n$, if we accept Postulate 1 and the argumentation based on the strong law of large numbers and symmetry, then we are in a situation that is equivalent to a "classical probability" setting, in which we have a very large population of balls, a fraction p_i of which is marked "i", $i = 1, ..., n$, and one ball will be chosen at random. In this way we are able to extract heuristic meaning from Born's probabilities in collapse quantum mechanics, even in one-shot situations. And these can then be applied, for instance, in decision problems, in which one wants to maximize some expected utility.

There is an irony here. We are so used to applying probabilities in science and elsewhere, and the mathematical theory of probability based on Kolmogorov's axioms is so powerful and beautiful, that we often forget how delicate this concept is philosophically. And we forget that collapse quantum mechanics requires us to be clear about it. No one will doubt that Postulate 1 is true in collapse quantum mechanics. But there is an intuitive feeling that there is more to probabilities than what is contained in Postulate 1. In other words, that Postulate 1 does not exhaust the power of probabilities in collapse quantum mechanics. A criticism of the discussion above and that in Section 3, along these lines, would need to be clear about what additional predictions collapse quantum mechanics is supposed to make, beyond those contained in Postulate 1.

There is no reason why the concept of probabilities, when applied in real life, must have a single meaning. This observation goes by the name of "pluralistic view of probabilities" (see Chapters 8 and 9 of [9]) and seems to me to be well supported by the variety of ways we use the notion of probabilities. For instance, in making a decision in a one-shot situation, it is natural (and well supported by axiomatic mathematical decision theory; see, e.g., [19], or Chapter 3 of [10]) to attribute subjective probabilities (in addition to utilities) to each possible outcome. In the scheme proposed by Savage [15], for instance, these probabilities emerge from the axioms, and do not have to be related to any "objective probabilities". Such subjective probabilities are of great use, but it is not obvious that they must conceptually, or numerically be the same as the probabilities that appear in the axioms of collapse quantum mechanics, when the decision involves the outcome of a quantum experiment. It seems to me that the intuitive feeling mentioned in the last paragraph, that there is more information in collapse probabilities than contained in the impossibility of events of probability 0, results from confusion between different concepts of probability, which concern different realms of application.

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