Free products and Bass-Serre rigidity

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Definition (Slicing)

The *slicing* ***affiliated*** with an $\mathcal{R}$-invariant partition $\coprod_{j\in J} V_j$ of the domain $\mathrm{D}(\mathcal{R})$ is the free product decomposition

$$\mathcal{R} = \mathop{*}_{j\in J} \mathcal{R}|V_j.$$

Definition (Sliding)

Let $U \subset \mathrm{D}(\mathcal{R})$ be a *complete section* of $\mathcal{R}$.
A *sliding* of $\mathcal{R}$ to U consists in
– a smooth treeable subrelation $\mathbf{T} < \mathcal{R}$ defined on $\mathrm{D}(\mathcal{R})$ with fundamental domain U and in
– the corresponding free product decomposition

$$\mathcal{R} = \mathcal{R}|U * \mathbf{T}.$$

Definition (Inessential Free Product Decomposition)

A free product decomposition $\mathcal{R} = *_{j \in J} \mathcal{R}_j$, of a (countable) standard p.m.p. equivalence relation on X is called *inessential* if there is

1. an $\mathcal{R}$-invariant partition $X = \coprod_{j \in J} V_j$
2. in each V_j a complete section U_j for $\mathcal{R}|V_j$ s.t.

$$\mathcal{R}|U_j = \mathcal{R}_j|U_j$$

Definition ($\mathcal{FI}$ Equivalence Relation)

A countable standard Borel equivalence relation $\mathcal{R}$ is *freely indecomposable* ($\mathcal{FI}$) if any free product decomposition $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2 * \cdots * \mathcal{R}_i * \cdots$ is inessential.

Example

If $\mathcal{R}$ is treeable (for instance hyperfinite) and $\mathcal{FI}$ then it is smooth.

Definition

A countable group is called *measurably freely indecomposable* ($\mathcal{MFI}$) if all its free p.m.p. actions are freely indecomposable ($\mathcal{FI}$).

Proposition (SOE/ME Invariance)

Being $\mathcal{FI}$ is an SOE invariant. Being $\mathcal{MFI}$ is an ME invariant.

(obvious for OE)

Theorem (Alvarez-G.)

Every non-amenable countable group Γ with $\beta_1(\Gamma) = 0$ is $\mathcal{MFI}$.

For instance infinite property (T) groups (It is worth noting that infinite Kazhdan's property (T) groups also are $\mathcal{MFI}$ from Adams-Spatzier 90), direct products of infinite groups, lattices in $\mathrm{SO}(p, q)$ $(p.q \neq 2)$, lattices in $\mathrm{SU}(p, q)$ groups with an infinite finitely generated normal subgroup of infinite index, groups with an infinite normal subgroup with the relative property (T), amalgamated free products of groups with $\beta_1 = 0$ over an infinite subgroup, mapping class groups,...

Theorem (Alvarez-G.)

If $\mathcal{R}$ is a nowhere hyperfinite p.m.p. standard equivalence relation on (X, μ) with $\beta_1(\mathcal{R}) = 0$, then it is $\mathcal{FI}$.

Theorem (Alvarez-G.)

Infinite countable $\mathcal{MFI}$ *groups*

– $(\Gamma_i)_{i\in I}$, $I = \{1, 2, \cdots, n\}$

– $(\Lambda_j)_{j\in J}$, $J = \{1, 2, \cdots, m\}$, $n, m \in \mathbb{N}^* \cup \{\infty\}$

Two p.m.p. free stably orbit equivalent actions

$$(\mathop{*}_{i\in I} \Gamma_i) \overset{\alpha}{\curvearrowright} (X, \mu) \overset{\mathrm{SOE}}{\sim} (\mathop{*}_{j\in J} \Lambda_j) \overset{\beta}{\curvearrowright} (Y, \nu)$$

Assume the restrictions $\alpha|\Gamma_i$ *and* $\beta|\Lambda_j$ *are* ***ergodic***, $\forall i, j$,
THEN $n = m$ *and there is a bijection of the indices* $\theta : I \longrightarrow J$ *s.t. the restrictions are SOE*

$$\alpha|\Gamma_i \overset{\mathrm{SOE}}{\sim} \beta|\Lambda_{\theta(i)}$$

Compare corollaries in [IPP05], [CH08]

Kazhdan-like + ICC-like [Ioana-Peterson-Popa 2005] ($\mathcal{FT}$)

ICC non amen. direct products [Chifan-Houdayer 2008] (primality)

Measure Equivalence rigidity

Theorem (Alvarez-G., ME Bass-Serre rigidity)

Infinite countable $\mathcal{MFI}$ groups
– $(\Gamma_i)_{i\in I}$, $I = \{1, 2, \cdots, n\}$
– $(\Lambda_j)_{j\in J}$, $J = \{1, 2, \cdots, m\}$, $n, m \in \mathbb{N}^* \cup \{\infty\}$
– $\Gamma_0 \overset{\mathrm{ME}}{\sim} \mathbf{F}_p, \Lambda_0 \overset{\mathrm{ME}}{\sim} \mathbf{F}_q$ *ME w. some free groups* $p, q \in \mathbb{N} \cup \{\infty\}$
If their free products are measure equivalent,

$$\mathop{*}_{i\in I} \Gamma_i * \Gamma_0 \overset{\mathrm{ME}}{\sim} \mathop{*}_{j\in J} \Lambda_j * \Lambda_0$$

then there are two maps $\theta : I \to J$ *and* $\theta' : J \to I$ *such that:*

$$\Gamma_i \overset{\mathrm{ME}}{\sim} \Lambda_{\theta(i)} \quad \textit{and} \quad \Lambda_j \overset{\mathrm{ME}}{\sim} \Gamma_{\theta'(j)}$$

Observe: we do not assume the index $\kappa = 1$. Would we do so, we would not get $\kappa = 1$ in the conclusion. The groups Γ_0, Λ_0 do not appear in the

Bass-Serre rigidity

Theorem (Alvarez-G., Bass-Serre rigidity)

Assume Θ *witnesses*

$$\mathcal{R} = \mathop{*}_{p\in P} \mathcal{R}_p * \mathbf{T} \overset{\mathrm{SOE}}{\sim} \mathcal{R}' = \mathop{*}_{p\in P} \mathcal{R}'_p * \mathbf{T}'$$

w. each factor $\mathcal{R}_p$ *and* $\mathcal{R}'_p$ *is* $\mathcal{FI}$ *and aperiodic on its domain; and w.* $\mathbf{T}$ *and* $\mathbf{T}'$ *are treeable.*
then $\exists$ *slicings of the factors:*
$\mathcal{R} = \mathop{*}_{p\in P} (\mathop{*}_{k\in K(p)} \mathcal{R}_p|X_k) * \mathbf{T}$ *and* $\mathcal{R}' = \mathop{*}_{p\in P} (\mathop{*}_{k'\in K'(p)} \mathcal{R}'_p|X'_{k'}) * \mathbf{T}'$
and $\exists$ *bijection* $\theta : \coprod_{p\in P} K(p) \to \coprod_{p\in P} K'(p)$ *between index sets s.t.,*

$$\mathcal{R}_p|X_k \overset{\mathrm{SOE}}{\sim} \mathcal{R}'_{\theta(k)}|X'_{\theta(k)}$$

$\forall k \in \coprod_{p\in P} K(p)$*, via some* $f'_k \Theta f_{\theta(k)}$*, with* $f_k \in [[\mathcal{R}]]$*,* $f'_{\theta(k)} \in [[\mathcal{R}']]$*.*

Playing with non-ergodicity

Combine with Monod-Shalom's ME result for direct products.

Corollary

Assume Γ is either
i) a finite direct product of non-trivial free products of torsion-free $\mathcal{MFI}$ groups Γ_i, or
ii) a free product of non-trivial finite direct products of torsion free groups Γ_i in the class C_{reg}.
If Γ is ME with Λ a group of the same kind, then the elementary pieces Γ_i of Γ define the same set of ME-classes as those of Λ.

This contruction can be iterated by taking free products of finite direct products of free products of finite direct products of free products of torsion-free $\mathcal{MFI}$ groups.
And similarly for ... of free products of finite direct products of ... of free products of finite direct products of torsion free groups in the class C_{reg}.

Corollary

Assume $\beta_1(\Gamma_p) = 0$, $\forall p$.
Assume $\Gamma_1 \overset{\mathrm{ME}}{\not\sim} \Gamma_p$ $(\forall\ p \neq 1)$ *and* $\beta_r(\Gamma_1) \neq 0, \infty$ *(for some* $r \geq 2$*).*
If α *and* α' *are two SOE p.m.p. free actions of* $(*_{p \in P} \Gamma_p * \mathbf{F}_n)$*, then they are in fact OE and the restrictions to* Γ_1 *are OE.*
In particular, they have the same measure space of ergodic components.

Example

Let $\Gamma_0 = \mathbf{F}_2$ and $\Gamma_1 = \mathbf{F}_3 \times \mathbf{F}_3$. Consider a one-parameter family of free p.m.p. action $\Gamma_0 * \Gamma_1 \overset{\alpha_s}{\curvearrowright} (X, \mu)$, s.t. the restriction $\alpha_s|\Gamma_1$ has two ergodic components of respective measures $s, 1-s$. The actions α_s are mutually non-stably orbit equivalent for $s \in [0, 1/2]$.

By using [IPP05] $\mathcal{FT}$ theorem:

Theorem (sample)

Let Γ_1, Γ_2 be non-ME, $\mathcal{MFI}$ groups. Assume $\beta_r(\Gamma_1) \neq 0, \infty$ for some $r > 1$. The crossed-product II$_1$ factors

$$M_1 \underset{A}{*} M_2$$

associated with the various ergodic relative property (T) free p.m.p. actions $\Gamma_1 * \Gamma_2 \overset{\sigma}{\curvearrowright} (X, \mu)$ are distinguished by the pairs

$$A \subset M_1$$

and in part. by the isomorphism class of the centers $(Z(M_1), \tau)$.

Relative property (T) and Outer automorphisms

Theorem (Törnquist 2006)

There is a free action of the free product $*_{i\in I}\Gamma_i \overset{\sigma}{\curvearrowright} (X,\mu)$ *whose restriction to the components is conjugated with any prescribed free action* $\Gamma_i \overset{\alpha_i}{\curvearrowright} (X,\mu)$

Theorem (G.)

Every non-trivial free product of countable infinite groups $*_{i\in I}\Gamma_i$ *admits uncountably many* von Neumann inequivalent *free ergodic p.m.p. actions* $(\sigma_t)_{t\in T}$ *s.t.*

- *they all have* relative property (T)
- *the restrictions to each factor* Γ_i *are conjugated with a* prescribed *free action*
- $\mathrm{Out}(\mathcal{R}_{\sigma_t}) = \{1\}$

Compare Popa-Vaes: groupes of the type $\mathbf{F}_\infty * \Lambda$
⇝ first examples of actions of $\mathbf{F}_p$ w. $\mathrm{Out}(\mathcal{R}_\sigma) = \{1\}$, $p \in 2, \cdots, \infty$

Lemma (Replacement lemma)

Let $\mathcal{S}_1$, $\mathcal{S}_2$ be p.m.p. aperiodic countable standard equivalence relations. For every integer $p \in \mathbb{N} \setminus \{0, 1\}$, there is a p.m.p. equivalence relation on X that decomposes as a free product of subrelations:

$$\mathcal{S} = \mathcal{S}_1 * \mathcal{S}'_2$$

where: $\mathcal{S}'_2 \overset{\mathrm{OE}}{\sim} \mathcal{S}_2$ in such a way that there is a Borel subset $Y \subset X$ of measure $\frac{1}{p}$ s.t. for any $\mathbf{F}_{p-1}$-action β on Y whose orbit equivalence relation $\mathbf{T}_\beta$ is in free product with $\mathcal{S}_1|Y * \mathcal{S}'_2|Y$, the equivalence relation $\mathcal{S} := \langle \mathcal{S}_1, \mathcal{S}'_2|Y, \mathbf{T}_\beta \rangle$ admits free product decompositions:

$$\mathcal{S}_\beta := \mathcal{S}_1 * \mathcal{S}'_2|Y * \mathbf{T}_\beta$$
$$= \mathcal{S}_1 * \mathcal{S}_{2,\beta}$$

where $\mathcal{S}_{2,\beta}$ is orbit equivalent with $\mathcal{S}_2$: $\mathcal{S}_{2,\beta} \overset{\mathrm{OE}}{\sim} \mathcal{S}_2$

In particular, if the $\mathbf{F}_{p-1}$-action β has the relative property (T)...