DOBRUSHIN-KOTECKÝ-SHLOSMAN THEOREM UP TO THE CRITICAL TEMPERATURE

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ABSTRACT. We develop a non-perturbative version of the Dobrushin-Kotecký-Shlosman theory of phase separation in the canonical 2D Ising ensemble. The results are valid for all temperatures below critical.

Key words and phrases: Ising model, phase separation, large and moderate deviations, local limit theorems, Wulff shape.

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1. Introduction

Dobrushin-Kotecký-Shlosman (DKS) Theorem [DKS], [DS] gives a rigorous probabilistic content to the assertion that pure phases are separated on the macroscopic scale along the boundary of the equilibrium crystal shape. Their results were formulated and proved in the context of the 2D Ising model at very low temperatures. Despite this particular setting it would be appropriate to talk in terms of the DKS Theory, rather than in terms of only one theorem with a very long proof (as the authors of [DKS] modestly did). For many of their ideas and insights will certainly find a way into both more general lattice models and higher dimensions. During several years following the publication of [DKS], however, the main efforts have been invested into attempts to relax their proof [Pf] and, later on, to get rid of the "very low temperature" assumption. It has been commonly believed that one needs low temperature solely in order to have an additional technical tool of convergent cluster expansions readily available, whereas the results themselves should remain qualitatively the same in the whole of the phase transition region. And indeed, in a series of articles [I1], [I2], [SS1], [SS3], [CGMS] and culminating in [PV] and [V] some sort of the DKS theory has been developed in the non-perturbative regime and pushed all the way to the critical temperature. These results, however, have been based on the integral type limit theorems and are, in parts, closer in spirit to the non-perturbative treatment of the 2D Bernoulli percolation [ACC] than to the local limit setting of the exact canonical ensemble in the original monograph [DKS]. Subsequently, the phase separation geometry has been much less pronounced, and the accent has been generally shifted to the precise leading surface order of integral estimates. Moreover, it seems that the local limit part of the DKS theory is the one to be the most robust and amenable as opposed to the skeleton coarse graining techniques, which are probably too much oriented to the two-dimensional lattice and nearest neighbour interactions. In this paper we try to fill in this gap and to extend the theory up to the critical temperature in the original setting of [DKS], i.e., in the exact canonical ensemble. The results we obtain are comparable in strength and scope to those appearing in the corresponding chapters of [DKS] and [DS]. It should be noted, however, that we cover only part of their results - a delicate analysis of the phase separation line for all subcritical temperatures is currently beyond our reach, and we have to appeal to exact solution dependent facts about the Ornstein-Zernike behaviour of the two-point correlations [MW] and about analytic properties of the surface tension [AA], while deriving the lower bound of Section 3. But this is the only point at which our results fail to be self-contained. It also should be clear that the basic philosophy was already perceived in the original ground breaking monograph [DKS], our main contribution has been merely to understand how to implement it for all temperatures below critical, that is using only qualitative facts about ferromagnetic Ising measures, e.g., the Markov property, FKG and other correlation inequalities.

The problem of defining the equilibrium crystal shape as the one which minimizes the interfacial surface energy was formulated on the turn of the century in [W]. In our setting of the 2D Ising model let $\beta > \beta_c$ be the inverse temperature, and let $\tau_\beta : \mathbb{S}^1 \mapsto \mathbb{R}_+$ be the corresponding anisotropic surface tension, see [DKS], [Pf] and [A] for the definitions and properties. Then the equilibrium crystal shape (Wulff shape) of the area v is defined to be a solution of the following isoperimetric type problem

$$\begin{aligned} \mathcal{W}_\beta(\partial V) \ &\triangleq \ \int_{\partial V} \tau_\beta(n_s) ds \ \longrightarrow \ \min \\ \text{Given: } \ \text{Area}(V) \ &= \ v \end{aligned}$$

where ds is the unit speed parametrization of the boundary ∂V and n_s is the normal to ∂V at the point s. A nice feature of the variational problem above is its scale invariance:

$$\mathcal{W}_\beta\big(\partial(aV)\big) \;=\; a\mathcal{W}_\beta\big(\partial V\big).$$

Consequently any dilatation of an optimal solution is itself optimal, and, therefore, one really should think in terms of shapes and, moreover, needs to specify solutions only at one prescribed value of the area v. A canonical way to pin down the solution is to define the unnormalized Wulff shape

$$\mathcal{K} \;=\; \bigcap_{n\in\mathbb{S}^1}\{x:\;(x,n)_{\mathbb{R}^2}\le\tau_\beta(n)\;\},$$

where $(\cdot,\cdot)_{\mathbb{R}^2}$ is the usual scalar product in $\mathbb{R}^2$, and then to scale it down to the unit area

$$\mathcal{K}_1 \;\triangleq\; \frac{1}{\sqrt{|\mathcal{K}|}}\mathcal{K}.$$

It is a known fact of the Brunn-Minkowski theory (c.f., for example [Sc]), that all optimal shapes of the unit area are just shifts of $\mathcal{K}_1$. We shall use the unit area as the reference scale and along with $\mathcal{K}_1$ also introduce a separate notation for the surface energy of the boundary $\partial\mathcal{K}_1$ respectively,

$$\omega_1 \;\triangleq\; \mathcal{W}_\beta\big(\partial\mathcal{K}_1\big).$$

In the next subsection we introduce some basic notation. The results and the underlying heuristics are described in Subsection 1.3. Finally, in the last subsection of the Introduction we try to outline the proof and, subsequently, to facilitate the orientation of the reader.

1.1. **Notation.** One of the main features of our approach is a uniform treatment of the appropriate family of subsets of $\mathbb{Z}^2$.

Lattice boxes: $\mathbb{Z}^2$ denotes the two-dimensional integer lattice. Given a point $x=(x_1,x_2)\in\mathbb{Z}^2$ we use $\|x\|_1\triangleq|x_1|+|x_2|$. For any set $A\subseteq\mathbb{Z}^2$ we define its (outer) boundary ∂A via

$$\partial A \;\triangleq\; \big\{x\in\mathbb{Z}^2\setminus A\;:\;\min_{y\in A}\|x-y\|_1=1\big\}.$$

Finally we define the closure of A as

$$\widehat{A} \;\triangleq\; A\cup\partial A \;=\; \big\{y\in\mathbb{Z}^2:\;\min_{z\in A}\|y-z\|_1\le1\big\}.$$

Our results are asymptotic with the microscopic size of the system tending to infinity. In order to avoid irrelevant complications related to the finite box boundary effects, we choose our basic sequence of boxes $\Lambda_N\subset\mathbb{Z}^2$ as:

$$\Lambda_N=N\mathcal{K}_1\cap\mathbb{Z}^2,$$

where, as before, $\mathcal{K}_1$ is the unit volume Wulff shape. More generally, we shall work with the following family $\mathcal{D}_N$ of "boxes" in $\mathbb{Z}^2$:

$$A\in\mathcal{D}_N\iff aN^2\le|A|\le N^2\text{ and }|\partial A|\le RN\log N,$$

where a and R are two respectively very small and very large numbers, which are fixed throughout the article.

Measures: The results are obtained on different scales, which are quantified by the value of the large contour parameter $s(N)$. There are two typical scales for $s(N)$:

1. For $K = K(\beta)$ sufficiently large we define the basic scale
$$s(N) = K \log N,$$
which corresponds to the maximal probable size of contours inside Λ_N in the pure phase.
2. Intermediate scales
$$s(N) = N^b (\log N)^\lambda; \; b \in (0,1), \; \lambda \in \mathbb{R}.$$

Given a large contour parameter $s(N)$, we define an $s(N)$-restricted phase or, equivalently, the phase of $s(N)$-small contours as the corresponding Ising Gibbs measure, conditioned on the event that no configuration has a $\pm$ contour with the diameter exceeding $s(N)$.

For a set $A \in \mathcal{D}_N$ we use $\mathbb{P}^{\beta}_{A,-,g}$,$\mathbb{P}^{\beta,s}_{A,-,g}$ and $\langle \cdot \rangle^{\beta}_{A,-,g}$, $\langle \cdot \rangle^{\beta,s}_{A,-,g}$, to denote measures and expectations with minus boundary conditions, at the inverse temperature β and the magnetic field g in the unrestricted phase and in the phase of $s(N)$-small contours, respectively . We also use subindex N instead of Λ_N, whenever A is the Λ_N box itself, and we drop any finite box subindex while talking about infinite volume "$-$" state. Similarly, we drop the magnetic field subindex g, whenever $g = 0$. Finally, for each $\beta\beta_c$, we use m^* to denote the spontaneous magnetization of the extremal Gibbs state,
$$m^* = m^*(\beta) = -\langle \sigma(0) \rangle^{\beta}_{-} > 0,$$
and χ to denote its susceptibility,
$$\chi = \chi(\beta) = \sum_{x \in \mathbb{Z}^2} \langle \sigma(0); \sigma(x) \rangle^{\beta}_{-} < \infty.$$

Magnetization: The space of spin configurations on $A \subset \mathbb{Z}^2$ is denoted by $\Omega_A \triangleq \{-1,1\}^A$. Given a box $A \subset \mathbb{Z}^2$ we set
$$M_A = \sum_{x \in A} \sigma(x),$$
to denote the total magnetization on A. In the case $A = \Lambda_N$, we use shortcuts Ω_N and M_N respectively.

One of the main objects of this article is to give precise asymptotics on the probabilities of the deviation of M_N from $-m^* N^2$,
$$\mathbb{P}^{\beta}_{N,-}\left(M_N = -m^* N^2 + a_N \right),$$
in the whole of the low temperature region $\beta > \beta_c$, and, most importantly, to use results on such asymptotics in order to describe the phenomenon of the phase separation in the canonical ensemble
$$\mathbb{P}^{\beta}_{N,-}\left(\bullet \,\middle|\, M_N = -m^* N^2 + a_N \right).$$

As in [DKS] our approach to this problem is built up upon uniform local limit type estimates on the deviations of M_A; $A \in \mathcal{D}_N$. Depending on the context it will be convenient to formulate such estimates either directly in terms of the deviation of M_A from $-m^*|A|$, or in terms of its deviations from the averages under current measures
$$\mathbb{M}_A \triangleq \langle M_A \rangle^{\beta}_{A,-} \qquad \text{or} \qquad \mathbb{M}^s_A \triangleq \langle M_A \rangle^{\beta,s}_{A,-}.$$

We shall see (c.f., Remark 2.2.2) , however, that uniformly in $A \subset \mathbb{Z}^2$,

$$\begin{aligned} \Big| \mathbb{M}_A + m^*|A| \Big| \ &\leq \ c_1(\beta)\big|\partial A\big|, \\ &\text{and} \\ \Big| \ \mathbb{M}_A - \mathbb{M}^s_A \ \Big| \ &\leq \ c_1(\beta)\big|A\big|\mathrm{e}^{-c_2(\beta)s(N)}. \end{aligned} \tag{1.1.1}$$

In particular, things are uniformly under control, once we restrict attention to the domains $A \in \mathcal{D}_N$.

Of course, local limit estimates make sense only for the admissible values of the magnetization. Thus, for a deviation a_N we write

$$a_N \ \in \ \mathcal{M}_A$$

if, depending on the context, either $-m^*|A| + a_N$ or $\mathbb{M}_A + a_N$ or $\mathbb{M}^s_A + a_N$ lie in the range of M_A, that is in between $-|A|$ and $|A|$ and equal to $|A|$ mod(2).

We, furthermore, concentrate on the positive values of a_N,

$$a_N \in \mathcal{M}^+_A,$$

which point inside the phase transition region. Indeed, the probabilities

$$\mathbb{P}^\beta_{A,-}\big(\ M_A = b_N \ \big)$$

exhibit, in the language of [DS], the classical limit behaviour, as soon as $b_N \leq \mathbb{M}_A$, and the corresponding asymptotics were thoroughly worked out in the latter article[1]. Let us state a consequence of their results in the form we need it here:

For any $A \subset \mathbb{Z}^2$, non-positive magnetic field $g \leq 0$ and a number $a \leq 0$, the following Gaussian estimate is true:

$$\mathbb{P}^\beta_{A,-,g}\big(\ M_A \ = \ \langle M_A \rangle^\beta_{A,-,g} + a \ \big) \ \leq \ \exp\big\{ -c_3(\beta)\frac{a^2}{|A|} \ \big\}. \tag{1.1.2}$$

Contours and skeletons: Our contours are always self avoiding objects, constructed according to one of the two possible splitting rules (see [DKS]). All contours lie on the edges of the dual lattice $\mathbb{Z}^2_* \stackrel{\Delta}{=} (1/2, 1/2) + \mathbb{Z}^2$.

Given a $\pm$ contour γ and a large contour parameter $s(N)$, a set of dual vertices $S = (u_1, ..., u_n)$ is called an $s(N)$-skeleton of γ if

1. All vertices of S lie on γ.
2. $s(N)/2 \leq \|u_i - u_{i+1}\| \leq 2s(N)$; $\forall \ i = 1, ..., n$, where we have identified $u_{n+1} \equiv u_1$ and $\| \bullet \|$ stands for the supremum norm; $\|(a,b)\| = \max\{|a|, |b|\}$.
3. The Hausdorff distance $d_{\mathbb{H}}$ between γ and the polygonal line $\mathbf{P}(S)$ through the vertices of S satisfies
$$d_{\mathbb{H}}\big(\gamma, \mathbf{P}(S)\big) \ \leq \ s(N).$$

If S satisfies the above conditions, we say that γ and S are compatible and write $\gamma \sim S$. Note that a contour might have many compatible skeletons. Note also that any $s(N)$-large contour, i.e., one whose diameter is at least $s(N)$, has a compatible $s(N)$-skeleton.

We shall consider contours and their skeletons on different $s(N)$-scales. For each large contour parameter $s(N)$ fixed and for any given spin configuration σ we use a generic notation $\Gamma = \Gamma(\sigma) =$

[1] The first part of their Theorem 1.5.1 is true for any $\beta \neq \beta_c$

$(\gamma_1, ..., \gamma_n)$ for a set of all $s(N)$-large contours of σ. Each collection Γ of $s(N)$-large contours splits Λ_N into the disjoint union of its "$-$' and "+" components, $\Lambda_N = B \cup C$ respectively. We use

$$\mathrm{Vol}_+(\Gamma(\sigma)) \triangleq |C|$$

for the cardinality of the "+" component $|C|$.

A collection of $s(N)$-skeletons $\mathfrak{S} = (S_1, ..., S_n)$ is then said to be compatible with σ (or with $\Gamma(\sigma)$), which we denote as $\mathfrak{S} \sim \Gamma(\sigma)$, if $S_i \sim \gamma_i;\ i = 1, ..., n$.

The skeleton language is the main coarse graining tool - the surface tension is produced in the probabilistic estimates, once the events are started to be expressed in their skeleton approximation. The surface tension of a collection $\mathfrak{S} = (S_1, ..., S_n)$ is defined in a standard way:

$$\mathcal{W}_\beta(\mathfrak{S}) \;=\; \sum_{i=1}^{n} \mathcal{W}_\beta(\mathbf{P}(S)),$$

where, as before, $\mathbf{P}(S)$ is the polygon through the vertices of the skeleton S.

Though the contours are self avoiding objects, the polygonal lines through the vertices of their skeletons are, in general, not. Moreover, two polygons corresponding to two different skeletons of the same configurations might intersect as well. Thus, it is not immediately clear what should be the interiour of a collection $\mathfrak{S}$. This difficulty was overcome in [DKS], and we stick to their definition of the phase volume. Moreover, we use $\mathfrak{S}_+$ to denote the corresponding "+" phase component of $\mathfrak{S}$. As it was proved in [DKS], given any family of $s(N)$-large contours Γ and a compatible collection of $s(N)$-large skeletons $\mathfrak{S};\ \mathfrak{S} \sim \Gamma$,

$$\big| C \Delta \mathfrak{S}_+ \big| \;\le\; c(\beta) \mathcal{W}_\beta\big(\mathfrak{S}\big) s(N), \tag{1.1.3}$$

where C is the "+" component of Λ_N under the Γ-splitting. Note, by the way, that $\mathfrak{S}_+$ is a disjoint union of connected (and simply connected if we restrict attention to exterior large contours) polygonal subsets of $\mathbb{R}^2$, and, therefore, $\mathcal{W}_\beta(\partial\mathfrak{S}_+)$ is well defined. Moreover,

$$\mathcal{W}_\beta(\mathfrak{S}_+) \;\triangleq\; \mathcal{W}_\beta(\partial\mathfrak{S}_+) \;=\; \mathcal{W}_\beta(\mathfrak{S}). \tag{1.1.4}$$

1.2. **Heuristics and results.** The typical phase picture under the low temperature, $\beta > \beta_c$, pure (minus) state could be loosely described as a sea of "$-$" spins with a homogeneous archipelago of "+" islands, some of which might contain "$-$" lakes, etc. The size of these islands inside Λ_N does not typically exceed $K \log N$, and their density is such that the mean magnetization produced is close to $-m^*$. One can think of two principal mechanisms behind a shift of the magnetization from its averaged value in the corresponding pure phase:

1. A homogeneous shift in the density of "+" islands, without, however, modifying their typical sizes.
2. A creation of abnormally huge islands of "+" phase. In particular, the a_N shift of the magnetization should correspond to the excess area $a_N/2m^*$ of those islands.

The first scenario corresponds to the Gaussian fluctuations and its probabilistic price for the $a_N \in \mathcal{M}_A^+$ shift of the magnetization should be close to

$$\exp\big(-\frac{a_N^2}{2\chi|A|}\big).$$

The phenomenon of the phase separation, of course, manifests itself in the second scenario. Its probabilistic price should be related to the surface tension of the optimal shape "huge islands" configuration, which, if there is no additional restriction on the size of those huge islands (e.g., if the estimates are performed in the unrestricted phase), leads to a value close to

$$\exp\big(-\sqrt{\frac{a_N}{2m^*}}\omega_1\big).$$

A comparison between the two expressions above indicates that the Gaussian contribution should win, whenever $a_N \ll N^{4/3}$, whereas the creation of huge islands should become a dominant factor as soon as $a_N \gg N^{4/3}$, which is the regime where the phase separation should be observed.

Remark 1.2.1. The case of critical deviations $a_N \sim N^{4/3}$ requires both a more accurate heuristics and, respectively, more refined rigorous estimates. We relegate the corresponding discussion to a future publication.

Another issue we do not work out in all the details here is the precise geometry of the phase picture in the canonical ensemble in the $s(N)$-restricted phase. Instead, we confine ourselves only to a derivation of imprecise bounds in this regime, which, nonetheless, capture the leading exponential order of decay of the corresponding probabilities and sharpen all the previous estimates [I2], [SS3] and [PV] of this sort.

We attempt in this paper to develop a non-perturbative ($\forall\ \beta > \beta_c$) exact theory of the phase separation, which would be comparable in scope to the low temperature results obtained in [DKS] and [DS] using the method of cluster expansions.

For any $\delta > 0$ fixed we distinguish between small moderate deviation values of $a_N \in \mathcal{M}_A^+$,

$$a_N \ \ll \ N^{4/3-\delta},$$

and large moderate deviation values of $a_N \in \mathcal{M}_A^+$,

$$a_N \ \gg \ N^{4/3+\delta}.$$

Theorems A,B and C were proved in [DKS] and [DS] only for sufficiently large values of the inverse temperature β. Our main result is their validity up to the critical temperature, i.e., for any value $\beta > \beta_c$.

Theorem A. *Let $\delta \in (0, 2/3)$ and assume that $a_N \in \mathcal{M}_N^+$ satisfies*

$$a_N \ \sim \ N^{\frac{4}{3}+\delta}.$$

Then, for every $\beta > \beta_c$ fixed,

$$\big(\sqrt{\frac{a_N}{2m^*}}\omega_1\big)^{-1} \log \mathbb{P}_{N,-}^{\beta}\big(\ M_N \ = \ -m^* N^2 \ + a_N \ \big) \ = \ -1 \ + \ \mathrm{o}\big(N^{-\frac{\delta}{2}}\big). \tag{1.2.1}$$

Moreover, if K is large enough, with the $\mathbb{P}_{N,-}^{\beta}(\ \cdot\ |M_N = -N^2 m^* + a_N)$-probability converging to 1 as $N \to \infty$:

1. *There is exactly one exteriour $K \log N$-large contour γ.*
2. *This γ satisfies*

$$\min_x d_{\mathbb{H}}\big(\ \sqrt{\frac{2m^*}{a_N}}\gamma, x + \partial\mathcal{K}_1\big) \ \ll \ N^{-\frac{\delta}{4}}. \tag{1.2.2}$$

Theorem A deals with the case of large moderate deviations; $a_N \ll N^2$, proper. The estimates on the right hand sides of (1.2.1) and (1.2.2) can be improved at various particular values of δ chosen. Similarly, with an additional care one is able to rule out all $K \log N$-large non-Wulff contours and not only the external ones as we assert here. This would, however, give rise to rather messy formulas, and, as we feel, might obscure the common logic of the proof. This is also the reason why we decided not to address here the critical regime, described in the Remark 1.2.1.

We apply the theory in the full strength only in the traditionally interesting case of large deviations $a_N \sim N^2$.

Theorem B. *Let the inverse temperature $\beta > \beta_c$ be fixed, and let the sequence $\{a_N\}$; $a_N \in \mathcal{M}_N^+$ be such that the limit*

$$\lim_{N\to\infty} \frac{a_N}{N^2} \ \in\ (0, 2m^*(\beta))$$

exists. Then,

$$\Big(\sqrt{\frac{a_N}{2m^*}}\omega_1\Big)^{-1} \log \mathbb{P}^{\beta}_{N,-}\big(\ M_N \ =\ -m^* N^2\ + a_N\ \big) \ =\ -1\ +\ O\big(N^{-1/2}\log N\big).$$

Moreover, if K is large enough, with the $\mathbb{P}^{\beta}_{N,-}(\ \cdot\ |M_N = -N^2 m^* + a_N)$-probability converging to 1 as $N \to \infty$:

1. *There is exactly one $K \log N$-large contour γ.*
2. *This γ satisfies*

$$\min_x d_{\mathbb{H}}\Big(\ \sqrt{\frac{2m^*}{a_N}}\gamma, x + \partial\mathcal{K}_1\Big)\ \le\ c_1(\beta) N^{-1/4}\sqrt{\log N}. \tag{1.2.3}$$

Comparing Theorem B with the corresponding assertion (1.9.4) in Theorem 1.9 in [DKS], we find that our bound (1.2.3) on the typical Hausdorff distance to the dilatation of the Wulff shape even slightly improves their low temperature estimate. On the other hand, the best we can do in order to control the area of the symmetric difference between the interiour of γ and the optimal deterministic shape is to apply (1.2.3), which yields

$$\min_x \Big|\sqrt{\frac{2m^*}{a_N}}\mathrm{int}(\gamma)\,\Delta\big(x + \mathcal{K}_1\big)\Big|\ \le\ c_2(\beta) N^{-1/4}\sqrt{\log N}.$$

The above estimate falls short of the $N^{-4/5}(\log N)^{\kappa}$ order in the corresponding result ([DKS], (1.9.3)). This should not be too surprising, since their estimate is based on a more refined (though still not optimal) analysis of the fluctuations of phase separation lines and related fluctuations of the phase volume. Though such a refinement is apparently feasible in our setting as well, we decided not to perform it here. A much more challenging task would be to understand and develop a non-perturbative counterpart of the optimal results on the fluctuations of the phase separation line about Wulff shapes [DH], but this, as we have already mentioned, is beyond the scope of our work.

Finally, in the case of small moderate deviations we obtain, uniformly in $A \in \mathcal{D}_N$, the following result:

Theorem C. *Let $\delta \in (0, 4/3)$ be fixed.*

Then, for every $\beta > \beta_c$ there exist a value of $K = K(\beta) < \infty$, such that uniformly in $A \in \mathcal{D}_N$ and $a_N \in \mathcal{M}_A^+$ satisfying

$$a_N\ \le\ N^{4/3-\delta},$$

the following two estimate hold:

$$\mathbb{P}^{\beta}_{A,-}\big(\, M_A \,=\, \mathbb{M}_A \,+\, a_N \,\big) \;=\; \frac{1}{\sqrt{2\pi\chi|A|}}\exp\big(-\frac{a_N^2}{2\chi|A|}\big)\big(\,1 \,+\, \mathrm{o}(1)\big), \tag{1.2.4}$$

and

$$\mathbb{P}^{\beta}_{A,-}\Big(\,\exists\; K\log N\text{-}\mathit{large}\;\;\pm\;\;\mathit{contour}\;\big|\; M_A = \mathbb{M}_A + a_N\,\Big) \;= \mathrm{o}(1). \tag{1.2.5}$$

Remark 1.2.2. Given local limit asymptotics (1.2.4), one can derive various results on the statistical properties of configurations inside and outside the unique "Wulff" contour asserted in Theorems A and B, putting, in this way, more flesh on the corresponding notion of the phase separation. We refer to Subsection 1.10 of [DKS] for the statement and to Subsections 6.3 and 6.4 for respectively the proofs of such results.

1.3. **Notes on the proofs.** There are two main ingredients of the theory:

1. Coarse graining of contours.
2. Uniform local limit estimates on moderate deviations over domains $A \in \mathcal{D}_N$ under various restricted phases $\mathbb{P}^{\beta,s}_{A,-}$.

Coarse graining is imperative for the production of the (macroscopic quantity) surface tension and it is performed in terms of skeletons: Important geometric events we deal with here are roughly of the following type:

$$\big\{\text{There is a } \pm \text{ contour close to the boundary of a certain deterministic shape}\big\}.$$

The point is that one-contour Peierls estimates never capture the precise order of decay of the probabilities of such events. In other words, the probability of having a contour close to some shape is substantially larger than the probability of each particular contour contributing to the event. Contrary to this the probability to observe certain skeleton already integrates the entropy of the number of various contours compatible with this skeleton. Most of the coarse graining estimates we use were obtained before, and we refer to [Pf] for a detailed discussion.

On every $s(N)$-scale,

$$\mathbb{P}^{\beta}_{N,-}\big(\,\mathfrak{S}\,\big) \;\le\; \exp\big(\;-\mathcal{W}_{\beta}(\mathfrak{S}_+)\;\big), \tag{1.3.1}$$

and a scaling computation (c.f., for example, [Pf]), to which we always refer as to the usual skeleton computation, gives rise to the following important "Energy" estimate on any $s(N)$ scale : There exists a constant $c_1 = c_1(\beta)$, such that

$$\mathbb{P}^{\beta}_{N,-}\big(\,\mathcal{W}_{\beta}(\mathfrak{S}) \ge r\big) \;\le\; \sum_{\mathcal{W}_{\beta}(\mathfrak{S})\ge r} \mathbb{P}^{\beta}_{N,-}\big(\,\mathfrak{S}\big) \;\le\; \exp\{\,-r(1-\frac{c_1\log N}{s(N)})\}. \tag{1.3.2}$$

Applying this bound on the basic $K\log N$ scale, and noting that if $\mathfrak{S}$ is a $K\log N$ collection which is compatible with some family of $K\log N$-large contours $\Gamma = (\gamma_1,...,\gamma_n)$, then

$$|\Gamma| \;\triangleq\; \sum_{i=1}^{n}|\gamma_i| \;\le\; K\log N\frac{\mathcal{W}_{\beta}(\mathfrak{S}_+)}{\min_n \tau_{\beta}(n)},$$

we infer the following simple but nonetheless useful consequence:

There exists $c_2 = c_2(\beta)$, such that for any $r > 0$ and for an event $\mathcal{C} \in \Omega_A$ with $\mathbb{P}^{\beta}_{A,-}(\mathcal{C}) \geq \mathrm{e}^{-r}$,

$$\mathbb{P}^{\beta}_{A,-}\left(\left|\Gamma\right| > c_2 r \log N \;\middle|\; \mathcal{C} \right) \;=\; \mathrm{o}(1). \tag{1.3.3}$$

In particular, since, as we shall see later, local limit estimates of Section 2 readily imply a lower bound of the form

$$\mathbb{P}^{\beta}_{N,-}\left(M_N = -m^* N^2 + a_N \right) \;\geq\; e^{-c_3(\beta)\sqrt{a_N}}$$

uniformly in large moderate deviations $a_N \in \mathcal{M}^+_N$, one can control the length of large $\pm$-contours or, equivalently, the length of the boundaries of the "+" and "−" components in the corresponding splitting of Λ_N. This, by the way, explains our definition of the family $\mathcal{D}_N$.

Local limit estimates are stated uniformly in domains from $\mathcal{D}_N$, which, as we just remarked, includes "−" components of admissible families Γ of large contours. In the phase separation regime or, equivalently, at large moderate values of deviations $a_N \sim N^{4/3+\delta}$, the only "stable" large contours are those of the maximal linear size $\sqrt{a_N}$. A mathematical interpretation and derivation of this fact comprises two steps: first we show that subcritical contours, i.e., those whose diameter is much smaller than $N^{2/3}$ do not appear in the unrestricted phase in any circumstances. This ought to be clear on the heuristical level: suppose that subcritical contours produce some $\tilde{a}_N$ shift in the magnetization. Then for small moderate values of $\tilde{a}_N$, the surface tension price for this should be much higher than the corresponding quadratic term in Gaussian estimates in the basic $K \log N$ restricted phase. On the other hand, for large moderate values of $\tilde{a}_N$ it would be much less expensive to produce such shifts by means of contours of the linear size $\sim \sqrt{\tilde{a}_N}$, and then to compensate a possible disbalance of magnetization in the restricted $K \log N$ phase inside and outside these contours. A rigorous implementation, therefore, strongly depends on accurate local limit bounds in the phase of $K \log N$-small contours, which we subsequently derive in Section 2.

Large unstable contours, in their turn, are ruled out by an isoperimetric type argument in Section 5. The efficiency of this argument, however, depends on both on sharp coarse graining estimates, in particular on the lower bound of Section 3, and on sharp local estimates on various $s(N)$ scales.

Many such estimates are derived according to the following pattern:

In order to estimate the probability of a certain event $\mathcal{C}$ under $\mathbb{P}^{\beta}_{A,-}$, we choose a certain large contour parameter $s(N)$ and make a skeleton decomposition of $\mathcal{C}$,

$$\mathcal{C} \;=\; \bigcup_{\mathfrak{S}} \mathcal{C} \cap \{\sigma : \sigma \sim \mathfrak{S}\}.$$

For each collection of skeletons $\mathfrak{S}$ we control the length and the shape of any $\Gamma \sim \mathfrak{S}$ in terms of $\mathcal{W}_\beta(\mathfrak{S})$ through (1.3.3) and (1.1.3) respectively. Any $\mathfrak{S}$-compatible collection Γ, on the other hand, decouples $\mathbb{P}^{\beta}_{N,-}$ into the product of the "+" and "−" states over the corresponding components in the induced decomposition of Λ_N.

In the case $\mathcal{C} = \{M_A = -m^*|A| + a_N\}$, each admissible collection of large contours Γ leads to the following decomposition:

$$\mathcal{C} \cap \{\sigma : \Gamma(\sigma) = \Gamma\} \;=\; \bigcup_{b_N + c_N = a_N - \Delta(B,C)} \left\{ \Gamma \;;\; M_B = -m^*|B| + b_N \;;\; M_C = m^*|C| + c_N \right\}, \tag{1.3.4}$$

where, as usual $A = B \cup C$ is the Γ-induced decomposition of A into the respectively "$-$" and "$+$" components, while

$$\begin{aligned}\Delta(B,C) &= m^*\big|A\big| - m^*\big|B\big| + m^*\big|C\big| + \big|\partial_+\Gamma\big| - \big|\partial_-\Gamma\big| \\ &= 2m^*\big|C\big| + (m^*+1)\big|\partial_+\Gamma\big| + (m^*-1)\big|\partial_-,\Gamma\big|\end{aligned}$$

where $\partial_+\Gamma$ and $\partial_-\Gamma$ are the set of sites in A where the presence of Γ forces the spins to be $+1$ and -1, respectively. Similar splitting is valid for the event $\mathcal{C} = \{M_A = \mathbb{M}_A + a_N\}$, and we shall use in this case (1.1.1) and (1.3.3) in order to control $b_N + c_N$ in terms of a_N and $|C|$. We then use coarse graining estimates, e.g., (1.3.2), to control the probability of skeletons and local limit estimates to control the magnetization inside and outside compatible contours.

The paper is organized in the following way: Section 2 is devoted to the local limit estimates over various domains $A \subset \mathbb{Z}^2$. These results are the backbone of the theory. In particular, they imply a uniform (in $A \in \mathcal{D}_N$) sharp lower bound

$$\mathbb{P}^\beta_{A,-}\big(\ M_A = \mathbb{M}_A + a_N\ \big) \ \geq\ \frac{1}{\sqrt{2\pi\chi|A|}}\exp\big(-\frac{a_N^2}{2\chi|A|}\big)\big(1+\mathrm{o}(1)\big) \tag{1.3.5}$$

in the case of small moderate deviations $a_N \ll N^{4/3-\delta}$.

Then, in Section 3, we use a combination of the skeleton coarse graining techniques and the local limit estimates of Section 2 in order to derive sharp lower bounds

$$\mathbb{P}^\beta_{N,-}\big(\ M_N = -m^* + a_N\ \big) \ \geq\ \exp\big(-\sqrt{\frac{a_N}{2m^*}}\omega_1 - c_4\sqrt[4]{a_N}\log N\big), \tag{1.3.6}$$

on the large moderate deviations $a_N \gg N^{4/3+\delta}$.

The central result of Section 4 asserts that no subcritical, i.e., with the diameter $\ll N^{2/3}$, $K\log N$-large contours ever appear in the canonical ensemble

$$\mathbb{P}^\beta_{N,-}\big(\ \bullet \ \big|\ M_N = -m^*N^2 + a_N\ \big),$$

regardless of whether a_N is in the regime of small or large moderate deviations. In order to prove this, the probabilities of events we want to rule out are tested against sharp lower bounds (1.3.5) and (1.3.6) of two previous sections via the skeleton type decompositions similar to (1.3.4). As a byproduct we derive the full statement of Theorem C.

The proofs of Theorems A and B are concluded in Section 5. Combining the lower bound of Section 3 with the energy bound (1.3.2) and with the ubiquitous local limit estimates, we argue on the grounds of a simple isoperimetric stability statement, that with the exception of one large "Wulff" contour no other $N^{2/3-\nu}$-large contours appear in the typical configuration in the canonical ensemble. Since contours, whose diameter lies in the interval $[K\log N, N^{2/3-\nu}]$, were already ruled out in Section 4, this implies the one-contour assertions in Theorems A and B, and, with some additional work, their full statements.

Constants. The constants $K = K(\beta)$ in the definition of the basic phase $K\log N$ and R, a in the definition of the family of domains $\mathcal{D}_N$ are fixed throughout the article (regarding the choice of K see Theorem C and Remark 2.3.2).

We use also finite positive constants $c_1, c_2, \ldots$ .Their values are updated with each subsection. Depending on the context the value of these constants might depend on the inverse temperature β and on the cutoff value δ in the large/small moderate deviation setting, but, unless mentioned explicitly, not on anything else. In particular they are always independent of N, of the current domain $A \subset \mathbb{Z}^2$ and the deviation $a_N \in \mathcal{M}_A$ under consideration.

2. Local Estimates for Moderate Deviations

2.1. **Structure of local estimates.** Let $A \in \mathcal{D}_N$ and $a_N \in \mathcal{M}_A^+$. The main object of this section is to give precise estimates on the probability

$$\mathbb{P}_{A,-}^{\beta,s}\big(M_A = \mathbb{M}_A^s + a_N \big).$$

The classical approach to such estimates is to find the value of magnetic field

$$g = g(A, s(N), a_N),$$

such that the expected magnetization under the g-tilted state is precisely what we want,

$$\langle M_A \rangle_{A,-,g}^{\beta,s} = \mathbb{M}_A^s + a_N, \tag{2.1.1}$$

and, then, to rewrite the $\mathbb{P}_{A,-}^{\beta,s}$-probability in terms of the $\mathbb{P}_{A,-,g}^{\beta,s}$ one:

$$\begin{aligned}\mathbb{P}_{A,-}^{\beta,s}\big(M_A = \mathbb{M}_A^s + a_N\big) &= \exp\big\{ -(\mathbb{M}_A^s + a_N)g + \log\big\langle e^{gM_A}\big\rangle_{A,-}^{\beta,s}\big\}\, \mathbb{P}_{A,-,g}^{\beta,s}\big(M_A = \langle M_A\rangle_{A,-,g}^{\beta,s}\big) \\ &= \exp\Big\{-\int_0^g\int_r^g \langle M_A; M_A\rangle_{A,-,h}^{\beta,s} dh dr\Big\}\, \mathbb{P}_{A,-,g}^{\beta,s}\big(M_A = \langle M_A\rangle_{A,-,g}^{\beta,s}\big).\end{aligned} \tag{2.1.2}$$

Pushing the analogy with the classical case further, we encounter three types of problems :

1. Give a sufficiently precise estimate on $g = g(A, s(N), a_N)$ in (2.1.1).
2. Given such an estimate on g derive sufficiently sharp estimates on the semi-invariants of the family $\{\mathbb{P}_{A,-,r}^{\beta,s}\}$ for $r \in (0, g)$.
3. Prove a local CLT under $\mathbb{P}_{A,-,g}^{\beta,s}$.

All three problems are, of course, inter-related, and in the classical case this procedure leads to Gaussian estimates in any moderate deviations regime.

In our case, however, unless the values of a_N and s are further qualified, this approach is in general doomed. Indeed, if , for example, there is no restriction on the size of large contours, i.e., the constraint $s(N)$ does not appear at all, then the mean magnetization $\langle M_A\rangle_{A,-,g}^{\beta}$ is extremely sensitive to the changes of g of order $1/N$, and there is essentially a jump from the "−" to "+" phase within this range of the magnetic field. Moreover, as it was explained in the introduction, for $a_N \gg N^{4/3}$ the asymptotics of

$$\mathbb{P}_{A,-}^{\beta}\big(M_A = \mathbb{M}_A + a_N\big)$$

is not expected to be Gaussian at all.

Therefore, the important thing for us is to understand how the magnetization $\langle M_A\rangle_{A,-,g}^{\beta,s}$ and other semi-invariants of $\mathbb{P}_{A,-,g}^{\beta,s}$ change with g in the phase of $s(N)$-small contours. Such questions were investigated in [SS2], and in the next subsection we state and explain a version of the corresponding results in the latter article. Then we proceed to derive precise estimates in what happens to be the most important regime of $K\log N$-small contours. Finally, in the last subsection, we use these $K\log N$-phase moderate deviations results to obtain useful upper bounds on various other $s(N)$-scales.

2.2. **Estimates in cutoff ensembles.** The breaking of the classical limit behaviour in the $s(N)$-restricted phase manifests itself by the jump of the magnetization and by the explosion of the susceptibility, which, in their turn, are related to the appearance of abnormally large $\pm$-contours. On the heuristic level it is clear what should be the critical order of the magnetization g, at which those large contours should start to be favoured: for a $\pm$ contour of the linear size $s(N)$ one wins $\sim s^2 g$ on the level of magnetization and loses $\sim s$ on the level of surface energy. These two terms start to be comparable when $sg \sim 1$. Therefore no particular deviation from the classical behaviour should be expected as far as

$$gs(N) \ \ll \ 1 \tag{2.2.1}$$

Lemma 2.2.1 below is a mathematical counterpart of these intuitive considerations. It also generalizes the corresponding results in [SS2] (see Lemmas 2.3.4 and 2.3.5 there).

Let us first introduce some additional notation:
For a finite subset $B \subseteq \mathbb{Z}^2$ and $k \in \mathbb{R}_+$ we define

$$\Lambda_B(k) \ = \ \{y \in \mathbb{Z}^2 : \ \min_{x \in B} \|y - x\| \ < \ k\}.$$

If $B = \{x_1, x_2, ..., x_n\}$ we use notations $\Lambda_{x_1 x_2 \dots x_n}$ for Λ_B. Similarly for a local function ϕ we use

$$\Lambda_\phi \ \triangleq \ \Lambda_{\mathrm{supp}\phi},$$

where $\mathrm{supp}\phi$ is the support of ϕ. We use $r(\phi) \triangleq |\mathrm{supp}\phi|$ to denote the cardinality of the support of ϕ. Finally, for two local functions ϕ and ψ we let $d(\phi, \psi)$ denote the distance between the corresponding supports.

We state results in the asymptotic form as $N \to \infty$. Thus, the condition (2.2.1) should be understood in the sense that the sequence of nonnegative magnetic fields $g = g(N)$ satisfies

$$\lim_{N \to \infty} g(N)s(N) \ = \ 0. \tag{2.2.2}$$

Finally the estimates we give deteriorate with the cardinality of supports of local functions. In order to give the estimates in the uniform way we fix a number M and impose the following restriction:

$$r(\phi) \ + \ r(\psi) \ \le \ M. \tag{2.2.3}$$

Lemma 2.2.1. *For any $\beta > \beta_c$ and $M \in \mathbb{N}$ there exists a constant $c = c(M, \beta)$, such that for any sequence $(s(N), g(N))$ satisfying* (2.2.2) *one can choose two constants c_1 and c_2, such that*

$$\left| \langle \phi \rangle_{A,-,h}^{\beta,s} \ - \ \langle \phi \rangle_{\Lambda_\phi(s) \cap A,-,h}^{\beta} \right| \ \le \ c_1 \|\phi\| e^{-cs(N)}, \tag{2.2.4}$$

and

$$\left| \langle \phi ; \psi \rangle_{A,-,h}^{\beta,s} \right| \ \le \ c_2 \|\phi\| \|\psi\| e^{-cs(N) \wedge d(\phi,\psi)}, \tag{2.2.5}$$

uniformly in $N \in \mathbb{N}$, magnetic fields $h \in [0, g(N)]$, domains $A \subset \mathbb{Z}^2$ and local functions ϕ, ψ : $\Omega_A \mapsto \mathbb{R}$ satisfying the support size constraint (2.2.3).

Remark 2.2.2. For the value of M fixed, the Lemma gives means for a uniform control of semi-invariants up to the order M under $\mathbb{P}_{A,-,g}^{\beta,s}$. This proves to be a key to a classical treatment of both terms in the right hand side of (2.1.2) already in the case $M = 3$. Note also that the control estimates (1.1.1) on the expected values $\mathbb{M}_A$ and $\mathbb{M}_A^s$ at zero value of the magnetic field instantly follow from (2.2.4) and the fact (c.f. [CCS] and Section 1.3 in [SS2]) that for any $A, B \subseteq \mathbb{Z}^2$ and any local function ϕ with $\mathrm{supp}\phi \subseteq A \cap B$ and the cardinality of the support $|\mathrm{supp}\phi| = M$, there exist $c_3 = c_3(M, \beta)$ and $c_4 = c_4(M, \beta)$, such that

$$\left| \langle\phi\rangle^{\beta}_{A,-} - \langle\phi\rangle^{\beta}_{B,-} \right| \leq c_3 \|\phi\| \mathrm{e}^{-c_4 d(\mathrm{supp}\phi, A\Delta B)}, \tag{2.2.6}$$

where $d(C, D)$ is the $\|\cdot\|$-distance between the sets C and D.

As in [SS2] in the heart of the proof lies an estimate on the exponential decay of certain connectivity functions (Proposition 2.2.3 below):

Let us first of all recall the notions of $+$ and $+*$ connectedness: two sites $x, y \in \mathbb{Z}^2$ are called neighbours ($*$-neighbours respectively), if $\|x-y\|_1 = 1$ (respectively $\|x-y\| = 1$), where, as before, $\|\cdot\|_1$ and and $\|\cdot\|$ are respectively the l_1 and the supremum lattice norms.

A sequence of sites $x_1, ..., x_n$ is called a connected (respectively $*$-connected) chain if each pair (x_i, x_{i+1}); $i = 1, ..., n-1$, is a pair of neighbours ($*$-neighbours respectively). Finally, a set $B \subseteq \mathbb{Z}^2$ is $+$ ($+*$ respectively) connected to a set $C \subseteq \mathbb{Z}^2$, if there exists a connected ($*$-connected) chain of sites $x_1, ..., x_n$, such that $x_1 \in B$, $x_n \in C$ and $\sigma(x_i) = 1$; $i = 1, .., n$. The corresponding event is denoted as $\{B \xrightarrow{+} C\}$ (respectively $\{B \xrightarrow{+*} C\}$). The notions of $-$ and $-*$ connectedness are defined in a completely similar way.

Proposition 2.2.3. *Let the sequence $(s(N), g(N))$ be as in the conditions of Lemma 2.2.1 . Then, there exists a positive constant $c_5 = c_5(\beta)$, such that uniformly in $k \in \mathbb{R}_+$, magnetic fields $h \in [0, g(N)]$ and domains $A \subseteq \mathbb{Z}^2$,*

$$\mathbb{P}^{\beta,s}_{A,-,h}\left(x \xrightarrow{+} \Lambda_x(k)^c \right) \leq e^{-c_5 k}, \tag{2.2.7}$$

as soon as N is large enough.

Proof. There is nothing to prove when $k > s(N)$. Indeed, in this case the very restriction of the $s(N)$-small phase rules out the possibility of x being $+$ connected to the boundary $\partial\Lambda_x(k)$. Let us start by proving the assertion in the case $k \in [s/2, s]$. At this stage we almost literally follow the logic of the corresponding proof in [SS2] (Lemma 2.3.4):

Let $\alpha = (\gamma_1, ..., \gamma_n)$ be the collection of all exteriour $s(N)$-small contours, such that

$$\mathrm{int}(\gamma_i) \cap \partial\Lambda_x(3s(N)) \neq \emptyset, \qquad i = 1, ..., n,$$

where ∂B is used to denote the outer boundary of a lattice domain $B \subset \mathbb{Z}^2$. Define the random box Λ_α via

$$\Lambda_\alpha = \Lambda_x\big(3s(N)\big) \setminus \bigcup_{\gamma\in\alpha} \widehat{\mathrm{int}(\gamma)},$$

where $\widehat{\mathrm{int}(\gamma)}$ is the closure of $\mathrm{int}(\gamma)$. Note that due to the restriction on contours to be $s(N)$-small, $\Lambda_x(k) \subseteq \Lambda_\alpha$ for each $k \leq s(N)$. Note also that the event corresponding to α does not depend on the spins inside the box Λ_α.

Employing the decomposition with respect to all possible realizations of Λ_α, we obtain

$$\mathbb{P}^{\beta,s}_{A,-,h}\left(x \xrightarrow{+} \Lambda_x(k)^c \right) = \sum_\alpha p_\alpha \mathbb{P}^{\beta,s}_{\Lambda_\alpha\cap A,-,h}\left(x \xrightarrow{+} \Lambda_x(k)^c \right), \tag{2.2.8}$$

where $\{p_\alpha\}$ is a collection of probabilistic weights; $\sum_\alpha p_\alpha = 1$, whose precise values are of no importance for us.

Notice, that the $s(N)$-restriction played the crucial role in the above reduction. Once, however, (2.2.8) is established, we do not need the phase of $s(N)$-small contours any more: Since the event {all contours are $s(N)$ − small} is non-increasing, one can take advantage of the FKG properties of Ising measures and develop the right hand side of (2.2.8) further as,

$$\begin{aligned}\mathbb{P}^{\beta,s}_{A,-,h}\big(\, x \xrightarrow{+} \Lambda_x(k)^c \,\big) \;&\leq\; \sum_{\alpha} p_\alpha \mathbb{P}^{\beta}_{\Lambda_\alpha\cap A,-,h}\big(\, x \xrightarrow{+} \Lambda_x(k)^c \,\big)\\ &\leq\; \mathbb{P}^{\beta}_{\Lambda_x(3s(N)),-,h}\big(\, x \xrightarrow{+} \Lambda_x(k)^c \,\big)\end{aligned}$$

On the other hand, $\big|\Lambda_x\big(3s(N)\big)\big| \leq 36s(N)^2$. Since, due to the condition (2.2.2), the magnetic field $h \in [0,g]$ is uniformly under control, this means that the logarithm of the Radon-Nikodym derivative

$$\log\left|\frac{d\mathbb{P}^{\beta}_{\Lambda_x(3s),-,h}}{d\mathbb{P}^{\beta}_{\Lambda_x(3s),-,0}}\right|$$

is $\mathrm{o}\big(s(N)\big)$ uniformly in $h \in [0,g]$. Therefore, again uniformly in $h \in [0,g]$,

$$\mathbb{P}^{\beta,s}_{\Lambda_x(3s),-,h}\big(\, x \xrightarrow{+} \Lambda_x(k)^c \,\big) \;\leq\; \mathrm{e}^{\mathrm{O}(s(N))}\mathbb{P}^{\beta}_{\Lambda_x(3s),-,0}\big(\, x \xrightarrow{+} \Lambda_x(k)^c \,\big).$$

By the results on the exponential decay of connectivities at zero value of the magnetic field [CCS] the latter quantity is bounded above by some $e^{-c_6(\beta)k}$, and the claim of the Proposition follows for k in the range $k \in [s/2, s]$.

Let us now pick any value of $k \in (0, s/2)$. The problem with smaller values of k is that from the first glance the exponential decay in the zero magnetic field phase might be overshot by the value of the Radon-Nikodym derivative over boxes of the $s(N)$ linear size. This, however, can be circumvented in the following way:

We pick a finite sequence of numbers $k_1, k_2, ..., k_n$; $n = n(k)$ and construct the corresponding sequence of boxes $\Lambda_i \triangleq \Lambda_x(k_i)$, where

$$k_1 \equiv k \qquad k_{i+1} = 2k_i \qquad \text{and} \qquad n(k) = \max_{2^i k < 3s(N)} i.$$

Now notice that,

$$\begin{aligned}&\mathbb{P}^{\beta}_{\Lambda_x(3s),-,h}\big(\, x \xrightarrow{+} \Lambda_x(k)^c \,\big)\\ &\quad\leq\; \mathbb{P}^{\beta}_{\Lambda_x(3s),-,h}\big(\, x \xrightarrow{+} \Lambda_x(k)\ ;\big\{\Lambda_{n-1} \xrightarrow{+} \Lambda_n^c\big\}^c\big) \;+\; \mathbb{P}^{\beta}_{\Lambda_x(3s),-,h}\big(\Lambda_{n-1} \xrightarrow{+} \Lambda_n^c\big). \qquad (2.2.9)\end{aligned}$$

Each configuration $\sigma \in \big\{\Lambda_{n-1} \xrightarrow{+} \Lambda_n^c\big\}^c$ contains a $-*$ connected loop inside $\Lambda_n \setminus \Lambda_{n-1}$ around Λ_{n-1}. Thus, with a proper splitting argument we might hope to reduce the computation of the probability of $\{x \xrightarrow{+} \Lambda_x(k)^c\}$ in a first term on the right hand side of (2.2.9) to a box of the size smaller (roughly one half) than that of $\Lambda_x(3s)$. This is very advantageous, since the impact of the magnetic field through the Radon-Nikodym derivative diminishes in this way. The second term on the right hand side of (2.2.9) is already treatable by the first part of the proof. We then decompose probabilities in the smaller box exactly in the same fashion as in (2.2.9) and continue in this way until we reduce a computation to uniform estimates over boxes B; $\Lambda_1 \subseteq B \subseteq \Lambda_2$.

We now pass to the formal setting: Define a sequence of random domains Λ_{α_i} via

$$\Lambda_{\alpha_n} \equiv \Lambda_x\big(3s(N)\big) \qquad \text{and} \qquad \Lambda_{\alpha_i} \triangleq \Lambda_{i+1} \setminus \widehat{D_{\alpha_i}};\ i = 1, ..., n-1,$$

where random domains D_{α_i} above are given by

$$D_{\alpha_i} \ \stackrel{\Delta}{=}\ \{y \in D_{i+1} :\ y \xrightarrow{+} \Lambda_{i+1}^c\}.$$

Now observe that

$$\left\{\Lambda_i \xrightarrow{+} \Lambda_{i+1}^c\right\}^c \implies \Lambda_i \subseteq \Lambda_{\alpha_i} \subseteq \Lambda_{i+1}. \tag{2.2.10}$$

Also observe that, as before, for any $B \subseteq \Lambda_{i+1}$ the event $\left\{\Lambda_{\alpha_i} = B\right\}$ does not depend on the value of spins inside B. Unfolding (2.2.9) and splitting with respect to the different realizations of Λ_{α_i}; $i = 1, ..., n$, we obtain:

$$\begin{aligned}
&\mathbb{P}^{\beta}_{\Lambda_x(3s),-,h}\left(\ x \xrightarrow{+} \Lambda_x(k)^c\ \right) \\
&\leq \max_{\alpha_1,\dots,\alpha_{n-1}} \left\{\ \mathbb{P}^{\beta}_{\Lambda_{\alpha_1},-,h}\left(\ x \xrightarrow{+} \Lambda_x(k)^c\ \right)\ +\ \sum_{i=2}^{n} \mathbb{P}^{\beta}_{\Lambda_{\alpha_i},-,h}\left(\ \Lambda_{i-1} \xrightarrow{+} \Lambda_i^c\ \right)\ \right\},
\end{aligned} \tag{2.2.11}$$

where the maximum is over all possible realizations of boxes $\Lambda_{\alpha_1}, ..., \Lambda_{\alpha_{n-1}}$, which satisfy the inclusion on the right hand side of (2.2.10)

Estimating the Radon-Nikodym derivative with respect to the zero field measures we infer from (2.2.11), (2.2.10) and the FKG inequality,

$$\mathbb{P}^{\beta}_{\Lambda_x(3s),-,h}\left(\ x \xrightarrow{+} \Lambda_x(k)^c\ \right)\ \leq\ e^{4k^2h}\ \mathbb{P}^{\beta}_{-}\left(\ x \xrightarrow{+} \Lambda_x(k)^c\ \right)\ +\ \sum_{i=2}^{n} e^{4k_i^2h}\ \mathbb{P}^{\beta}_{-}\left(\ \Lambda_{i-1} \xrightarrow{+} \Lambda_i^c\ \right).$$

The result now follows by the exponential decay properties of $\mathbb{P}^{\beta}_{-}$ and the assumption (2.2.2). □

Remark 2.2.4. Notice that the only relevant feature of the box $\Lambda_x(3s)$ we have used in the last part of the proof is the upper bound on its cardinality, $|\Lambda_x(3s)| \leq 36s^2$. A straightforward rerun of the arguments above leads to the following general estimate:

Fix a number $t \in \mathbb{R}$, and let $s(N)$ and $g(N)$ obey (2.2.2). Then, there exists $c_7 = c_7(\beta, t)$, such that once N is large enough,

$$\mathbb{P}^{\beta}_{B,-}\left(\ x \stackrel{+}{\to} \Lambda_x(k)^c\ \right)\ \leq\ \exp\left(-\, c(\beta, t)k\right). \tag{2.2.12}$$

uniformly in domains $B \subset \mathbb{Z}^2$ satisfying $|B| \leq ts^2$, $x \in B$, magnetic fields $h \in [0, g]$ and $k \in \mathbb{N}$.

Proof of Lemma 2.2.1. Given Proposition 2.2.3, we can now closely follow, for proving (2.2.4), the corresponding arguments in [SS2]. Since, however, the setting, notation and scaling of the latter article is different from ours, we present here the rest of the proof in full details for the sake of completeness.

It is enough to consider only the case of monotone non-decreasing local ϕ and ψ . With $A \subset \mathbb{Z}^2$ being fixed, set

$$E_\phi\ =\ \left\{\ \Lambda_\phi\left(\frac{s(N)}{8M}\right) \xrightarrow{+} \Lambda_\phi\left(\frac{s(N)}{4M}\right)^c\ \right\}\ \subseteq \Omega_A,$$

where M is the maximal support size in (2.2.3).

By the Proposition 2.2.3,

$$\mathbb{P}^{\beta,s}_{A,-,h}\left(\ E_\phi\ \right)\ \leq\ c_8 \mathrm{e}^{-c_9 s(N)},$$

uniformly in $A \subseteq \mathbb{Z}^2$ and $h \in [0, g(N)]$. On the other hand every extended configuration $\sigma \cup \{-1\}^{\partial A}$; $\sigma \in E^c_\phi$, contains a closed $-*$ loop of spins inside $\widehat{A} \cap \Lambda_\phi(\frac{s(N)}{4M})$ surrounding $\widehat{A} \cap \Lambda_\phi(\frac{s(N)}{8M})$, where, as before, $\widehat{A} \triangleq A \cup \partial A$ is the closure of A. Thus, splitting E_ϕ^c; $E_\phi^c = \cup \alpha$, according to the realizations of the random set

$$\Lambda_\alpha \triangleq \Lambda_\phi(\frac{s(N)}{4M}) \setminus \widehat{D_\alpha} \qquad \text{and} \qquad D_\alpha \triangleq \{y : y \xrightarrow{+} \Lambda_\phi(\frac{s(N)}{4M})^c\},$$

we obtain:

$$\langle \phi; E_\phi^c \rangle_{A,-,h}^{\beta,s} = \sum_\alpha p_\alpha \langle \phi \rangle_{\Lambda_\alpha \cap A,-,h}^{\beta,s},$$

where $p_\alpha = \mathbb{P}_{A,-,h}^{\beta,s}(\alpha)$ and, of course, $\sum p_\alpha = \mathbb{P}_{A,-,h}^{\beta,s}(E_\phi^c)$.

By construction, each realization of Λ_α satisfies

$$\Lambda_\phi(\frac{s(N)}{8M}) \subseteq \Lambda_\alpha \subseteq \Lambda_\phi(\frac{s(N)}{4M}). \tag{2.2.13}$$

Consequently the super-index $s = s(N)$ can be trivially omitted in $\mathbb{P}_{\Lambda_\alpha,-,h}^{\beta,s}$; indeed, the diameter of Λ_α is less than s for each α. But the unrestricted phase already enjoys FKG inequality. Subsequently, using the monotonicity assumption on ϕ, we infer:

$$\langle \phi \rangle_{\Lambda_\phi(\frac{s(N)}{8M}) \cap A,-,h}^{\beta} \le \langle \phi \rangle_{\Lambda_\alpha \cap A,-,h}^{\beta} \equiv \langle \phi \rangle_{\Lambda_\alpha \cap A,-,h}^{\beta,s} \le \langle \phi \rangle_{\Lambda_\phi(s(N)) \cap A,-,h}^{\beta}.$$

As before, splitting the event $\{\mathrm{supp}(\phi) \xrightarrow{+} \Lambda_\phi(\frac{s(N)}{8M})^c\}^c$ with respect to the random + connected component of $\Lambda(\frac{s(N)}{8M})^c$, we obtain from the FKG inequality,

$$\begin{aligned} 0 &\le \langle \phi \rangle_{\Lambda_\phi(s) \cap A,-,h}^{\beta} - \langle \phi \rangle_{\Lambda(\frac{s(N)}{8M}) \cap A,-,h}^{\beta} \\ &\le \|\phi\| \mathbb{P}_{\Lambda_\phi(s) \cap A,-,h}^{\beta} \Big(\mathrm{supp}(\phi) \xrightarrow{+} \Lambda_\phi(\frac{s(N)}{8M})^c \Big) \le \|\phi\| e^{-c_{10} s(N)}, \end{aligned} \tag{2.2.14}$$

where the last inequality follows from (2.2.12) in Remark 2.2.4. This proves the first assertion (2.2.4) of Lemma 2.2.1.

The covariance estimate (2.2.5) is a consequence of (2.2.4) and the decay property (2.2.7). Indeed, proceeding as in (2.2.14) above, we infer from the estimate (2.2.4),

$$\langle \phi; \psi \rangle_{A,-,h}^{\beta,s} = \langle \phi\psi \rangle_{\Lambda_{\phi,\psi}(\frac{s}{M}) \cap A,-,h}^{\beta} - \langle \phi \rangle_{\Lambda_{\phi,\psi}(\frac{s}{M}) \cap A,-,h}^{\beta} \langle \psi \rangle_{\Lambda_{\phi,\psi}(\frac{s}{M}) \cap A,-,h}^{\beta} + \mathrm{O}\big(\|\phi\|\|\psi\| \mathrm{e}^{-c_{11} s(N)}\big),$$

where $\Lambda_{\phi,\psi} \triangleq \Lambda_\phi \cup \Lambda_\psi$. Note that the diameter of each connected component of $\Lambda_{\phi,\psi}(\frac{s(N)}{M})$ does not exceed, by the condition (2.2.3), $s(N)$. Thus, the problem boils down to the following: Prove that uniformly in connected domains B; $\mathrm{diam}(B) \le s(N)$, monotone functions ϕ and ψ satisfying (2.2.3) and magnetic fields $h \in [0, g]$,

$$\langle \phi; \psi \rangle_{B,-,h}^{\beta} = \langle \phi\psi \rangle_{B,-,h}^{\beta} - \langle \phi \rangle_{B,-,h}^{\beta} \langle \psi \rangle_{B,-,h}^{\beta} \le c_{12} \|\phi\|\|\psi\| \mathrm{e}^{-c_{13} d(\phi,\psi)}. \tag{2.2.15}$$

Define the "median" set $M(\phi, \psi)$ between the two supports as

$$M(\phi,\psi) \triangleq \Big\{ z : \Big| \min_{x \in \mathrm{supp}\phi} \|z - x\| - \min_{y \in \mathrm{supp}\psi} \|z - y\| \Big| \le 1 \Big\},$$

and consider the event

$$E_{\phi,\psi} \triangleq \{\mathrm{supp}\phi \xrightarrow{+} M(\phi,\psi)\} \bigcup \{\mathrm{supp}\psi \xrightarrow{+} M(\phi,\psi)\}.$$

Using the exponential decay of connectivities (2.2.12), one readily obtains that

$$\mathbb{P}^{\beta}_{B,-,h}\big(E_{\phi,\psi}\big) \;\leq\; c_{14}\mathrm{e}^{-c_{15}d(\phi,\psi)}.$$

On the complement of $E_{\phi,\psi}$ the supports of ϕ and ψ are necessarily separated by a $-*$ connected chain, and, using the splitting decomposition with respect to the random domains

$$B_\alpha \;\triangleq\; B\setminus\widehat{D_\alpha} \qquad \text{and} \qquad D_\alpha \;\triangleq\; \big\{z : z \stackrel{+}{\longrightarrow} M(\phi,\psi)\big\},$$

we readily obtain that $E^c_{\phi,\psi}$ render a non-positive contribution to the left hand side of (2.2.15). Lemma 2.2.1 is completely proven.

2.3. **Estimates in the $K\log N$-Phase.** We are going to give an estimate on $g = g(A, s(N), a_N)$ in (2.1.1) on the basic scale $s(N) = K\log N$ and for the moderate values of deviations $a_N \in \mathcal{M}^+_N$ satisfying

$$a_N \;\ll\; \frac{N^2}{\log N}. \tag{2.3.1}$$

Remark 2.3.1. The above condition on the range of a_N is of technical nature: The requirement $a_N \ll N^2/\log N$ is related to the condition (2.2.2) and imposes no limitations on the investigation of the unrestricted phase $\big\langle\cdot\big\rangle^{\beta}_{N,-}$, which, in the end of all, is our primary object. Indeed, as we shall see later, large a_N shifts of the empirical magnetization in the unrestricted phase lead to creation of large contours on an appropriate scale and are, thereby, reduced to moderate deviations of the magnetization of much smaller order inside and outside those contours.

On the other hand, the non-positive values of a_N; $a_N \leq 0$, correspond to the deviations outside the phase transition region. In particular the choice of g in (2.1.1) is, in this case, also non-positive. Since the event $\mathcal{A}_s \triangleq \{$ There are no $s(N)$ – large $\pm$ contours$\}$ is non-increasing,

$$\mathbb{P}^{\beta}_{A,-,g}\big(\;\mathcal{A}_s\;\big) \;\geq\; \mathbb{P}^{\beta}_{A,-}\big(\;\mathcal{A}_s\;\big) \;\geq\; 1-\mathrm{e}^{-c_1 s(N)},$$

where the first inequality follows from the FKG properties of Ising measures, whereas the second one is a consequence of the energy estimate (1.3.2). It is, then, easy to see that the $a_N \leq 0$ case essentially boils down to the corresponding classical regime of deviations in the unrestricted ensemble, which was summarized in (1.1.2). In order to have a convenient reference we restate the latter estimate here in the general case of $s(N)$-restricted phases:

For any $A\subset\mathbb{Z}^2$, large contour parameter $s(N)$, magnetic field $g\leq 0$ and any $a\leq 0$, the following Gaussian upper bound is valid:

$$\mathbb{P}^{\beta,s}_{A,-,g}\big(\;M_A \;=\; \big\langle M_A\big\rangle^{\beta,s}_{A,-,g} + a\;\big) \;\leq\; \exp\big(\;-c_2(\beta)\frac{a^2}{|A|}\big). \tag{2.3.2}$$

We turn now to the moderate deviations in the phase transition range (2.3.1).

On any scale $s = s(N)$, we have:

$$\begin{aligned}\big\langle M_A\big\rangle^{\beta,s}_{A,-,g} \;&=\; \big\langle M_A\big\rangle^{\beta,s}_{A,-} \;+\; \int_0^g \big\langle M_A;M_A\big\rangle^{\beta,s}_{A,-,r}dr\\ &=\; \big\langle M_A\big\rangle^{\beta,s}_{A,-} \;+\; g\big\langle M_A;M_A\big\rangle^{\beta,s}_{A,-} \;+\; \int_0^g\int_0^r \big\langle M_A;M_A;M_A\big\rangle^{\beta,s}_{A,-,h}dhdr.\end{aligned} \tag{2.3.3}$$

By (2.2.4) in Lemma 2.2.1,

$$\begin{aligned} \left\langle M_A ; M_A \right\rangle_{A,-}^{\beta,s} &= \sum_{x,y\in A} \left\langle \sigma(x);\sigma(y)\right\rangle_{A,-}^{\beta,s} \\ &= \sum_{x,y\in A} \left\langle \sigma(x);\sigma(y)\right\rangle_{A\cap\Lambda_{xy}(s),-}^{\beta} \ +\ \mathrm{O}\big(\mathrm{e}^{-c_3(\beta)s(N)}|A|^2\big). \end{aligned} \tag{2.3.4}$$

Remark 2.3.2. The second summand in the bottom line above dictates the choice of the parameter $K = K(\beta)$ in the definition of the basic $s(N) = K\log N$ scale. Namely we pick up K to be so large that $Kc_3(\beta) \gg 1$. As a result the second term on the right hand side above is $\mathrm{o}(1)$.

From (2.2.6) we obtain, that the $\mathbb{P}^{\beta}_{A\cap\Lambda_{xy}(s),-}$ covariances are related to the corresponding infinite volume covariances via

$$\Big|\left\langle \sigma(x);\sigma(y)\right\rangle_{-}^{\beta} \ -\ \left\langle \sigma(x);\sigma(y)\right\rangle_{A\cap\Lambda_{xy}(s),-}^{\beta}\Big| \ \leq\ \exp\big(-c_4(\beta)d(\partial A,\{x,y\})\wedge s(N)\big)$$

uniformly in domains $A\subset\mathbb{Z}^2$. Consequently,

$$\Big|\ \left\langle M_A;M_A\right\rangle_{A,-}^{\beta,s} \ -\ |A|\chi\ \Big| \ \leq\ c_4(\beta)\big|\partial A\big|. \tag{2.3.5}$$

On the other hand, for each g satisfying condition (2.2.2), the covariance estimate (2.2.5) implies a similar bound on higher order semi-invariants (see Appendix B of[ML]). Therefore we have:

$$\Big|\int_0^g\int_0^r \left\langle M_A;M_A;M_A\right\rangle_{A,-,h}^{\beta,s} dhdr\ \Big| \ \leq\ c_5(\beta)g^2\big|A\big|. \tag{2.3.6}$$

Since, by the definition , $|\partial A|\leq RN\log N$ for every $A\in\mathcal{D}_N$, the bounds (2.3.5) and (2.3.6) lead to the following estimate on the solution $g = g(A,s,a_N)$ of (2.1.1):

$$g \ =\ \frac{a_N}{\chi|A|}\ +\ \mathrm{O}\big(\frac{a_N}{N^3}(\log N\vee\frac{a_N}{N})\big). \tag{2.3.7}$$

Notice , that in the $s = K\log N$ phase the range restriction $a_N \ll N^2/\log N$ is, then, precisely translated into the smallness condition (2.2.2); $\lim g(N)s(N) = 0$, on the magnetic field $g(N) = g(A,s,a_N)$.

Using the above results we can already estimate the crucial expression in the exponent on the right hand side of (2.1.2):

$$\begin{aligned} &\int_0^g\int_r^g \left\langle M_A;M_A\right\rangle_{A,-,h}^{\beta,s} dhdr \\ &= -\frac{g^2}{2}\left\langle M_A;M_A\right\rangle_{A,-}^{\beta,s} \ -\ \int_0^g\int_r^g\int_0^h \left\langle M_A;M_A;M_A\right\rangle_{A,-,q}^{\beta,s} dqdhdr \\ &= -\frac{g^2}{2}\left\langle M_A;M_A\right\rangle_{A,-}^{\beta,s}\ +\ \mathrm{O}\big(g^3|A|\big) \\ &= -\frac{a_N^2}{2\chi|A|}\ +\ \mathrm{O}\big(\frac{a_N^2}{N^3}(\log N\vee\frac{a_N}{N})\big). \end{aligned} \tag{2.3.8}$$

We switch now to the $K\log N$ scale proper (with some fixed choice of $K = K(\beta)$ as it was specified in the Remark 2.3.2).

Lemma 2.3.3. *Let $s(N) = K\log N$. Given a sequence of numbers $\hat{a}_N$ satisfying* (2.3.1)*, the following asymptotic (in N) estimate;*

$$\mathbb{P}^{\beta,s}_{A,-}\big(\ M_A\ = \mathbb{M}^s_A + a_N\big)\ = \\ = \ \frac{1}{\sqrt{2\pi\chi|A|}}\exp\big\{-\frac{a_N^2}{2\chi|A|} + O\big(\frac{a_N^2}{N^3}(\log N\vee\frac{a_N}{N})\big)\big\}\big(1\ +\ \mathrm{o}(1)\big), \tag{2.3.9}$$

holds uniformly in $A\in\mathcal{D}_N$ and $a_N\in\mathcal{M}^+_A\cap[0,\hat{a}_N]$.

Proof. The prefactor near the exponent is due to the local CLT, which we prove in the next subsection. The expression in the exponent follows by the substitution of (2.3.8) into the formula (2.1.2). □

Remark 2.3.4. Note that in the range of small moderate deviations $a_N \ll N^{4/3-\delta}$ the correction term $\mathrm{O}\big(\frac{a_N^2}{N^3}(\log N\vee\frac{a_N}{N})\big)$ is $\mathrm{o}(1)$ and, hence, can be omitted.

2.4. **Local CLT estimate.** We continue our investigation of the basic phase of $s(N) = K\log N$ small contours and remain in the the framework of the previous subsection. Our main task here is to justify the form of the prefactor in (2.3.9). Let $A\in\mathcal{D}_N$ and assume that $a_N\in\mathcal{M}^+_A$ complies with the range restriction (2.3.1). We choose $g = g(A, K\log N, a_N)$ according to the formula (2.3.7), so that $\mathbb{M}^s_A + a_N$ is produced as the mean magnetization under $\mathbb{P}^{\beta,s}_{A,-,g}$, i.e., so that (2.1.1) is satisfied. As it was already noted in the lines following (2.3.7) such choice of g is compatible with the condition (2.2.2). It would be convenient to state our CLT result regardless of a_N.

Lemma 2.4.1. *Assume that (the sequence of) magnetic fields $\bar{g}(N)\geq 0$ satisfies* (2.2.2)*. Then, uniformly in domains $A\in\mathcal{D}_N$ and in magnetic fields $g\in[0,\bar{g}(N)]$, such that*

$$\big\langle M_A\big\rangle^{\beta,s}_{A,-,g}\ \in\ \mathcal{M}^+_A,$$

the following asymptotic ($N\to\infty$) estimate is valid:

$$\mathbb{P}^{\beta,s}_{A,-,g}\big(\ M_A\ =\ \big\langle M_A\big\rangle^{\beta,s}_{A,-,g}\ \big)\ =\ \frac{1}{\sqrt{2\pi\chi|A|}}\big(\ 1\ +\ \mathrm{o}(1)\ \big) \tag{2.4.1}$$

Lemma 2.4.1 is a local CLT type result. We are following here the idea of [DT] to prove it from an integral CLT estimate combined with decoupling properties of finite range Gibbs state.

Set

$$\widehat{M_A}\ =\ \frac{M_A\ -\ \big\langle M_A\big\rangle^{\beta,s}_{A,-,g}}{\sqrt{\chi|A|}}$$

By the Fourier inversion formula,

$$\mathbb{P}^{\beta,s}_{A,-,g}\big(\ M_A\ =\ \big\langle M_A\big\rangle^{\beta,s}_{A,-,q}\ \big)\ =\ \mathbb{P}^{\beta,s}_{A,-,g}\big(\ \widehat{M_A}\ =\ 0\ \big)\ =\ \frac{1}{2\pi\sqrt{\chi|A|}}\int_{-\pi\sqrt{\chi|A|}}^{\pi\sqrt{\chi|A|}}\big\langle \mathrm{e}^{it\widehat{M_A}}\big\rangle^{\beta,s}_{A,-,g}dt.$$

Using the identity $1 = 1/\sqrt{2\pi}\int \mathrm{e}^{-t^2/2}dt$, we obtain that for each choice of a large number $r>0$,

$$2\pi\sqrt{\chi|A|}\ \Big|\ \mathbb{P}^{\beta,s}_{A,-,g}\big(\ M_A\ =\ \langle M_A\rangle^{\beta,s}_{A,-,g}\ \big)\ -\ \frac{1}{\sqrt{2\pi\chi|A|}}\ \Big|\ \leq\ \mathrm{I}_1\ +\ \mathrm{I}_2\ +\ \mathrm{I}_3, \tag{2.4.2}$$

where

$$\mathrm{I}_1\ =\ \Big|\int_{-r}^{r}\big(\mathrm{e}^{-\frac{t^2}{2}}\ -\ \langle \mathrm{e}^{it\widehat{M_A}}\rangle^{\beta,s}_{A,-,g}\big)dt\ \Big|,\qquad \mathrm{I}_2\ =\ \int\limits_{r<|t|<\pi\sqrt{\chi|A|}}\Big|\langle \mathrm{e}^{it\widehat{M_A}}\rangle^{\beta,s}_{A,-,g}\Big|dt,$$

and finally, $\mathrm{I}_3\ =\ \int_{|t|>r}\mathrm{e}^{-\frac{t^2}{2}}dt$. Clearly, the last term satisfies $\mathrm{I}_3\leq c_1\mathrm{e}^{-r^2/2}$. We are going to show that also for each r fixed,

$$\lim_{N\to\infty}\mathrm{I}_1\ =\ 0\qquad\text{and}\qquad \mathrm{I}_2\ \leq\ \mathrm{e}^{-c_2r^2\wedge s(N)} \tag{2.4.3}$$

uniformly in $A\in\mathcal{D}_N$ and $g\in[0,\bar{g}]$.

Let us start with I_1 part of (2.4.3). This is an integral CLT estimate. In order to prove it we shall follow the approach of [ML]. Namely, it happens to be a consequence of the following convergence result on the level on log-moment generating functions:

$$\lim_{N\to\infty}\log\langle \mathrm{e}^{t\widehat{M_A}}\rangle^{\beta,s}_{A,-,g}\ =\ \frac{t^2}{2} \tag{2.4.4}$$

uniformly on compact subsets of $\mathbb{R}$, $A\in\mathcal{D}_N$ and $g\in[0,\bar{g}]$. Indeed, once (2.4.4) is established, the corresponding part of (2.4.3) is implied by the results of Appendix C in [ML] and an easy compactness argument. In its turn (2.4.4) is a consequence of the estimates on semi-invariants of Lemma 2.2.1: Expanding $\langle \mathrm{e}^{t\widehat{M_A}}\rangle^{\beta,s}_{A,-,g}$ up to the third order terms,

$$\log\langle \mathrm{e}^{t\widehat{M_A}}\rangle^{\beta,s}_{A,-,g}\ =\ \frac{t^2}{2}\langle\widehat{M_A};\widehat{M_A}\rangle^{\beta,s}_{A,-,g}\ +\ \int_0^t\int_0^h\int_g^{g+q/\sqrt{\chi|A|}}\langle\widehat{M_A};\widehat{M_A};\widehat{M_A}\rangle^{\beta,s}_{A,-,\tau}d\tau dq dh$$

$$=\frac{t^2}{2\chi|A|}\sum_{x,y\in A}\langle\sigma(x);\sigma(y)\rangle^{\beta,s}_{A,-,g}\ +\ \mathrm{O}\Big(\ \frac{t^3}{N^3}\max_{|g-g'|\leq t/\sqrt{\chi|A|}}\Big|\sum_{x,y,z\in A}\langle\sigma(x);\sigma(y);\sigma(z)\rangle^{\beta,s}_{A,-,g'}\Big|\ \Big). \tag{2.4.5}$$

In order to estimate the second term in the right hand side above notice, that the magnetic field g' also satisfies (2.2.2) in the whole range of its definition, more precisely,

$$K\log N\max_{A\in\mathcal{D}_N}\max_{|g'-g|\leq t/\sqrt{\chi|A}}g'\ \leq\ \bar{g}K\log N+\frac{tK\log N}{N\sqrt{\chi}}\ll\ 1,$$

on any compact interval $t\in[0,T]$, provided only that N is sufficiently large. Consequently, as with (2.3.6), it follows from (2.2.5), that

$$\max_{A\in\mathcal{D}_N}\max_{|g-g'|\leq t/\sqrt{\chi|A|}}\Big|\sum_{x,y,z\in A}\langle\sigma(x);\sigma(y);\sigma(z)\rangle^{\beta,s}_{A,-,g'}\Big|\ \leq\ c_3(\beta)N^2.$$

Similarly, the covariance term in (2.4.5) can be further expanded as

$$\langle M_A; M_A\rangle^{\beta,s}_{A,-,g} = \langle M_A; M_A\rangle^{\beta,s}_{A,-} + \int_0^g \langle M_A; M_A; M_A\rangle^{\beta,s}_{A,-,h} dh$$
$$= \langle M_A; M_A\rangle^{\beta,s}_{A,-} + \mathrm{O}\big(gN^2\big).$$

On the other hand, the zero-field covariance above was already estimated in (2.3.5). We, thus, infer:

$$\max_{|t|\leq T}\max_{A\in\mathcal{D}_N} \Big|\ \log\big\langle \mathrm{e}^{t\widehat{M_A}}\big\rangle^{\beta,s}_{A,-,g} - \frac{t^2}{2}\ \Big| = \mathrm{O}\big(\frac{T^3}{N} + T^2\bar{g}(N)\big),$$

and (2.4.4) follows.

Finally, the I_2 part of (2.4.3) is implied by the following upper bound on on characteristic functions $\big\langle \mathrm{e}^{it\widehat{M_A}}\big\rangle^{\beta,s}_{A,-,g}$:

For all $A\in\mathcal{D}_N$, $g\in[0,\bar{g}]$ and $t\in\mathbb{R}$,

$$\Big|\ \big\langle \mathrm{e}^{it\widehat{M_A}}\big\rangle^{\beta,s}_{A,-,g}\ \Big| \leq \mathrm{e}^{-c_4t^2} \vee \mathrm{e}^{-c_5(\beta)s(N)}, \tag{2.4.6}$$

as we prove next. The proof of (2.4.6) is inspired by the conditional argument of [DT]. With the basic large contour parameter $s(N) = K\log N$ fixed, let us define a sub-lattice $\tilde{\mathbb{Z}}^2 = [\frac{s(N)}{2}]\mathbb{Z}^2\subset\mathbb{Z}^2$ and consider the covering of A by the family of boxes $\{\Lambda_x(s(N)/4)\}_{x\in\tilde{\mathbb{Z}}^2}$.

First of all, notice that,

$$\bigcup_{x\in\tilde{\mathbb{Z}}^2:\ \Lambda_x(\frac{s(N)}{4})\cap\partial A\neq\emptyset} \big|\Lambda_x(\frac{s(N)}{4})\big| \leq \frac{s^2}{4}\big|\partial A\big|,$$

which, since A was assumed to be in $\mathcal{D}_N$, is bounded above by $R(K\log N)^2/4$. On the other hand,

$$\bigcup_{x\in\tilde{\mathbb{Z}}^2:\ \Lambda_x(\frac{s(N)}{4})\cap A\neq\emptyset} \big|\Lambda_x(\frac{s(N)}{4})\big| \geq |A| \geq aN^2.$$

Thus,

$$\bigcup_{x\in\tilde{\mathbb{Z}}^2:\ \Lambda_x(\frac{s(N)}{4})\subset A} \big|\Lambda_x(\frac{s(N)}{4})\big| \geq \frac{|A|}{2}.$$

Consequently, setting,

$$A_x = A\cap\Lambda_x\big(\frac{s(N)}{8}\big);\qquad x\in\tilde{\mathbb{Z}}^2,$$

we obtain

$$A\in\mathcal{D}_N \implies \sum\big|\,A_x\,\big| \geq \frac{a}{8}N^2. \tag{2.4.7}$$

Define now the event $E\subseteq\Omega_A$ as,

$$E = \bigcup_{x\in A\cap\tilde{\mathbb{Z}}^2} \Big\{\ \Lambda_x\big(\frac{s(N)}{8}\big) \stackrel{+}{\longrightarrow} \Lambda_x\big(\frac{s(N)}{4}\big)^c\ \Big\}.$$

By virtue of the Proposition 2.2.3 (and again, provided that K in the definition of $s(N) = K \log N$ was picked large enough) ,

$$\mathbb{P}^{\beta,s}_{A,-,g}\left(E \right) \;\leq\; \mathrm{e}^{-c_6 s(N)}. \tag{2.4.8}$$

For each $\sigma \in E^c$ the extended configuration $\sigma \cup \{-1\}^{\partial A}$ necessarily contains a $-*$ closed loop of spins inside $\widehat{A} \cap \Lambda_x(\frac{s(N)}{4})$ surrounding $\widehat{A} \cap \Lambda_x(\frac{s(N)}{8}))$ for each $x \in A \cap \tilde{\mathbb{Z}}^2$.

We now introduce a domain splitting of the event E^c, one of the type we have used so often in Subsection 2.2: We set $E^c = \cup \alpha$, where α labels all possible realizations of random domains $\{A_x^\alpha\}_{x \in A \cap \tilde{\mathbb{Z}}^2}$,

$$A_x^\alpha \;\triangleq\; A \cap \Lambda_x\left(\frac{s(N)}{4}\right) \setminus \widehat{D_x^\alpha} \qquad \text{and} \qquad D_x^\alpha \;\triangleq\; \left\{ y \in \Lambda_x\left(\frac{s(N)}{4}\right) :\; y \xrightarrow{+} \Lambda_x\left(\frac{s(N)}{4}\right)^c \right\}$$

Performing the domain decomposition of E^c, note, that since $\mathrm{diam}(A_x^\alpha) \leq s/2$, the super-index s can be dropped, and, as in the proof of Lemma 2.2.1, we obtain:

$$\left| \left\langle \mathrm{e}^{it\widehat{M_A}} \,;\, E^c \right\rangle^{\beta,s}_{A,-,g} \right| \;\leq\; \sum_\alpha p_\alpha \prod_{x \in A \cap \tilde{\mathbb{Z}}^2} \left| \left\langle \mathrm{e}^{it\frac{M_{A_x^\alpha}}{\sqrt{\chi |A|}}} \right\rangle^{\beta}_{A_x^\alpha,-,g} \right|. \tag{2.4.9}$$

Since there are only characteristic functions of unrestricted ensembles on the right hand side above, we, at this stage, can simply apply the conditional variance argument of [DT] (see the derivation of (1.17) in the latter paper) and, thereby, to conclude that uniformly in α, $|t| \leq \sqrt{\chi |A|}$ and $x \in \tilde{\mathbb{Z}}^2$,

$$\left| \left\langle \mathrm{e}^{it\frac{M_{A_x^\alpha}}{\sqrt{\chi |A|}}} \right\rangle^{\beta}_{A_x^\alpha,-,g} \right| \;\leq\; \mathrm{e}^{-c_7 t^2 \frac{|A_x^\alpha|}{|A|}}.$$

Since $A_x \subseteq A_x^\alpha$, the latter estimate , combined with (2.4.8), (2.4.7) and (2.4.9), implies the bound (2.4.6) and hence the claim of the lemma.

2.5. **Upper bound in the $s(N)$-phase of small contours.**

The results of this subsection are used to derive super-surface order local estimates, which we shall need throughout the paper. In particular they are crucial for the isoperimetric stability argument of Section 5.

Lemma 2.5.1. *Let the large contour parameter $s(N) \gg \log N$ be fixed. There exists a constant $c = c(\beta) > 0$, such that for all $N > 0$, $A \in \mathcal{D}_N$ and all $a_N \in \mathcal{M}_A^+$,*

$$\mathbb{P}^{\beta,s}_{A,-}\left(M_A = \mathbb{M}_A^s + a_N \right) \;\leq\; \exp\left(-c \frac{a_N^2}{N^2} \wedge \frac{a_N}{s(N)} \right). \tag{2.5.1}$$

Remark 2.5.2. Lemma 2.5.1 provides an, in a certain sense, optimal generalization of various estimates in the phase of small contours (see [I2], [SS3] and [PV]), which lie in the heart of all previous weak integral results on the phase separation up to the critical temperature T_c. Actually we need (2.5.1) only in the case of large moderate deviations $a_N \gg N^{4/3}$. In the case of small moderate deviations $a_N \ll N^{4/3}$ a much more precise statement will be derived in Subsection 4.1 independently of (2.5.1). Also the techniques of the latter subsection readily imply an asymptotic bound of the form (2.5.1) in the critical case $a_N \sim N^{4/3}$ as well. Thus, there is no loss to assume from the beginning that

$$a_N \;\gg\; N^{\frac{4}{3}}. \tag{2.5.2}$$

Proof. The idea of the proof is simple: either a volume of order $a_N/2m^*$ is exhausted by $K\log N$-large contours, which, in the $\mathbb{P}^{\beta,s}_{A,-}$-restricted phase, should have a surface tension price with the exponent of order $a_N/s(N)$, or the $K\log N$-large contours cover a volume much less than $a_N/2m^*$, and the remaining excess in magnetization ought to be compensated in the $K\log N$ restricted phase, where we can subsequently apply the estimates of Subsection 2.3.

So let $\Gamma(\sigma)$ be the collection of all $K\log N$-large contours of σ . Recall, that $\mathrm{Vol}_+(\Gamma(\sigma))$ was set to denote the area of the "+" component in the Γ-induced decomposition of A. We write:

$$\mathbb{P}^{\beta,s}_{A,-}\big(\, M_A \;=\; \mathbb{M}^s_A \;+\; a_N \,\big) \;\leq \mathbb{P}^{\beta,s}_{A,-}\big(M_A = \mathbb{M}^s_A + a_N \;;\; \mathrm{Vol}_+(\Gamma(\sigma)) \;<\; \frac{a_N}{4m^*}\,\big)$$
$$+\; \mathbb{P}^{\beta,s}_{A,-}\big(\,\mathrm{Vol}_+(\Gamma(\sigma)) \;\geq\; \frac{a_N}{4m^*}\,\big). \tag{2.5.3}$$

By the estimates on the phase volume (1.1.3) on the $K\log N$ scale:

$$\mathrm{Vol}_+(\Gamma(\sigma)) \;\geq\; \frac{a_N}{4m^*} \;\Longrightarrow\; \big|\mathfrak{S}_+\big| \;\geq\; \frac{a_N}{4m^*} \;-\; c_1 K\log N\mathcal{W}_\beta\big(\mathfrak{S}_+\big) \tag{2.5.4}$$

for any collection of $K\log N$ skeletons $\mathfrak{S}\sim\Gamma$. We claim that in view of $s(N)$-restriction, (2.5.4) implies, in addition, that

$$\mathcal{W}_\beta\big(\mathfrak{S}_+\big) \;\geq\; c_2\frac{a_N}{s(N)}, \tag{2.5.5}$$

as will be shown next.

Indeed, the moment we assume, for example, that $\mathcal{W}_\beta\big(\mathfrak{S}_+\big) < a_N/s(N)$, we immediately infer from (2.5.4) and the choice $s(N)\gg\log N$, that

$$\big|\mathfrak{S}_+\big| \;>\; \frac{a_N}{8m^*}. \tag{2.5.6}$$

Since, $\Gamma\sim\mathfrak{S}$ is forced by $\mathbb{P}^{\beta,s}_{A,-}$ to comply with the $s(N)$-restriction on the size of its contours, i.e.,

$$\Gamma = \big(\gamma_1,...,\gamma_N\big) \;\Rightarrow\; \max_i \mathrm{diam}(\gamma_i) \;<\; s(N),$$

the diameter and the area of each connected component of $\mathfrak{S}_+$ do not exceed $2s(N)$ and $4s(N)^2$ respectively.

In any case, (2.5.6) implies, by the isoperimetric inequality, that

$$\mathcal{W}_\beta\big(\mathfrak{S}_+\big) \;\geq\; c_3\sqrt{a_N}.$$

If $a_N/8m^*\leq 4s^2$, then

$$\sqrt{a_N} \;=\; \frac{a_N}{s(N)}\sqrt{\frac{s^2}{a_N}} \;\geq\; c_4\frac{a_N}{s(N)}.$$

Otherwise, if $a_N\geq 32m^*s(N)^2$, then it is easy to see that $\mathcal{W}_\beta(\mathfrak{S}_+)$ is always larger than or equal to the total surface energy of $[a_N/32m^*s(N)^2]$ Wulff droplets of the maximal permitted area $4s(N)^2$ (it is obvious that the surface energy is minimized on a collection of Wulff shaped droplets, and a simple computation shows that the transfer of "mass" from a smaller to a larger droplet decreases the surface energy). Since the surface tension of each of these droplets equals to $2s(N)\omega_1$, we obtain,

$$\mathcal{W}_\beta\big(\mathfrak{S}_+\big) \;\geq\; 2\omega_1\big[\frac{a_N}{32m^*s(N)^2}\big]s(N) \;\geq\; c_5\frac{a_N}{s(N)},$$

and (4.2.11) follows.

From (2.5.5) and the energy inequality (1.3.2) on the $K\log N$ scale (see also (1.1.4)) the second term on the right hand side of (2.5.3) is bounded above by $\exp\bigl(-c_6 a_N/s(N)\bigr)$.

We proceed to investigate the first term on the right hand side of (2.5.3): let Γ satisfy $|C| = \mathrm{Vol}_+(\Gamma(\sigma)) < a_N/4m^*$, where C is the plus component of the corresponding Γ-induced decomposition of A; $A = B \cup C$. Since, by the definition, Γ is the collection of *all* $K\log N$ large contours, the following factorization of $\mathbb{P}^{\beta,s}_{A,-}$ is valid:

$$\mathbb{P}^{\beta,s}_{A,-}\left(\,\bullet\, \mid \Gamma \right) \;=\; \mathbb{P}^{\beta,K\log N}_{B,-}\left(\,\bullet\, \right)\mathbb{P}^{\beta,K\log N}_{C,+}\left(\,\bullet\, \right).$$

Using the corresponding analog of the decomposition (1.3.4),

$$\begin{aligned} &\left\{\, M_A = \mathbb{M}^s_A + a_N \;;\; \Gamma \,\right\} \\ &\quad= \bigcup_{b_N + c_N = a_N - \Delta(B,C)} \left\{\, \Gamma \;;\; M_B = \mathbb{M}^{K\log N}_B + b_N \;;\; M_C = -\mathbb{M}^{K\log N}_C + c_N \,\right\}, \end{aligned} \tag{2.5.7}$$

where, of course, the compensator $\Delta(B,C)$ is given by

$$\Delta(B,C) \;=\; \mathbb{M}^{K\log N}_B \;-\; \mathbb{M}^{K\log N}_C \;-\; \mathbb{M}^s_A \;+\; \bigl|\partial_+\Gamma\bigr| \;-\; \bigl|\partial_-\Gamma\bigr|.$$

By Lemma 2.2.1 and (1.1.1), if K is large enough,

$$\Bigl|\; \Delta(B,C) \;-\; 2m^*|C| \;\Bigr| \;\le\; c_7\bigl|\partial C\bigr| \;=\; c_7\bigl|\Gamma\bigr|. \tag{2.5.8}$$

In view of the energy estimate (1.3.2) there is no loss to assume that any Γ-compatible collection of $K\log N$ skeletons $\mathfrak{S}$ satisfies $\mathcal{W}_\beta(\mathfrak{S}) \le a_N/s(N)$. Consequently,

$$\bigl|\Gamma\bigr| \;\le\; c_8 K\log N\mathcal{W}_\beta(\mathfrak{S}) \;\ll\; a_N,$$

which, by virtue of the assumption on the volume $|C| \le a_N/4m^*$ and by the estimate (2.5.8), gives the following bound on $\Delta(B,C)$:

$$a_N \;-\; \Delta\bigl(B,C\bigr) \;\ge\; \frac{a_N}{3}.$$

Therefore, the decomposition (2.5.7), in fact, implies:

$$\begin{aligned} &\mathbb{P}^{\beta,s}_{A,-}\left(M_A \;=\; \mathbb{M}^s_A \;+\; a_N \;\middle|\; \Gamma \right) \\ &\le \max_{b_N \ge a_N/6} \mathbb{P}^{\beta,K\log N}_{B,-}\left(M_B \;=\; \mathbb{M}^{K\log N}_B \;+\; b_N \right) \bigvee \max_{c_N \ge a_N/6} \mathbb{P}^{\beta,K\log N}_{C,-}\left(M_C \;=\; \mathbb{M}^{K\log N}_C \;-\; c_N\right), \end{aligned}$$

where we have used the flip symmetry of Ising measures to rewrite the probability on C in the minus state setting. By the Gaussian bound (2.3.2),

$$\max_{c_N \ge a_N/6} \mathbb{P}^{\beta,K\log N}_{C,-}\left(M_C \;=\; \mathbb{M}^{K\log N}_C \;-\; c_N\right) \;\le\; \exp\bigl(-c_9\frac{a_N^2}{N^2}\bigr).$$

As far as the "$-$" component B is concerned, we can, of course try to apply the local limit estimates of the previous subsection. Since, however, we are interested only in coarse upper bounds, the following usual large deviation type bound suffices,

$$\begin{aligned} &\log \mathbb{P}^{\beta,K\log N}_{B,-}\left(M_B \;=\; \mathbb{M}^{K\log N}_B \;+\; b_N \right) \\ &\quad\le\; -\max_g \Bigl\{\; g\bigl(\mathbb{M}^{K\log N}_B + b_N\bigr) \;-\; \log\bigl\langle \mathrm{e}^{gM_B}\bigr\rangle^{\beta,K\log N}_{B,-} \;\Bigr\}. \end{aligned} \tag{2.5.9}$$

For the values of magnetic field g satisfying $g \log N \ll 1$ one is entitled to use Lemma 2.2.1 and, perform the corresponding expansion of the log-moment generating function, to estimate

$$\begin{aligned}\log \left\langle \mathrm{e}^{gM_B} \right\rangle_{B,-}^{\beta, K\log N} \;&=\; g\mathbb{M}_B^{K\log N} + \frac{g^2}{2}\left\langle M_B ; M_B \right\rangle_{B,-}^{\beta, K\log N} \;+\; \mathrm{O}\big(g^3 N^2\big) \\ &\leq\; g\mathbb{M}_B^{K\log N} \;+\; c_{10} g^2 N^2,\end{aligned}$$

where the $\mathrm{O}\big(g^3 N^2\big)$ term above follows from (2.2.5) (as (2.3.6) did), whereas the last inequality is a consequence of (2.2.5) and the fact that $|B| \leq |A| \leq N^2$.

Substituting the above bound into (2.5.9) we obtain,

$$\max_{b_N \geq a_N/6} \log \mathbb{P}_{B,-}^{\beta, K\log N}\big(M_B \;=\; \mathbb{M}_B^{K\log N} \;+\; b_N \big) \;\leq\; -\max_{g \ll 1/\log N}\Big(\frac{g a_N}{6} \;-\; c_{10} g^2 N^2 \Big).$$

For the values of $a_N \ll N^2/\log N$ the right hand side above is of order a_N^2/N^2. Otherwise an admissible choice of $g = 1/s(N)$ leads to the upper bound of order $a_N/s(N)$, which is again compatible with the assertion of the lemma. The proof is, thereby, concluded. □

3. Lower Bound

3.1. **The estimate.** With the exception of the references to the results of [CCS] this is the only part of the paper, where we use the FK representation. Likewise, this is the only place, where we digress from purely probabilistic considerations and use results obtained in the framework of exact solutions. Namely, we rely on two facts about the 2D Ising model: first of all it is proved in [AA], that the Wulff shape has everywhere positive radius of curvature, more precisely for any $\beta > \beta_c$, the surface tension τ_β enjoys the following positive stiffness property:

$$\min_{n\in\mathbb{S}^1}\Big(\ \tau_\beta''(n)\ +\ \tau_\beta(n)\ \Big)\ \triangleq\ R(\beta)\ >\ 0. \tag{3.1.1}$$

Secondly [MW], for any dual inverse temperature $\beta^* < \beta_c$, the two point correlation functions of the (unique) infinite volume Gibbs distribution are subject to an Ornstein-Zernike type correction formula:

$$\exp\big(-\|u\|_2 m_{\beta^*}(\frac{u}{\|u\|_2})\big)\ \geq\ \big\langle\sigma(u)\sigma(0)\big\rangle_{\beta^*}\ \geq\ \exp\big(-\|u\|_2 m_{\beta^*}(\frac{u}{\|u\|_2})\ -\ c_1(\beta^*)\log\|u\|_2\big), \tag{3.1.2}$$

where $\|\cdot\|_2$ is the Eucledian norm on $\mathbb{R}^2$ and m_{β^*} is the directionally dependent mass gap at β^*, which is related by the Krammer-Wannier duality (see [Pf] for more on this) to the surface tension of the direct model $m_{\beta^*}\equiv\tau_\beta$ as soon as

$$e^{2\beta}\ =\ \tanh\beta^*. \tag{3.1.3}$$

Both (3.1.1) and (3.1.2) play an important role in our approach to the lower bound. We would like to remark, however, that the technique of exact solutions itself does not seem to be indispensable. At least in the case of self-avoiding walks an alternative probabilistic treatment is developed in [I3], whereas various correction formulas similar to (3.1.2) were obtained in [A1] and in [A2] for two and three dimensional Bernoulli percolation and in general framework of multidimensional subadditive arrays respectively. Moreover, (3.1.2) itself was recently established in [A3] for a class of lattice models including the one we we consider here.

Theorem 3.1.1. *For every $\beta > \beta_c$ there exists a number $N_0 = N_0(\beta) < \infty$ and a constant $c_2 = c_2(\beta) > 0$, such that for all $N \geq N_0$ and all $a_N \geq N^{4/3}$ in the range of M_N; $a_N \in \mathcal{M}_N^+$, the following estimate holds:*

$$\mathbb{P}^\beta_{N,-}\big(M_N = -N^2m^* + a_N\big)\ \geq\ \exp\big\{-\sqrt{\frac{a_N}{2m^*}}\omega_1 - c_2(\beta)\sqrt[4]{a_N}\log N\big\}. \tag{3.1.4}$$

Remark 3.1.2. In fact an appropriate version of the above lower bound holds for the whole range of $a_N \in \mathcal{M}_N^+$. For small moderate values of a_N, however, the leading asymptotics of $\mathbb{P}^\beta_{N,-}\big(M_N = -N^2m^* + a_N\big)$ stems from the Gaussian estimate (2.3.9) in the $K\log N$ phase and is, therefore, different from the one on the right hand side of (3.1.4). Thus, the lower bound (3.1.1) becomes sharp only for large moderate values of a_N.

Notice also, that a completely similar lower bound is, of course, valid for the deviations from the true average $\mathbb{P}^\beta_{N,-}\big(M_N = \mathbb{M}_N + a_N\big)$, whenever a_N satisfies conditions of Theorem 3.1.1.

3.2. **Outline of the proof.** Note first of all that using results of Section 2, one can trivially derive a lower bound, which would capture the right surface order. Indeed, pick a $\pm$ contour γ, such that $|\mathrm{int}(\gamma)| = a_N/2m^* + \mathrm{O}(1)$ and $|\gamma| \sim \sqrt{a_N}$. Then, using local limit estimates of the previous section inside and outside of γ, one readily obtains:

$$\mathbb{P}^{\beta}_{N,-}\big(M_N = -N^2 m^* + a_N \big) \ \geq\ \exp\big(- c(\beta)\sqrt{a_N}\big). \tag{3.2.1}$$

As it was mentioned in the introduction, an important consequence of (3.2.1) is that by the observation (1.3.3), one can, from now on, restrict attention only to those collections of large contours, which have admissible total length, i.e., of order $\sqrt{a_N}\log N$ at most.

The logic behind the lower bound is transparent: in order to induce the a_N-shift of the magnetization M_N from its expected value under the $\mathbb{P}^{\beta}_{N,-}$ measure, one tries to "chop out" an island of area close to $a_N/2m^*$ and of the optimal shape, and to enforce a typical "+"-phase configuration, i.e., of the mean magnetization close to m^*, over this island. Thus, the proof of the lower bound boils down to the following procedure: we choose a certain parameter w_N and let T_N be the tube of the width w_N around the required dilatation $\sqrt{a_N/2m^*}\mathcal{K}_1$ of the unit volume Wulff shape. We shall let $\mathcal{E}_N$ denote the event that there is a $\pm$ contour inside T_N around $\sqrt{a_N/2m^*}\mathcal{K}_1 \setminus T_N$, and $\mathcal{E}^K_N$ be the subset of $\mathcal{E}_N$ in which there is a unique such contour, all other contours being $K\log N$-small, and the large contour has its length bounded above by $K\sqrt{a_N}\log N$. Our estimate will be based on

$$\mathbb{P}^{\beta}_{N,-}\big(M_N = -N^2 m^* + a_N \big) \ \geq\ \mathbb{P}^{\beta}_{N,-}\big(\mathcal{E}^K_N \big)\mathbb{P}^{\beta}_{N,-}\big(M_N = -N^2 m^* + a_N \,\big|\, \mathcal{E}^K_N \big). \tag{3.2.2}$$

This splits the problem into two: to an estimate on $\mathbb{P}^{\beta}_{N,-}\big(\mathcal{E}^K_N \big)$ and, provided that the area of the tube $\big|T_N\big| \sim \sqrt{a_N}w_N \ll a_N$, to local limit estimates on $\mathbb{P}^{\beta}_{N,-}\big(M_N = -N^2 m^* + a_N \,\big|\, \mathcal{E}^K_N \big)$. The latter problem happens to fall in the framework of the previous section.

Clearly, the terms in the product on the right hand side of (3.2.2) are of competing nature. Indeed, an estimate on $\mathbb{P}^{\beta}_{N,-}\big(\mathcal{E}^K_N \big)$ should be a coarse grained one, and it is on this stage that the surface tension τ_β enters the picture. In other words, the width w_N of the tube T_N is intimately related to the large skeleton scale $s(N)$, we choose to coarse grain. In fact, it happens that the optimal choice without going into detailed analysis of the fluctuations of the phase separation line is of order $w_N \sim \sqrt{s(N)\log N}$. We shall, then, employ the Gaussian estimates of the previous section to conclude that

$$\mathbb{P}^{\beta}_{N,-}\big(M_N = -N^2 m^* + a_N \,\big|\, \mathcal{E}^K_N \big) \ \geq\ \exp\Big(-c_3(\beta) w_N^2 \Big). \tag{3.2.3}$$

Note that the estimate above deteriorates with $s(N)$. On the other hand, due to the Ornstein-Zernike correction formula (3.1.2), the estimate on $\mathbb{P}^{\beta}_{N,-}\big(\mathcal{E}^K_N \big)$ becomes better at large values of the large contour parameter $s(N)$. Namely, we shall prove that on the $s(N)$ skeleton scale, i.e., with the corresponding choice of the tube width $w_N \sim \sqrt{s(N)\log N}$,

$$\mathbb{P}^{\beta}_{N,-}\big(\mathcal{E}^K_N \big) \ \geq\ \exp\Big\{ -\sqrt{\frac{a_N}{2m^*}}\omega_1\big(1 - c_3(\beta)\frac{\log N}{s(N)} \big)\Big\}. \tag{3.2.4}$$

Consequently, the final assertion (3.1.4) is an outcome of an optimization in terms of the large skeleton scale $s(N)$.

There are several possibilities to choose "scissors" for the implementation of the first part of the above program. The most straightforward one is to try to investigate directly the random line representation of contours in $\mathcal{E}_N$. Such an approach was pursued in [Pf], [I1], [PV] and [V]. The estimates obtained in the latter work are essentially equivalent to those we derive here. It is our opinion, however, that this direct approach leads to unnecessary complications, related to the fact that one is somehow compelled to move against the current and to reverse the natural direction of the inequalities related to the random line representation. Contrary to this, we shall follow [SS1] and use indirect "FK-scissors" to chop out domains of the desirable shapes. As we shall see later the immediate advantage of this indirect approach is that the natural inequalities (FKG inequalities) become in this way friendly and point precisely in the direction we need.

3.3. **FK representation.** We refer to [ACCN], [CCS], [Pi] and [SS1] for the definition of the FK measures and the corresponding discussions of their properties.

For any E-finite set of nearest neighbour edges of $\mathbb{Z}^2$; $E \subset \mathcal{E}^2$, we use $\mathfrak{G}_E = (V_E, E)$ to denote the corresponding subgraph of $(\mathbb{Z}^2, \mathcal{E}^2)$, where V_E is the set of all the vertices of edges from E. Equally for any finite set of vertices $B \subset \mathbb{Z}^2$ we define E_B to be the set of all the edges of $\mathcal{E}^2$ with both endpoints in B, and write $\mathfrak{G}_B = (B, E_B)$ for the corresponding subgraph. Thus, given an inverse temperature $\beta \in (0, \infty)$, the notation $\mathbb{P}^{\beta}_{E,f}$, $\mathbb{P}^{\beta}_{E,+}$ and $\mathbb{P}^{\beta}_{E,-}$ (respectively $\mathbb{P}^{\beta}_{B,f}$, $\mathbb{P}^{\beta}_{B,+}$ and $\mathbb{P}^{\beta}_{B,-}$) are reserved for finite volume Ising measures on $\mathfrak{G}_E$ (respectively on $\mathfrak{G}_B$) with free, plus and minus boundary conditions. Similarly, $\mu^{\beta}_{E,w}$ and $\mu^{\beta}_{E,f}$ (respectively $\mu^{\beta}_{B,w}$ and $\mu^{\beta}_{B,f}$) are used to denote the wired and free FK measures on $\mathfrak{G}_E$ (respectively on $\mathfrak{G}_B$) at the percolation parameter

$$p \;=\; p(\beta) \;\triangleq\; 1 \;-\; \mathrm{e}^{-2\beta}.$$

It is convenient to use the above graph notation, once the dual quantities come into the play: let $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$ be the dual lattice and $\mathcal{E}^2_*$ the corresponding set of dual edges. To each direct edge $e \in \mathcal{E}^2$ there corresponds exactly one dual edge e^*; $e \perp e^*$, which intersect e in the sense of geometric embedding of both lattices into $\mathbb{R}^2$. Thus, given a finite set of direct edges $E \subset \mathcal{E}^2$, we define its dual $E^* \subset \mathcal{E}^2_*$ via

$$E^* \;=\; \left\{ \, e^* \in \mathcal{E}^2_* : \; \mathrm{e}^* \perp e \text{ for some } e \in E \, \right\}.$$

The notion of duality between subsets of $\mathbb{Z}^2$ and $\mathbb{Z}^2_*$ is defined, then, through the above notion of the edge duality: given $B \in \mathbb{Z}^2$ we define its dual $B^* \subset \mathbb{Z}^2_*$ via $B^* = V_{E^*_B}$.

We proceed by recalling the dual correspondence between the set of direct bond configurations $\Omega_E = \{0,1\}^E$ and the set of dual bond configurations $\Omega_{E^*} = \{0,1\}^{E^*}$,

$$n^*(e^*) \;=\; 1 \,-\, n(e) \;\; \text{for each pair } e \perp e^*.$$

As a result any probability measure on Ω_E automatically induces and, in fact, can be identified with the corresponding measure on Ω_{E^*}. In particular [CCS], the direct wired measure $\mu^{\beta}_{B,w}$ corresponds in this way to the dual free measure $\mu^{\beta^*}_{B^*,f}$, where β^* is precisely the dual temperature given by the Krammer-Wannier relation (3.1.3). Similarly, the direct free measure $\mu^{\beta}_{B,f}$ corresponds to the dual wired one $\mu^{\beta^*}_{B^*,w}$. It should be stipulated, though, that the notion of duality is pronounced in the FK language much stronger than on the usual level of partition functions: not only both measures in duality are played on the same probability space, they, as we have already mentioned, are actually identified, and we shall frequently switch from the direct to the dual picture in the course of studying probabilities of occurrence of various geometric events related to the FK-percolation.

So let a large skeleton parameter $s(N)$ be fixed,

$$s(N) \gg \log N, \tag{3.3.1}$$

and set $w_N = M\sqrt{s(N)\log N}$, where $M = M(\beta)$ is a large enough number to be specified later. Let also $a_N \in \mathcal{M}_N^+$ be in the range of large moderate deviations; $a_N \sim N^{4/3+\delta}$. The tube $\widetilde{T_N}$ around the appropriate dilatation of the boundary of the Wulff shape is defined via:

$$\widetilde{T_N} = \{ x : d\big(x, \sqrt{\frac{a_N}{2m^*}}\partial\mathcal{K}_1\big) \le \frac{1}{2}w_N \}. \tag{3.3.2}$$

Below is our main probabilistic estimate on chopping out domains in terms of the "FK-scissors":

Lemma 3.3.1. *Set*

$$\mathcal{C}_N = \{ \exists \textit{ loop of open dual bonds inside } \widetilde{T_N} \textit{ around } \sqrt{\frac{a_N}{2m^*}}\mathcal{K}_1 \setminus \widetilde{T_N} \}.$$

Then,

$$\mu^{\beta}_{\Lambda_N,w}(\mathcal{C}_N) = \mu^{\beta^*}_{\Lambda_N^*,f}(\mathcal{C}_N) \ge \exp\{ -\sqrt{\frac{a_N}{2m^*}}\omega_1 - c_4(\beta)\frac{\sqrt{a_N}\log N}{s(N)} \}. \tag{3.3.3}$$

Proof. Let $S = (u_1, ..., u_n)$ be an $s(N)$-skeleton around $\sqrt{a_N/2m^*}\partial\mathcal{K}_1$, i.e., assume that S is an $s(N)$-skeleton and that $d(u_i, \sqrt{a_N/2m^*}\partial\mathcal{K}_1) < 1$; $i = 1, ..., n$. As in [I1], using the positive stiffness of τ_β, it is not hard to see that the Hausdorff distance between the boundary $\sqrt{a_N/2m^*}\partial\mathcal{K}_1$ and the polygonal line $\mathbf{P}(S)$ through the vertices of S is of order

$$d_{\mathbb{H}}\big(\sqrt{\frac{a_N}{2m^*}}\partial\mathcal{K}_1, \mathbf{P}(S) \big) \sim \frac{s(N)^2}{\sqrt{a_N}}.$$

Consequently, if

$$s(N)^3 \ll a_N \log N, \tag{3.3.4}$$

one can think about the $\widetilde{T_N}$ tube as being actually drawn around $\mathbf{P}(S)$ itself.

In what follows we adopt a convenient construction in [V], based on the sharp triangle inequality of [I1].

For each pair of successive vertices $\{u_i, u_{i+1}\}$ in S, let us define the set $B_i \subset \mathbb{Z}^2_*$ via

$$B_i = \{ z \in \mathbb{Z}^2_* : \|u_i - z\|_2 + \|z - u_{i+1}\|_2 \le \|u_i - u_{i+1}\|_2 + M'\log N \}, \tag{3.3.5}$$

where $\|\cdot\|_2$ is the usual Euclidean norm on $\mathbb{R}^2$ and M' is some (big) number. Clearly, given $M' > 0$ one can can always find $M > 0$, such that the following inclusion is true:

$$B_i \subset \widetilde{T_N}; \quad i = 1, ..., n . \tag{3.3.6}$$

On the other hand, once (3.3.6) is satisfied, the FKG property of random cluster measures implies:

$$\mu^{\beta^*}_{\Lambda_N^*,f}(\mathcal{C}_N) \;\geq \mu^{\beta^*}_{\Lambda_N^*,f}\big(\bigcap_i \{u_i \xleftrightarrow{\mathrm{FK}} u_{i+1} \text{ inside } B_i\}\big) \;\geq\; \prod_i \mu^{\beta^*}_{\Lambda_N^*,f}\big(\{u_i \xleftrightarrow{\mathrm{FK}} u_{i+1} \text{ inside } B_i\}\big)$$

$$\geq\; \prod_i \mu^{\beta^*}_{B_i,f}\big(\{u_i \xleftrightarrow{\mathrm{FK}} u_{i+1} \text{ inside } B_i\}\big) \;=\; \prod_i \big\langle \sigma(u_i)\sigma(u_{i+1})\big\rangle^{\beta^*}_{B_i,f}. \tag{3.3.7}$$

By the results on the random line representation of the pair correlations [PV], for any set $B \subset \mathbb{Z}^2_*$ and for any two points $u, v \in B$ one can compare the finite volume correlation $\big\langle\sigma(u)\sigma(v)\big\rangle^{\beta^*}_{B,f}$ with the infinite volume one $\big\langle\sigma(u)\sigma(v)\big\rangle^{\beta^*}_{f}$ via

$$\big\langle\sigma(u)\sigma(v)\big\rangle^{\beta^*}_{B,f} \;\geq\; \big\langle\sigma(u)\sigma(v)\big\rangle^{\beta^*}_{f} \;-\; c_5\big|\partial B\big| \max_{z\in\partial B} \big\langle\sigma(u)\sigma(z)\big\rangle^{\beta^*}_{f} \big\langle\sigma(z)\sigma(v)\big\rangle^{\beta^*}_{f},$$

where c_5 is a combinatorial constant, which depends only on the dimension and on the notion of the boundary ∂B chosen. In our case we define ∂B to be the outer boundary of B,

$$\partial B \;=\; \big\{\; z \in B^c : \; \min_{x\in B} \|z - x\| = 1 \;\big\}. \tag{3.3.8}$$

The positive stiffness condition implies [I1], [V] the following form of the sharp triangle inequality: for any three distinct points $u, v, z \in \mathbb{Z}^2_*$,

$$\|u-z\|_2\tau_\beta\big(\frac{u-z}{\|u-z\|_2}\big) \;+\; \|v-z\|_2\tau_\beta\big(\frac{v-z}{\|v-z\|_2}\big) \;-\; \|u-v\|_2\tau_\beta\big(\frac{u-v}{\|u-v\|_2}\big)$$
$$\geq\; \frac{c_6}{R(\beta)}\big(\; \|u-z\|_2 \;+\; \|v-z\|_2 \;-\; \|u-v\|_2 \;\big). \tag{3.3.9}$$

By the Ornstein-Zernike correction formula (3.1.2) and by the very construction of the sets B_i we, thus, infer from (3.3.9) that for $i = 1, .., n$,

$$\max_{z\in\partial B} \big\langle\sigma(u_i)\sigma(z)\big\rangle^{\beta^*}_{f} \big\langle\sigma(u_{i+1})\sigma(z)\big\rangle^{\beta^*}_{f} \;\leq\; \big\langle\sigma(u_i)\sigma(u_{i+1})\big\rangle^{\beta^*}_{f} \exp\big(-\frac{c_6}{R(\beta)} M' \log N \;+\; c_7 \log N\;\big).$$

Consequently, choosing M' sufficiently large, which, as we have already remarked, amounts to choosing M in (3.3.2) large enough, we are able to conclude from (3.3.8) and (3.3.7):

$$\mu^{\beta^*}_{\Lambda_N^*,f}\big(\;\mathcal{C}_N\;\big) \;\geq\; \big(1 + \mathrm{o}(1)\big) \prod_i \big\langle\sigma(u_i)\sigma(u_{i+1})\big\rangle^{\beta^*}_{f}.$$

Since the distance between any two neighbouring vertices of S lies in the interval $[s(N), 3s(N)]$, and the number of all such pairs of neighbours is of the order $\sqrt{a_N}/s(N)$, the claim of the lemma follows by another application of the Ornstein-Zernike lower bound (3.1.2) to each term in the product above. Notice, that in this argument the error which results from approximating ω_1 by a Riemann sum is negligible provided

$$s(N) \;\ll\; \sqrt[4]{a_N}\sqrt{\log N}. \tag{3.3.10}$$

□

3.4. **Proof of the lower bound.** We start by proving (3.2.4). Recall that the tube width w_N was defined in the previous subsection as

$$w_N \;=\; M\sqrt{s(N)\log N},$$

and we now formally define the tube T_N itself via

$$T_N \;=\; \left\{\; x :\; d(x, \sqrt{\frac{a_N}{2m^*}}\partial\mathcal{K}_1) \;\le\; w_N \right\}$$

and

$$\mathcal{E}_N \;=\; \left\{\; \text{There is a } \pm \text{ contour inside } T_N \text{ around } \sqrt{\frac{a_N}{2m^*}}\mathcal{K}_1 \setminus T_N \;\right\}.$$

In order to prove the desired lower bound on the $\mathbb{P}^{\beta}_{N,-}$-probability of $\mathcal{E}^K_N$, we are going to construct a certain event $\widetilde{\mathcal{E}_N}$ stated in terms of the FK percolation geometry, such that

$$\mu^{\beta}_{\Lambda_N,w}(\;\widetilde{\mathcal{E}_N}\;) \;\ge\; \exp\Big(\;-\sqrt{\frac{a_N}{2m^*}}\omega_1\big(1 - c_7(\beta)\frac{\log N}{s(N)}\big)\Big), \tag{3.4.1}$$

$$\mathbb{P}^{\beta}_{N,-}(\;\mathcal{E}_N\;) \;\ge\; \frac{1}{4}\mu^{\beta}_{\Lambda_N,w}(\;\widetilde{\mathcal{E}_N}\;) \tag{3.4.2}$$

and

$$\mathbb{P}^{\beta}_{N,-}(\;\mathcal{E}^K_N\;) \;\ge\; \frac{1}{2}\mathbb{P}^{\beta}_{N,-}(\;\mathcal{E}_N\;). \tag{3.4.3}$$

The FK event $\widetilde{\mathcal{E}_N}$ we chose is

$$\widetilde{\mathcal{E}_N} \;=\; \left\{\; \exists \text{ two disjoint FK loops of occupied direct bonds inside } T_N \text{ around } \sqrt{\frac{a_N}{2m^*}}\mathcal{K}_1 \setminus T_N \right\}.$$

We would like to recall at this stage a useful way [ES] to construct $\mathbb{P}^{\beta}_{N,-}$ from the wired FK measure $\mu^{\beta}_{\Lambda_N,w}$. This comprises two steps: first play a bond configuration $n \in \{0,1\}^{E_{\Lambda_N}}$ under $\mu^{\beta}_{\Lambda_N,w}$, and second paint independently each maximal connected component of n into $+1$ or -1 with the probability $1/2$ each, if this component is disconnected from $\partial\Lambda_N$, while assigning identical -1 spin to the boundary cluster.

Since the inner loop of $\widetilde{\mathcal{E}_N}$ is, clearly, disjoint from the boundary $\partial\Lambda_N$, it will costs us exactly probability $1/2$ to paint it into $+1$, which provides a $+$ connected circuit of spins inside T_N. The outer loop of $\widetilde{\mathcal{E}_N}$ may or may not be connected to $\partial\Lambda_N$, in either case the probability to paint it into -1 (independently from the inner loop!) is at least $1/2$. The presence of both $+$ and $-$ connected circuits of spins inside T_N implies $\mathcal{E}_N$, and (3.4.2) follows.

To prove (3.4.3) we partition $\mathcal{E}_N \setminus \mathcal{E}^K_N$ into three events which we define next. $\mathcal{B}_1$ will denote the event that inside the outermost contour which surrounds $\sqrt{\frac{a_N}{2m^*}}\mathcal{K}_1 \setminus T_N$ there is some $K\log N$-large contour. $\mathcal{B}_2$ will denote the event that outside the innermost contour which surrounds $\sqrt{\frac{a_N}{2m^*}}\mathcal{K}_1 \setminus T_N$ there is some $K\log N$-large contour. And $\mathcal{B}_3$ will denote the event that there is some contour with length larger than $K\sqrt{a_N}\log N$.

Standard conditioning arguments show that when K is sufficiently large $\mathbb{P}^{\beta}_{N,-}\left(\mathcal{B}_1\middle|\mathcal{E}_N\right) = \mathrm{o}(1)$, and $\mathbb{P}^{\beta}_{N,-}\left(\mathcal{B}_2\middle|\mathcal{E}_N\right) = \mathrm{o}(1)$. Regarding $\mathcal{B}_3$, one can just use (1.3.3) and a trivial lower bound on $\mathbb{P}^{\beta}_{N,-}\left(\mathcal{E}_N\right)$ to show that $\mathbb{P}^{\beta}_{N,-}\left(\mathcal{B}_3\middle|\mathcal{E}_N\right) = \mathrm{o}(1)$ as well. The inequality (3.4.3) is now immediate from

$$\begin{aligned}\mathbb{P}^{\beta}_{N,-}\left(\, \mathcal{E}^K_N \,\right) \;&\geq\; \mathbb{P}^{\beta}_{N,-}\left(\, \mathcal{E}_N \,\right)\mathbb{P}^{\beta}_{N,-}\left(\, \mathcal{E}^K_N \;\middle|\; \mathcal{E}_N\right)\\ &\geq\; \mathbb{P}^{\beta}_{N,-}\left(\mathcal{E}_N\right)\Big(\, 1 \,-\, \mathbb{P}^{\beta}_{N,-}\left(\mathcal{B}_1\middle|\mathcal{E}_N\right) \,-\, \mathbb{P}^{\beta}_{N,-}\left(\mathcal{B}_2\middle|\mathcal{E}_N\right) \,-\, \mathbb{P}^{\beta}_{N,-}\left(\mathcal{B}_3\middle|\mathcal{E}_N\right)\,\Big).\end{aligned}$$

In order to prove (3.4.1) we use the estimate of Lemma 3.3.1 and the construction similar to the one already employed in [SS1]: for each bond configuration of $\mathcal{C}_N$ (as in the statement of Lemma 3.3.1) let us define a (random) set

$$\mathcal{D} \;=\; \Big\{\, x :\; x \overset{\mathrm{FK}}{\longleftrightarrow} \Lambda_N \setminus \big(\sqrt{\frac{a_N}{2m^*}}\mathcal{K}_1 \cup \widetilde{T_N}\big)\,\Big\}.$$

Furthermore, let us split $\mathcal{C}_N$ according to the realization of $\mathcal{G} \triangleq \Lambda_N \setminus \mathcal{D}$:

$$\mathcal{C}_N \;=\; \bigcup_{\alpha} \mathcal{G}_\alpha,$$

where, with a slight abuse of notations, we use $\mathcal{G}_\alpha$ to denote both the subset of $\Lambda_N \setminus \mathcal{D}$ and the corresponding event $\{\mathcal{G} = \mathcal{G}_\alpha\}$. Note that by the very virtue of $\mathcal{C}_N$,

$$\sqrt{\frac{a_N}{2m^*}}\mathcal{K}_1 \setminus \widetilde{T_N} \;\subseteq\; \mathcal{G}_\alpha \;\subseteq\; \sqrt{\frac{a_N}{2m^*}}\mathcal{K}_1 \cup \widetilde{T_N}. \tag{3.4.4}$$

Note also that the event $\mathcal{G}_\alpha$ does not depend on the bond configuration on the edges of $\mathfrak{G}_{\mathcal{G}_\alpha}$.

Let us define now the event $\mathcal{R}^{in}_N$ via

$$\mathcal{R}^{in}_N \;=\; \Big\{\, \text{There is an occupied dual FK chain across } T_N \cap \big(\sqrt{\frac{a_N}{2m^*}}\mathcal{K}_1 \setminus \widetilde{T_N}\big)\,\Big\}.$$

By the FKG inequality and the results of [CCS], for any realization of $\mathcal{G}_\alpha$, we have:

$$\mu^{\beta^*}_{\mathcal{G}_\alpha,w}\left(\, \mathcal{R}^{in}_N \,\right) \;\leq\; \mu^{\beta^*}_{\sqrt{\frac{a_N}{2m^*}}\mathcal{K}_1\setminus\widetilde{T_N},w}\left(\, \mathcal{R}^{in}_N \,\right) \;\leq\; c_8\sqrt{a_N}\exp\left(\, -c_9(\beta)M\sqrt{s(N)\log N}\,\right),$$

which is $\mathrm{o}(1)$ as soon as M is large enough, provided that (3.3.1) holds. Therefore, using the decoupling properties [K], [Pi] of the FK measures and the duality,

$$\begin{aligned}\mu^{\beta}_{\Lambda_N,w}\left(\, \mathcal{C}_N\,;\, \mathcal{R}^{in}_N\,\right) \;&=\; \sum_{\alpha} \mu^{\beta}_{\Lambda_N,w}\left(\, \mathcal{G}_\alpha\,;\, \mathcal{R}^{in}_N\,\right)\\ &=\; \sum_{\alpha} \mu^{\beta}_{\Lambda_N,w}\left(\, \mathcal{G}_\alpha\,\right)\mu^{\beta^*}_{\mathcal{G}_\alpha,w}\left(\, \mathcal{R}^{in}_N\,\right) \;=\; o\left(\, \mu^{\beta}_{\Lambda_N,w}\left(\, \mathcal{C}_N\,\right)\,\right).\end{aligned} \tag{3.4.5}$$

Exactly in the same fashion one might define

$$\mathcal{R}^{out}_N \;=\; \Big\{\, \text{There is an occupied dual FK chain across } T_N \setminus \big(\sqrt{\frac{a_N}{2m^*}}\mathcal{K}_1 \cup \widetilde{T_N}\big)\,\Big\},$$

and, with the obvious modifications in the definitions of the random sets $\mathcal{D}$ and $\mathcal{G}$, obtain:

$$\mu^{\beta}_{\Lambda_N,w}\left(\, \mathcal{C}_N\,;\, \mathcal{R}^{out}_N\,\right) \;=\; o\left(\, \mu^{\beta}_{\Lambda_N,w}\left(\, \mathcal{C}_N\,\right)\,\right). \tag{3.4.6}$$

Since

$$\widetilde{\mathcal{E}_N} \supset \left(\mathcal{R}_N^{in} \cup \mathcal{R}_N^{out} \right)^c ,$$

the estimates (3.4.5),(3.4.6) and (3.3.3) readily imply (3.4.1), and the proof of the lower bound (3.2.4) on the occurrence of the $\pm$ contour close to the required dilatation of the Wulff shape is, thereby, complete.

It remains, therefore, to perform the second step of the proof of the Theorem 3.1.1, which is to give a local limit type estimate on

$$\min_{\gamma} \mathbb{P}_{N,-}^{\beta} \left(M_N = -N^2 m^* + a_N \mid \mathcal{E}_N^{K,\gamma} \right), \tag{3.4.7}$$

where $\mathcal{E}_N^{K,\gamma}$ partitions $\mathcal{E}_N^K$ according to what the single $K \log N$-large contour γ in $\mathcal{E}_N^K$ is. And then to optimize the combined estimate in terms of the large skeleton scale $s(N)$ chosen.

Each $\pm$ contour γ in (3.4.7) splits Λ_N into the disjoint union of the inner component C and the outer component B; $\Lambda_N = B \cup C$, and the following decomposition, which we repeatedly use throughout the paper is valid:

$$\mathbb{P}_{N,-}^{\beta} \Big(M_N = -m^* N^2 + a_N \mid \mathcal{E}_N^{K,\gamma} \Big) = \sum_{b_N + c_N = a_N - \Delta(B,C)} \mathbb{P}_{B,-}^{\beta, K \log N} \Big(M_B = -m^* |B| + b_N \Big) \times \mathbb{P}_{C,-}^{\beta, K\log N} \Big(M_C = m^* |C| + c_N \Big),$$

where $\Delta(B,C) = 2m^* \big|C\big| + (m^*+1)\big|\partial_+\gamma\big| + (m^*-1)\big|\partial_-\gamma\big|$.

However, by the definition, γ lies in the $w_N = M\sqrt{s(N)\log N}$ tube around $\sqrt{a_N/2m^*}\partial\mathcal{K}_1$. Consequently, any realization of the inner domain C satisfies:

$$\big| \, a_N \; - \; 2m^*|C| \, \big| \; \leq \; c_{10}\sqrt{a_N} w_N \; \leq \; c_{11}\sqrt{a_N s(N) \log N}.$$

Moreover, by the definition of $\mathcal{E}_N^K$, $\big|\partial C\big| \leq 2K\sqrt{a_N}\log N$. Combined with the above bound on the volume of C, this, in view of (3.3.10), implies that $C \in \mathcal{D}_{\sqrt{a_N}}$. Furthermore, since $\big|\partial_+\gamma\big| \vee \big|\partial_-\gamma\big| \leq 2\big|\partial C\big|$,

$$\Big| \, a_N \; - \; \Delta(B,C) \, \Big| \; \leq \; c_{12}\sqrt{a_N s(N)\log N}$$

as well. As a result Gaussian estimates of the previous section ((2.3.2) and Lemma 2.3.3 whose hypothesis are satisfied thanks to (3.3.10)) apply, and we, thereby, obtain:

$$\min_{\gamma} \mathbb{P}_{N,-}^{\beta} \left(M_N = -N^2 m^* + a_N \, \middle| \mathcal{E}_N^{K,\gamma} \right) \geq \exp\left(- c_{13} \frac{a_N s(N) \log N}{N^2} \right) \geq \exp\left(- c_{13} s(N) \log N \right).$$

Together with (3.2.2) and (3.2.4) this yields the following bound:

$$\mathbb{P}_{N,-}^{\beta} \Big(M_N = -m^* N^2 + a_N \Big) \geq \exp\left(- \sqrt{\frac{a_N}{2m^*}}\omega_1 \; - \; c_{14} \log N \Big(\frac{\sqrt{a_N}}{s(N)} + s(N) \Big) \right). \tag{3.4.8}$$

In view of the right hand side of (3.4.8) above we see that the optimal scale $s(N)$ corresponds to the minimal available order of

$$\frac{\sqrt{a_N}}{s(N)} \; + \; s(N),$$

which is attained at $s(N) \sim \sqrt[4]{a_N}$. Note that such an optimal choice is always compatible with the restrictions (3.3.4), (3.3.1) and (3.3.10) which we were implicitly assuming in the course of the proof.

Substituting $s(N) \sim \sqrt[4]{a_N}$ into the estimate (3.4.8), we arrive to the conclusion of the Theorem 3.1.1.

Remark 3.4.1. The estimate on the optimal scale could be further refined if we minimize a more exact expression

$$\frac{\sqrt{a_N}}{s(N)} + \frac{a_N s(N)}{N^2}.$$

4. Subcritical Contours

In this section we show that contours γ with

$$K \log N \;\leq\; \mathrm{diam}(\gamma) \;\ll\; N^{\frac{2}{3}}$$

do not appear with the $\mathbb{P}^{\beta}_{A,-}(\,\cdot\,|M_A = \mathbb{M}_A + a_N)$-probability tending to one essentially in the whole range of $A \in \mathcal{D}_N$ and $a_N \in \mathcal{M}_A$.

In the small moderate deviation case this assertion is part of the claim of Theorem C, which we prove in the first subsection. The case of large moderate deviations $a_N \gg N^{4/3+\delta}$ is studied in Subsection 4.2.

4.1. **Proof of Theorem C.** So assume that $\delta \in (0, 4/3)$ is fixed, and $a_N \ll N^{4/3-\delta}$. We write,

$$\begin{aligned} \mathbb{P}^{\beta}_{A,-}\big(M_A = \mathbb{M}_A + a_N\big) \;&=\; \mathbb{P}^{\beta, K\log N}_{A,-}\big(M_A = \mathbb{M}_A + a_N\big)(1 + \mathrm{o}(1)) \\ &+\; \mathbb{P}^{\beta}_{A,-}\big(M_A = \mathbb{M}_A + a_N;\; \exists\; K\log N - \text{large contour}\big). \end{aligned} \tag{4.1.1}$$

By the second of inequalities (1.1.1),

$$\big|\mathbb{M}_A \;-\; \mathbb{M}^{K\log N}_A\big| \;=\; \mathrm{o}(1),$$

as soon as K is sufficiently large, which we always assume. Thus, by virtue of the results on the moderate deviations in the $K \log N$ restricted phase, i.e., by Lemma 2.3.3, the first term in (4.1.1) equals to

$$\mathbb{P}^{\beta, K\log N}_{A,-}\big(M_A = \mathbb{M}_A + a_N\big) \;=\; \frac{1}{\sqrt{2\pi\chi|A|}}\exp\Big\{\,-\frac{a_N^2}{2\chi|A|}\Big\}\big(1 \;+\; \mathrm{o}(1)\big), \tag{4.1.2}$$

uniformly in all A and a_N satisfying the conditions of the theorem (see Remark 2.3.4 after the statement of Lemma 2.3.3).

Thus, the claim of the Theorem follows once we show that the second term in (4.1.1) is negligible with respect to the above expression.

Notice that the existence of a $K \log N$-large contour implies that the set of $K \log N$-large skeletons is not empty. Thus,

$$\mathbb{P}^{\beta}_{N,-}\Big(\,\cdot\,;\;\exists\; K\log N\text{-large contour}\Big) \;\leq\; \sum_{\mathfrak{S}\neq\emptyset}\mathbb{P}^{\beta}_{N,-}\Big(\,\cdot\,;\;\mathfrak{S}\,\Big),$$

and the main step of the proof will be to derive upper bounds on $\mathbb{P}^{\beta}_{N,-}\big(M_A = \mathbb{M}_A + a_N; \mathfrak{S}\big)$ for different collections $\mathfrak{S}$ of $K \log N$-large skeletons.

The expression (4.1.2) provides a lower bound for $\mathbb{P}^{\beta}_{A,-}\big(M_A = \mathbb{M}_A + a_N\big)$ and we shall repeatedly use it in order to rule out various improbable events. Specifically, by the energy estimate (1.3.2) there exists $c_1 < \infty$, such that the contribution to the right hand side of(4.1.1) of all $K \log N$-collections $\mathfrak{S}$, which do not comply with the energy bound

$$\mathcal{W}_\beta(\mathfrak{S}_+) \;\leq\; \frac{a_N^2}{2\chi|A|}\big(1 + \frac{c_1}{K}\big) \tag{4.1.3}$$

is negligible.

Remark 4.1.1. In particular, the second term on the right hand side of (4.1.1) is always negligible whenever $a_N^2/N^2 = \mathrm{O}(1)$ (provided that K is large enough). Consequently, it remains to study the case of $\delta \in (0, 1/3)$ only.

By the isoperimetric inequality, we are entitled to disregard any collection of $K\log N$-large skeletons $\mathfrak{S}$, unless

$$\big|\mathfrak{S}_+\big| \;\leq\; \big(\frac{1}{\omega_1}\mathcal{W}_\beta(\mathfrak{S}_+)\big)^2 \;\ll\; a_N N^{-3\delta}. \tag{4.1.4}$$

Let now $\mathfrak{S}$ be a collection of $K\log N$-large skeletons which satisfies both energy and (hence) volume constraints above. As usual, each $\Gamma\sim\mathfrak{S}$ splits A into the disjoint union of the "$-$" and "$+$" components; $A \;=\; B\cup C$. Using the corresponding decomposition $M_A = M_B + M_C$ and the flip symmetry of the $\mathbb{P}_C$ measures, we estimate;

$$\begin{aligned}\mathbb{P}^{\beta}_{A,-}\big(\, M_A \;=\; \mathbb{M}_A + a_N \;;\; \mathfrak{S}\big) \;&\leq \mathrm{e}^{-\mathcal{W}_\beta(\mathfrak{S}_+)} \times \\ \times \max_{\Gamma\sim\mathfrak{S}} N^2 \max_{b_N+c_N=a_N-\Delta(B,C)} &\mathbb{P}^{\beta,K\log N}_{B,-}\big(\, M_B \;=\; \mathbb{M}^{K\log N}_B + b_N\big)\times \\ &\times\mathbb{P}^{\beta,K\log N}_{C,-}\big(\, M_C \;=\; \mathbb{M}^{K\log N}_C - c_N\big),\end{aligned} \tag{4.1.5}$$

where the compensator $\Delta(B,C)$ is given this time by

$$\Delta\big(B,C\big) \;=\; \mathbb{M}^{K\log N}_B \;-\; \mathbb{M}^{K\log N}_C \;-\; \mathbb{M}_A \;+\; \big|\partial_+\Gamma\big| \;-\; \big|\partial_-\Gamma\big|.$$

As usual, by the very notion of skeletons and by the phasevolume estimate (1.1.3), both the area $|C|$ and the boundary $|\partial C|$ of the microscopic "$+$" phase component are controlled in terms the restrictions (4.1.3) and (4.1.4) on the $K\log N$-large skeleton collections $\mathfrak{S}$. Specifically,

$$\big|\partial C\big| \;\leq\; c_1 K\log N\big|\partial\mathfrak{S}_+\big| \;\ll\; N\log N,$$

which means that the "$-$" component B belongs to $\mathcal{D}_N$. Thus, by Lemma 2.3.3,

$$\mathbb{P}^{\beta,K\log N}_{B,-}\big(\, M_B = \mathbb{M}^{K\log N}_B + b_N\,\big) \;\leq\; \exp\big(-\frac{b_N^2}{2\chi|B|}\big). \tag{4.1.6}$$

On the other hand, the area of the "$+$" component C is bounded above by

$$\big|C\big| \;\leq\; \big|\mathfrak{S}_+\big| \;+\; c_2 K\log N\mathcal{W}_\beta\big(\mathfrak{S}_+\big) \;\ll\; a_N N^{-3\delta}, \tag{4.1.7}$$

as it readily follows from (4.1.3), (4.1.4) and the choice of δ in the range $\delta\in(0,1/3)$ (see Remark 4.1.1 above).

Let us now inspect the right hand side of (4.1.5) more closely: If $c_N\leq 0$, then $b_N\geq a_N-\Delta(A,B)$ which, by (4.1.6), implies

$$\begin{aligned}\mathbb{P}^{\beta,K\log N}_{B,-}\big(M_B = \mathbb{M}^{K\log N}_B + b_N\big)&\mathbb{P}^{\beta,K\log N}_{C,-}\big(M_C = \mathbb{M}^{K\log N}_C - c_N\big)\\ &\leq\; \exp\big(-\frac{(a_N-\Delta(B,C))^2}{2\chi|B|}\big).\end{aligned} \tag{4.1.8}$$

For c_N outside the phase transition region; $c_N>0$, we use the Gaussian estimate (2.3.2):

$$\mathbb{P}^{\beta,K\log N}_{C,-}\big(\, M_C = \mathbb{M}^{K\log N}_C - c_N\,\big) \;\leq\; \exp\big(-c_3(\beta)\frac{c_N^2}{|C|}\big).$$

In this case we shall also need the fact that for large N, since $|B| \leq |A|$ and $|C| \ll |A|$,

$$
\begin{aligned}
\frac{b_N^2}{2\chi|B|} \; + \; c_3\frac{c_N^2}{|C|} \; &\geq \; \frac{(b_N+c_N)^2}{2\chi|A|} \; - \; 2\frac{b_N c_N}{\chi|A|} \\
&\geq \; \frac{(a_N-\Delta(B,C))^2}{2\chi|A|} \; - \; 2\frac{(a_N-\Delta(B,C))|C|}{\chi|A|} \\
&\geq \; \frac{(a_N)^2}{2\chi|A|} \; - \; c_4\frac{a_N|C|}{|A|},
\end{aligned}
$$

where we have used Lemma 2.2.1 and (2.2.6) to bound $\Delta(B,C) \leq c_5|C|$ in the last inequality.
Combined with (4.1.8) this gives

$$
\max_{b_N+c_N=a_N-\Delta(B,C)} \mathbb{P}_{B,-}^{\beta,K\log N}\big(M_B = \mathbb{M}_B^{K\log N}+b_N\big)\mathbb{P}_{C,-}^{\beta,K\log N}\big(M_C = \mathbb{M}_C^{K\log N}-c_N\big)
$$
$$
\leq \; \exp\big(-\frac{(a_N)^2}{2\chi|A|} \; + \; c_4\frac{a_N|C|}{|A|}\big). \quad (4.1.9)
$$

By the estimates (4.1.4), (4.1.7) and (4.1.3),

$$
\begin{aligned}
\frac{|C|}{\mathcal{W}_\beta(\mathfrak{S}_+)} \; &\leq \; c_6\Big(\mathcal{W}_\beta(\mathfrak{S}_+) \; + \; K\log N\Big) \\
&\leq \; c_7\mathcal{W}_\beta(\mathfrak{S}_+) \; \leq \; c_8\frac{a_N^2}{N^2}.
\end{aligned}
$$

Thus,

$$
\frac{a_N|C|}{|A|} \; = \; \mathcal{W}_\beta(\mathfrak{S}_+)\frac{|C|}{\mathcal{W}_\beta(\mathfrak{S}_+)}\frac{a_N}{|A|} \; \leq \; c_9\mathcal{W}_\beta(\mathfrak{S}_+)\frac{a_N^3}{N^4} \; \ll \; N^{-3\delta}\mathcal{W}_\beta(\mathfrak{S}_+).
$$

Consequently, the right hand side of the (4.1.9) is bounded above by

$$
\exp\big(-a_N^2/2\chi|A| + N^{-3\delta}\mathcal{W}_\beta(\mathfrak{S}_+)\big).
$$

Back to the decomposition (4.1.5), we obtain:

$$
\mathbb{P}_{A,-}^{\beta}\big(\, M_A \; = \; \mathbb{M}_A + a_N \; ; \; \mathfrak{S}\big) \; \leq \; N^2 \mathrm{e}^{-\mathcal{W}_\beta(\mathfrak{S}_+)(1-N^{-3\delta})}\exp\big(-\frac{a_N^2}{2\chi|A|}\big).
$$

Comparing the expression on the right hand side above with (4.1.2), we see that it can be further bounded above by

$$
c_{10}N^3 e^{-\frac{1}{2}\mathcal{W}_\beta(\mathfrak{S}_+)}\mathbb{P}_{A,-}^{\beta,K\log N}\big(M_A = \mathbb{M}_A + a_N\big).
$$

However, for large enough values of K,

$$
N^3\sum_{\mathfrak{S}\neq\emptyset} e^{-\frac{1}{2}\mathcal{W}_\beta(\mathfrak{S}_+)} \; = \; \mathrm{o}(1),
$$

and the claim of Theorem C follows.

4.2. **Large moderate deviations.**

Lemma 4.2.1. *Assume that the value of $a_N \in \mathcal{M}_N$ is in the domain of large or large moderate deviations; $a_N \geq N^{\frac{4}{3}+\delta}$.*

Then, for large enough K, $\forall\, \epsilon > 0$,

$$\mathbb{P}^{\beta}_{N,-}\big(\exists \text{ exteriour contour } \gamma : \ diam(\gamma) \in (K\log N, N^{\frac{2}{3}-\epsilon}) \ \big|\ M_N = -N^2 m^* + a_N \big) \ = \ o(1). \tag{4.2.1}$$

Remark 4.2.2. By (1.1.1) exactly the same conclusion is valid under the conditions of the lemma for $\mathbb{P}^{\beta}_{N,-}\big(\,\cdot \mid M_N = \mathbb{M}_N + a_N \,\big)$.

Proof. With $\epsilon > 0$ fixed we choose a large contour parameter $s(N)$ of the form

$$s(N) \ = \ N^{\frac{2}{3}-\eta}\ , \qquad \eta \in (0,\epsilon). \tag{4.2.2}$$

We are going to prove that with $\mathbb{P}^{\beta}_{N,-}\big(\ \cdot \ \big|\ M_N = -m^* N^2 + a_N \big)$-probability tending to one all $K\log N$-large exteriour contours of σ are $s(N)$ large as well.

So let $\Gamma(\sigma)$ be the collection of all $s(N)$-large exteriour contours of σ. As usual we consider the Γ-induced decomposition $\Lambda_N = B \cup C$ of Λ_N into respectively "$-$" and "$+$" components. By the lower bound (3.1.4) and by the energy estimate (1.3.2) applied on the $K\log N$-scale we routinely restrict attention only to the case of collections of large contours Γ of admissible length; $|\Gamma| \leq RN\log N$ so that $B \in \mathcal{D}_N$. Pick now a $\xi > 0$, such that

$$N^{1+\xi} \ \ll \ N^{\frac{4}{3}}. \tag{4.2.3}$$

Assume that $|C| = \mathrm{Vol}_+(\Gamma(\sigma))$ satisfies,

$$\Big| \frac{a_N}{2m^*} \ - \ \mathrm{Vol}_+(\Gamma(\sigma)) \Big| \ \ll \ N^{1+\xi}, \tag{4.2.4}$$

Remark 4.2.3. In the case of large deviations; $a_N \sim N^2$, (4.2.4) immediately implies that the "$+$" component $C \in \mathcal{D}_N$ as well. Consequently, a straightforward modification of the proof below, which is entirely built upon the uniform estimates of Theorem C, enables one to drop the adjective "exteriour" in the statement of Lemma 4.2.1.

We claim that under assumption (4.2.4) with $\mathbb{P}^{\beta}_{N,-}(\ \cdot\ | M_N = -N^2 m^* + a_N)$-probability close to one there are no $K\log N$-large exteriour contours other than those belonging to Γ. Indeed, such contours can appear only inside B, and, as in the proof of Theorem C, we can write for events from Ω_B,

$$\begin{aligned} &\mathbb{P}^{\beta}_{N,-}\big(\ \cdot\ ; M_N = -N^2 m^* + a_N \ \big|\Gamma\big) \\ &= \sum_{b_N + c_N = a_N - \Delta(B,C)} \mathbb{P}^{\beta,s}_{B,-}\big(\ \cdot\ ;\ M_B = \mathbb{M}^s_B + b_N\big)\ \mathbb{P}^{\beta}_{C,-}\big(M_C = \mathbb{M}_C - c_N\big), \end{aligned} \tag{4.2.5}$$

where the corresponding value of the compensator $\Delta(B,C)$ is given by

$$\Delta\big(B,C\big) \ = \ N^2 m^* + \mathbb{M}^s_B - \mathbb{M}_C + (m^*+1)\big|\partial_+\Gamma\big| + (m^*-1)\big|\partial_-\Gamma\big|,$$

and

$$\Big|\Delta\big(B,C\big) - 2m^*\big|C\big|\Big| \ \leq \ c_1(\beta)\big|\partial C\big| \ \leq \ c_1 RN\log N, \tag{4.2.6}$$

where the last two inequalities follow respectively from (1.1.1) and the admissibility of contour lengths $|\Gamma|$ under consideration.

Furthermore, for each $b_N \geq 0$, we have by the upper bound of Lemma 2.5.1,

$$\mathbb{P}^{\beta,s}_{B,-}\left(M_B = \mathbb{M}^s_B + b_N\right) \;\leq\; \exp\Big(-c_1 \frac{b_N^2}{N^2} \wedge \frac{b_N}{s(N)}\Big).$$

In particular, there exists ν small enough, such that

$$\mathbb{P}^{\beta,s}_{B,-}\left(M_B = \mathbb{M}^s_B + b_N\right) \;\leq\; \mathrm{e}^{-c_2 N^{2\xi+\nu}},$$

for each b_N satisfying $b_N > N^{4/3-\nu}$.

On the other hand, by Chebyshev inequality,

$$\mathbb{P}^{\beta}_{C,-}\left(\left|M_C - \mathbb{M}_C\right| \;>\; N^{1+\xi} \right) \;=\; \mathrm{o}(1).$$

Consequently, it follows from the decomposition (4.2.5) and the local limit estimate of Lemma 2.3.3, that for each $s(N)$-collection of contours Γ satisfying (4.2.4),

$$\begin{aligned}\mathbb{P}^{\beta}_{N,-}\left(M_N = -N^2 m^* + a_N \;\middle|\; \Gamma\right) \;&\geq\; \frac{1}{2N^{1+\xi}} \min_{\substack{b_N \in \mathcal{M}_B \\ |b_N| \leq N^{1+\xi}}} \mathbb{P}^{\beta,s}_{B,-}\left(M_B = \mathbb{M}^s_B + b_N\right) \\ &\geq \frac{c_3}{N^{1+\xi}} \exp\big(-c_4 N^{2\xi}\big).\end{aligned}$$

Thus we conclude that for any such Γ,

$$\mathbb{P}^{\beta}_{N,-}\left(M_B \geq \mathbb{M}^s_B + N^{4/3-\nu} \;\middle|\; \Gamma; M_N = -N^2 m^* + a_N\right) \;=\; \mathrm{o}(1),$$

provided that $\nu = \nu(\xi,\eta) > 0$ is sufficiently small.

At this stage we can evoke Theorem C and assert that in the remaining range of values $b_N \leq N^{4/3-\nu}$,

$$\mathbb{P}^{\beta,s}_{B,-}\left(\exists\; K \log N\text{-large contour} \;\middle|\; M_B = \mathbb{M}^s_B + b_N\right) \;=\; \mathrm{o}(1)$$

uniformly in all families Γ of $s(N)$-large contours satisfying the volume condition (4.2.4) (and having the admissible length $|\Gamma| \leq RN \log N$).

It, therefore, suffices to verify the following statement:

Lemma 4.2.4. *Suppose that $a_N \in \mathcal{M}^+_N$ is in the range of large moderate deviations; $a_N \geq N^{4/3+\delta}$, $\delta > 0$. Suppose also that the large contour parameter $s(N)$; $\log N \ll s(N) \ll N^{2/3}$, and $\xi \in (0,1/3)$ satisfy:*

$$N^{\frac{2}{3}} \;\gg\; N^{2\xi} \;\gg\; \frac{\sqrt{a_N}}{s(N)} \log N \vee \frac{N^2}{a_N} \vee \sqrt[4]{a_N} \log N \vee s(N).$$

Let $\Gamma = \Gamma(\sigma)$ denote the collection of exteriour $s(N)$-large contours. Then for each $\nu > 0$,

$$\mathbb{P}^{\beta}_{N,-}\left(\left|\frac{a_N}{2m^*} \;-\; Vol_+(\Gamma(\sigma))\right| \;>\; \nu N^{1+\xi} \;\middle|\; M_N = -N^2 m^* + a_N\right) \;=\; \mathrm{o}(1). \tag{4.2.7}$$

Remark 4.2.5. Note that by choosing η in (4.2.2) small enough, we can apply this lemma in our setting above, and so complete the proof of Lemma 4.2.1 .

Proof of Lemma 4.2.4. As before we employ the skeleton decomposition of the event

$$\mathcal{C} \triangleq \{ \big| \frac{a_N}{2m^*} - \mathrm{Vol}_+(\Gamma(\sigma)) \big| \ > \ \nu N^{1+\xi} \}$$

on the $s(N)$-scale,

$$\mathbb{P}^{\beta}_{N,-}(\ \mathcal{C}\ ) \ \leq \ \sum_{\mathfrak{S}} \mathbb{P}^{\beta}_{N,-}(\ \mathcal{C}\ ;\ \mathfrak{S}\ ).$$

It happens to be convenient to fix a parameter $r_N \ll \sqrt{a_N}$, the precise range of values for which we specify later on, and to distinguish between high; $\mathcal{W}_\beta(\mathfrak{S}_+) \geq (\sqrt{a_N/2m^*} - r_N)\omega_1$, and low; $\mathcal{W}_\beta(\mathfrak{S}_+) \leq (\sqrt{a_N/2m^*} - r_N)\omega_1$, energy collections $\mathfrak{S}$.

Let us start with the high energy case:

There are at most N^2 terms in the sum on the right hand side of (4.2.5). Thus, applying for each term in this sum either Lemma 2.5.1 inside the phase transition region or Gaussian estimates (2.3.2) outside, we infer that

$$\mathbb{P}^{\beta}_{N,-}\big(\ M_N = -N^2 m^* + a_N \ \big|\ \Gamma\big) \ \leq \ \exp\big(-c_5 N^{2\xi} \wedge \frac{N^{1+\xi}}{s(N)}\big) \ \leq \ \mathrm{e}^{-c_6 N^{2\xi}},$$

for each Γ violating (4.2.4). Therefore, by the energy estimate (1.3.2),

$$\begin{aligned}
&\sum_{\mathcal{W}_\beta(\mathfrak{S}_+) \geq (\sqrt{\frac{a_N}{2m^*}} - r_N)\omega_1} \mathbb{P}^{\beta}_{N,-}\big(M_N = -N^2 m^* + a_N \ ;\ \mathfrak{S}\ ;\ \big|\frac{a_N}{2m^*} - \mathrm{Vol}_+(\Gamma(\sigma))\big| > \nu N^{1+\xi}\big) \\
&\leq \sum_{\mathcal{W}_\beta(\mathfrak{S}_+) \geq (\sqrt{\frac{a_N}{2m^*}} - r_N)\omega_1} \mathrm{e}^{-\mathcal{W}_\beta(\mathfrak{S}_+)} \max_{\substack{\Gamma \sim \mathfrak{S} \\ \left|\frac{a_N}{2m^*} - \mathrm{Vol}_+(\Gamma)\right| > \nu N^{1+\xi}}} \mathbb{P}^{\beta}_{N,-}\big(M_N = -N^2 m^* + a_N \ \big|\ \Gamma\ \big) \qquad (4.2.8) \\
&\leq \ \exp\Big(\ -(\sqrt{\frac{a_N}{2m^*}} - r_N)\omega_1(1 - \frac{c_7 \log N}{s(N)})\ - c_8 N^{2\xi}\Big).
\end{aligned}$$

Our next step amounts to a careful choice of the parameters r_N and ξ in (4.2.8):

By the lower bound (3.1.4) the right hand side of (4.2.8) is negligible with respect to $\mathbb{P}^{\beta}_{N,-}(M_N = -N^2 m^* + a_N)$, as soon as

$$N^{2\xi} \ \gg \ r_N \vee \frac{\sqrt{a_N}\log N}{s(N)} \vee \sqrt[4]{a_N}\log N. \qquad (4.2.9)$$

In other words, for any choice of parameters ξ, r_N and $s(N)$, satisfying (4.2.3) and (4.2.9), the occurrence of high energy $\mathcal{W}_\beta(\mathfrak{S}+) \geq \omega_1(\sqrt{a_N/2m^*} - r_N)$ collections is ruled out modulo $\{M_N = -N^2 m^* + a_N\}$ once (4.2.4) is violated.

As a result, it remains to show that

$$\sum_{\mathcal{W}_\beta(\mathfrak{S}_+) < (\sqrt{\frac{a_N}{2m^*}} - r_N)\omega_1} \mathbb{P}^{\beta}_{N,-}\big(M_N = -N^2 m^* + a_N \ ;\ \mathfrak{S}\ ;\ \big|\frac{a_N}{2m^*} - \mathrm{Vol}_+(\Gamma(\sigma))\big| > \nu N^{1+\xi}\big) \qquad (4.2.10)$$

is also negligible with respect to $\mathbb{P}^{\beta}_{N,-}(M_N = -N^2 m^* + a_N)$.

To this end, note, first of all, that by the isoperimetric inequality and (4.2.9), for each $\mathfrak{S}$, such that

$$\mathcal{W}_\beta(\mathfrak{S}_+) \ \le\ \Big(\sqrt{\frac{a_N}{2m^*}}\ -\ r_N\Big)\omega_1,$$

the following bound on $|\mathfrak{S}_+|$ takes place:

$$\big|\mathfrak{S}_+\big| \ \le\ \Big(\frac{1}{\omega_1}\mathcal{W}_\beta(\mathfrak{S}_+)\Big)^2 \ \le\ \frac{a_N}{2m^*} -\ c_9 r_N\sqrt{a_N}. \tag{4.2.11}$$

Since on the $s(N)$-scale,

$$\big|\mathrm{Vol}_+(\Gamma(\sigma))\ -\ |\mathfrak{S}_+|\big| \ \le\ c_{10}\mathcal{W}_\beta(\mathfrak{S}_+)s(N),$$

whenever Γ and $\mathfrak{S}$ are compatible, we conclude from (4.2.11) that also

$$\mathrm{Vol}_+(\Gamma(\sigma)) \ \le\ \frac{a_N}{2m^*}\ -\ c_{11}r_N\sqrt{a_N}, \tag{4.2.12}$$

provided that

$$r_N \ \gg\ s(N). \tag{4.2.13}$$

Now, the low droplet energy expression in (4.2.10) is bounded above by

$$\sum_{\mathcal{W}_\beta(\mathfrak{S}_+)<(\sqrt{\frac{a_N}{2m^*}}-r_N)\omega_1} \mathbb{P}^\beta_{N,-}\big(M_N = -N^2m^* + a_N\ ;\ \mathfrak{S}\ \big)$$
$$\le \sum_{\mathcal{W}_\beta(\mathfrak{S}_+)<(\sqrt{\frac{a_N}{2m^*}}-r_N)\omega_1} \mathrm{e}^{-\mathcal{W}_\beta(\mathfrak{S}_+)} \max_{\Gamma\sim\mathfrak{S}} \mathbb{P}^\beta_{N,-}\big(M_N = -N^2m^* + a_N\ \big|\ \Gamma\ \big).$$

By the decomposition (4.2.5) each Γ-term in this sum is, in its turn, bounded above by

$$N^2 \max_{b_N+c_N=a_N-\Delta(B,C)} \mathbb{P}^{\beta,s}_{B,-}\big(M_B = \mathbb{M}^s_B + b_N\big)\ \mathbb{P}^\beta_{C,-}\big(M_C = \mathbb{M}_C - c_N\big),$$

A careful look at (4.2.12) and (4.2.6) reveals that this max is attained for positive values of c_N, such that $\mathbb{M}_C - c_N$ is outside the phase transition region. In such a case one always has a Gaussian bound on the $\mathbb{P}^\beta_{C,-}$-probabilities, and, using Lemma 2.5.1 to bound $\mathbb{P}^{\beta,s}_{B,-}$-term, we conclude that the above expression does not exceed

$$\max_{b_N+c_N=a_N-\Delta(B,C)} \exp\Big(\ -c_{12}\Big(\frac{b_N^2}{N^2}\wedge\frac{b_N}{s(N)}\ +\ \frac{c_N^2}{|C|}\Big)\Big)$$
$$\le\ \exp\Big(-c_{13}\frac{\big(a_N-\Delta(B,C)\big)^2}{N^2}\wedge\frac{\big(a_N-\Delta(B,C)\big)}{s(N)}\ \Big).$$

Since for each low energy collection $\mathfrak{S}$, $a_N \gg s(N)\mathcal{W}_\beta(\mathfrak{S}_+)$, and since, in addition, $|\mathfrak{S}_+| \le \big(\mathcal{W}_\beta(\mathfrak{S}_+)/\omega_1\big)^2$, a crude estimate , based on (4.2.6) and (1.1.3), yields,

$$a_N\ -\ \Delta\big(B,C\big)\ \ge\ a_N\ -\ 2m^*\big|\mathfrak{S}_+\big|\ -\ c_{14}\mathcal{W}_\beta(\mathfrak{S}_+)s(N)\ \ge\ c_{15}\Big(\frac{a_N}{2m^*}\omega_1^2\ -\ (\mathcal{W}_\beta(\mathfrak{S}))^2\Big).$$

In order to simplify notation let us introduce

$$\mathcal{W}_\beta(a_N)\ \triangleq\ \sqrt{\frac{a_N}{2m^*}}\omega_1.$$

Subsequently, each $\mathfrak{S}$ term in (4.2.10) is bounded above by

$$\exp\Big\{ -\mathcal{W}_\beta(\mathfrak{S}_+) \;-\; c_{16}\frac{\big(\mathcal{W}_\beta(a_N)^2-\mathcal{W}_\beta(\mathfrak{S}_+)^2\big)^2}{N^2}\wedge\frac{\mathcal{W}_\beta(a_N)^2-\mathcal{W}_\beta(\mathfrak{S}_+)^2}{s(N)}\Big\}. \tag{4.2.14}$$

We proceed to study both terms in the wedge product in (4.2.14):

For each $\mathfrak{S}$ satisfying

$$\mathcal{W}_\beta(\mathfrak{S}_+)\;<\;\mathcal{W}_\beta(a_N)\;-\;r_N\omega_1,$$

we obtain:

$$\begin{aligned}
&\mathcal{W}_\beta(\mathfrak{S}_+)\;+\;c_{16}\frac{\big(\mathcal{W}_\beta(a_N)^2-\mathcal{W}_\beta(\mathfrak{S}_+)^2\big)^2}{N^2}\\
&=\;\mathcal{W}_\beta(a_N)\;+\;c_{16}\frac{(\mathcal{W}_\beta(a_N)-\mathcal{W}_\beta(\mathfrak{S}_+))^2(\mathcal{W}_\beta(a_N)+\mathcal{W}_\beta(\mathfrak{S}_+))^2}{N^2}\;-\;\big(\mathcal{W}_\beta(a_N)\;-\;\mathcal{W}_\beta(\mathfrak{S}_+)\big)\\
&\qquad\ge\;\mathcal{W}_\beta(a_N)\;+\;r_N\big(c_{16}\frac{r_Na_N}{N^2}\;-\;1\big),
\end{aligned}$$

which is bounded below by

$$\sqrt{\frac{a_N}{2m^*}}\omega_1\;+\;c_{16}\frac{r_N^2a_N}{N^2}, \tag{4.2.15}$$

provided that

$$r_N\;\gg\;\frac{N^2}{a_N}. \tag{4.2.16}$$

Similarly,

$$\begin{aligned}
\mathcal{W}_\beta(\mathfrak{S}_+)\;+\;c_{16}\frac{\mathcal{W}_\beta(a_N)^2-\mathcal{W}_\beta(\mathfrak{S}_+)^2}{s(N)}\;&\ge\;\mathcal{W}_\beta(a_N)\;+\;r_N\big(\frac{c_{16}}{s(N)}\mathcal{W}_\beta(a_N)\;-\;1\big)\\
&>\;\mathcal{W}_\beta(a_N)\;+\;c_{17}\,\frac{r_N\sqrt{a_N}}{s(N)}.
\end{aligned} \tag{4.2.17}$$

Substituting final estimates obtained (4.2.15) and (4.2.17) into the upper bound (4.2.14), we infer that each $\mathfrak{S}$ term in the decomposition (4.2.10) can be bounded above by

$$N^2\#\big\{\mathfrak{S}:\;\mathcal{W}_\beta(\mathfrak{S}_+)\le\sqrt{\frac{a_N}{2m^*}}\omega_1-r_N\big\}\exp\big(\;-\sqrt{\frac{a_N}{2m^*}}\omega_1\;-\;c_{18}\frac{r_N^2a_N}{N^2}\wedge\frac{r_N\sqrt{a_N}}{s(N)}\big).$$

However, the number of $s(N)$ collections, whose surface tension does not exceed some value r, is , by the usual skeleton computation, bounded above by

$$\#\big\{\mathfrak{S}:\;\mathcal{W}_\beta(\mathfrak{S}_+)\le r\big\}\;\le\;\mathrm{e}^{c_{19}r\log N/s(N)}.$$

By (4.2.9) there is always room to choose $r_N\gg\sqrt{a_N}\log N/s(N)$. On the other hand, such a choice of r_N leads to

$$\frac{r_N\sqrt{a_N}}{s(N)}\wedge\frac{r_N^2a_N}{N^2}\;=\;r_N\big(\;\frac{\sqrt{a_N}}{s(N)}\wedge\frac{r_Na_N}{N^2}\big)\;\gg\;r_N\;\gg\;\frac{\sqrt{a_N}}{s(N)}\log N.$$

Indeed, $\sqrt{a_N}/s(N)\gg1$ by the very choice of $a_N\gg N^{4/3}$ and $s(N)\ll N^{2/3}$, whereas the condition $r_Na_N/N^2\gg1$ was explicitly stipulated in (4.2.16).

Therefore,

$$\sum_{\mathcal{W}_\beta(\mathfrak{S}_+)<(\sqrt{\frac{a_N}{2m^*}}-r_N)\omega_1} \mathbb{P}^\beta_{N,-}\big(M_N = -N^2m^* + a_N \ ; \ \mathfrak{S}\ \big) \ \ll \ \exp\Big(\ -\sqrt{\frac{a_N}{2m^*}}\omega_1 \ - \ c_{20}r_N\ \Big), \tag{4.2.18}$$

which is, by the lower bound (3.1.4), negligible with respect to $\mathbb{P}^\beta_{N,-}(M_N = -N^2m^* + a_N)$, and the claim of the lemma follows.

Note that the requirements (4.2.3), (4.2.9), (4.2.13) and (4.2.16) are consistent, provided that the hypothesis of the lemma are satisfied. $\square$

5. Proof of the Main Result

The proof of Theorems A and B comprises a certain simple isoperimetric estimate, which we establish in the first subsection, and an implementation of this inequality based on the results we have obtained earlier.

5.1. **An Isoperimetric Estimate.**

Lemma 5.1.1. *Assume that,*

$$\mathcal{W}_\beta(\mathfrak{S}_+) \leq \sqrt{\frac{a_N}{2m^*}}\omega_1 \ + \ t_N \tag{5.1.1}$$

and

$$\left|\mathfrak{S}_+\right| \geq \frac{a_N}{2m^*} \ - \ k_N. \tag{5.1.2}$$

Let $\mathfrak{S}_+^1$ be the largest connected component of $\mathfrak{S}_+$ and $\mathfrak{S}_+^2$ be the rest;

$$\mathfrak{S}_+ \ = \ \mathfrak{S}_+^1 \cup \mathfrak{S}_+^2,$$

Then, the following bounds on the volume of $\mathfrak{S}_+^2$ and on the energy $\mathcal{W}_\beta(\mathfrak{S}_+^2)$ hold:

$$\left|\mathfrak{S}_+^2\right| \ \leq \ c_1 a_N \Big(\frac{t_N}{\sqrt{a_N}} \ + \ \frac{k_N}{a_N} \Big)^2. \tag{5.1.3}$$

and

$$\mathcal{W}_\beta(\mathfrak{S}_+^2) \ \leq \ c_2\sqrt{a_N}\Big(\frac{t_N}{\sqrt{a_N}} \ + \ \frac{k_N}{a_N} \Big). \tag{5.1.4}$$

Moreover, if $\mathfrak{S}_+^1$ is in addition simply connected, then

$$\min_x d_{\mathbb{H}}\Big(\sqrt{\frac{2m^*}{a_N}}\partial\mathfrak{S}_+^1, x + \partial\mathcal{K}_1\Big) \ \leq \ c_3\sqrt{\frac{t_N}{\sqrt{a_N}} \ + \ \frac{k_N}{a_N}}. \tag{5.1.5}$$

Proof. The interesting case is, of course,

$$\frac{t_N}{\sqrt{a_N}} \ + \ \frac{k_N}{a_N} \ \ll \ 1, \tag{5.1.6}$$

which, by virtue of the isoperimetric inequality, means that the ratio

$$x \ \triangleq \ \frac{\left|\mathfrak{S}_+^2\right|}{\left|\mathfrak{S}_+\right|} \ \ll \ 1$$

as well.

By the isoperimetric inequality and (5.1.2),

$$\begin{aligned} \mathcal{W}_\beta(\mathfrak{S}_+) \ &\geq \ \omega_1\Big(\ \sqrt{|\mathfrak{S}^1_+|} \ + \ \sqrt{|\mathfrak{S}^2_+|} \ \Big) \\ &\geq \ \sqrt{|\mathfrak{S}_+|}\Big(\sqrt{1-\frac{|\mathfrak{S}^2_+|}{|\mathfrak{S}_+|}} \ + \ \sqrt{\frac{|\mathfrak{S}^2_+|}{|\mathfrak{S}_+|}}\Big)\omega_1 \\ &\geq \ \Big(\sqrt{\frac{a_N}{2m*}} \ - \ k_N\Big)\Big(\sqrt{1-x} \ + \ \sqrt{x}\Big)\omega_1. \end{aligned} \tag{5.1.7}$$

Since for small values of x;

$$\sqrt{1 \ - \ x} \ + \ \sqrt{x} \ \geq \ 1 \ + \ \frac{1}{2}\sqrt{x},$$

we infer from (5.1.1) and (5.1.7) that

$$\sqrt{\frac{a_N}{2m^*}}\omega_1 \ + \ t_N \ \geq \ \sqrt{\frac{a_N}{2m^*}}\omega_1\big(1 \ - \ \frac{2m^*k_N}{a_N}\big)\big(1 \ + \ \frac{1}{2}\sqrt{x}\big).$$

As a result,

$$\sqrt{\frac{|\mathfrak{S}^2_+|}{|\mathfrak{S}_+|}} \ = \ \sqrt{x} \ \leq \ c_4\big(\ \frac{t_N}{\sqrt{a_N}} \ + \ \frac{k_N}{a_N} \ \big),$$

and, since by (5.1.1) and (5.1.6) $\big|\mathfrak{S}_+\big| < a_N/m^*$, (5.1.3) follows.

The second claim (5.1.4) is an easy consequence. Indeed, by the isoperimetric inequality, (5.1.2) and (5.1.3),

$$\begin{aligned} \mathcal{W}_\beta(\mathfrak{S}^1_+) \ &\geq \sqrt{|\mathfrak{S}^1_+|}\omega_1 \ = \ \sqrt{|\mathfrak{S}_+|\big(1 \ - \ \frac{|\mathfrak{S}^2_+|}{|\mathfrak{S}_+|}\big)} \\ &\geq \ \sqrt{\Big(\frac{a_N}{2m^*} \ - \ k_N\Big)\Big(1 \ - \ c_5\big(\frac{t_N}{\sqrt{a_N}} \ + \ \frac{k_N}{a_N}\big)^2\Big)} \ \omega_1 \\ &\geq \ \sqrt{\frac{a_N}{2m^*}}\omega_1\sqrt{1 \ - \ c_5\frac{k_N}{a_N}} \ \sqrt{1 \ - \ c_6\big(\frac{t_N}{\sqrt{a_N}} + \frac{k_N}{a_N}\big)^2}. \end{aligned}$$

Since $\mathcal{W}_\beta(\mathfrak{S}_+) = \mathcal{W}_\beta(\mathfrak{S}^1_+) + \mathcal{W}_\beta(\mathfrak{S}^2_+)$, (5.1.4) follows, using also (5.1.1).

Finally connected and simply connected $\mathfrak{S}^1_+$ satisfy the following generalization of Bonnensen inequality ([DKS], Section 9 of Chapter 2):

$$\min_x d_{\mathbb{H}}\big(\ \sqrt{\frac{1}{|\mathfrak{S}^1_+|}}\partial\mathfrak{S}^1_+, x + \partial\mathcal{K}_1\big) \ \leq \ \frac{c_7}{\sqrt[4]{a_N}}\sqrt{ \ \mathcal{W}_\beta(\mathfrak{S}^1_+) \ - \ \sqrt{|\mathfrak{S}^1_+|}\omega_1}.$$

Using assumption (5.1.1) to bound $\mathcal{W}_\beta(\mathfrak{S}^1_+)$ from above and assumption (5.1.2) together with the estimate (5.1.3) to bound $|\mathfrak{S}^1_+|$ from below, we conclude that the right hand side above is bounded by

$$\frac{c_7}{\sqrt[4]{a_N}}\sqrt{t_N + \frac{k_N}{\sqrt{a_N}}} = c_7\sqrt{\frac{t_N}{\sqrt{a_N}} + \frac{k_N}{a_N}},$$

which implies the stability estimate (5.1.4). □

5.2. **Proof of Theorem A.** Let us reformulate results of Sections 3 and 4 in the following way:

Fix $\delta \in (0, 2/3)$ and let $a_N \sim N^{4/3+\delta} \in \mathcal{M}_N^+$. Then with the $\mathbb{P}^{\beta}_{N,-}(\,\cdot\,|M_N = -N^2m^* + a_N)$-probability tending to one:

1. By Theorem 4.2.1 for large K and for each $\epsilon > 0$ fixed no exteriour contour γ with

$$K\log N < \mathrm{diam}(\gamma) \leq N^{\frac{2}{3}-\epsilon}$$

appears.

2. By the lower bound (3.1.4) and the energy estimate (1.3.2) if,

$$s(N) \gg \sqrt[4]{a_N}, \tag{5.2.1}$$

only $s(N)$-collections $\mathfrak{S}$ with

$$\mathcal{W}_\beta(\mathfrak{S}) \leq \sqrt{\frac{a_N}{2m^*}}\omega_1 + c_1\sqrt[4]{a_N}\log N \tag{5.2.2}$$

appear.

3. By Lemma 4.2.4 for any choice of $\xi < 1/3$ and a large contour parameter $s(N)$, such that $\log N \ll s(N) \ll N^{2/3}$ and

$$N^{2/3} \gg N^{2\xi} \gg \frac{\sqrt{a_N}\log N}{s(N)} \vee \frac{N^2}{a_N} \vee \sqrt[4]{a_N}\log N \vee s(N), \tag{5.2.3}$$

the family Γ of exteriour $s(N)$-large contours of σ satisfies

$$\left| \mathrm{Vol}_+(\Gamma(\sigma)) - \frac{a_N}{2m^*} \right| \ll N^{1+\xi}. \tag{5.2.4}$$

Therefore, by the phase volume estimate (1.1.3), any Γ-compatible collections of skeletons on the $s(N)$-scale satisfies

$$\left|\mathfrak{S}_+\right| \geq \frac{a_N}{2m^*} - N^{1+\xi} - c_2 s(N)\sqrt{a_N}. \tag{5.2.5}$$

Inequalities (5.2.2) and (5.2.5) set up the stage for the application of the isoperimetric estimate of Lemma 5.1.1 with

$$t_N = c_1\sqrt[4]{a_N}\log N \qquad \text{and} \qquad k_N = c_3 N^{1+\xi} \vee s(N)\sqrt{a_N}. \tag{5.2.6}$$

We, therefore, conclude from (5.1.4) that any connected exteriour component S of $\mathfrak{S}_+$, with the exception of the largest one, obeys the following bound:

$$\mathrm{diam}(S) \leq c_4\Big(\sqrt[4]{a_N}\log N + \frac{N^{1+\xi}}{\sqrt{a_N}} \vee s(N)\Big). \tag{5.2.7}$$

We shall suppose that for some $\eta > 0$,

$$s(N) \ll N^{\frac{2}{3}-\eta}. \tag{5.2.8}$$

Since $a_N \sim N^{4/3+\delta}$ with $\delta \in (0, 2/3)$, it follows from (5.2.7) then that for $\nu > 0$ small enough, also

$$\mathrm{diam}(S) \ll N^{\frac{2}{3}-\nu}.$$

Consequently all exteriour $s(N)$-large contours γ, with the exception of the largest one, satisfy $\mathrm{diam}(\gamma) \ll N^{2/3-\nu}$. On the other hand, by virtue of Lemma 4.2.1, $N^{2/3-\nu}$-small contours appear with $\mathbb{P}^{\beta}_{N,-}\big(\cdot \,\big|\, M_N = -m^* N^2 + a_N\big)$-probability tending to zero. Thus, in view of (5.2.4), there is exactly one $K \log N$-large exteriour contour γ.

As a result we conclude that with the $\mathbb{P}^{\beta}_{N,-}\big(\bullet \mid M_N = -m^* N^2 + a_N \big)$-probability tending to one, the collection $\mathfrak{S}_{ext}$ of all exteriour $s(N)$-large skeletons actually consists exactly of one skeleton $\mathfrak{S}_{ext} = \{S\}$; $S \sim \gamma$, where γ is the unique exteriour $s(N)$-large contour. Though γ is, by the definition, self-avoiding, the boundary $\partial\mathfrak{S}_+$ of the "+" phase component of $\mathfrak{S}_{ext}$ in general may not be. However, thanks to (5.1.4), the sum of the diameters of all possible small connected components of $\mathfrak{S}_+$ is under control.

Consequently, the Hausdorff distance between γ and $\partial\mathfrak{S}^1_+$ is bounded above as,

$$d_{\mathbb{H}}\big(\partial\mathfrak{S}^1_+, \gamma\big) \;\le\; 2s(N) \;+\; c_5\sqrt{a_N}\Big(\frac{t_N}{\sqrt{a_N}} \;+\; \frac{k_N}{a_N} \Big).$$

Specifying the values of t_N and k_N as in (5.2.6), we obtain:

$$d_{\mathbb{H}}\big(\partial\mathfrak{S}^1_+, \gamma\big) \;\le\; c_6\Big(\sqrt[4]{a_N}\log N \;+\; \frac{N^{1+\xi}}{\sqrt{a_N}} \vee s(N) \Big). \tag{5.2.9}$$

On the other hand, the distance

$$\min_x d_{\mathbb{H}}\Big(\partial\mathfrak{S}^1_+,\; x + \sqrt{\frac{a_N}{2m^*}}\partial\mathcal{K}_1\Big)$$

is already controlled by the stability estimate (5.1.5). Thus, substituting the values of t_N and k_N from (5.2.5) , we obtain from the stability estimate (5.1.5), the bound (5.2.9) on the Hausdorff distance between γ and $\partial\mathfrak{S}_+$ and the triangle inequality,

$$\begin{aligned}\min_x d_{\mathbb{H}}\Big(\sqrt{\frac{2m^*}{a_N}}\gamma, x + \partial\mathcal{K}_1\Big) \;&\le\; c_7\Big(\frac{\log N}{\sqrt[4]{a_N}} \;+\; \frac{s(N)}{\sqrt{a_N}} \vee \frac{N^{1+\xi}}{a_N} \Big) \;+ \\ &+\; c_8\sqrt{\frac{\log N}{\sqrt[4]{a_N}} \;+\; \frac{s(N)}{\sqrt{a_N}} \vee \frac{N^{1+\xi}}{a_N}} \;\le\; c_9\sqrt{\frac{\log N}{\sqrt[4]{a_N}} \;+\; \frac{s(N)}{\sqrt{a_N}} \vee \frac{N^{1+\xi}}{a_N}}.\end{aligned} \tag{5.2.10}$$

It is not difficult to optimize in the right hand side of the inequality (5.2.10) within the range of parameters ξ and $s(N)$ described in (5.2.3), (5.2.1) and (5.2.8). The answer, however, does not have a nice compact form, and we, therefore, simply observe that no matter what the value of $\delta \in (0, 2/3)$ is, it is always possible to choose admissible ξ and $s(N)$, such that

$$\frac{\log N}{\sqrt[4]{a_N}} \;+\; \frac{s(N)}{\sqrt{a_N}} \vee \frac{N^{1+\xi}}{a_N} \;\ll\; N^{-\frac{\delta}{2}}.$$

This implies the stability statement (1.2.2).

Finally, by the lower bound (5.2.5) on the volume of $\mathfrak{S}_+$, and the energy estimate (1.3.2),

$$\begin{aligned}\mathbb{P}^{\beta}_{N,-}\big(M_N = -m^*N^2 + a_N \big) &\leq \mathbb{P}^{\beta}_{N,-}\Big(|\mathfrak{S}_+| \geq \frac{a_N}{2m^*} - c_{10}N^{1+\xi} \vee s(N)\sqrt{a_N} \Big)\big(1+\mathrm{o}(1) \big) \\ &\leq \mathbb{P}^{\beta}_{N,-}\Big(\mathcal{W}_\beta(\mathfrak{S}_+) \geq \sqrt{\frac{a_N}{2m^*}}\,\omega_1 - c_{11}\frac{N^{1+\xi}}{\sqrt{a_N}} \vee s(N) \Big) \\ &\leq \exp\Big(-\sqrt{\frac{a_N}{2m^*}}\,\omega_1 + c_{12}\frac{N^{1+\xi}}{\sqrt{a_N}} \vee s(N) \vee \frac{\sqrt{a_N}}{s(N)}\log N \Big).\end{aligned} \tag{5.2.11}$$

It is, again, possible to optimize in the right hand side above within the admissible range of ξ and $s(N)$, specified in (5.2.3), (5.2.1) and (5.2.8). Note, however, that for any admissible choice of ξ,

$$\frac{N^{1+\xi}}{\sqrt{a_N}} \ll N^{\frac{2}{3}-\frac{\delta}{2}}.$$

On the other hand, the $s(N) \vee \sqrt{a_N}\log N/s(N)$ term is clearly minimized on the admissible scale

$$s(N) \sim \sqrt[4]{a_N}\sqrt{\log N} \sim N^{1/3+\delta/4}\sqrt{\log N}.$$

Therefore, whatever $\delta \in (0, 2/3)$ is, one can always find admissible ξ and $s(N)$, such that

$$\frac{N^{1+\xi}}{\sqrt{a_N}} \vee s(N) \vee \frac{\sqrt{a_N}}{s(N)}\log N = \mathrm{O}\big(N^{\frac{2}{3}-\frac{\delta}{4}}\big).$$

Consequently,

$$\mathbb{P}^{\beta}_{N,-}\big(M_N = -m^*N^2 + a_N \big) \leq \exp\Big(-\sqrt{\frac{a_N}{2m^*}}\big(1 + \mathrm{O}(N^{-\frac{3\delta}{4}}) \big) \Big).$$

Since by the lower bound (3.1.4),

$$\mathbb{P}^{\beta}_{N,-}\big(M_N = -m^*N^2 + a_N \big) \geq \exp\Big(-\sqrt{\frac{a_N}{2m^*}}\big(1 + \mathrm{O}(\frac{\log N}{\sqrt[4]{a_N}}) \big) \Big),$$

and for each $\delta \in (0, 2/3)$,

$$\frac{\log N}{\sqrt[4]{a_N}} \ll N^{-\delta/2},$$

the estimate (1.2.1) follows, and the proof of Theorem A is, thereby concluded.

5.3. **Proof of Theorem B.** The proof essentially amounts to a more careful look on the results we have already obtained: Let a_N satisfy the conditions of Theorem B. First of all, as it was already mentioned in the Remark 4.2.3, the results and techniques of Sections 4 and 5, can be easily adjusted to rule out all interiour $K\log N$ contours as well, i.e., with $\mathbb{P}^{\beta}_{N,-}\big(\bullet \mid M_N = -m^*N^2 + a_N \big)$ probability tending to one there is exactly one $K\log N$-large contour γ. Furthermore, substituting $a_N \sim N^2$ into various formulas of the previous subsection, we infer that for any choice of ξ and large contour parameter s (see (5.2.3), (5.2.1) and (5.2.8)), such that,

$$N^{2/3} \gg N^{2\xi} \gg \frac{N\log N}{s(N)} \vee N^{1/2}\log N \vee s(N), \qquad s(N) \gg N^{\frac{1}{2}}, \qquad s(N) \ll N^{\frac{2}{3}-\eta}, \tag{5.3.1}$$

for some $\eta > 0$, this unique $K \log N$-large contour γ satisfies (see (5.2.10)),

$$\min_x d_{\mathbb{H}}\Big(\sqrt{\frac{2m^*}{a_N}}\gamma, x + \partial\mathcal{K}_1\Big) \;\le\; c_1\sqrt{\frac{\log N}{N^{1/2}} \;+\; \frac{s(N) \vee N^\xi}{N}}, \tag{5.3.2}$$

which is bounded above by $c_2 N^{-1/4}\sqrt{\log N}$ for the admissible choice of $s(N) \sim N^{1/2}\log N$.

Finally, by the lower bound (3.1.4),

$$\mathbb{P}^\beta_{N,-}\big(\; M_N = -m^* N^2 + a_N \;\big) \;\ge\; \exp\Big(-\sqrt{\frac{a_N}{2m^*}}\omega_1 \;-\; c_3 N^{1/2}\log N\;\Big),$$

and, by the upper bound (5.2.11) written at $a_N \sim N^2$,

$$\mathbb{P}^\beta_{N,-}\big(\; M_N = -m^* N^2 + a_N \;\big) \;\le\; \exp\Big(-\sqrt{\frac{a_N}{2m^*}}\omega_1 \;+\; c_4 N^\xi \vee s(N) \vee \frac{N\log N}{s(N)}\;\Big).$$

The optimal choice of $s(N)$ for the upper bound above is given by $s(N) \sim N^{1/2}\sqrt{\log N}$, which is again an admissible value. Theorem B is completely proven.

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