PRACTICE PROBLEMS FOR MIDTERM 1
MATH 120A

1. Local Theorem

In all the problems below, a curve $\alpha(t)$ is called regular if $\alpha'(t) \neq 0$ and $\alpha''(t) \neq 0$.

Problem 1. a) Let I and J be two intervals on the real line $\mathbb{R}$. What is a reparametrization of a regular parametrized curve defined on I to a curve defined on J?

b) Prove that $f : (-1, 1) \to (-\infty, \infty)$ is a reparametrization;

c) Prove that the arc length of a regular parametrized curve is independent of reparametrization;

Problem 2. Let $\alpha(t) : I \to \mathbb{R}^2$ be a regular plane curve, such that there exists a vector $v \in \mathbb{R}^2$ with the property that $(\alpha(t) - v)$ is orthogonal to the tangent vector$T(t)$ for all t. Prove that $\alpha(t)$ lies on a circle.

Problem 3. Give an example of a curve $\alpha : I \to \mathbb{R}^3$ which is not regular and explain why it is not regular.

Problem 4. Reparametrize the curve $\alpha(t) = (e^t \text{cos} t, e^t \text{sin} t, e^t)$ by arc length.

Problem 5. Show that for the curve

$$\alpha(s) = (5/13\text{cos}(s),\ 8/13 - \text{sin}\ (s),\ -12/13\text{cos}(s))$$

the parametrization by s is a parametrization by arc length. For this curve, compute the Frenet trihedron at each point, and the curvature and torsion at each point.

Problem 6. Let $\alpha(s) = (x(s), y(s)) : I \to \mathbb{R}^2$ be a regular plane curve parametrized by the arc lengths. Prove that the curvature of α is given by

$$k(s) = |x'y'' - x''y'|;$$

Problem 7. Prove that if $\alpha(s) : [a, b] \to \mathbb{R}^3$ is a curve parametrized by the arc length such that $k(s) \equiv 0$ on $[a, b]$, then α is the interval on a straight line.

Problem 8. Prove that $k(s)\tau(s) = -\langle t'(s),\ b'(s)\rangle$.

Problem 9. Find a curve $\alpha(s)$ parametrized by the arc length such that $k(s) = 1/(1+s^2)$ and $\tau \equiv 0$.

Problem 10. Let $\beta(t)$ be a curve not necessarily parametrized by the arc length. Prove that $k(t) = |\beta' \times \beta''|/|\beta'|^3$, where prime denotes the derivative with respect to t .

2. Global Theory

Problem 11. Look through the problems in section 1.7 which are related to the isoperimetric inequality, four vertex theorem, and Cauchy-Crofton formula.

Here are some other problems.

Problem 12. Compute the rotation index of a given curve (draw a curve, with some self-intersections, and compute its rotation index).

Problem 13. a) What is a convex curve? (Give a definition).

b) Let $\alpha(s) : [0, l]$ be a convex simple closed curve. Can it happen that there are three points, $0 < s_1 < s_2 < s_3 < l$ such that the tangent lines at these points are parallel?

Problem 14. Let $A = (-1, 1)$ and $B(1, -1)$ be two points on the plane. Let $l \geq 2$. Find the curve $\alpha(s) : [0, l] \to \mathbb{R}^2$ such that $\alpha(0) = A$, $\alpha(l) = B$ and the area of the region bounded by the segments $\overline{OA}$, $\overline{OB}$ and the curve α is maximal.