# PRELIMINARY INFORMATION

This handout contains some notation and basics of set theory which is necessary to know for the purposes of math115ah.

QUANTIFIERS:

1. $\exists$ — "there exists" ;
2. ! — "unique";
3. $\forall$ — "for all";

Example: $\forall\; n \in \mathbb{Z}\; \exists!\; m \in \mathbb{Z}$ such that $n + m = 5$;

OTHER NOTATIONS:

- "$\doteq$ "— is defined to be;
- "iff" — if and only if;

SETS, FUNCTIONS AND RELATIONS:

A *set* is a collection of objects (called elements of the set). A set can be described in the following ways:

- By listing all the elements of the set. E.g., $A \doteq \{3, 4, 5\}$;
- By indicating all the properties of elements of the set. E.g., $A \doteq \{n \in \mathbb{Z} :\; 3 \leq n \leq 5\}$ describes the same set as above.
  The symbol $\emptyset$ denotes the empty set, i.e., the set which contains no elements.

Let $A$ and $B$ be sets. The following notation and terminology is commonly used:

- $x \in A$: $x$ is an element of $A$;
- $x \notin A$: $x$ is not an element of $A$;
- $A \subset B$: $A$ is a *subset* of $B$. I.e., each element of $A$ is an element of $B$;
- $A \subset B$ and $A \neq B$ is a *proper subset* of $B$. I.e., all the elements of $A$ are elements in $B$, but there are some element of $B$ which are not in $A$.
- $A = B$: $A$ and $B$ contain exactly the same elements. To prove that $A = B$, it is sufficient to prove that $A \subset B$ and $B \subset A$.
- $A \cup B = \{x :\; x \in A \,\text{or}\, x \in B\}$: the union of $A$ and $B$;
- $A \cap B = \{x :\; x \in A \text{ and } x \in B\}$: the intersection of $A$ and $B$;
- $A \setminus B = \{x :\; x \in A \text{ and} \, x \notin B\}$: the difference of $A$ and $B$;
- $A \times B = \{(a, b) |\; a \in A,\; b \in B\}$: Cartesian product of $A$ and $B$;
- $A$ and $B$ are *disjoint:* $A \cap B = \emptyset$;

Similar notation can be used when we are interested in intersection, union, or product of more then two sets.

A *function* $f : A \to B$ is a rule that assigns to each element of $A$ an element of $B$. In other words, a function $f$ is defined if $\forall\, a \in A \;\; \exists!\, b \in B$ such that $b = f(a)$. In this case, $A$ is called the *domain* of $f$, $B$ is called the *codomain* of $f$, and the set $\{f(a) : a \in A\}$ is called the *range* of $f$. For $a \in A$ the element $f(a) \in B$ is called the *image of* $A$. For $b \in B$ an element $a$ such that $b = f(a)$ is called the *preimage* of $b$ (sometimes denoted as $f^{-1}(b)$). Note that preimage of an element does not have to be unique (consider $A = B = \mathbb{R}$ and $f : A \to B$ defined by $f(x) = x^2$ for $x \in A$. Then for any $y > 0$ there are two preimages, equal to $\pm\sqrt{y}$ ).

There are functions with special properties which are of interest:

1. $f : A \to B$ is *onto* (or, *surjective)* if $f(A) = B$ (i.e., the image of $f$ is the whole set $B$). In other words, the function is onto if its range is equal to its codomain.
2. $f : A \to B$ is *one-to-one* (or, *injective)* if each element of $B$ has a unique preimage. In other words, $f$ is one-to-one if $f(x) = f(y)$ implies that $x = y$.

A function satisfying both of this properties is called an *isomorphism* of sets.

Suppose that for a function $f : A \to B$ there is a function $g : B \to A$ such that $(A \circ B)(y) = y$ for all $y \in B$ and $(B \circ A)(x) = x$ for all $x \in A$. (Here $\circ$ denotes the composition of functions). If such a function $g$ exists, it is unique and called the *inverse* of $f$ (denoted by $f^{-1}$ ). Show, as an exercise, that a function is invertible iff it is an isomorphism.

Here are some properties of invertible functions. Let $f : A \to B$ and $h : B \to C$ be invertible. Then

1. $(f^{-1})^{-1} = f$.
2. $(f \circ h)^{-1} = h^{-1} \circ f^{-1}$ .

Note that a function $f : A \to B$ can be described by its *graph* $\Gamma(f) \doteq \{(a, f(a)) \in A \times B\}$. Generalizing this observation, we come to the notion of relation.

Let $A$ and $B$ be sets. A *relation* $R$ between $A$ and $B$ is a subset of $A \times B$. One usually writes $aRb$ for $(a, b) \in R$. There are many examples of relations. Take $A = B = \mathbb{R}$ and $R$ to be the relation "less then" (denoted by $<$, as usual). Then a pair $(a, b) \in R$ iff $a < b$.

A relation $R \subset A \times A$ on $A$ is called an *equivalence relation* if it satisfies the following properties: $\forall a, b, c \in A$ we have

1. $(a, a) \in R$ (or, equivalently, $aRa$).................... *(reflexivity).*

2. $aRb \Rightarrow bRa$.......................................................(*symmetry*).
3. $aRb$ and $bRc \Rightarrow aRc$.........................................(*transitivity*).

Question: Why properties 2 and 3 do not imply property 1?

The simplest example of an equivalence relation is the *equality*. For any set $S$, say that $s_1, s_2 \in R$ are equivalent if $\{s_1\} = \{s_2\}$ (i.e., the subsets consisting of $s_1$ and $s_2$ respectively are equal). Under this relation, every element is equivalent only to itself. A more interesting example is the relation on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ defined by $(a,b)R(c,d)$ if $ad = bc$. (This just says that the fractions $a/b$ and $c/d$ are the same).

Equivalence relations in Linear algebra

The most common examples of equivalence relations in linear algebra are the following:

1. *Raw-equivalnce.* Two matrices $A, B \in M_{m\times n}(F)$ are called row-equivalent ($A \sim B$) if $A$ can be obtained from $B$ by a finite number of elementary row operations (see Handout 3). It is easy to check that this relation is reflexive, symmetric and transitive. One can check that this equivalence relation can be described in a few different ways as follows:
   - $A \sim B$ iff there if there is an invertible $m \times m$ matrix $P$ such that $B = PA$;
   - $A \sim B$ iff the homogeneous systems of linear equations $AX = 0$ and $BX = 0$ have the same solutions;
2. *Similarity.* Two matrices $A, B \in M_{n\times n}(F)$ are similar if there is an invertible matrix $Q \in M_{n\times n}(F)$ such that $B = Q^{-1}AQ$. Check that this is indeed an equivalence relation. We will learn in the course that $A$ and $B$ are similar iff they represent the same linear operator on $F^n$ with respect to two different bases.