Classification by countable structures

UCLA, March 17, 07

§1. Smooth/non-smooth:

Defn. Let E be an equivalence relation on a Polish (i.e. separable, allows a complete metric) X.

E is *smooth* if there is Borel

$$f: X \to \mathbb{R}$$

s.t.

$$x_1 E x_2 \iff f(x_1) = f(x_2).$$

Rmk: We can replace $\mathbb{R}$ by any other uncountable Polish space and have an equivalent definition.

f is "Borel" if pull backs of open sets are open.

Fact: E is smooth iff there are E-invariant Borel sets, $(B_n)_{n \in \mathbb{N}}$, s.t.

$$x_1 E x_2 \quad \text{iff} \quad \forall n \, (x_1 \in B_n \iff x_2 \in B_n).$$

E smooth is like saying

X/E can be thought as a subset of a "non-pathological" space.

Examples

E.g.1. Γ a countable abelian group.
$\mathcal{H}$ a finite dimensional Hilbert space.
X = Representations of Γ by unitary operators on $\mathcal{H}$
(= closed subset of $\prod_{\Gamma} \mathcal{H}$ $\therefore$ Polish).
E = unitary equivalance of representations.

E is smooth!
(Take a basis of eigenvectors.
Record the eigenvalues.)

E.g.2. For Γ a general countable group,
$\mathrm{Rep}_H(\Gamma)$, the irreducible representations of Γ on some separable Hilbert space H,
form a G_δ subset of $\prod_{\Gamma} \mathcal{H}$
$\therefore$ Polish.

Thoma: The irreducible representations of Γ are smooth (for every separable H)
iff
Γ is abelian-by-finite.

E.g.3. E_0: Eventual agreement on elements of $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$.
I.e. $\vec{x} E_0 \vec{y}$ iff $\exists N \, \forall m > N \; (x_m = y_m)$.
E_0 not smooth

Eg.4. Rank 1 torsion free abelian groups.
$\sim$ Subgroups of $\mathbb{Q}$

which is a closed subset of all subsets of $\mathbb{Q} = \mathcal{P}(\mathbb{Q}) \sim \{0,1\}^{\mathbb{Q}} = 2^{\mathbb{Q}}$

(Identify $A \subseteq \mathbb{Q}$ with its characteristic function $\chi_A \in 2^{\mathbb{Q}}$.)

$\cong_1$: Isomorphism on subgroups of $\mathbb{Q}$.

NOT smooth.

(Baer's complete invariant:
Functions from $\mathbb{P}$ to $\mathbb{N} \cup \{\infty\}$ up to certain finite perturbation .... cf E_0).

Eg.5. Irrational rotation:

$$T : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$$
$$x \longrightarrow x + \sqrt{2} \mod 1$$

Resulting equivalence relation:
Not smooth.

Eg.6. (Solecki)
For G a Polish group,
all its continuous actions on Polish spaces give rise to smooth orbit equivalence relations
iff
G is compact.

Eg.7. Measures on $[0,1]$ considered up to absolute continuity.
Not smooth.

In 1990, building on earlier work of Glimm and Effros, a remarkable dichotomy theorem was published:

Thm. (Harrington-Kechris-Louveau)

Let E be a Borel equivalence relation on a Polish space X.
Then exactly one of the following holds:

(I) E is smooth;

(II) there is Borel $f: 2^{\mathbb{N}} \to X$ $\qquad \prod_{\mathbb{N}} \{0,1\}$

s.t.

$$\vec{x}\, E_0\, \vec{y} \text{ iff } f(\vec{x})\, E\, f(\vec{y}).$$

$$\exists N \forall m > N (x_m = y_m)$$

<u>Rmk</u>: We often write (II) as "$E_0 \leq_B E$".
It more or less says that

$$2^{\mathbb{N}}/E_0 \hookrightarrow X/E$$

... something like an embedding of quotient structures.

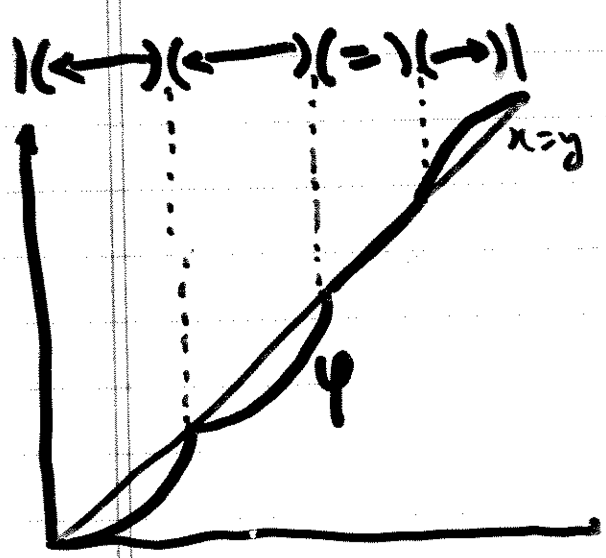
§2. Classification by countable structures

Before any formal definitions, a few loose examples where we seem to obtain some kind of classification without smoothness.

$\varphi_1 E \varphi_2$ iff $\exists \psi \in \mathrm{Hom}^+([0,1])$ $\psi \circ \varphi_1 \circ \psi^{-1} = \varphi_2$

Eg.1. $\mathrm{Hom}^+([0,1])$
Orientation preserving homeomorphisms of $[0,1]$ considered up to conjugacy.

Given $\varphi \in \mathrm{Hom}^+([0,1])$, enumerate the maximal open intervals on which φ is either increasing, decreasing, or the identity map.



Associate to φ the linear ordering consisting of this maximal open intervals. $L\varphi$.

Add to $L\varphi$ information about which of the possible 3 behaviours takes place on each interval.
Predicates: P_+, P_-, $P_=$

$L\varphi$: 4 points, arranged so:

$\overset{\bullet}{P_-}$ $\overset{\bullet}{P_-}$ $\overset{\bullet}{P_=}$ $\overset{\bullet}{P_+}$

Then: $L\varphi_1 \cong L\varphi_2$ iff φ_1, φ_2 conjugate.

DC.

Eg 2. (Giordano, Putnam, Skau)
Minimal homeomorphisms of $\{0,1\}^{\mathbb{N}}$ $(=_{df.} 2^{\mathbb{N}})$.

considered up to topological conjugacy of orbits.

These are completely classified by a corresponding countable ordered abelian group
(considered up to isomorphism).

Eg 3.
Compact, zero dimensional metric spaces considered up to homeomorphism.

Completely classified by the corresponding countable Boolean algebra of clopen sets considered up to $\cong$.

Eg 4. (Pontryagin duality)

Compact abelian metric groups considered up to topological & algebraic $\cong$.

Completely classified by (countable, discrete) dual group considered up to $\cong$.

To make this a bit more precise, need to define Polish spaces of countable structures. The general definition is a bit involved, but in some special cases:

Eg.1. Linear orderings on $\mathbb{N}$.
Identify a linear ordering on $\mathbb{N}$ with its characteristic function on $\mathbb{N}\times\mathbb{N}$.

$\underline{LO} = \{ f \in 2^{\mathbb{N}\times\mathbb{N}} : \{(n,m) \mid f(n,m)=1\}$ defines a linear ordering on $\mathbb{N}\}$.

Closed subset of $2^{\mathbb{N}\times\mathbb{N}}$
$\therefore$ Polish.

Eg.2. Group structures on $\mathbb{N}$ – and lets just agree that 1 will always be the identity.
Identify Γ with $\{(n,m,\ell) : n \cdot_\Gamma m = \ell\}$.
Then:
$Grp = \{ f \in 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} : \{(n,m) \mapsto \ell$ iff $f(n,m,\ell)=1\}$ defines a group structure on $\mathbb{N}$ with identity $=1\}$

is a G_δ subset of $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$
$\therefore$ Polish.

So: Now we can make it more precise:
Given E on X, is $E \leq_B \cong_{LO}$?
i.e. is there Borel $\varphi : X \to LO$
s.t. $x_1 E x_2$ iff $\varphi(x_1), \varphi(x_2)$ describe $\cong$ linear orders.

For the sake of completeness, here is the general definition.

Def$^{\underline{n}}$ A ((countable, relational) language $\mathcal{L}$ consists of a sequence of symbols $R_1, R_2, \ldots$
and associated arities $n_1, n_2, \ldots$

$\mathcal{M}$ is an element of $\mathrm{Mod}(\mathcal{L})$
if assigns to each R_i a subset of $\mathbb{N}^{n_i}$

Thus: $\mathrm{Mod}(\mathcal{L})$ can be identified with

$$\prod \mathcal{P}(\mathbb{N}^{n_i}) \sim \prod 2^{(\mathbb{N}^{n_i})}$$

ordered sequences of length n_i

$$2^{(\mathbb{N}^{n_i})} = \{0,1\}^{(\mathbb{N}^{n_i})}$$

Polish in the product topology

$\therefore$ $\mathrm{Mod}(\mathcal{L})$ is Polish in product topology.

S_∞

S_∞ = all permutations of $\mathbb{N}$ (full support possible!)

$\mathcal{M}_1 \cong \mathcal{M}_2$ if there exists $\sigma \in S_\infty$ s.t.
$\forall i \; \forall a_1, \ldots a_\ell \in \mathbb{N}$

$$(a_1, a_2, \ldots) \in R_i^{\mathcal{M}_1} \text{ iff } (\sigma(a_1), \ldots) \in R_i^{\mathcal{M}_2}$$

Def$^{\underline{n}}$. An equivalence relation E on Polish X admits <u>classification</u> by <u>countable structures</u>

if there is a Borel function $f: X \to \mathrm{Mod}(\mathcal{L})$
with $x_1 E x_2$ iff $f(x_1) \cong f(x_2)$.

Fact: There is no Borel F s.t. for any Borel E we have exactly one of:

(I) E classifiable by countable structures;

(II) $F \leq_B E$.

Rmks: (i) In other words, no analog of Glimm-Effros (Harrington-Kechris-Louveau) for classification by countable structures.

(ii) $F \leq_B E$ means: There is a Borel function f s.t.

$$x_1 F x_2 \text{ iff } f(x_1) E f(x_2).$$

(iii) Much (much, much) worse is true. Ilijas Farah showed there is not even a countable sequence of Borel equivalence relations,

$$F_1, F_2, \ldots$$

s.t. either (for any Borel E)

(I) E classifiable by countable structures,

or (II) for some i

$$F_i \leq_B E.$$

§3. Dynamical characterizations.

Defn Let G be a Polish group acting on Polish X continuously.
The action is *generically ergodic* if

(i) every orbit is dense
(ii) every orbit is meager.

Rmk: Polish groups appear all over the place:
U_∞ = unitary operators on ℓ^2
M_∞ = measure preserving transformations of $([0,1], \text{Lebesgue})$
S_∞ = permutations of $\mathbb{N}$

"Most" real life examples of equivalence relations seem to arise from the continuous action of a Polish group on a Polish space.

Defn. G acting on X. Use E_G for orbit equivalence relation.

Theorem: Let G act continuously on X, G, X Polish.
Assume E_G Borel.
Then exactly one of: (I) E_G smooth;
(II) there is a generically ergodic Polish G-space Y and continuous G-embedding
$\pi : Y \to X$

Defn Let G be a Polish group acting continuously on Polish $\mathbb{X}$.

The action is __turbulent__ if

(i) every orbit dense

(ii) every orbit meager

(iii) given $V \subseteq G$ an open nbhd. of 1_G

$\mathcal{U} \subseteq \mathbb{X}$ an open nbhd. of some $x \in \mathbb{X}$

and given $y \in \mathbb{X}$,

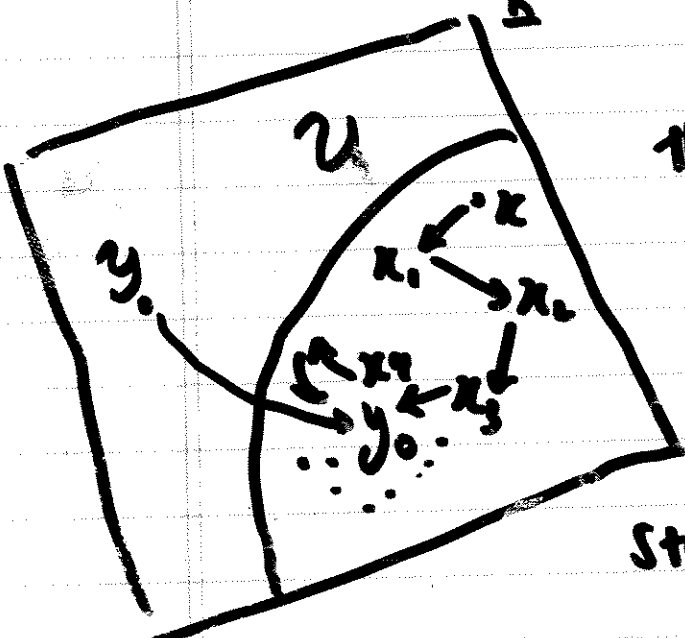
there exists $y_0 \, E_G \, y$ and $(g_i)_{i \in \mathbb{N}}$ in V

s.t. if we let $x_0 = x$

$x_1 = g_1 \cdot x_0$

$x_{i+1} = g_{i+1} \cdot x_i$

then $y_0 \in \overline{\{x_i \mid i \in \mathbb{N}\}}$



__In other words__:

We can approach a representation of $[y]_G$ using small jumps (in V!) staying inside $\mathcal{U}$.

__Thm__ Let G be a Polish group acting continuously on a Polish space $\mathbb{X}$.

Suppose E_G Borel.

Then exactly one of:

(I) E_G classifiable by countable structures

(II) there is a turbulent Polish G-space $\mathbb{Y}$ and continuous G-embedding $\pi: \mathbb{Y} \to \mathbb{X}$.

§4. Application.

1. (Kechris-Sofronidis):

Unitary operators on ℓ^2.

Equivalence relation of conjugacy.

Not classifiable by countable structures.

Equivalently: $M([0,1])$, Borel probability measures on $[0,1]$, considered up to absolute continuity (having the same null sets)

not classifiable by countable structures.

2. If $\mathbb{R}^{\mathbb{N}}$ (more generally: any countable product of lcsc groups) acts continuously on a Polish space X, then

$E_{\mathbb{R}^{\mathbb{N}}}$ classifiable by countable structures.

3. Measure preserving transformations of $([0,1], \lambda)$ considered up to conjugacy

not classifiable by countable structures.

(special cases are: Discrete spectrum.)

4. $\mathrm{Hom}([0,1]^2)$ considered up to conjugacy

not classifiable by countable structures.

5. Γ countable, not abelian by finite.

Irreducible representations of Γ on ℓ^2 considered up to unitary equivalence not classifiable by countable structures