FINAL EXAM HINTS AND SOLUTIONS

Problem 1. Compute the Euler characteristic of the surface $S = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^4 + z^6 = 10\}$.

SOLUTION. The Euler characteristic is invariant under a homeomorphism. It is easy to checkt that the map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\varphi(x, y, z) = (x, y^2, z^3)$ gives a homeomorphism of S and the sphere $S^2_{\sqrt{10}}$ with center $(0, 0, 0)$ and radius $\sqrt{10}$. Since $\chi(S^2) = 2$, we have $\chi(S) = 2$.

Problem 2. Prove that $(0, 0)$ is an isolated singular point of $X = (x^3 - 3xy^2)\partial_x + (y^3 - 3x^2y)\partial_y$, and compute the index of the vector field at this point.

SOLUTION. By solving the system of equations $x^3 - 3xy^2 = 0$, $y^3 - 3x^2y = 0$, we find that $(0, 0)$ is the only singular point. Thus, isolated. Computing the restriction of X to the circle $(x(t), y(t)) = (\cos t, \sin t)$ and using trigonometric identities, we get $X(t) = \cos t\partial_x + \sin(-3x)\partial_y$. Thus, $I(0, 0) = -3$. (Alternatively, make a drawing of the flow of vector field, and use it in your computation).

Problem 3. What is $T^{\#m}\#P^{\#n}$?

SOLUTION. If $n = 0$, we get $T^{\#m}$. If $n \neq 0$, use $T\#P = P^{\#3}$ to reduce the answer to $P^{\#n+2m}$.

Problem 4. Compute $\pi_1(S^2 \setminus \{n \text{ pts}\})$.

SOLUTION. $S^2 \setminus \{n \text{ pts}\}$ is homeomorphic to $\mathbb{R}^2 \setminus \{(n-1) \text{ pts}\}$. Thus, $\pi_1 = \mathbb{F}_{n-1}$, the free group on $(n - 1)$generators.

Problem 5. Describe an atlas of T^*M corresponding to an atlas of M.

SOLUTION. Let $\{U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{R}^n\}$ be an atlas of M. Note that $T^*M = \cup_{x \in M} T^*_x M = \cup T^*U_\alpha$. Since U_α is homeomorphic to a disc, T^*U_α is homeomorphic to $U_\alpha \times \mathbb{R}^n$, with the homeomorphism $\tilde{\varphi_\alpha}$ given by $\tilde{\varphi}(x, v) = (\varphi_\alpha(x), (v_1, \ldots, v_n))$, where $x \in U_\alpha$, $v = v_i dx^i \in T^*_x U_\alpha$ (here $x^1, \ldots, x^n$are coordinates on U_α). Thus, the required atlas of T^*M is given by $\{U_\alpha \times \mathbb{R}^n, \tilde{\varphi}_\alpha : V_\alpha \to \mathbb{R}^n \times \mathbb{R}^n\}$.

Problem 6. Let $f(x, y) = x^3 + xy + y^3 + 1$. For which of the following: $p_1 = (0, 0)$, $p_2 = (1/3, 1/3)$ and $p_3 = (-1/3, -1/3)$ the set $f^{-1}(f(p))$ is an embedded submanifold.

SOLUTION. Compute $df = (3x^2 + y)dx + (3y^2 + x)dy$. Since at p_1 and p_2 we have $df = 0$, these points do not correspond to an embedded submanifold. For p_3, it is easy to check that $df \neq 0$ for any $p \in f^{-1}(f(p))$. Thus, we get an embedded submanifold (diffeomorphic to $\mathbb{R}^1$).

Problem 7. Let $\omega \in \Omega^k(M \times N)$. Show that $\omega = \pi^*\alpha$ (where $\pi : M \times N \to N$ is the natural projection to the second factor) for some $\alpha \in \Omega^k(N)$ iff for any $X \in \mathfrak{X}^1(M \times N)$ such that $d\pi(X) = 0$ we have $\imath_X\omega = 0$ and $L_X\omega = 0$.

HINT. Use the formulas $\imath_X\omega(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1})$ and $L_X = \imath_X \circ d + d \circ \imath_X$, as well as the properies of pull-back.

Problem 8. Prove that $L_X(\omega(Y)) = (L_X\omega)(Y) + \omega(L_XY)$.

HINTS. 1st solution: $(L_X\omega)(Y) = (\imath_X d\omega)(Y) + (d\imath_X\omega)(Y) = d\omega(X,Y) + Y\omega(X) = X\omega(Y) - Y\omega(X) = -\omega([X,Y]) + Y\omega(X) = X\omega(Y) - \omega([X,Y])$. Using this formula, and $L_XY = [X,Y]$, we get the anwer.

Alternatively, write X, Y, ω in components, and use explicit formulas, but this is much longer.

Problem 9. Prove that the Jacoby identity for the Lie bracket of vector fields is equivalent to the derivation property of the Lie derivative with respect to the Lie bracket.

HINT. Use $[X,Y] = L_XY$.

Problem 10. Show that the form $\omega = \frac{1}{r^3}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)$ on $\mathbb{R}^3 \setminus (0,0,0)$ is closed but not exact.

SOLUTION. The three terms in ω are cyclic permutations of each other. Note that $d(\frac{1}{r^3}xdy \wedge dz) = \partial_x(\frac{x}{r^3})dx \wedge dy \wedge dz = \frac{y^2+z^2-2x^2}{r^5}dx \wedge dy \wedge dz$. Computing the other terms by cyclic permutations, and adding up, we get $d\omega = 0$.

Assume that $\omega = d\theta$ for some 1-form θ. Then by Stokes theorem we should have $\int_S \omega = \int_{\partial S} \theta$ for a surface S. Choose S to be the unit sphere. Then $\partial S = \emptyset$. Thus, the integral on the right is 0. By an explicit computation, you can see that the integral on the left is (proportional) to the area of the sphere, and is not zero. Contradiction. Thus, ω is not exact.