Handout 2

COMPLEX NUMBERS

For the purposes of algebra, the field of real numbers is not sufficient, for there are polynomials of nonzero degree with real coefficients that have no zeros in the field of real numbers (for example, $x^2 + 1$). It is often desirable to have a field in which any polynomial of nonzero degree with coefficients from that field has a zero in that field. It is possible to "enlarge" the field of real numbers to obtain such a field.

Definitions. *A* ***complex number*** *is an expression of the form $z = a + bi$, where a and b are real numbers called the* ***real part*** *and the* ***imaginary part*** *of z, respectively.*

The ***sum*** *and* ***product*** *of two complex numbers $z = a + bi$ and $w = c + di$ (where a, b, c, and d are real numbers) are defined, respectively, as follows:*

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$zw = (a + bi)(c + di) = (ac - bd) + (bc + ad)i.$$

Example 1

The sum and product of $z = 3 - 5i$ and $w = 9 + 7i$ are, respectively,

$$z + w = (3 - 5i) + (9 + 7i) = (3 + 9) + [(-5) + 7]i = 12 + 2i$$

and

$$zw = (3 - 5i)(9 + 7i) = [3\cdot 9 - (-5)\cdot 7] + [(-5)\cdot 9 + 3\cdot 7]i = 62 - 24i. \quad \blacklozenge$$

Any real number c may be regarded as a complex number by identifying c with the complex number $c + 0i$. Observe that this correspondence preserves sums and products; that is,

$$(c + 0i) + (d + 0i) = (c + d) + 0i \quad \text{and} \quad (c + 0i)(d + 0i) = cd + 0i.$$

Any complex number of the form $bi = 0 + bi$, where b is a nonzero real number, is called **imaginary**. The product of two imaginary numbers is real since

$$(bi)(di) = (0 + bi)(0 + di) = (0 - bd) + (b\cdot 0 + 0\cdot d)i = -bd.$$

In particular, for $i = 0 + 1i$, we have $i \cdot i = -1$.

The observation that $i^2 = i \cdot i = -1$ provides an easy way to remember the definition of multiplication of complex numbers: simply multiply two complex numbers as you would any two algebraic expressions, and replace i^2 by -1. Example 2 illustrates this technique.

Example 2

The product of $-5 + 2i$ and $1 - 3i$ is

$$\begin{aligned}(-5 + 2i)(1 - 3i) &= -5(1 - 3i) + 2i(1 - 3i) \\ &= -5 + 15i + 2i - 6i^2 \\ &= -5 + 15i + 2i - 6(-1) \\ &= 1 + 17i. \quad \blacklozenge\end{aligned}$$

The real number 0, regarded as a complex number, is an additive identity element for the complex numbers since

$$(a + bi) + 0 = (a + bi) + (0 + 0i) = (a + 0) + (b + 0)i = a + bi.$$

Likewise the real number 1, regarded as a complex number, is a multiplicative identity element for the set of complex numbers since

$$(a + bi)\cdot 1 = (a + bi)(1 + 0i) = (a\cdot 1 - b\cdot 0) + (b\cdot 1 + a\cdot 0)i = a + bi.$$

Every complex number $a + bi$ has an additive inverse, namely $(-a) + (-b)i$. But also each complex number except 0 has a multiplicative inverse. In fact,

$$(a + bi)^{-1} = \left(\frac{a}{a^2 + b^2}\right) - \left(\frac{b}{a^2 + b^2}\right)i.$$

In view of the preceding statements, the following result is not surprising.

Theorem D.1. *The set of complex numbers with the operations of addition and multiplication previously defined is a field.*

Proof. Exercise. ∎

Definition. *The* ***(complex) conjugate*** *of a complex number $a + bi$ is the complex number $a - bi$. We denote the conjugate of the complex number z by $\bar{z}$.*

Example 3

The conjugates of $-3 + 2i$, $4 - 7i$, and 6 are, respectively,

$$\overline{-3 + 2i} = -3 - 2i, \quad \overline{4 - 7i} = 4 + 7i, \quad \text{and} \quad \bar{6} = \overline{6 + 0i} = 6 - 0i = 6. \quad \blacklozenge$$

The next theorem contains some important properties of the conjugate of a complex number.

Theorem D.2. *Let z and w be complex numbers. Then the following statements are true.*

(a) $\overline{\overline{z}} = z$.
(b) $\overline{(z+w)} = \overline{z} + \overline{w}$.
(c) $\overline{zw} = \overline{z} \cdot \overline{w}$.
(d) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ *if* $w \neq 0$.
(e) *z is a real number if and only if* $\overline{z} = z$.

Proof. We leave the proofs of (a), (d), and (e) to the reader.
(b) Let $z = a + bi$ and $w = c + di$, where $a, b, c, d \in R$. Then

$$\overline{(z+w)} = \overline{(a+c) + (b+d)i} = (a+c) - (b+d)i$$
$$= (a - bi) + (c - di) = \overline{z} + \overline{w}.$$

(c) For z and w, we have

$$\overline{zw} = \overline{(a+bi)(c+di)} = \overline{(ac-bd) + (ad+bc)i}$$
$$= (ac - bd) - (ad + bc)i = (a - bi)(c - di) = \overline{z} \cdot \overline{w}.$$ ∎

For any complex number $z = a + bi$, $z\overline{z}$ is real and nonnegative, for

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2.$$

This fact can be used to define the absolute value of a complex number.

Definition. *Let $z = a + bi$, where $a, b \in R$. The* **absolute value** *(or* **modulus***) of z is the real number $\sqrt{a^2 + b^2}$. We denote the absolute value of z by $|z|$.*

Observe that $z\overline{z} = |z|^2$. The fact that the product of a complex number and its conjugate is real provides an easy method for determining the quotient of two complex numbers; for if $c + di \neq 0$, then

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd) + (bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.$$

Example 4

To illustrate this procedure, we compute the quotient $(1 + 4i)/(3 - 2i)$:

$$\frac{1+4i}{3-2i} = \frac{1+4i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{-5+14i}{9+4} = -\frac{5}{13} + \frac{14}{13}i.$$ ◆

The absolute value of a complex number has the familiar properties of the absolute value of a real number, as the following result shows.

Theorem D.3. *Let z and w denote any two complex numbers. Then the following statements are true.*

(a) $|zw| = |z| \cdot |w|$.
(b) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ *if* $w \neq 0$.
(c) $|z + w| \leq |z| + |w|$.
(d) $|z| - |w| \leq |z + w|$.

Proof. (a) By Theorem D.2, we have

$$|zw|^2 = (zw)\overline{(zw)} = (zw)(\overline{z} \cdot \overline{w}) = (z\overline{z})(w\overline{w}) = |z|^2|w|^2,$$

proving (a).
(b) For the proof of (b), apply (a) to the product $\left(\frac{z}{w}\right) w$.
(c) For any complex number $x = a + bi$, where $a, b \in R$, observe that

$$x + \overline{x} = (a + bi) + (a - bi) = 2a \leq 2\sqrt{a^2 + b^2} = 2|x|.$$

Thus $x + \overline{x}$ is real and satisfies the inequality $x + \overline{x} \leq 2|x|$. Taking $x = w\overline{z}$, we have, by Theorem D.2 and (a),

$$w\overline{z} + \overline{w}z \leq 2|w\overline{z}| = 2|w||\overline{z}| = 2|z||w|.$$

Using Theorem D.2 again gives

$$|z + w|^2 = (z + w)\overline{(z + w)} = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + w\overline{z} + z\overline{w} + w\overline{w}$$
$$\leq |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2.$$

By taking square roots, we obtain (c).
(d) From (a) and (c), it follows that

$$|z| = |(z + w) - w| \leq |z + w| + |-w| = |z + w| + |w|.$$

So

$$|z| - |w| \leq |z + w|,$$

proving (d). ∎

It is interesting as well as useful that complex numbers have both a geometric and an algebraic representation. Suppose that $z = a + bi$, where a and b are real numbers. We may represent z as a vector in the complex plane (see Figure D.1(a)). Notice that, as in R^2, there are two axes, the **real axis** and the **imaginary axis**. The real and imaginary parts of z are the first and second coordinates, and the absolute value of z gives the length of the vector z. It is clear that addition of complex numbers may be represented as in R^2 using the parallelogram law.

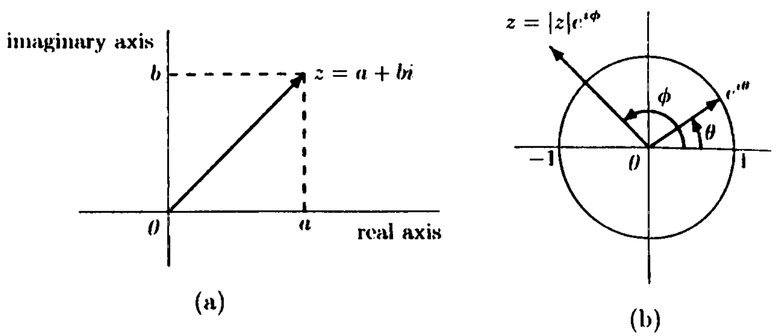

Figure D.1

In Section 2.7 (p.132), we introduce Euler's formula. The special case $e^{i\theta} = \cos\theta + i\sin\theta$ is of particular interest. Because of the geometry we have introduced, we may represent the vector $e^{i\theta}$ as in Figure D.1(b); that is, $e^{i\theta}$ is the unit vector that makes an angle θ with the positive real axis. From this figure, we see that any nonzero complex number z may be depicted as a multiple of a unit vector, namely, $z = |z|e^{i\phi}$, where ϕ is the angle that the vector z makes with the positive real axis. Thus multiplication, as well as addition, has a simple geometric interpretation: If $z = |z|e^{i\theta}$ and $w = |w|e^{i\omega}$ are two nonzero complex numbers, then from the properties established in Section 2.7 and Theorem D.3, we have

$$zw = |z|e^{i\theta} \cdot |w|e^{i\omega} = |zw|e^{i(\theta+\omega)}.$$

So zw is the vector whose length is the product of the lengths of z and w, and makes the angle $\theta + \omega$ with the positive real axis.

Our motivation for enlarging the set of real numbers to the set of complex numbers is to obtain a field such that every polynomial with nonzero degree having coefficients in that field has a zero. Our next result guarantees that the field of complex numbers has this property.

Theorem D.4 (The Fundamental Theorem of Algebra). *Suppose that $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial in $\mathsf{P}(C)$ of degree $n \geq 1$. Then $p(z)$ has a zero.*

The following proof is based on one in the book *Principles of Mathematical Analysis* 3d., by Walter Rudin (McGraw-Hill Higher Education, New York 1976).

Proof. We want to find z_0 in C such that $p(z_0) = 0$. Let m be the greatest lower bound of $\{|p(z)|: z \in C\}$. For $|z| = s > 0$, we have

$$|p(z)| = |a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0|$$

$$\begin{aligned} &\geq |a_n||z^n| - |a_{n-1}||z|^{n-1} - \cdots - |a_0| \\ &= |a_n|s^n - |a_{n-1}|s^{n-1} - \cdots - |a_0| \\ &= s^n[|a_n| - |a_{n-1}|s^{-1} - \cdots - |a_0|s^{-n}]. \end{aligned}$$

Because the last expression approaches infinity as s approaches infinity, we may choose a closed disk D about the origin such that $|p(z)| > m + 1$ if z is not in D. It follows that m is the greatest lower bound of $\{|p(z)|: z \in D\}$. Because D is closed and bounded and $p(z)$ is continuous, there exists z_0 in D such that $|p(z_0)| = m$. We want to show that $m = 0$. We argue by contradiction.

Assume that $m \neq 0$. Let $q(z) = \dfrac{p(z + z_0)}{p(z_0)}$. Then $q(z)$ is a polynomial of degree n, $q(0) = 1$, and $|q(z)| \geq 1$ for all z in C. So we may write

$$q(z) = 1 + b_k z^k + b_{k+1}z^{k+1} + \cdots + b_n z^n,$$

where $b_k \neq 0$. Because $-\dfrac{|b_k|}{b_k}$ has modulus one, we may pick a real number θ such that $e^{ik\theta} = -\dfrac{|b_k|}{b_k}$, or $e^{ik\theta}b_k = -|b_k|$. For any $r > 0$, we have

$$\begin{aligned} q(re^{i\theta}) &= 1 + b_k r^k e^{ik\theta} + b_{k+1}r^{k+1}e^{i(k+1)\theta} + \cdots + b_n r^n e^{in\theta} \\ &= 1 - |b_k|r^k + b_{k+1}r^{k+1}e^{i(k+1)\theta} + \cdots + b_n r^n e^{in\theta}. \end{aligned}$$

Choose r small enough so that $1 - |b_k|r^k > 0$. Then

$$\begin{aligned} |q(re^{i\theta})| &\leq 1 - |b_k|r^k + |b_{k+1}|r^{k+1} + \cdots + |b_n|r^n \\ &= 1 - r^k[|b_k| - |b_{k+1}|r - \cdots - |b_n|r^{n-k}]. \end{aligned}$$

Now choose r even smaller, if necessary, so that the expression within the brackets is positive. We obtain that $|q(re^{i\theta})| < 1$. But this is a contradiction. ∎

The following important corollary is a consequence of Theorem D.4 and the division algorithm for polynomials (Theorem E.1).

Corollary. *If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree $n \geq 1$ with complex coefficients, then there exist complex numbers $c_1, c_2, \cdots, c_n$ (not necessarily distinct) such that*

$$p(z) = a_n(z - c_1)(z - c_2)\cdots(z - c_n).$$

Proof. Exercise. ∎

A field is called **algebraically closed** if it has the property that every polynomial of positive degree with coefficients from that field factors as a product of polynomials of degree 1. Thus the preceding corollary asserts that the field of complex numbers is algebraically closed.