Global aspects of ergodic group actions

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Introduction

In this talk, I will discuss some aspects of recent work concerning the global structure of the space of measure preserving actions and their associated cohomology. This is part of a forthcoming book, the current version of which is in my web page at Caltech. (It will appear this year in the Math. Surveys and Monographs series of the AMS.)

Introduction

- Study of measure preserving actions of a countable discrete group Γ on a standard measure space (X, μ). Denote by $A(\Gamma, X, \mu)$ the space of such actions. This can be also viewed as the space of homomorphisms of Γ into the group of automorphisms $\mathrm{Aut}(X, \mu)$ of the measure space.
- One part of this work is concerned with the global structure of this space including, in particular, problems related to the classification of measure preserving actions under various notions of equivalence (unitary equivalence, conjugacy, orbit equivalence, etc.). Another part deals with the study of the cohomology of such actions and I will discuss this aspect in this talk.

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Cocycles

Definition

Let $a \in A(\Gamma, X, \mu)$ be a given action and G another countable group. A cocycle of a with values in G (or with target G) is a Borel map $\alpha : \Gamma \times X \to G$, so that writing $a(\gamma, x) = \gamma \cdot x$, we have the cocycle identity: $\alpha(\gamma\delta, x) = \alpha(\gamma, \delta \cdot x)\alpha(\delta, x)$, μ-a.e.(x). We denote by $Z^1(a, G)$ the set of cocycles.

Example

The trivial cocycle: $\alpha(\gamma, x) = 1$.

Example

A "homomorphism": $\alpha(\gamma, x) = \varphi(\gamma)$, where $\varphi : \Gamma \to G$ is a homomorphism.

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Cocycles induced by homomorphisms of equivalence relations.
Denote by E_a the equivalence relation induced by a and let $b \in A(G, Y, \nu)$ be a *free* action of G with associated equivalence relation E_b. Let $f : X \to Y$ be a homomorphism of E_a into E_b. Then f gives rise to the cocycle α_f given by $\alpha_f(\gamma, x) = g$, where $f(\gamma \cdot x) = g \cdot f(x)$.

This last example provides the link for applications of the theory of cocycles to ergodic theory (orbit equivalence) and to descriptive set theory (reducibility hierarchy of Borel equivalence relations). It is crucial in proving *rigidity results*, which is one of the motivations for studying the structure of cocycles.

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Two cocycles $\alpha, \beta \in Z^1(a, G)$ are cohomologous or equivalent, in symbols, $\alpha \sim \beta$, if there is a Borel map $p : X \to G$ such that

$$\beta(\gamma, x) = p(\gamma \cdot x)\alpha(\gamma, x)p(x)^{-1}.$$

Example

If $\alpha = \alpha_f$ is induced by a homomorphism f of the equivalence relation E_a into E_b and $\beta \sim \alpha$, then $\beta = \alpha_g$ for another homomorphism g, so that $g(x) E_b f(x)$.

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If for instance a_f is cohomologous to a "simple" kind of cocycle (e.g., a homomorphism of the acting groups or a cocycle taking values in a "small" subgroup of G), one can use this information to rule out the existence of such a homomorphism of the equivalence relations or establish connections between the groups and the actions. Such "cocycle reduction" results are crucial in the work of Zimmer, Furman, Popa, ..., in ergodic theory and Adams-K, Thomas, Hjorth-K, ... in descriptive set theory.

Definition

The quotient space $H^1(a, G) = Z^1(a, G)/\sim$ is called the (1st)-cohomology space of the action a (relative to G.)

Definition

A cocycle is a coboundary if it is cohomologous to the trivial cocycle. The set of coboundaries is denoted by $B^1(a, G)$.

When G is abelian, $Z^1(a, G)$ is an abelian group, $B^1(a, G)$ a subgroup and $H^1(a, G) = Z^1(a, G)/B^1(a, G)$ is the (1st)-cohomology group. We will however be interested here in the non-abelian case.

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Cohomology

The cohomology classes are the orbits of a canonical action. Denote by $L(X, \mu, G)$ the group of G-valued random variables (under pointwise multiplication). This acts on $Z^1(a, G)$ by

$$f \cdot \alpha(\gamma, x) = f(\gamma \cdot x)\alpha(\gamma, x)f(x)^{-1}$$

and the orbits are the cohomology classes.

The group $L(X, \mu, G)$ is a Polish group under the topology of convergence in measure. Similarly the space $Z^1(a, G)$ is a Polish space under the topology of convergence in measure (for each fixed element of Γ). The above action is then continuous, so cohomology is an equivalence relation induced by a continuous action of a Polish group on a Polish space. It can then be analyzed by using methods from topological dynamics and descriptive set theory.

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Smooth versus rough classification

Theorem (Hjorth)

The cohomology equivalence relation is Borel.

Proposition

The closures of the cohomology classes form a partition of the space $Z^1(a, G)$.

We now divide $Z^1(a, G)$ into two disjoint sets, each invariant under the above partition: $\mathrm{SMOOTH}(a, G)$ and $\mathrm{ROUGH}(a, G)$, where $\mathrm{SMOOTH}(a, G)$ consists of all closed cohomology classes.

Theorem

$\mathrm{SMOOTH}(a, G)$, $\mathrm{ROUGH}(a, G)$ are Borel sets and the cohomology relation is smooth on $\mathrm{SMOOTH}(a, G)$ and non-smooth on $\mathrm{ROUGH}(a, G)$.

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Amenable group actions

For actions of amenable groups, there is an extensive literature concerning the structure of their cocycles (Bezuglyi, Danilenko, Fedorov, Golodets, Schmidt, Sinelshchikov, Zimmer). In particular we have the following result that is essentially due to Parthasarathy-Schmidt:

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Amenable group actions

We also have a converse.

Proposition

If the action of Γ is free and ergodic, then the following are equivalent:
i) Γ is amenable,
ii) For any G, every cohomology class in $Z^1(a, G)$ is dense.

Strongly ergodic actions

Definition

An action $a \in A(\Gamma, X, \mu)$ is strongly ergodic if it does not admit non-trivial almost invariant sets, i.e., there is no sequence of Borel sets A_n, whose measures stay away from 0,1, such that $\mu(\gamma \cdot A_n \triangle A_n) \to 0, \forall \gamma \in \Gamma$.

Example

If Γ is amenable, no action of Γ is strongly ergodic.

Example

(Schmidt, Connes-Weiss) Γ has Kazhdan's property (T) iff every ergodic action of Γ is strongly ergodic.

As it turns out the dichotomy strongly/non-strongly ergodic is crucial for the behavior of the cohomology relation. This was first pointed out by Schmidt in the case of abelian target groups G.

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The non-strongly ergodic case

Theorem

If the action a is ergodic but non-strongly ergodic, then $B^1(a, G)$ is contained in the rough part of $Z^1(a, G)$, for every non-trivial G. Moreover, the action of $L(X, \mu, G)$ on each cohomology class closure contained in the rough part is turbulent and thus cohomology cannot be classified by countable structures.

Remark: Special cases of this non-classification result for amenable group actions were earlier proved by Hjorth.

Remark: As opposed to the case where the acting group is amenable or where the target group is abelian, where (for such actions) every cocycle is in the rough part, there are examples of such actions with nonabelian target groups in which both the rough and the smooth part are nonempty and in fact each contains continuum many cohomology class closures.

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Definition

A group G is rough if for every ergodic but not strongly ergodic action a every cocycle in $Z^1(a, G)$ is in the rough part.

Thus every abelian group is rough. What are the rough groups? I do not know the complete answer but here is a partial result:

Theorem

Every weakly commutative group is rough and every rough group is inner amenable. Moreover in the case G is not inner amenable, every orbit equivalence cocycle of a free ergodic action is in the smooth part.

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Popa has recently shown that Bernoulli actions of property (T) and many other groups have the very strong property that every cocycle to any countable group G is cohomologous to a homomorphism. This has many applications in ergodic theory, operator algebras as well as descriptive set theory. Popa calls this property *cocycle superrigidity*. (It implies, for example, that, in many situations, the equivalence relation induced by the Bernoulli action completely determines the group and the action.)

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An action a is G-superrigid if every cocycle into G is cohomologous to a homomorphism.

Corollary

If the action a is weakly mixing but non-strongly ergodic, then within each cohomology class closure contained in the rough part, the generic cocycle is not cohomologous to a homomorphism, i.e., in a sense, a is "generically not G-superrigid", for any G.

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There are also implications concerning global properties of actions of groups. Recall that *every* ergodic action of a property (T) group is G-cocycle superrigid for certain G, for example, torsion-free HAP groups. Also many *non-property* (T) groups have some actions, e.g., Bernoulli actions, that are G-cocycle superrigid for every G. However this happens, in some sense, very rarely.

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If Γ does not have property (T), then the generic action of Γ is not G-cocycle superrigid for any non-trivial G, in fact it is even "generically not G-superrigid".

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Next we have the following connectedness result.

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If the action a is non-strongly ergodic, then each cohomology class closure is path connected.

So, for example, for any action a of an amenable group $Z^1(a, G)$ is path connected. I do not know if $Z^1(a, G)$ is path connected for any non-strongly ergodic a. Also I do not know if the rough part of any such a is path connected. This will not be the case, for example, if there are actions which have only countably many orbit closures in the rough part.

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The strongly ergodic case

We have seen that for non-strongly ergodic actions the cohomology relation is quite complicated. One might expect that the opposite happens for strongly ergodic actions. This is indeed the case but with an interesting twist: It depends on the structure of the target group, so we have an additional dichotomy here.

Definition

A group G satisfies the minimal condition on centralizers if there is no strict infinite descending sequence $C_0 > C_1 > \dots$ of centralizers (under inclusion).

This class of groups is quite extensive. It includes the abelian groups and the linear groups and is closed under subgroups, finite products and finite extensions. In particular, it contains the free groups.

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Theorem

Let G satisfy the minimal condition on centralizers. Then for any ergodic action a, the following are equivalent:
i) The action a is strongly ergodic.
ii)The cohomology equivalence relation is smooth, i.e., $\mathrm{SMOOTH}(a, G) = Z^1(a, G)$.

Thus strong ergodicity implies that the cohomology relation is simple, if the target group satisfies the minimal condition on centralizers. In particular, it is somewhat surprising that cocycles taking values in free groups can in principle be classified up to cohomology, if the action is strongly ergodic.

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The strongly ergodic case

The role of the minimal condition on centralizers is not accidental, as the following result indicates.

Theorem

Let a be a strongly ergodic, weakly mixing action of the free group F_∞. Then for any group G the following are equivalent:
i) The cohomology relation in $Z^1(a, G)$ is smooth, i.e., $\mathrm{SMOOTH}(a, G) = Z^1(a, G)$.
ii) G satisfies the minimal condition on centralizers.

To summarize: Cohomology is complicated for non-strongly ergodic actions. For strongly ergodic actions, it is simple if the target group satisfies the minimal condition on centralizers but may be complicated if this condition fails.

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The property (T) case

When the acting group has property (T), the structure of cocycles is very simple in view of the following result.

Theorem (Popa)

If Γ has property (T), then for any ergodic action a of Γ, every cohomology class in $Z^1(a, G)$, for any G, is clopen, thus there are only countably many cohomology classes.

Remark: The property (T) groups are the only groups for which there are countably many cohomology classes.

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If Γ has property (T), then for any ergodic action a of Γ, every cohomology class in $Z^1(a, G)$, for any G, is clopen, thus there are only countably many cohomology classes.

Remark: The property (T) groups are the only groups for which there are countably many cohomology classes.

The property (T) case

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Let $a \in A(\Gamma, X, \mu)$ be a given action. The full group of E_a, $[E_a]$, is the group of all $T \in \mathrm{Aut}(X, \mu)$ with $T(x) E_a x$, a.e. The normalizer of E_a, $N[E_a]$, is the group of automorphisms of E_a, i.e., the group of all $T \in \mathrm{Aut}(X, \mu)$ with $x E_a y \Leftrightarrow T(x) E_a T(y)$, a.e. The quotient group $\mathrm{Out}(E_a) = N[E_a]/[E_a]$ is called the outer automorphism group of E_a. Jones-Schmidt considered the problem of understanding the dynamical properties of the action that would guarantee that the outer automorphism is Polish (under a natural quotient topology). It is natural to consider here the role of strong ergodicity. However they have shown that, in general, strong ergodicity does not imply that the outer automorphism group is Polish. In their counterexample the acting group Γ does not satisfy the minimal condition on centralizers. We in fact have the following positive answer to this question.

Another application

Theorem

Assume that the group Γ satisfies the minimal condition on centralizers and is ICC. Then for any free, strongly ergodic action of Γ, the outer automorphism group is Polish.