PRACTICE PROBLEMS FOR MIDTERM 2

MATH 120A

The midterm will cover the material on regular surfaces, up to (and including) the second fundamental form.

Problem 1. a) Give the book's definition of a regular surface.

b) For each of the following subsets $S \subset \mathbb{R}^3$ determine whether S is a regular surface. If it is, give a proof and describe the shape of S geometrically (e.g., give a sketch of how the surface looks like). If not, explain why not:

1. $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2/a^2 + y^2/b^2 + z^2/c^2 = 1\}$ for some $a, b, c \in \mathbb{R}$.
2. $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 - y^2 = 1\}$.
3. $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 - y^2 = z\}$.
4. $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2,\ z > 0\}$.
5. $S = \{(x,y,z) \in \mathbb{R}^3 \mid -1 < x < 1,\ -1 < y < x,\ z = 0\}$.
6. $S = \{(x,y,z) \in \mathbb{R}^3 \mid -1 < x, y, z < 1\}$.
7. $S = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = 0,\ 0 < z < 1\}$.

Problem 2. a) Give the definition of a diffeomorphism between two regular surfaces.

b) Show that the paraboloid $S_1 = \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$ is diffeomorphic to the sphere without north pole, $S_2 = S^2 \setminus \{(0,0,1)\}$. (Hint: Show that S_1 is diffeomorphic to the x, y-plane. Show that S_2 is diffeomorphic to the plane. Use the composition of the first of these maps with the inverse of the second one to construct a diffeomorphism of S_1 and S_2).

Problem 3. Describe the parametrization of the sphere minus the north pole by the inverse of the stereographic projection (derive the formula for the parametrization, and prove that it is indeed a parametrization).

Problem 4. Let $\varphi : S_1 \to S_2$ be a differentiable map between two regular surfaces.

a) Give the definition of the critical point of φ;

b) Do problem 5 on page 65.

Problem 5. a) Give the definition of the tangent plane of a regular surface S at a point p.

b) Find the equation of the tangent plane to the surface $x^2 + y^2 = z$ at the point $(1, 1, 2)$.

Problem 6. # 7 on page 89.

Problem 7. a) What is the first fundamental form of a regular surface?

b) Find the coefficients of the first fundamental form of the unit sphere minus the north pole in the parametrization of the (inverse of) the stereographic projection.

c) Compute the lengths of the curve $\alpha(t) = (\cos t, \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\sin t)$, where $t \in (0, \pi/2)$. using the first quadratic form you have found in part b).

Problem 8. Using the first fundamental form of the sphere, compute the area of the "triangle" formed by the parts of the meridians connecting the points

* $(0, 0, 1)$ and $(1, 0, 0)$,
* $(0, 0, 1)$ and $(0, 1, 0)$,

and the part of the equator connecting the points $(1, 0, 0)$ and $(0, 1, 0)$.

Problem 9. Describe the region of the unit sphere covered by the image of the Gauss map of the parabaloid $z = x^2 + y^2$.

See also all the homework problems up to (and including section 3.2).