A Note on Percolation in a Voronoi Competition–Growth Model

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Abstract

We study a model in which two entities (e.g., plant species, political ideas, ...) compete for space on a plane, starting from randomly distributed seeds, and growing deterministically at possibly different rates. An entity which forms an infinite cluster is considered to dominate over the other (which then cannot percolate). We analyze the occurence of such a form of domination in situations in which one entity starts from a much larger density of seeds than the other one, but the latter one grows at a much faster rate than the former one. The model studied here generalizes the problem of Voronoi percolation.

1 Introduction

Suppose that initially the seeds of two plant species A and B are randomly distributed on the plane and that the region occupied by each species starts to grow radially with constant speed v_A and v_B for the species A and B, respectively, starting from their seeds, so that after a relatively short time interval s, each seed A, or B, will be the center of a ball of radius sv_A, or sv_B, respectively. The growth of each species into empty land is

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normal to the tangent to the boundary of the occupied region, and we suppose also that land already occupied by one species cannot be invaded by the other one. We can think that a species which percolates, i.e., forms an infinite cluster, dominates the other one (from [Z] we know that at most one of the species can percolate). It is natural to ask whether one of them will ultimately percolate, and under which conditions.

Motivated in part by this fairly caricatural biological picture, we will consider in this note a site percolation problem on a family of random partitions of $\mathbb{R}^2$. The heuristic definition given above, with some additional notation, suffices to understand the statement of our result (the proposition below). The rigorous definition of the model is given in the next section together with the proof of the result.

Denote by $\mathcal{P}$ the set of initial points that we assume chosen according to a Poisson point process of rate one. Declare each point of type A (resp. B) with probability α (resp. $1-\alpha$), independently. Around each seed $x \in \mathcal{P}$ a cell starts to grow and compete for space as described above.

This defines a time dependent partition of the plane in three kinds of regions: type A, type B and empty regions. Let us denote by A_t the union of all cells of type A at time $t > 0$ and similarly for B_t. Write also $E_t = \mathbb{R}^2 \setminus (A_t \cup B_t)$ for the empty region at time t. Thus the model on which we discuss percolation questions is a dynamical system $\{A_t, B_t, E_t\}_{t \geq 0}$ with randomness only present in the initial state given by the positions of seeds $\mathcal{P}$.

In any finite region the growth eventually stops and a final partition is obtained into type A and B cells. Note that the time to reach this final partition for a given finite region $\Lambda \subset \mathbb{R}^2$ is random (depends on $\mathcal{P}$). The final partition in $\mathbb{R}^2$ is the one that agrees in any finite region with the final partition there.

Let $r = v_A/v_B$ be the ratio of the two velocities. The occurrence of A or B percolation in the final random partition depends on the relationship between the initial densities of the species, α, and their growth rate ratio r. Without loss of generality we may take $v_A = 1$ and $v_B = r^{-1}$, and to emphasize the symmetry between A's and B's we will sometimes use the parameter $R = r/(1+r) \in [0,1]$.

If $v_A = v_B$, the corresponding final partition is known as Poisson Voronoi Tesselation of the plane ([M]) which has been studied in many different contexts. It is the only case in which the cells are simple to describe. The cell corresponding to $x \in \mathcal{P}$ is given by

$$C(x|\mathcal{P}) = \{y \in \mathbb{R}^2 \colon \|x-y\| \leq \|x'-y\| \text{ for all } x' \in \mathcal{P}\}. \tag{1}$$

For this case the percolation problem was posed in [GG] and [BS1]. The percolation model here is self-dual, in that a square must either be crossed by a line of A's in the horizontal direction or else by a line of B's in the vertical direction; the two possibilities being equally likely when $\alpha = 1/2$. This leads to the conjecture that B percolates for $\alpha < 1/2$, and A percolates for $\alpha > 1/2$. (If a proper version of the RSW lemma from discrete percolation theory (see, e.g., Theorem 9.70 in [G]) as well as a version of the Menshikov-Aizenman-Barsky result on sharpness of the phase transition (see, e.g., Chapter 3 in [G]) were available here, one would be able to prove it.) Currently it is only known that for some $1/2 \leq \alpha_c < 1$, A percolates when $\alpha > \alpha_c$ and B percolates when $\alpha < 1 - \alpha_c$ ([Z]).

The basic percolation problem in our more general setting involves the analysis of the "phase diagram" in terms of these two parameters, that is, characterizing the regions of values of α and R for which one has A percolation, B percolation, and the region on which no percolation occurs. The same kind of heuristics that is used for the usual Voronoi percolation suggests the conjecture that there is a single *critical line* in this diagram separating two regions, one on which A percolates, while on the other B percolates.

Of course, A percolates if both parameters favor strongly enough this situation, that is, for α and R both close to 1, and similarly for the B percolation when α and R are both close to 0.

A more interesting region is where α and R compete with each other: α near 1 and R small or vice versa. This is the region we analyze here and obtain bounds for the *critical line* (or *critical region* if that is the case).

Heuristically, to decide which species percolates, one can compare the typical time needed for A's to close the gap between their adjacent nuclei and for B's to do the same. Such times are of the order of $\alpha^{-1/2}/(2v_A)$ and $(1-\alpha)^{-1/2}/(2v_B)$, respectively. Equating these expressions one obtains the heuristic critical line

$$\frac{1-\alpha}{\alpha} = r^2 \ .$$

Such a naive computation, based on averages, neglects the fluctuations in the distributions of nuclei and the interference between the growth of the two species, and can therefore be seen as a sort of "mean field" approximation. Nevertheless the proposition below shows that there is some truth to this heuristics when r is close to zero. By switching A's and B's one can find analogous statements for r very large. Figure 1 shows solid lines corresponding to the rigorous bounds obtained in the proposition and a dashed line corresponding to the critical line of this "mean field" approximation.

Proposition *There exist positive finite constants c and C, $c \leq C$, such that for small enough r*

(1) A percolates if $1-\alpha < c\, r^2$*, and*

(2) B percolates if $1-\alpha > C\, r^2$*.*

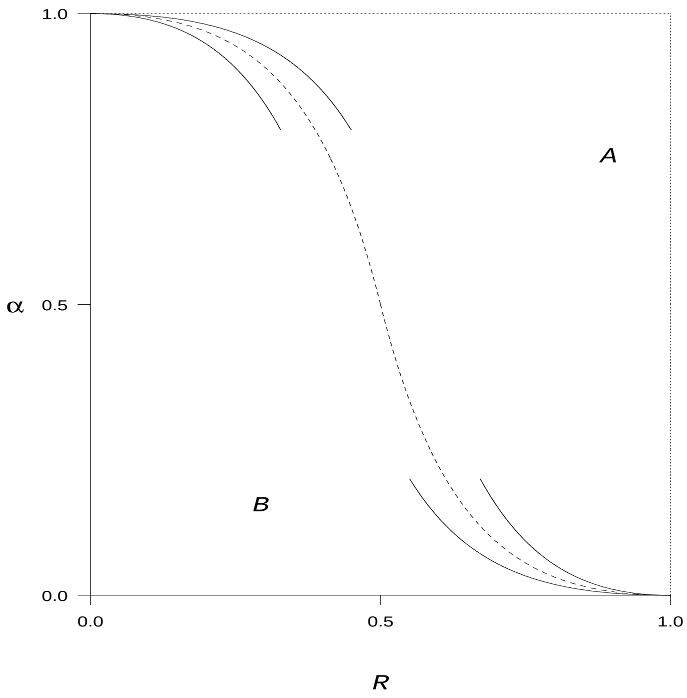

Figure 1: Phase diagram (R, α).

It seems to us that the model discussed here, referred to as *Voronoi Competition-Growth model*, is both natural and interesting in itself and incorporates nicely the balance between growth and competition for space. Besides the open questions already discussed about the whole phase diagram of this model in its final state ($t \to \infty$) one could also ask about the time dependence of its properties. An interesting open problem is the analysis of the behavior of the (deterministic) time needed for the occurrence of percolation.

A related system is defined in [M] in which different growth velocities are obtained by changing the Euclidean distance $\|x-y\|$ in (1) by $\|x-y\|/v(x)$, where $v(x)$ is the isotropic growth velocity of the cell around seed $x \in \mathcal{P}$. This leads to a quite different competition-growth system as a fast-growing seed can dominate new, previously empty, territory by "flying over" an area already dominated by slow-growing seeds and the resulting cells can be disconnected. Another system presented in [M] is the Johnson and Mehl model in which the growth velocities are the same for all seeds but they start to grow at random times.

We also mention a recent algorithm to obtain Voronoi partitions using the so called *wavefront approach* ([DK]) which also adopts a dynamical point of view. There, a line sweeps across the plane, say with constant velocity, leaving behind "the best possible approximation" of Voronoi partition knowing the positions of the points already swept.

2 Proof of the Proposition

We start with the definition of the dynamical system $\{A_t, B_t, E_t\}_{t \geq 0}$ for each random initial configurations of seeds $\mathcal{P}$.

Without loss of generality lets us assume r small, that is $v_B = r^{-1} \geq v_A = 1$.

For two points x and y in $I\!R^2$ let $l_{x,y}$ denote a smooth non-self-intersecting curve from x to y ; each $z \in l_{x,y}$ is specified by the length of the curve from x to z; denote this length by $l_{x,y}(z)$. Write also $|l_{x,y}|$ for the length of the whole curve.

Define now a B-neighborhood of a curve $l_{x,y}$:

$$\mathcal{N}_B(l_{x,y}) = \cup_{z \in l_{x,y}} \mathcal{B}_{\rho(z)}(z) \tag{2}$$

where $\mathcal{B}_{\rho(z)}(z)$ denotes the ball of radius $\rho(z)$ centered in z and

$$\rho(z) = l_{x,y}(z)/v_B = r l_{x,y}(z). \tag{3}$$

Let $\mathcal{P}$, the set of seeds, and its subsets $\mathcal{P}_A$ of type A seeds and $\mathcal{P}_B = \mathcal{P} \setminus \mathcal{P}_A$ of type B seeds, be given.

For each $t \geq 0$, we say that a point $y \in B_t$ if and only if there exists a smooth non-self-intersecting curve $l_{x,y}$ from $x \in \mathcal{P}_B$ to y with $|l_{x,y}| \leq t/r$ so that $\mathcal{N}_B(l_{x,y}) \cap \mathcal{P}_A = \emptyset$.

After defining the area dominated by the fast growing seeds, B_t, we set: $y \in A_t$ if and only if there exists a smooth non-intersecting curve $l_{x,y} \subset I\!R^2 \setminus B_t$ from $x \in \mathcal{P}_A$ to y with $|l_{x,y}| \leq t$. Of course then E_t will be the complement of $A_t \cup B_t$.

Note that this approach does not work if we wanted to start defining A_t and then B_t. Basically the difference is that slow growing seeds can help each other: a slow growing seed can succeed in invading a certain point of the plane because other slow growing seeds can block an invasion by a fast growing one.

We now proceed to the **proof of the proposition:**

Let us analyze first the case (1). We will need to rescale the space twice.

First we rescale the space $\mathbb{R}^2$ by placing a grid of mesh $\frac{1}{\sqrt{2}}\kappa$ on it, where κ is chosen so that at time zero the probability of having at least one nucleus in a square $\frac{1}{\sqrt{2}}\kappa \times \frac{1}{\sqrt{2}}\kappa$ is greater than $p_{c,0}$, the critical value of the independent site percolation model. That is, κ satisfies $e^{-\frac{\kappa^2}{2}} < 1 - p_{c,o}$.

Let us suppose for the moment that $\alpha = 1$, so that all nuclei are of species A. Let the system evolve up to time κ. At this time every nucleus A would be the center of a ball of radius κ. If at time zero a $\frac{1}{\sqrt{2}}\kappa \times \frac{1}{\sqrt{2}}\kappa$ square contained a nucleus A, then at time κ that square would be entirely covered by A, even if we disregard invasion by A's from outside this square. In the rescaled space, that square would be an A square, and we can think of a (independent) site percolation model on $\mathbb{Z}^2$, where each $\frac{1}{\sqrt{2}}\kappa \times \frac{1}{\sqrt{2}}\kappa$ square plays the role of a site. The choice of κ and standard facts from percolation theory guarantee that the probability that a box of size $\kappa r^{-1} \times \kappa r^{-1}$ has a circuit of A squares which surrounds the $\frac{1}{2}\kappa r^{-1} \times \frac{1}{2}\kappa r^{-1}$ box at its center is at least $1 - c_1 \exp\{-c_2 r^{-1}\}$, for some positive finite constants c_1 and c_2.

To take into account the presence of B's, we rescale the space again. The blocks in this construction are squares of side-length $3\kappa r^{-1}$ (see Figure 2). In this construction, neighboring blocks will intersect each other. Indeed, we place one such block centered at each point of $\frac{1}{2}\kappa r^{-1}\mathbb{Z}^2$. We say that a block in this construction is good in case at time zero it has the following properties.

(i) It contains enough nuclei A inside the square of side-length κr^{-1} concentric with it to assure that in the absence of B's, at time κ this square would contain a circuit of A's surrounding the square of side-length $\frac{1}{2}\kappa r^{-1}$ also concentric with it.

(ii) It contains no nucleus B.

Clearly the geometry of the blocks and conditions (i) and (ii) above were chosen so that the B's cannot prevent the A's from forming the desired circuit described in (i) at time κ. Clearly also percolation of good blocks in this construction implies percolation of A's at time κ.

Our arguments above showed that (i) occurs with arbitrarily large probability when r is small enough. To also assure a large probability for (ii) we choose $(1-\alpha)$ small enough. It is enough to make the choice so that

$$\exp\{-(1-\alpha)(3\kappa r^{-1})^2\} > 1 - \frac{1}{2}(1 - p_{c,6}) \ ,$$

where $p_{c,6}$ is the critical value for the 6–dependent site percolation model (the 6–dependence comes from the intersection between neighboring blocks, and we are using the ℓ^∞-norm in describing the range of dependence). This is equivalent to requiring $1-\alpha < c\, r^2$, with $c = (3\kappa)^{-2}\log(1-\frac{1}{2}(1-p_{c,6}))^{-1}$.

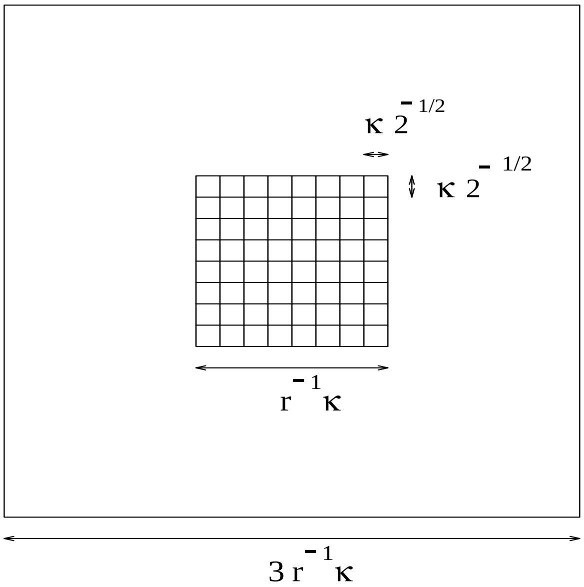

Figure 2: Blocks in the second rescaling procedure.

We turn now to case (2). Again we will need to rescale the space twice.

Here the proof involves basically two steps. First one verifies that even if the initial density of A seeds is very large it can not percolate before a suitable short enough time and that, at this time, the space not occupied by A forms an infinite cluster. The second step controls how large one has to take v_B so that B can, during that time, grow and dominate enough of this non-A infinite cluster and percolate.

First we cover $\mathbb{R}^2$ with a grid of mesh κ, where κ is chosen so small that, at time zero, the probability of having any nucleus in a square $3\kappa \times 3\kappa$ is smaller than a quantity ε which we will specify later on. That is $1-\exp\{-9\kappa^2\} < \varepsilon$.

Let us suppose for the moment that the nuclei of B do not grow, while those of A grow with their speed $v_A = 1$. In this construction, we will say that a square $\kappa \times \kappa$ which at time κ is completely empty of A is a "good $\kappa \times \kappa$ square".

In this situation, the probability that a $\kappa \times \kappa$ square is good is bounded below by $1-\varepsilon$, and for distinct such squares, these events have a ℓ^∞ range of dependence equal to 2 when indexed by the sites in the lattice defined by the squares. The actual growth of the B's can only hinder that of the A's, so that we have a useful comparison between the actual model and the modified one just discussed.

To take into account the presence of the B's, we rescale the space again. The basic geometric objects in this second rescaling procedure are rectangles $\frac{1}{2}\kappa r^{-1} \times \frac{1}{8}\kappa r^{-1}$. We will say that such a rectangle is good in case at time zero it has the following two properties.

(I) There are so few A's inside the rectangle and within l^∞-distance 2κ of it that in the modified model in which B's do not grow, at time κ at least $1/6$ of the $\kappa \times \kappa$ squares from the first rescaling procedure which lie inside our rectangle are connected to both smaller sides of the rectangle by paths of $\kappa \times \kappa$ squares completely free of A's (i.e., good squares) and which also fulfill the following conditions. The $\kappa \times \kappa$ squares in the path are supposed to lie entirely inside the rectangle, except possibly for the ones at the extremes of the path, which may be partially outside it. Moreover the path is such that inside the union of the $\kappa \times \kappa$ squares which belong to it there is a smooth curve of length at most $(3/4)\kappa r^{-1}$ joining the two smaller sides of the rectangle. We will call these good $\kappa \times \kappa$ squares "doubly connected".

(II) There is at least one nucleus B inside one of the $\kappa \times \kappa$ doubly connected squares.

In order to estimate the probability of the event defined in (I) above, we will use oriented percolation. This is to control not only the existence of a path of empty space inside some region but also its length and thus make sure it can be invaded by B in the allotted time with the velocity given by the proposition.

Consider as sites of a rescaled lattice the $\kappa \times \kappa$ squares, and declare a site to be occupied when it corresponds to a good $\kappa\times\kappa$ square. First we consider oriented percolation in which each site can only be connected to the site to its right and to the site above it. Call this NE-oriented percolation. Given a site $s = (x_s, y_s)$ in our rescaled lattice, let E_s^{NE} be the event that it belong to an infinite NE-oriented cluster contained in the region $\{(x,y) \in \mathbb{Z}^2 \colon y \geq y_s, x_s \leq x \leq x_s + (y - y_s)/16\}$. Consider next oriented percolation on the same rescaled system, but with the orientation in both directions reversed. Call this SW-oriented percolation. Given a site $s = (x_s, y_s)$ in our rescaled lattice, let E_s^{SW} be the event that it belong to an infinite SW-oriented cluster contained in the region $\{(x,y) \in \mathbb{Z}^2 \colon y \leq y_s, x_s - (y_s - y)/16 \leq x \leq x_s\}$. Set also $E_s = E_s^{\mathrm{NE}} \cap E_s^{\mathrm{SW}}$. Standard Peierls type estimates (for finite-range-dependent oriented percolation) show that if ε is small enough (which amounts to taking κ small enough), then the probability of E_s can be made as close to one as we want. How large we need it to be will be clarified in the next paragraph.

Consider inside of our $\frac{1}{2}\kappa r^{-1} \times \frac{1}{8}\kappa r^{-1}$ rectangle the concentric rectangle with half the linear dimensions of our rectangle. Note that when r is small, this smaller rectangle contains more than $1/5$ of the $\kappa \times \kappa$ squares which are contained in the larger rectangle.

Also, if a $\kappa \times \kappa$ square which lies inside of the smaller rectangle and corresponds to the site s of the rescaled lattice has the event E_s happening, then elementary geometry tells us that it is a doubly connected $\kappa \times \kappa$ square. Since the probability of E_s does not depend on s and the number of $\kappa \times \kappa$ squares that may provide this double connection grows as r decreases we conclude (via a standard use of ergodicity) that the probability of the event defined in (I) above converges to 1 as $r \to 0$.

Conditioning on the event defined in (I), the event defined in (II) will have probability bounded below by $1 - \exp\{-(1-\alpha)\frac{1}{120}\kappa^2 r^{-2}\}$, when r is small. Therefore by making $(1-\alpha)r^{-2}$ large enough, we can make the probability that the rectangle is good arbitrarily large for small r.

Note that if a rectangle is good, then, if r is small, at time κ the region occupied by B will contain a curve which joins its smaller sides. (B has time to make it, and A cannot prevent it.)

There are various ways in which one can use the good rectangles to set up a rescaling procedure. One possibility is as follows. The blocks that correspond to the rescaled sites are squares $\frac{1}{2}\kappa r^{-1} \times \frac{1}{2}\kappa r^{-1}$. Of relevance are the four rectangles $\frac{1}{2}\kappa r^{-1} \times \frac{1}{8}\kappa r^{-1}$ which are contained in this square and share a complete side with it. We say that the square is good in case all these four rectangles are good. A good square $\frac{1}{2}\kappa r^{-1} \times \frac{1}{2}\kappa r^{-1}$ will have a ring of empty $\kappa \times \kappa$ squares formed by the paths on the four good rectangles. The rescaled lattice is defined by placing one such square $\frac{1}{2}\kappa r^{-1} \times \frac{1}{2}\kappa r^{-1}$ centered at each point of $(\frac{1}{4}\kappa r^{-1})\mathbb{Z}^2$. When r is small, the distribution of good $\frac{1}{2}\kappa r^{-1} \times \frac{1}{2}\kappa r^{-1}$ squares defines a 2–dependent random field, since the goodness of a $\frac{1}{2}\kappa r^{-1} \times \frac{1}{8}\kappa r^{-1}$ rectangle depends only on the distribution at time zero of nuclei inside of it or at most at ℓ^∞-distance 2κ from it. Therefore percolation of good squares is assured when r is small. The percolation of these intersecting good squares ensures that the rings of empty paths inside them form an infinite cluster, therefore implying percolation of B at time κ in our model.

$\diamond$

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