Coordinatewise decomposition and dichotomy results in descriptive set theory

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A function $f : X \to Y$ is *Borel* if

$$\forall B \subseteq Y \ (B \text{ is open } \Rightarrow f^{-1}(B) \text{ is Borel}).$$

Definition

Suppose that $S \subseteq X \times Y$, G is a group, and $f : S \to G$ is a function. A *coordinatewise decomposition* of f is a pair (u, v), where $u : X \to G$ and $v : Y \to G$, such that

$$\forall (x, y) \in S \; (f(x, y) = u(x)v(y)).$$

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Definition

A coordinatewise decomposition is *Borel* if both u and v are Borel.

Remark

For the sake of simplicity, we will assume that $2 \leq |G| \leq \aleph_0$ and $\forall x \in X \, \forall y \in Y \ (1 \leq |S_x|, |S^y| \leq \aleph_0)$.

Question (Kłopotowski-Nadkarni-Sarbadhikari-Srivastava)

Suppose that X and Y are Polish spaces and $S \subseteq X \times Y$ is Borel. Under what circumstances does every Borel function from S into G admit a Borel coordinatewise decomposition?

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We consider first the purely combinatorial version of the question.

Remark

For notational convenience, assume that X and Y are disjoint.

Definition

We use $\mathcal{G}_S$ to denote the graph on the set $Z = X \cup Y$ given by

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Proposition

The following are equivalent:

1. Every $f : S \to G$ admits a coordinatewise decomposition;
2. The graph $\mathcal{G}_S$ is acyclic.

Proof of $\neg(2) \Rightarrow \neg(1)$

Suppose that $\langle x_0, y_0, x_1, \ldots, x_n \rangle$ is a $\mathcal{G}_S$-cycle, and fix any function $f : S \to G$ with the property that

$$\prod_{i<n} f(x_i, y_i) f(x_{i+1}, y_i)^{-1} \neq 1_G.$$

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$$\prod_{i<n} f(x_i, y_i) f(x_{i+1}, y_i)^{-1} \neq 1_G.$$

If (u, v) is a coordinatewise decomposition of f, then

$$\prod_{i<n} f(x_i, y_i) f(x_{i+1}, y_i)^{-1} = \prod_{i<n} u(x_i) v(y_i) v(y_i)^{-1} u(x_{i+1})^{-1},$$

which equals 1_G, contradicting our choice of f. □

Proof of (2) $\Rightarrow$ (1)

Fix a set $Z_0 \subseteq X$ which intersects every connected component of $\mathcal{G}_S$ in exactly one point, and let

$$Z_{n+1} = \{z \in Z \setminus \bigcup_{m \leq n} Z_m : \exists z' \in Z_n \, ((z, z') \in \mathcal{G}_S)\}.$$

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Suppose that $f : S \to G$, and define

$$[u|Z_0](x) = 1_G.$$

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Suppose that $f : S \rightarrow G$, and define

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Suppose now that we have defined $u|Z_{2n}$, and set

$$[v|Z_{2n+1}](y) = u(x)^{-1} f(x, y),$$

where $x \in Z_{2n}$ and $(x, y) \in \mathcal{G}_S$. Define $u|Z_{2n+2}$ similarly. □

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Theorem

The following are equivalent:

1. *Every Borel function* $f : S \to G$ *admits a Borel coordinatewise decomposition;*
2. $\mathcal{G}_S$ *is acyclic and* E_S *admits a Borel transversal.*

Remark

We have essentially already given the proof of (2) $\Rightarrow$ (1).

Coordinatewise decomposition

Global decomposability

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Definition

Let E_0 denote the equivalence relation on $2^{\mathbb{N}}$ given by

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Definition

Suppose that E_1 and E_2 are equivalence relations on X_1 and X_2. An *embedding of* E_1 *into* E_2 is an injection $\pi : X_1 \rightarrow X_2$ with

$$\forall x, y \in X_1\ (xE_1y \Leftrightarrow \pi(x)E_2\pi(y)).$$

Remark

By the Glimm-Effros dichotomy, we can assume that there is a Borel embedding $\pi : 2^{\mathbb{N}} \to Z$ of E_0 into E_S.

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Definition

Fix $g_0 \in G$ with $g_0 \neq 1_G$, and define $\rho_0 : E_0 \to G$ by setting $\rho_0(x, y) = g$ if and only if there exists $n \in \mathbb{N}$ such that

$$\forall m \geq n\ (x(m) = y(m)) \text{ and } g = g_0^{\sum_{m<n} x(m) - \sum_{m<n} y(m)}.$$

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Remark

The map ρ_0 is a *cocycle*: $x E_0 y E_0 z \Rightarrow \rho_0(x, z) = \rho_0(x, y)\rho_0(y, z)$.

Definition

Let E_{ρ_0} be the subequivalence relation of E_0 given by

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A set $B \subseteq Z/E$ is *Borel* if it is Borel when viewed as a subset of Z.

Claim

E_0/E_{ρ_0} does not admit a Borel transversal.

Definition

By the Lusin-Novikov uniformization theorem, there is a Borel cocycle $\rho : E_S \rightarrow G$ such that

$$\forall (x, y) \in E_0 \ (\rho(\pi(x), \pi(y)) = \rho_0(x, y)).$$

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Claim

The function w witnesses that ρ is a *Borel coboundary*, i.e.,

$$\forall (x, y) \in E_S \ (\rho(x, y) = w(x) w(y)^{-1}).$$

Claim

For each $g \in G$, the set $B_g = w^{-1}(g)/E_\rho$ is a Borel *partial transversal* of E_S/E_ρ, i.e., it intersects each equivalence class of E_S/E_ρ in at most one point.

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Claim

By appealing again to the Lusin-Novikov uniformization theorem, we can build from the sets B_g a Borel transversal of E_S/E_ρ.

Remark

Since ρ was chosen to ensure that there is no such transversal, this completes the proof of the theorem. $\square$

Remark

Next, we consider the circumstances under which a given function $f : S \subseteq X \times Y \to G$ admits a coordinatewise decomposition.

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Definition

The *weight* of a $\mathcal{G}_S$-cycle $\gamma = \langle x_0, y_0, \ldots, x_n \rangle$ is given by

$$w(\gamma) = \prod_{i<n} f(x_i, y_i) f(x_{i+1}, y_i)^{-1}.$$

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Proposition

The following are equivalent:

1. There is a coordinatewise decomposition of f;
2. The weight of every $\mathcal{G}_S$-cycle is 1_G.

Proof of (1) $\Rightarrow$ (2)

Suppose that (u, v) is a coordinatewise decomposition of f, and observe that if $\gamma = \langle x_0, y_0, x_1, \ldots, x_n \rangle$ is $\mathcal{G}_S$-cycle, then

$$\begin{aligned} w(\gamma) &= \prod_{i<n} f(x_i, y_i) f(x_{i+1}, y_i)^{-1} \\ &= \prod_{i<n} u(x_i) v(y_i) v(y_i)^{-1} u(x_{i+1})^{-1}, \end{aligned}$$

which equals 1_G. $\square$

Proof of (2) $\Rightarrow$ (1)

Fix a set $Z_0 \subseteq X$ which intersects every connected component of $\mathcal{G}_S$ in exactly one point, and let

$$Z_{n+1} = \{z \in Z \setminus \bigcup_{m \leq n} Z_m : \exists z' \in Z_n \ ((z, z') \in \mathcal{G}_S)\}.$$

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$$[u|Z_0](x) = 1_G.$$

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From this point forward, we assume that f admits a coordinatewise decomposition, and examine the circumstances under which f admits a Borel coordinatewise decomposition.

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Since the weight of every $\mathcal{G}_S$-cycle is 1_G, there is a unique extension of f to a cocycle $\rho_f : E_S \to G$.

Proposition

The following are equivalent:

1. There is a Borel coordinatewise decomposition of f;
2. ρ_f is a Borel coboundary.

Proof of (1) $\Rightarrow$ (2)

A straightforward induction shows that if (u, v) is a Borel coordinatewise decomposition of f, then

$$w = u \sqcup v^{-1}$$

witnesses that ρ_f is a Borel coboundary. $\square$

Proof of (1) $\Rightarrow$ (2)

A straightforward induction shows that if (u, v) is a Borel coordinatewise decomposition of f, then

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witnesses that ρ_f is a Borel coboundary. □

Proof of (2) $\Rightarrow$ (1)

If $w : Z \to G$ witnesses that ρ_f is a Borel coboundary, then

$$(u, v) = (w|X, (w|Y)^{-1})$$

is a Borel coordinatewise decomposition of f. □

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This reduces the problem to finding the circumstances under which a cocycle $\rho : E \to G$ is a Borel coboundary.

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Proposition

The following are equivalent:

1. ρ is a Borel coboundary;
2. E/E_ρ admits a Borel transversal.

Definition

A function $f : X_1/E_1 \longrightarrow X_2/E_2$ is *Borel* if its graph is Borel, when thought of as a subset of $X_1 \times X_2$.

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Theorem

Suppose that G is torsion-free. Exactly one of the following holds:

1. *E/E_ρ admits a Borel transversal;*
2. *There is a Borel embedding of E_0 into E/E_ρ.*

Remark

The proof follows closely the usual Glimm-Effros style arguments.

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Theorem

Suppose that G is torsion-free. Exactly one of the following holds:

1. *f admits a Borel coordinatewise decomposition;*
2. *There is a Borel embedding of E_0 into E_S/E_{ρ_f}.*

Remark

The analog of this theorem fails badly if G is not torsion free.

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However, there are still basis theorems which describe the circumstances under which a cocycle is a Borel coboundary, and which therefore describe the circumstances under which a Borel function admits a Borel coordinatewise decomposition.

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However, there are still basis theorems which describe the circumstances under which a cocycle is a Borel coboundary, and which therefore describe the circumstances under which a Borel function admits a Borel coordinatewise decomposition.

Remark

From this point forward, we focus on the case that G is finite.

Definition

Recall that, given $g_0 \in G \setminus \{1_G\}$, we obtain a cocycle $\rho_G : E_0 \to G$ by setting $\rho_G(x, y) = g$ if and only if there exists $n \in \mathbb{N}$ such that

$$\forall m \geq n \ (x(m) = y(m)) \text{ and } g = g_0^{\sum_{m<n} x(m) - \sum_{m<n} y(m)}.$$

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$$\forall m \geq n \; (x(m) = y(m)) \text{ and } g = g_0^{\sum_{m<n} x(m) - \sum_{m<n} y(m)}.$$

Definition

For $G = \mathbb{Z}/p\mathbb{Z}$, this defines a cocycle $\rho_p = \rho_G$.

Proposition

Suppose that $\rho : E \rightarrow G$ is not a Borel coboundary. Then there is at most one prime p such that E/E_ρ Borel embeds into E_0/E_{ρ_p}.

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Theorem

Exactly one of the following holds:

1. *E/E_ρ admits a Borel transversal;*
2. *There is a prime p such that E_0/E_{ρ_p} Borel embeds into E/E_ρ.*

Theorem

Suppose that X and Y are Polish spaces, $S \subseteq X \times Y$ is a Borel set with countable sections, G is a non-trivial countable group, and $f : S \to G$ is a Borel function which admits a coordinatewise decomposition. Then exactly one of the following holds:

1. *f admits a Borel coordinatewise decomposition;*
2. *Either (a) E_0 Borel embeds into E_S/E_{ρ_f}, or (b) there is a prime p such that E_0/E_{ρ_p} Borel embeds into E_S/E_{ρ_f}.*

Remark

The special case of the basis theorem for finite groups falls out of a proof of a series of much more general results.

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These results give also a complete classification of equivalence relations of the form E_0/E, where E is of finite index below E_0.

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Remark

These results give also a complete classification of equivalence relations of the form E_0/E, where E is of finite index below E_0.

Remark

Equivalently, we obtain a classification of Borel equivalence relations on $2^{\mathbb{N}}/E_0$ whose classes are of finite cardinality.

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The classification problem associated with such equivalence relations is smooth.

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In fact, one can associate with each Borel equivalence relation on $2^{\mathbb{N}}/E_0$ whose classes are of size n a family of subgroups of S_n which completely determines its isomorphism type.

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Remark

This invariant describes also the ways of assigning structures to the classes of E/E_0 in a Borel way, and the proof gives a family of dichotomy theorems which characterize the circumstances under which such assignments exist.

Theorem

Up to Borel isomorphism, there are exactly two Borel equivalence relations on $2^{\mathbb{N}}/E_0$ whose classes are of cardinality two. In order of Borel embeddability, they are: (1) the one which admits a Borel transversal, and (2) the one which does not.

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Theorem

Up to Borel isomorphism, there are exactly five Borel equivalence relations on $2^{\mathbb{N}}/E_0$ whose classes are of cardinality three.

Remark

The family of such equivalence relations is not linearly ordered under Borel embeddability.

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There is also maximal one. It is characterized by the fact that it admits no non-trivial Borel assignments of structures.

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There are also two incompatible such equivalence relations.

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One is generated by a Borel action of $\mathbb{Z}/3\mathbb{Z}$ on $2^{\mathbb{N}}/E_0$, but does not admit a Borel transversal.

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The other admits a Borel transversal, but is not generated by a Borel action of a countable group on $2^{\mathbb{N}}/E_0$.

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The other admits a Borel transversal, but is not generated by a Borel action of a countable group on $2^{\mathbb{N}}/E_0$.

Remark

There are over fifty Borel equivalence relations on $2^{\mathbb{N}}/E_0$ whose classes are of cardinality four!

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These papers are available at:

http://www.math.ucla.edu/∼bdm/papersandsuch.html.