PRACTICE PROBLEMS FOR MIDTERM 1 (MATH 115AH)

Problem 1. Prove that if m is not a prime number, then $\mathbb{Z}_m$ is not a field.

Problem 2. Let p be a prime number. Computer the multiplicative inverse of $(p+1)/2$ in the field $\mathbb{Z}_p$.

Problem 3. Let p be a prime. What is the number of elements in the field $\mathbb{Z}_p$? What is the number of elements in the vector space $(\mathbb{Z}_p)^n$? What is the dimension of $\mathbb{Z}_p^n$ over $\mathbb{Z}_p$.

Problem 4. Let $P_3(\mathbb{R})$ be the vector space of polynomials of degree at most 3 with real coefficients. Let

$$W = \{f \in P_3(\mathbb{R}) : f(0) = 2f'(0)\}.$$

(a) Show that W is a subspace of $P_3(\mathbb{R})$.
(b) Find a basis of W.
(c) Find the dimension of W.

Problem 5. Let $\{v_1, \ldots, v_n\}$ be a linearly independent set of vectors in V. Let $\{u_1, \ldots, u_m\}$ be another linearly independent set of vectors in V. Suppose that $n < m$. Show that the vectors $\{v_1, \ldots, v_n\}$ can not form a basis of V.

Problem 6. Let $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ be given by

$$T(f(x)) = (f(0), f'(0))$$

and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$U(a, b) = (a + b, a - b)$$

Let $\alpha = \{1, x, x^2\}$ be a basis of $P_2(\mathbb{R})$ and $\beta = \{(1,0), (0,1)\}$ be a basis of $\mathbb{R}^2$. Compute the matrix $[U \circ T]_\alpha^\beta$ of the composition of T and U.

Problem 7. True or False. For each of the following statements, indicate if it is true or false. This problem will be graded as follows: you will receive n points for a correct answer, 0 points if there is no answer, and $-n$ points if the answer is wrong.

1. The set of polynomials of degree exactly 3 is not a vector space.

2. The set $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 1\}$ is a subspace of $\mathbb{R}^3$.
3. A subset of a linearly dependent set is linearly dependent.
4. If $\dim(V) = n$, any generating set of V contains at least n vectors.
5. If a set of vectors S generates vectors space V, any vector in V can be written as a linear combination of vectors in S in a unique way.
6. A linear transformation $T : V \to V$ carries linearly independent subsets of V into linearly independent subsets of V.
7. In a vector space V the equality $av = aw$ for $a \in F$, $v, w \in V$ implies that $v = w$.
8. If W_1 and W_2 are subspaces of a vector space V, then the intersection $W_1 \cap W_2$ is a subspace iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.
9. If $S_1 \subset S_2$ are subsets of a vector space V and S_1 is linearly independent, then S_2 is also linearly independent.
10. For any $a \in \mathbb{R}$, the set of real-values functions $W = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : f(a) = 0\}$ is a subspace of the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of all real-values functions on the line.
11. If S is a subset of a vector space V, then $\mathrm{span}(S)$ is the intersection of all subspaces of V that contain S.
12. If a vector space V is generated by a finite set S, then some subset of S is a basis of V.
13. The dimension of the space $M_{2\times 3}(F)$ over F is 5.

Problem 8. Let V be the set of all pairs (x, y), where x is a real number and y is a positive real number. Define addition on V by

$$(x, y) + (x, x') = (x + x', y \cdot y')$$

and scalar multiplication by

$$c(x, y) = (cx, y^c) \qquad \text{for } c \in \mathbb{R}$$

Let $\overrightarrow{0} = (0, 1)$.

1. Show that V is a vector space with these operations.
2. Find the dimension of V.
3. Let n be the dimension of V which you found in part 2 of this problem. Construct an explicit isomorphism from V to $\mathbb{R}^n$.

Problem 9. Let W_1 and W_2 be subspaces of a vector space V. Prove that the following conditions are equivalent:

(1) each vector x in V can be uniquely written in the form $x = x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$;

(2) $W_1 \cap W_2 = \{\overrightarrow{0}\}$ and $V = W_1 + W_2$, where $W_1 + W_2 = \{w_1 + w_2,\, w_1 \in W_1,\, w_2 \in W_2\})$.

(If either of this condtions is satisfied, $V = W_1 \oplus W_2$).

Problem 10. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$. Prove that T is an isomorphism and find T^{-1}.

Problem 11. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear transformation given by $T(A) = A^t$, the transpose of A. Let $U : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation given by

$$U\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + 2bx + 3cx^2$$

Let $\alpha = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ be a basis of $M_{2\times 2}(\mathbb{R})$ and $\beta = \{1, x, x^2\}$ be a basis of $P_2(\mathbb{R}^2)$. Find the matrix $[U \circ T]_\alpha^\beta$ of the composition of linear transformations T and U.

Problem 12. Prove that vectors (a, b) and (c, d) in $\mathbb{C}^2$ are linearly dependent iff $ad = bc$.

Problem 13. Let V be a vector space, $\dim(V) = 4$. Show that if W_1 , W_2 are both subspaces of dimension 3, then there is a non-trivial intersection of W_1 and W_2.

Problem 14. Find a linear functional f on the vector space $P_3(\mathbb{R}) = \{p(t) = \sum_{i=0}^{3} a_i \cdot t^i \,|\, a_i \in \mathbb{R}\}$ such that

$$\begin{aligned} f(1) &= 1 \\ f(x^3 + 2x) &= 1 \\ f(x^3 + 3x^2) &= 2 \\ f(x^2 + 5x) &= 6 \end{aligned}$$

How many linearly independent linear functionals with this property can you find?

Problem 15. Let v and u be vectors in V such that $\{v\}^0 = \{u\}^0 \in V'$, where S^0 denotes the annihilator of a set $S \subset V$. Prove that $v = ku$ for some $k \in F$.

Problem 16. Prove that for any $v \in V$, $\dim(V) = n$, there exists a linear functional f such that $f(v) = \alpha$ for a given $\alpha \in F$. How many linearly independent linear functionals with this property can you find?

Problem 17. Is there a finite-dimensional vector space V with a subspace W such that W has a unique complement in V ? If yes, give an example. If not, explain why it can not exist.

Problem 18. a) For a finite-dimensional vector space V, is it true that a linear transformation is onto iff it is one-to-one? Prove that it is true, or give a counterexample.

b) For a vector space of sequences $\mathbb{R}^\infty$ with entries in $\mathbb{R}$, give an example of a linear transformation on $\mathbb{R}^\infty$ which is onto, but not one-to-one, and another linear transformation which is one-to-one but not onto.

In addition, please review also the problems related to invariant subspaces and projections, rank and null space, similar to the last homework assignment.