A BRIEF REVIEW OF MATRICES AND DETERMINANTS

Here are some basic facts about matrices and their determinants, which you should be familiar with for the purposes of math 115ah. (Most of this material was covered in the lower division linear algebra class. Please refer to textbooks for more details).

1. Elementary row (column) operations: The following operations on a matrix are called the elementary raw (column) operations:
   (a) interchanging two rows (columns);
   (b) multiplying a row (column) by a non-zero scalar;
   (c) replacing one of the raws by its sum with a different raw;
2. Rank: The rank of a matrix is the maximal number of its linearly independent columns. For example, the rank of the matrix
   $$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 2 & 0 & 4 \end{pmatrix}$$
   is $\text{rank}(A) = \dim(\text{span}\{(1\,0\,2), (0\,3\,0), (2\,3\,4)\} = 2$, because $(2\,3\,4) = 2 \cdot (1\,0\,2) + 1 \cdot (0\,3\,0)$.
   Recall that to each matrix $A \in M_{n\times m}(F)$ we can assign a linear transformation $L_A : F^m \to F^n$ defined as the left multiplication of a column in F^m by A. It turns out that $\text{rank}(A) = \text{rank}(L_A)$ (where $\text{rank}(L_A)$ is the dimension of the range of L_A, as it is usually defined for linear transformations).
   A matrix $A \in M_{m\times n}(F)$ with $\text{rank}(A) = r$ can be transformed by elementary matrix operations into the matrix
   $$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix},$$
   where I_r is the identity, and O_1, O_2, O_3 are the zero matrices of appropriate sizes.
3. Inverse: For a matrix A, a matrix B (of the same dimension) is called an inverse of A if $AB = BA = I$. If such a matrix B exists (which is not necessarily true), then it is unique and is denoted by A^{-1}.
   An important technical result is that any invertible matrix (i.e., a matrix which has an inverse) is a product of elementary matrices

(which are the matrices encoding the elementary row and column operations described above).

Let A be an (invertible) $n \times n$ matrix. Take the augmented matrix $C = (A \,|\, I_n)$. Multiplying by A^{-1} on both sides, we obtain $A^{-1}C = (A^{-1}A |\, A^{-1}) = (I_n \,|\, A^{-1})$. In practice, to obtain an inverse of a matrix A, take the corresponding augmented matrix $C = (A \,|\, I_n)$ and transform it by elementary operations into a matrix of the form $(I_n \,|\, B)$. The matrix B obtained in this way will be A^{-1}. (Think what will happen if you will apply this procedure to a matrix which is not invertible).

Example 1. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. To find A^{-1}, we write

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \mapsto \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right) \mapsto \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right)$$
$$\mapsto \left(\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right)$$

Hence, $A^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$

Similar matrices: Matrices A and B (of the same size) are called similar if there exists a matrix Q such that $B = Q^{-1}AQ$. For example, as we will see in the course, if $T : V \to V$ is a linear operator on V and β, β' are two bases of V, then the matrices of T with respect to these bases are similar, $[T]_{\beta'} = Q^{-1}[T]_{\beta}Q$. Similarity of matrices is an equivalence relation.

Determinants

1. Determinants of 2×2 matrices: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\mathrm{DET}(A) \equiv |A| \doteq ad - bc$.
   Note that

   - Determinant, considered as a function $\det : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ is not linear, since $\det(A + B) \neq \det(A) + \det(B)$.
   - $\det(A) \neq 0$ iff A is invertible. In this case, for A as above, $A^{-1} = \frac{1}{\det(A)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
   - One can interpret the determinant of a 2×2 matrix as follows. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be linearly independent

vectors in $\mathbb{R}^2$. These vectors determine a parallelogram in the usual way. Define the orientation of the pair of vectors to be

$$O(u,v) = \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|} = \pm 1$$

Then the area of the parallelogram defined by u, v is just

$$\text{Area}\begin{pmatrix} u \\ v \end{pmatrix} = O\begin{pmatrix} u \\ v \end{pmatrix} \cdot \det\begin{pmatrix} u \\ v \end{pmatrix} = \left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|$$

.

2. DETERMINANTS OF $n \times n$ MATRICES. For an $n \times n$ matrix A, let $\tilde{A}_{ij}$ be the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from the original matrix. Then the determinant of A is defined recursively by

$$\det(A) \doteq \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}).$$

That is, using the definition of the determinant for 2×2 matrices (given above) we define determinant for 3×3 matrices, and so on. After the determinant is defined for matrices of size $(n-1) \times (n-1)$, the formula above defines it for $n \times n$ matrices.

PROPERTIES OF THE DETERMINANT:

- If B is obtained from A by switching two rows (columns), $\det(B) = -\det(A)$.
- If B is obtained from A by multiplying a row (a column) by a constant k.,then $\det(B) = k \cdot \det(A)$.
- If B is obtained from A by adding a multiple of row i to row j, then $\det(B) = \det(A)$.
- If 2 rows (or columns) of a matrix are identical, the determinant is equal to 0.
- The determinant of an upper triangular matrix is the product of its diagonal entries.

  This property gives another method of computing the determinant: first, use Gauss elimination to reduce matrix to is upper-triangular form, then compute the product of the diagonal entries to obtain the determinant.
- $\det(AB) = \det(A) \cdot \det(B)$.
- A is invertible iff $\det(A) \neq 0$. If $\det(A) \neq 0$, then $\det(A^{-1}) = (\det(A))^{-1}$.

- $\det(A^t) = \det(A)$.

A CHARACTERIZATION OF THE DETERMINANT: Recall that a function $\delta : M_{n\times n}(F) \to F$ is called *n-linear* if it is a linear function of each row when the remaining rows are fixed, that is

$$\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\cdot\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}.$$

The determinant defined above has this property. Some other functions with this property are, for example, $\delta_1(A) = A_{1j}A_{2j}\cdots A_{nj}$ (the product of entries of a column), $\delta_2(A) = A_{11}\ldots A_{nn}$ (the product of diagonal entries).

Recall that a function $\delta : M_{n\times n}(F) \to F$ is called *alternating* if $\delta(A) = 0$ for any matrix A which has a pair of identical adjacent rows.

Theorem. (CHARACTERIZATION OF THE DETERMINANT) *If $\delta : M_{n\times n}(F) \to F$ is a function which is n-linear, alternating and satisfies $\delta(I_n) = 1$, then δ is the determinant.*