CORRESPONDENCES

(preliminary version)

by

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INTRODUCTION

The first operator algebra analogue of the rigidity phenomena in representation of groups and ergodic theory was discoverd by Connes in [ ] : He showed that the type II_1 factor $M=L(G)$ of a discrete group G with property T of Kazhdan has discrete automorphism group Aut M/Int M. As a consequence these factors were proved to have countable fundamental group $\mathcal{F}(M)$ ([ ]), a fact that may be viewed as a typical and specific rigidity result for operator algebra. Then in [ ] Connes defined the property T for arbitrary type II_1 factors, in a way that makes equivalent the property T of a group G and of its von Neumann algebra L(G). To do this Connes considered the concept of correspondences between von Neumann algebras: these are normal M-N Hilbert bimodules over pairs of von Neumann algebras or, equivalently, representations of the algebraic tensor product $M \otimes N$, normal on $M \otimes 1$ and $1 \otimes N$. If regarded as representations correspondences get suitable notions of equivalence, topology etc. In turn, from the theory of bimodules one get operations: of composition (or tensor product) and of inducing from smaller algebras to larger ones, both important in any representation theory. Yet, A. Connes pointed out a third and a fourth point of view of looking to these objects: he noted that a correspondence may be regarded as a *-isomorphism of N into some amplification of M and, also, that the Stinespring dilation associates in a natural way a correspondence to a normal completely positive map and vice-versa.

Since this notion is so rich in interpretations and embrasses so many mathematical aspects it is natural to expect that it may be used to get some new insight in von Neumann algebras, especially in the study of type II_1 factors. And indeed Connes' correspondences provided operator algebras with the right setting for studying II_1 factors with property T and for proving rigidity results about them in[ ],[ ].

The main purpose of this paper is to continue this study and to prove more rigidity results on factors of type II_1. The same way property T can be thought of as characterizing the opposite extreme from amenability in the case of groups and ergodic theory, it has all reasons to be so considered for operator algebras too. It is therefore natural to try to study amenability phenomena in operator algebra from the correspondences point of view and this is another purpose of our paper.

Let's now explain in more detail the content of this paper.

To develop the two mentioned directions (amenability and rigidity) we need the technical background on correspondences, all due to Connes, but most of it unpublished. So we begin with an expository part intended to fill in this gap. We mention that, aside few exceptions, we work only with finite von Neumann algebras. In this first part of the paper, besides the basic definitions, operations and properties, we also introduce some new notions and prove new technical results from which we mention here a

necessary and sufficient condition, in terms of correspondences, for two subalgebras to be inner conjugate.

In the chapter on amenability (Ch. 3) we first discuss Connes' classical results on the injective II_1 factor by using correspondences. Then we introduce a notion of amenability of an algebra M relative to a subalgebra $B\subset M$: it means the existence of a certain Følner type condition of the algebra M relative to its subalgebra $B\subset M$. We call this an amenable inclusion. We prove several equivalent descriptions of it and we give some sufficient conditions for an inclusion to be amenable. For instance we show that $B\subset M$ is amenable iff any normal derivation of M into a dual Banach M-bimodule X* which vanishes an B must be inner. A typical example of amenable inclusion $B\subset M$ is when M is the crossed product of B by the (B-cocycle) action of an amenable group.

The last chapter deals with rigidity. In the first part we define the relative property T for a II_1 factorM with respect to a subalgebra of it $B\subset M$. The same notion is independently considered by Anantharam-Delaroche in[ ] . In the case B=C this notion coincides with Connes' property T for M while in the case $B=A\subset M$ is a Cartan subalgebra it is equivalent with Zimmer's property T of the corresponding measured equivalence relation. We call such an inclusion $B\subset M$ rigid and if M itself has the property T then we call M a rigid factor. Then we prove some basic technical properties of rigid inclusions and describe how they behave to certain natural operations such as tensor products, crossed products, basic construction etc. (some of these results are independently obtained in[ ]). Section 4.2 contains the main technical result of this chapter (4.2.1): it shows that if N is a rigid subfactor of a separable type II_1 factor M and if φ is a normal completely positive map from N into some finite algebra M_o, close on a certain finite subset of elements in N to a *-morphism $\rho: N \to M_o$ then ψ and ρ are uniformly close. Section 4.3 contains a discussion of Connes-Jones result that rigid II_1 factors cannot be embedded in the algebra $L(\mathbb{F}_2)$ of the free group $\mathbb{F}_2$. In section 4.4 we prove a theorem that generalizes the rigidity results of Connes [ ] and respectively Zimmer [ ]: if $B \subset M$ is a rigid inclusion and $B \subset M_n \subset M$ is an increasing sequence of von Neumann algebras generating M then, from a certain n_o, the sequence M_n must be stable (in a certain sense). Connes' theorem is when M itself is rigid (i.e. $B=\mathbb{C}$) and was checked independently by Bion-Nadal in [ ]. Zimmer's result is when B=A is a Cartan subalgebra (cf. [ ]). Then, also in section 4.4, we prove a technical result showing that if two rigid subfactors of a type II_1 factor are close on a certain finite set of elements then they are "almost" inner conjugate (4.4.3).

In the last sections we prove the main rigidity results. We show that the set of rigid subfactors of a type II_1 factor is poor. Then we show that the presence of a rigid subfactor with small relative commutant (e.g. finite dimensional) in a separable type II_1 factor M already determines certain rigidity properties of that factor: M must have countable fundamental group $\mathcal{F}(M)$ and countable set of indices of subfactors $\mathcal{J}(M)$. Finally we consider a new type of rigidity result, not considered until now in operator algebra or in ergodic theory. Namely we compare the restrictions of a measured equivalence relation which contains a free ergodic action of a discrete group with property T and show that most of them are not orbit equivalent. Translated into operator algebra terms and combined with the construction of [ ] this statement

shows the existence of a separable type II_1 factor with uncountable many nonconjugate Cartan subalgebras.

The main results of the paper have been announced at the XI^{th} Conference in Operator Theory, 2-12 June 1986 and in a note circulated as INCREST preprint.

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CH.1 DEFINITIONS AND BASIC PROPERTIES

§1.1 Definition of correspondences

1.1.1. FIRST DEFINITION. Let N and M be von Neumann algebras. A <u>correspondence between</u> N <u>and</u> M (or <u>from</u> N <u>to</u> M) is an N-M Hilbert w*-bimodule $\mathcal{H}$, i.e. $\mathcal{H}$ is a Hilbert space and there are separately weakly continuous bilinear product maps $N \times \mathcal{H} \ni (y,\xi) \mapsto y\xi \in \mathcal{H}$, $\mathcal{H} \times M \ni (\xi,x) \mapsto \xi x \in \mathcal{H}$ such that $1_N\xi=\xi 1_M=\xi$, $y_1(y_2\xi)=(y_1y_2)\xi$, $(\xi x_1)x_2=\xi(x_1x_2)$ and $(y\xi)x=y(\xi x)$, for all $y,y_1,y_2\in N$, $x,x_1,x_2\in M$.

In particular the product maps are norm continuous and in fact they give rise to mutually commuting normal unital *-representations of N and M° (the opposite of M) on the Hilbert space $\mathcal{H}$. So, alternatively, a correspondence may be regarded as follows

1.1.2. SECOND DEFINITION. A <u>correspondence between</u> N <u>and</u> M is a pair of mutually commuting, normal, unital, *-representations π_N and π_{M° of N and respectively M° on the same Hilbert space $\mathcal{H}$.

1.1.3. THIRD DEFINITION. A <u>correspondence between</u> N <u>and</u> M is a unital *-representation of the algebraic tensor product $N \otimes M^\circ$, which is normal when restricted to both $N=N\otimes 1$ and $M^\circ = 1 \otimes M^\circ$.

There is yet another equivalent way of defining correspondences. This forth point of view will be helpful in several technical situations, for enalrging our intuition and also for justifying certain notions and statements in this paper.

1.1.4. FORTH DEFINITION. Suppose M is a factor. A correspondence from N to M is a normal, unital $*$-morphism ρ of N into an amplification of M.

Before any other comment on this last definition, let us recall what the amplification M_α of a factor M is: if M is a properly infinite factor and α is a cardinal then M_α is the factor $M\otimes\mathcal{B}(\mathcal{K})$ where $\mathcal{K}$ is a Hilbert space of dimension α (in particular if α is at most countable then $M_\alpha\cong M$); if M is a type II_1 factor then either $\alpha\in(0,\infty)$ or α is an infinite cardinal; if α is infinite then $M_\alpha=M\otimes\mathcal{B}(\mathcal{K})$ as before; if α is finite and $n\geq\alpha$ is an integer then $M_\alpha=p(M\otimes\mathcal{B}(\mathcal{K}))p$, where $\dim\mathcal{K}=n$ and the normalized trace of p is α/n; if M is a factor of type I_n then α is necessarily a cardinal (finite or infinite) and $M_\alpha=M\otimes\mathcal{B}(\mathcal{K})$ with $\alpha=\dim\mathcal{K}$.

To see that 1.1.4 defines the same objects as the preceding definitions note that if M is a properly infinite factor and π_N, π_{M° are as in 1.1.2 then by Tomita-Takesaki theory $\pi_{M^\circ}(M^\circ)'$ is isomorphic to an amplification of M, say M_α, and via this isomorphism π_N becomes a normal unital $*$-morphism of N into M_α. Conversely given a normal unital $*$-morphism ρ of N into M_α, by Tomita-Takesaki theory M_α may be represented on a Hilber space so that its commutant be isomorphic to M° and this gives π_N and π_M°. If M is a finite factor then the same arguments work, by the classical results of Murray and von Neumann (see e.g.[ ] Chap.7).

The Hilbert bimodule (1.1.1) associated to a $*$-morphism ρ (1.1.4) will be denoted bu $L(\rho)$.

Definitions 1.1.1 and 1.1.4 taken together show that the

name "correspondence" is better suited to call our objects than the more neutral name of "bimodule". Indeed, the presence of a $*$-morphism from N into an amplified M_α of M seem to relate the two algebras in some way and to reveal some conections between N and M.

1.1.5. EQUIVALENCE CLASSES AND Corr (N,M). Two correspondences $\mathcal{K}$, $\mathcal{K}'$ between N and M are equivalent if they are equivalent as N-M bimodules, or, in terms of 1.1.3, if the corresponding representations of $N\otimes M^\circ$ are (unitarily) equivalent. If we regard a correspondence as a $*$-morphism ρ (1.1.4) then this equivalence translates as follows: $\rho:N\to M_\alpha$, $\rho':N\to M_{\alpha'}$ are equivalent iff $\alpha=\alpha'$ and there is a unitary element u in M_α such that $\rho'=(\mathrm{Ad}u)\circ\rho$. We leave it as an exercise to check this observation.

We denote by $\sim$ the equivalence and by $\hat{\mathcal{K}}$ the class of $\mathcal{K}$ under this equivalence relation and by Corr (N,M) the set of all classes of correspondences; if no confusion is possible we'll sometimes use the same notation $\mathcal{K}$ for a correspondence and its class.

1.1.6. SUBCORRESPONDENCES AND SUBEQUIVALENCE. Given a correspondence $\mathcal{K}$ between N and M, a subcorrespondence $\mathcal{K}_o$ of $\mathcal{K}$ is a Hilbert subspace of $\mathcal{K}$, stable for the actions of N and M ($N\mathcal{K}_oM\subset\mathcal{K}_o$). In other words a subcorrespondence is a subrepresentation (of $N\otimes M^\circ$).

A correspondence $\mathcal{K}'$ between N and M is subequivalent to $\mathcal{K}$, or is contained in $\mathcal{K}$, if it is equivalent to a subcorrespondence of $\mathcal{K}$. We write this $\mathcal{K}'\leq\mathcal{K}$ (or $\hat{\mathcal{K}}'\leq\hat{\mathcal{K}}$).

The following two exemples of correspondences will play an important role in the sequel:

1.1.7. THE TRIVIAL AND THE COARSE CORRESPONDENCES. Let M be a von Neumann algebra and let $L^2(M)$ be the Hilbert space of the standard representation of M. By Tomita-Takesaki theory M acts on $L^2(M)$ by left and right multiplication. $L^2(M)$ with this bimodule structure is called <u>the identity correspondence</u> of M and it is unique up to equivalence. <u>The coarse correspondence</u> between N and M is the Hilbert space $\mathcal{H}_{co}=L^2(N)\ \bar{\otimes}\ L^2(M)$ with bimodule structure given by the left action of N on the first Hilbert space $(L^2(N))$ and by the right action of M on the second $(L^2(M))$. Equivalently $\mathcal{H}_{co}$ may be regarded as the Hilbert space of Hilbert-Schmidt operators from $L^2(N)$ into $L^2(M)$, with its obvious N-M bimodule structure.

The identity correspondence will play here the role of the trivial representation for groups while the coarse correspondence will be the analogue of the regular representation for groups.

We note here the trivial but important fact that if M is finite and countable decomposable then a correspondence $\mathcal{H}$ of M contains the identity correspondence $\mathcal{H}_{id}$ if and only if there exists a separating central vector for M in $\mathcal{H}$, i.e. $\xi\in\mathcal{H}$ with $x\xi=0$ iff $x=0$ and $x\xi=\xi x$ for all $x\in M$. We call such a vector a <u>central vector</u> for M.

§1.2 The correspondence $\mathcal{H}_\phi$ associated to a completely positive map ϕ

As we mentioned before, definition 1.1.4 suggests that the existence of a correspondence between two von Neumann algebras already relates them in some way. This is not necessarily the

case: a coarse correspondence exists between any two von Neumann algebras and it doesn't relate them at all. This is because the corresponding $*$-morphism of N into $M_\infty=M\ \bar{\otimes}\ \mathcal{B}(\mathcal{H})$ sends N into the amplification part $\mathcal{B}(\mathcal{H})$ of M_∞. In fact as we'll see in this section, to give a correspondence between two von Neumann algebras is the same as to give a normal completely positive map between them.

For the sake of technical simplicity, at this point we make the assumption that the von Neumann algebras N and M are finite and countable decomposable and from now on we'll only consider correspondences between such von Neumann algebras. Moreover we fix on N and M normal finite faithful traces τ_N and respectively τ_M, with $\tau_N(1)=1=\tau_M(1)$.

1.2.1. STINESPRING DILATION AND CONSTRUCTION OF $\mathcal{H}_\phi$. Let $\phi:N\to M$ be a normal completely positive map. In this section we show how one associates to ϕ a correspondence $\mathcal{H}_\phi$ and vice-versa.

Define on the linear space $\mathcal{H}_o=N\otimes M$ the sesquilinear form $\langle y_1\otimes x_1, y_2\otimes x_2\rangle_\phi=\tau_M(\phi(y_2^*y_1)x_1x_2^*)$, $y_1,y_2\in N$, $x_1,x_2\in M$. Let $\mathcal{H}_\phi$ be the completion of $\mathcal{H}_o/\sim$ ($\sim$ is here the equivalence modulo the null space of $\langle,\rangle_\phi$ in $\mathcal{H}_o$). Since ϕ is completely positive, if

$p=\sum_{k=1}^{n} y_k\otimes x_k$ for some $y_i\in N$, $x_j\in M$, then $N\ni y\mapsto \sum_{i,j}\tau(\phi(y_j^*yy_i)x_ix_j^*)$

is a positive normal functional on N of norm $\langle p,p\rangle_\phi$. Similarly, $M^o\ni x^o\to\sum_{i,j}\tau(\phi(y_j^*y_i)x_ixx_j^*)$ is normal, positive, of norm $\langle p,p\rangle_\phi$. Moreover N and M act on $\mathcal{H}_o$ on the left and respectively on the right by $ypx=y(\sum y_k\otimes x_k)x=\sum yy_k\otimes x_kx$. These two actions commute and by the above remarks we have $\langle yp,yp\rangle_\phi=\langle y^*yP,p\rangle_\phi\le\|y^*y\|\langle p,p\rangle_\phi=\|y\|^2\langle p,p\rangle$ and $\langle px,px\rangle_\phi\le\|x\|^2\langle p,p\rangle$ for $y\in N$, $x\in M$. Thus the above actions of N and M on $\mathcal{H}_0$ pass to $\mathcal{H}_0/\sim$ and then extend to commuting actions on $\mathcal{H}_\phi$. By the normality of the forms $y \mapsto \langle yp, p\rangle_\phi$, $x \mapsto \langle px, p\rangle_\phi$ the product actions are w-continuous.

These show that $\mathcal{H}_\phi$ with the above N-M bimodule structure is a correspondence between N and M. Moreover ϕ can be recuperated from $\mathcal{H}_\phi$ as follows: let $T: L^2(M,\tau_M) \to \mathcal{H}_\phi$, $T(x)=\hat{1} \otimes x$. Then $\langle T^*\pi_N(y)Tx_1, x_2\rangle_{\tau_M} = \langle \pi_N(y)(1 \otimes x_1), 1 \otimes x_2\rangle_p = \tau_M(\phi(y)x_1x_2^*) = \langle \phi(y)x_1, x_2\rangle_{\tau_M}$ so that $\phi(y)=T^*\pi_N(y)T$, where π_N is the representation of N on $\mathcal{H}_\phi$ associated to the left action of N on $\mathcal{H}_\phi$.

More generally, given a correspondence $\mathcal{H}$ between N and M, let $\xi \in \mathcal{H}$, $||\xi||=1$, be such that $\langle \xi xx^*, \xi\rangle \leq c\tau_M(xx^*)$ for some $c>0$. Let $T: L^2(M,\tau_M) \to \mathcal{H}$, $T(x)=\xi x$. Then T is a bounded operator that satisfies $\langle T^*\pi_N(y)Tx_1x, x_2\rangle_{\tau_M} = \langle \pi_N(y)(\xi x_1 x), \xi x_2\rangle = \langle \pi_N(y)\xi x_1, \xi x_2 x^*\rangle = \langle Jx^*J(T^*\pi_N(y)T)x_1, x_2\rangle_{\tau_M}$, which shows that $\phi(y)=T^*\pi_N(y)T$ commutes with the right multiplication on $L^2(M,\tau)$ with elements of M. In other words $\phi(y) \in (M')'=M$ and thus ϕ is a normal completely positive map from N into M. Now, if we denoty by $\mathcal{H}_0=\overline{\text{span}}\ N\xi M$ the subcorrespondence of $\mathcal{H}$ generated by ξ, then $U: \mathcal{H}_\phi \to \mathcal{H}_0$, $U(y \otimes x)=y\xi x$ satisfies

$$||\sum y_j \otimes x_j||_\phi^2 = \sum_{i,j} \tau(\phi(y_j^*y_i)x_ix_j^*) = \sum_{i,j} \tau((T^*\pi_N(y_j^*y_i)T)x_ix_j^*) =$$

$$= \sum_{i,j} \langle \pi_N(y_j^*y_i)Tx_i, Tx_j\rangle = \sum_{i,j} y_j^*y_i\xi x_i, \xi x_j\rangle =$$

$$= ||\sum y_j\xi x_j||^2 = ||U(\sum y_j \otimes x_j)||^2 .$$

This shows that $\mathcal{H}_\phi$ is equivalent to $\mathcal{H}_0$.

Note that any vector $\eta \in \mathcal{H}$ is a norm limit of "bounded" vectors $\xi \in \mathcal{H}$, i.e. so that $\langle \xi xx^*, \xi\rangle \leq c\tau_M(xx^*)$ for some $c>0$, as considered before. Hence it follows by a maximality argument that any

correspondence $\mathcal{H}$ is a direct sum of cyclic correspondences associated to completely positive maps as above. In fact we have the following more general useful observation:

1.2.2. LEMMA. If $\mathcal{H}$ is a correspondence between N and M then let $\mathcal{H}_0^o = \{\xi \in \mathcal{H} \mid \exists c>0$ such that $N \ni y \longmapsto \langle y\xi, \xi\rangle$ is majorised by $c\tau_N$ and $M \ni x \longmapsto \langle \xi x, \xi\rangle$ is majorised by $c\tau_M\}$. Then $\mathcal{H}_0^o$ is a dense vector subspace in $\mathcal{H}$ and $N\mathcal{H}_0^o M \subset \mathcal{H}_0^o$.

Proof. It is clear that $\mathcal{H}_0^o$ is a vector space. If $\xi \in \mathcal{H}_0^o$ and $x_0 \in M$ then $\langle y^*y\xi x_0, \xi x_0\rangle = \|y\xi x_0\|^2 \leq \|x_0\|^2 \|y\xi\|^2 = \|x_0\|^2 \langle y^*y\xi, \xi\rangle \leq \leq c\|x_0\|^2 \tau_N(y^*y)$ for some $c>0$. Also $\langle \xi x_0xx^*, \xi x_0\rangle \leq c'\tau_M(x_0xx^*x_0^*) = c'\tau_M(x^*x_0^*x_0x) \leq c'\|x_0\|^2\tau_M(x^*x)$, for some $c'>0$. This shows that $\mathcal{H}_0^o M \subset \mathcal{H}_0^o$. Similary $N\mathcal{H}_0^o \subset \mathcal{H}_0^o$. Mow if $\xi \in \mathcal{H}$ let $Y_0 \in L^1(N,\tau_N)_+$, $X_0 \in L^1(M,\tau_M)_+$ be so that $\langle y\xi, \xi\rangle = \tau_N(yY_0)$, $\langle \xi x, \xi\rangle = \tau_M(X_0x)$. Then X_0, Y_0 may be regarded as positive summable operators affiliated with M respectively N. Using the spectral decompositions of X_0 and Y_0 we can find increasing sequences of projections $e_n \in M$, ~~$f_n \in N$ with~~ $f_n Y_0 \in N$, $X_0 e_n \in$ $e_n \uparrow 1$, $f_n \uparrow 1$. Then $f_n\xi e_n \to \xi$ and it is easy to see that $f_n\xi e_n \in \mathcal{H}_0^o$ for each n.

Q.E.D.

1.2.3. EXAMPLES: The case ϕ is a *-morphism. If $\phi: N \to M$ is a trace preserving *-morphism of N into M then $\mathcal{H}_\phi = \phi(1)L^2(M,\tau)$ with the bimodule structure given by $y\cdot\xi\cdot x = \phi(y)\xi x$. Indeed the map $\phi(1)L^2(M,\tau) \ni \phi(1)x \to 1 \otimes x_{\,\theta}$ is a well defined isometry, satisfying $\|\sum_i \phi(y_i)x_i\|_2^2 = \|\sum_i y_i \otimes x_i\|_\theta^2$ which shows that it gives a bimodule equivalence. In particular this shows that if we assume the *-morphism ϕ is unital and denote it by ρ then with the notation of 1.1.4 we have $\mathcal{H}_\rho = L(\rho)$. Moreoever if $\phi = \theta$ is an automorphism of M then the class $\mathcal{H}_\theta$ corresponds to the class of θ in Aut M/int M.

Note that if $\phi = id_M$ then $\mathcal{H}_{id}$ is just the identity correspondence of M.

1.2.4. EXAMPLES: The case ϕ is a conditional expectation. If $B \subset M$ is a von Neumann subalgebra and E_B is the trace preserving conditional expectation of M onto B then the associated correspondence between M and B is the M-B bimodule $L^2(M,\tau)$, with M acting by left multiplication and B by right multiplication.

If $E_B: M \to B \subset M$ is regarded as a completely positive map from M to M then we denote the associated correspondence of M by $\mathcal{H}_B$. The bimodule structure can be described explicitely as follows: let M be represented by its left action on $L^2(M,\tau)$, let y be the canonical conjugation on $L^2(M,\tau)$; denote by $M_1 = JB'J$ and by e_B the orthogonal projection of $L^2(M,\tau)$ onto $L^2(B,\tau)$ (regarded as the closure of B as a vector subspace of $L^2(M,\tau)$). It is well known that e_B is the extension of E_B to $L^2(M,\tau)$. Then $M_1 = (M \cup (e_B))''$ and in fact, because $e_B x e_B = E_B(x)e_B$, we have $M_1 = \overline{\text{span}}\{\sum_{i=1}^n x_i e_B y_i \mid n \geq 1,\ x_i, y_i \in M\}$. Moreover,

since B is finite, M_1 is semifinite and, e_B having central support 1, there is a unique normal semifinite faithful trace Tr on M_1 so that $Tr(xe_B) = \tau(x)$. The details of this construction may be found in[ ] and in[ ]. In the terminology of [ ], M_1 is called the extension of M by B and the above construction, the basic construction for the inclusion $B \subset M$. Now, let $L^2(M_1,Tr)$ be the Hilbert space of the GNS construction for Tr, in other words $L^2(M_1,Tr)$ is the Hilbert space of the standard representation of (M_1,Tr). By restriction to M it becomes an M-M bimodule. This is $\mathcal{H}_B$. Indeed, the asignement $\sum y_i \otimes x_i \leftrightarrow \sum y_i e_B x_i$ clearly defines an equivalence of M-M bimodules between $\mathcal{H}_\phi$ and $L^2(M_1,Tr)$.

The above interpretation of $\mathcal{H}_B$ as a Hilbert algebra will be very helpful for us.

Related to suequivalence of correspondences of the form $\mathcal{H}_B$ let's note here the following facts.

1.2.5. PROPOSITION. (i) Let $N \subset M$ be type II_1 factors. Then $\underbrace{N' \cap M}_{\text{finite dimensional}}$ and $\mathcal{H}_N \supset \mathcal{H}_{id}$ implies $[M:N] < \infty$. If $[M:N] < \infty$ then $\mathcal{H}_N \supset \mathcal{H}_{id}$.

(ii) If $B \subset M$ are arbitrary finite von Neumann algebras then $\mathcal{H}_B \subset \mathcal{H}_{co}$ iff B is atomic.

Proof. (i) Suppose $[M:N] < \infty$ and let $(m_j)_j$ be an "orthonormal basis" of M over N as in[ ]. Then $0 \neq \xi = \sum m_j e_N m_j^* \in \mathcal{H}_N$ is central for M and thus $\mathcal{H}_{id} \subset \mathcal{H}_N$. Conversely if for some $\xi \in \mathcal{H}_N$ we have $x\xi = \xi x$, $x \in M$ then if we interpret $\xi \in \mathcal{H}_N = L^2(M_1,Tr)$ as a square summable operator affiliated with M_1 (the extension of M by N) it follows that x commutes with $|\xi|$ and with its spectral decomposition. Thus there exists a $\overset{\text{nonzero}}{\text{finite}}$ projection in M_1 that commutes with M. But since $\underbrace{N' \cap M}_{\text{is finite dimensional}}$, $\underbrace{M' \cap M_1}_{\text{is also finite dimensional}}$ and M_1 being properly infinite this is a contraction (by [ ]).

(ii) Suppose B is an atomic algebra. Let e_i be the atoms of $\mathcal{Z}(B)$ and $\{e^i_{kl}\}_{1\le k,l\le n_i}$ the matrix units of Be_i. Let $\xi=\sum_{i,k,l}\alpha^i_{kl}e^i_{kl}\otimes e^i_{kl}$, where $\alpha^i_{kl}=\quad ||e^i_{kl}||_2^{-1}$. Since $\sum_{i,k,l}(\alpha^i_{kl})^2=1$, $\xi\in\mathcal{H}_B$. Let $u,v\in M$ be unitary elements.

$$\mathrm{Tr}((ve^i_{kl}\otimes e^i_{kl})(ue^j_{pr}\otimes e^j_{pr}))=$$

$$=\delta_{pr}\delta_{ij}\delta_{lr}\tau(e^i_{lk}u\quad)\tau(\quad ve^i_{kl}).$$

Thus:

$$\mathrm{Tr}(\xi^*u\xi v)=$$

$$=\sum_{i,k,p}\tau(e^i_{kk})\frac{\tau(ue^i_{pk})\tau(ve^i_{kp})}{\tau(e^i_{kk})^2}$$

But $E_B(u)=\sum_{i,k,l}\frac{\tau(ue^i_{kl})}{\tau(e^i_{kk})}e^i_{lk}$, so that $\tau(E_B(u)v)=\sum_{i,k,l}\frac{\tau(ue^i_{kl})\ (ve^i_{lk})}{\tau(e^i_{kk})}$

which shows that $\mathrm{Tr}(\xi^*u\xi v)=\tau(E_B(u)v)$ and thus proves that $\mathcal{H}_B=\overline{M\xi M}\subset\mathcal{H}_{co}$.

To prove the converse implication, suppose on the contrary that B has a completely nonatomic part, say Be_o, for some $0\neq e_o\in\mathcal{Z}(B)$, and that $\mathcal{H}_B\subset\mathcal{H}_{co}$. It follows that there exists a Hilbert-Schmidt operator, say T_o, on $L^2(M,\tau)$, such that $bT_o=T_ob$ for all $b\in B\subset\mathcal{B}(L^2(M,\tau))$ and $e_oT_o\neq 0$. But a completely nonatomic von Neumann algebra cannot commute with a nonzero compact operator. This contradiction completes the proof.

Q.E.D.

1.2.6. LEMMA. Let M be a finite factor and $M_o\subset M$ a matrix subalgebra of M, $1_{M_o}=1_M$. Let $\Phi:M\to M$ be a normal completely positive map such that $\Phi=E_{M_o}\circ\Phi\circ E_{M_o}$. Then $\mathcal{H}_\Phi\subset\mathcal{H}_{co}$.

Proof. Let $\Phi_o:M_o\to M_o$, $\Phi_o=E_{M_o}\circ\Phi|_{M_o}$. If ξ_o is the cyclic unit vector of the coarce correspondence of M_o, $\mathcal{H}^o_{co}$, then $T(y\xi_ox)=y\xi_{\Phi_o}x$ defines a unique bounded operator from the finite dimensional Hilbert space $\mathcal{H}^o_{co}$ onto $\mathcal{H}_{\Phi_o}$, which is clearly an M_o-bimodule morphism. Taking the partial isometry in the polar decomposition of T instead of T, we may suppose $T\,T^*=\mathrm{id}_{\mathcal{H}_{\Phi_o}}$. Thus $T^*\mathcal{H}_{\Phi_o}\subset\mathcal{H}^o_{co}$ is a subcorrespondence of $\mathcal{H}^o_{co}$ equivalent to $\mathcal{H}_{\Phi_o}$. We now take $M_1=M_o'\cap M$ and note that the coarce correspondence of M satisfies $\mathcal{H}_{co}=\mathcal{H}^1_{co}\overline{\otimes}\mathcal{H}^o_{co}$, where $\mathcal{H}^1_{co}$ is the coarse correspondence of $M_o'\cap M$, and that $\mathcal{H}_\Phi=\mathcal{H}^1_{co}\overline{\otimes}\mathcal{H}_{\Phi_o}$. Thus, since $\mathcal{H}_{\Phi_o}\subset\mathcal{H}^o_{co}$, $\mathcal{H}_\Phi\subset\mathcal{H}_{co}$.

Q.E.D.

As concerning equivalence of correspondences of the form $\mathcal{H}_B$, it is trivial to observe that if $B_o, B \subset M$ are inner conjugate, i.e. there exists a unitary element $u \in M$ with $uB_o u^* = B$, then $\mathcal{H}_{B_o} \sim \mathcal{H}_B$. It seems that the converse implication is true in the most interesting situations. We can prove it for regular subalgebras and subfactors of finite index.

1.2.7. THEOREM. Let M be a finite factor, $B_o, B \subset M$ von Neumann subalgebras of M. If $\mathcal{H}_{B_o} \sim \mathcal{H}_B$. then in each of the following situations B_o and B are inner conjugate in M:

(i) B_o, B are Cartan subalgebras of M;

(ii) B_o, B are subfactors with $B_o' \cap M = \mathbb{C}$, $B' \cap M = \mathbb{C}$ and $\mathcal{N}(B_o)'' = M$, $\mathcal{N}(B)'' = M$;

(iii) B_o, B are subfactors of finite index in M.

Proof. We give separate proofs for each situation. The common property of these three cases that will help us in the proofs is the existence of nice orthonormal basis of M with respect to B (cf.[ ], [ ] respectively [ ]).

§1.3 Operations with correspondences

Until now we considered more or less explicitly two trivial operations with correspondences: restriction (if $\mathcal{H}$ is a correspondence between N and M and $N_o \subset N$, $M_o \subset M$ are von Neumann subalgebras then the restriction of $\mathcal{H}$ to N_o-M_o is just $\mathcal{H}$ with its $N_o(\subset N)$-$M_o(\subset M)$ bimodule structure) and direct sum. We now consider some other important operations.

1.3.1. COMPOSITION (OR TENSOR PRODUCT). Let $\mathcal{H}$ be a correspondence between N and P and $\mathcal{K}$ a correspondence between P and M. We define the composition correspondence $\mathcal{H} \circ \mathcal{K}$ (or the tensor product correspondence $\mathcal{H} \otimes_P \mathcal{K}$) as follows:

Let $\mathcal{K}_o = \{\eta \in \mathcal{K} \mid P \ni z \to \langle z\eta, \eta\rangle$ is majorised by $c\tau_P$ for some $c > 0\}$. Note that $\overline{\mathcal{K}_o} = \mathcal{K}$. Define on $\mathcal{H} \otimes \mathcal{K}_o$ a sesquilinear form by $\langle \xi \otimes \eta, \xi' \otimes \eta'\rangle = \langle \xi p, \xi'\rangle$ where $p \in P$ is the Radon-Nykodim derivative of the normal form $P \ni z \to \langle z\eta, \eta'\rangle$ with respect to the trace τ_P. Then the Hilbert space $\mathcal{H} \circ \mathcal{K}$ is the completion of $\mathcal{H} \otimes \mathcal{K}_o/\sim$ and the N-M bimodule structure is given by $y(\xi \otimes \eta)x = y\xi \otimes \eta x$.

Indeed, the proof of the positivity of the above defined sesquiliniar form is the same as in[ ],[ ]. For convenience we sketch here the proof in the case P acts in standard form on $\mathcal{K}$ (by left multiplication). If $\eta_1, \dots, \eta_n \in \mathcal{K}_o$, $\xi_1, \dots, \xi_n \in \mathcal{H}$ then let p_{ij} be the Radon-Nykodym derivative of $P \ni z \mapsto \langle z\eta_i, \eta_j\rangle$. We have $\langle \sum_i \xi_i \otimes \eta_i, \sum_j \xi_j \otimes \eta_j\rangle = \sum_{i,j} \langle \xi_i p_{ij}, \xi_j\rangle$. But $\mathcal{K} = L^2(P, \tau_P)$, $\mathcal{K}_o = P$ so that there are some elements $z_i \in P$, $z_i = \eta_i$, such that $\tau(z p_{ij}) = \tau(z z_i z_j^*)$, thus $p_{ij} = z_i z_j^*$ and $\sum_{i,j} \langle \xi_i p_{ij}, \xi_j\rangle = \sum_{i,j} \langle \xi_i z_i, \xi_j z_j\rangle = \| \xi_i z_i \|_2^2 \geq 0$. It is clear by the above computations that left multiplication by elements in N and right multiplication by elements in M preserve the null space of the above positive sesquilinear form on $\mathcal{X} \otimes \mathcal{K}_o$ and that they implement unital *-morphisms of N respectively M on $\mathcal{X} \otimes \mathcal{K}_o/\sim$. Moreover, since $N \ni y \longmapsto \sum_{i,j} \langle y\xi_i z_i, \xi_j z_j \rangle$ and $M \ni x \longmapsto \sum_{i,j} \langle \xi_i (z_i x), \xi_j z_j \rangle$ are normal these representations are normal and they clearly commute. We used here the fact that since P acts standardly on $\mathcal{K}$, M may be considered as a subalgebra of the right action of P on $\mathcal{K} = L^2(P, \tau_p)$ and the multiplication $z_i x$ has this meaning.

It is easy to see that we could have defined $\mathcal{X} \underset{P}{\otimes} \mathcal{K}$ starting from $\mathcal{X}_o = \{\xi \in \mathcal{X} \mid P \ni z \longmapsto \langle \xi z, \xi \rangle$ is majorised by $c\tau_p$ for some $c > 0\}$ and by letting $\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle q\eta, \eta' \rangle$ where $q \in P$ is the Radon--Nykodim derivative of $P \ni z \longmapsto \langle \xi z, \xi' \rangle$ with respect to τ_p. That indeed this definition coincides with the first one is equivalent to the fact that $\tau_p(qp) = \tau_p(pq)$ for all $p, q \in P$.

The next two results are trivial consequences of the definitions:

1.3.2. PROPOSITION. The composition of correspondences is associative.

1.3.3. LEMMA. If $\mathcal{X}_o \subset \mathcal{X}$ are correspondences from N to P and $\mathcal{K}_o \subset \mathcal{K}$ correspondences from P to M then $\mathcal{X}_o \circ \mathcal{K}_o \subset \mathcal{X} \circ \mathcal{K}$ as correspondences from N to M.

As we have seen (1.1.4 and 1.1.5) correspondences may be viewed as *-morphisms. It is desirable to see what composition

means in this context. It is what we expect to be:

1.3.4. PROPOSITION. Let N, P, M be finite countable decomposable von Neumann algebras (as usual) and assume P and M are factors. Let $\rho: N \to P_\alpha$, $\pi: P \to M_\beta$ unital normal *-morphisms of N into an amplification P_α of P and respectively of P into an amplification M_β of M. Denote by π_α the α amplification of π as a unital *-morphism of P_α into $(M_\beta)_\alpha$ (which is uniquely defined up to inner perturbations). Then $L(\rho) \circ L(\pi) = L(\pi_\alpha \circ \rho)$.

Proof. Assume for simplicity that $\alpha = \beta = 1$ (the proof of the general case is the same but the formalism is unsignificantly more complicated). Then $L(\rho) = L^2(P, \tau_p)$ with bimodule structure $y \cdot \xi \cdot p = \rho(y)\xi p$ and $L(\pi) = L^2(M, \tau_M)$ with $p \cdot \eta \cdot x = \pi(p)\eta x$. Define a linear map from the algebraic tensor product $P \otimes M$ into $L^2(M, \tau)$ by $p \otimes x \to \pi(p)x$, (we regard P as a vector subspace of $L^2(P, \tau)$ and M as a vector subspace of $L^2(M, \tau)$). It is easily seen that via this map the sesquilinear form defined at 1.3.1 on $P \otimes M (\subset L^2(P) \bar{\otimes} L^2(M))$ transforms in the usual scalar product form on $L^2(M)$. Moreover we have $y_o(p \otimes x)x_o = \rho(y_o)p \otimes xx_o \to \pi(\rho(y_o)p)xx_o = (\pi \circ \rho)(y_o)(\pi(p)x)x_o$ wich is the bimodule structure of $L(\pi \circ \rho)$. Q.E.D.

1.3.5. INDUCED CORRESPONDENCES. A very important operation in various representation theories (e.g. for groups) is that of inducing from "smaller" objects to larger ones. We also have such a concept here equaly important for this theory.

Let $N_o \subset N$, $M_o \subset M$ be finite countable decomposable von Neumann algebras and $\mathcal{X}_o$ a correspondence form N_o to M_o. Then <u>the correspondence induced by</u> $\mathcal{X}_o$ <u>from the pair</u> N_o, M_o <u>to the pair</u> N, M is by definition the N-M correspondence $L^2(N) \underset{N_o}{\otimes} \mathcal{X}_o \underset{M_o}{\otimes} L^2(M)$, where $L^2(N)$ is regarded here as a left N module and right N_o module and $L^2(M)$ as a left M_o module and right M module. We denote this correspondence by ${}^N\mathcal{X}_o^M$.

The next properties are trivial consequences of the definition:

1.3.6. PROPOSITION. (i) If $N_0 \subset N_1 \subset N$, $M_0 \subset M_1 \subset M$ and $\mathcal{H}$ is a correspondence between N_0 and M_0 then ${}^{N}\mathcal{H}^{M} = {}^{N}({}^{N_1}\mathcal{H}^{M_1})^{M}$.

(ii) If $B \subset M$ then $\mathcal{H}_B = {}^{M}(L^2(B))^{M}$.

1.3.7. THE ADJOINT CORRESPONDENCE. Let $\mathcal{H}$ be a correspondence from N to M. Let $\bar{\mathcal{H}}$ be the conjugate Hilbert space of $\mathcal{H}$, i.e. $\bar{\mathcal{H}} = \mathcal{H}$ as a set, the sum of vectors in $\bar{\mathcal{H}}$ is the same as in $\mathcal{H}$ but $\lambda \cdot \xi = \bar{\lambda}\xi$ and $\langle \xi, \eta \rangle_{\bar{\mathcal{H}}} = \langle \eta, \xi \rangle_{\mathcal{H}}$. We denote by $\bar{\xi}$ the vector ξ as an element of the Hilbert space $\bar{\mathcal{H}}$. Define on $\bar{\mathcal{H}}$ an M-N bimodule structure given by $x\bar{\xi}y = \overline{y^*\xi x^*}$. It is easy to verify that $\bar{\mathcal{H}}$ thus defined is a correspondence from M to N. We call it the <u>adjoint of the correspondence</u> $\mathcal{H}$.

The following proposition relates this operation (of adjointness) with the preceding ones. The next proposition will show that $\overline{\mathcal{H}_\phi} = \mathcal{H}_{\phi^*}$ whenever this makes sense.

1.3.8. PROPOSITION. (i) $\bar{\bar{\mathcal{H}}} = \mathcal{H}$.

(ii) $\overline{\mathcal{H} \circ \mathcal{K}} = \bar{\mathcal{K}} \circ \bar{\mathcal{H}}$.

(iii) $({}^{N}\mathcal{H}^{M})^{-} = {}^{M}\bar{\mathcal{H}}^{N}$.

Proof. (i), (ii) follow directly by the definitions and remarks in 1.3.1. Then by (ii) we have $(L^2(N) \underset{N_0}{\otimes} \mathcal{H} \underset{M_0}{\otimes} L^2(M))^{-} =$

$= L^2(M) \underset{M_0}{\otimes} \bar{\mathcal{H}} \underset{N_0}{\otimes} L^2(N)$ which proves (iii). Q.E.D.

1.3.9. PROPOSITION. Let $\phi: N \to M$ be a normal completely positive map. Consider N and M as vector subspaces of $N_* = L^1(N, \tau_N)$ and respectively $M_* = L^1(M, \tau_M)$ in the obvious way and denote $\phi^*: M \to N_*$ where $\phi^*(x)$ is the Radon-Nykodim derivative of the normal form

$N \ni y \to \tau(\phi(y)x)$ with respect to τ_N.

(i) If there exists $c>0$ such that $\tau_M \circ \phi \le c\tau_N$ then $\phi^*(M) \subset N$ and $\phi^*: M \to N$ is completely positive and normal. Moreover, in this case, as applications from N to M and respectively M to N, ϕ and ϕ^* can be extended to bounded operators from $L^2(N)$ to $L^2(M)$ and respectively from $L^2(M)$ to $L^2(N)$, also denoted by ϕ and ϕ^*, so that ϕ^* is the adjoint operator (in the usual sense) of ϕ.

(ii) Under the hypothesis and with the notations of (i) we have $(\mathcal{H}_\phi)^{-} = \mathcal{H}_{\phi^*}$.

Proof. Since $\phi^*(1)$ satisfies $\tau_N(y\phi^*(1)) = \tau_M(\phi(y)1) = \tau_M(\phi y))$, by the hypothesis of (i) it follows that $\phi^*(1) \in N$ and thus by the obvious positivity of ϕ^*, $\phi^*(M) \subset N$. To see that ϕ^* is completely positive note that $\sum_{i,j} \tau(\phi^*(x_i^* x_j) y_j^* y_i) = \sum_{i,j} \tau(x_i^* x_j \phi(y_j^* y_i)) \ge 0$.

Now we have by Kadison's inequality $||\phi(y)||_2^2 = \tau_M(\phi(y^*)\phi(y))$ "Φ" $\le \tau_M(\phi(y^*y)) \le c\tau_N(y^*y) = c||y||_2^2$ which completes the proof of (i).

(ii) The identification of the bimodules $\mathcal{H}_{\phi^*}$ and $\overline{\mathcal{H}_\phi}$ is given by $\mathcal{H}_{\phi^*} \supset M \otimes N \ni x \otimes y \mapsto (y^* \otimes x^*)^{-} \in (N \otimes M)^{-} \subset \bar{\mathcal{H}}_\phi$. Q.E.D.

1.3.10. OTHER OPERATIONS. If $\mathcal{H}_1 \in \mathrm{Corr}(N_1, M_1)$, $\mathcal{H}_2 \in \mathrm{Corr}(N_2, M_2)$ then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is in an obvious way a correspondence between $N_1 \bar{\otimes} N_2$ and $M_1 \bar{\otimes} M_2$. We call this the trivial tensor product of the correspondences $\mathcal{H}_1$, $\mathcal{H}_2$.

If N and M are factors and $\mathcal{H} \in \mathrm{Corr}(N,M)$, $0<\alpha,\beta<\infty$ then let $n \in \mathbb{N}$, $n>\alpha,\beta$ and $L^2(M_n, Tr) \otimes \mathcal{H}$ as a correspondence between $M_n \otimes N$ and $M_n \otimes M$ with M_n acting standardly on $L^2(M_n, Tr)$. Let $f \in M_n \otimes M$, $e \in M_n \otimes N$ be projections with $\tau(f)=\beta/n$, $\tau(e)=\alpha/n$. Then we denote by $\mathcal{H}_\alpha^\beta = f(L^2(M_n) \otimes \mathcal{H})e$ with its obvious bimodule structure of $f(M_n \otimes N)f$ on the left and $e(M_n \otimes M)e$ on the right. It is easily seen that the class of $\mathcal{H}_\alpha^\beta$ depends only on α, β and not on the choice of e and f. We call $\mathcal{H}_\alpha^\beta$ the $\alpha-\beta$ amplification of $\mathcal{H}$. If $\alpha=\beta$ we denote $\mathcal{H}_\alpha^\alpha = \mathcal{H}_\alpha$.

§1.4. Index of correspondences and stable equivalence of factors

1.4.1. THE INDEX OF $\mathcal{H}$. Suppose N and M are finite factors and let $\mathcal{H}$ be a correspondence from N to M. Then we define the index of $\mathcal{H}$ to be the number $\dim_N \mathcal{H} \cdot \dim_{M^\circ} \mathcal{H}$ and denote it $\dim_{N,M} \mathcal{H}$ or simply $\dim \mathcal{H}$ if no confusion is possible (here $\dim_N \mathcal{H}$, $\dim_{M^\circ} \mathcal{H}$ are the coupling constants of N and respectively M° in their representation on $\mathcal{H}$). Note that if the correspondence is given as in 1.1.4 by a *-morphisms ρ of N into an amplification M_α of M with $\alpha \in (0,\infty]$ then the index of $L(\rho)$ is infinite if $\alpha=\infty$ and is equal to Jones' index $[M_\alpha : \rho(N)]$ if $\alpha<\infty$ (i.e. when M_α is finite). Thus it follows by Jones' results in [ ] that $\dim_{N,M} \mathcal{H}$ can only take the values $\{4\cos^2\frac{\pi}{n},\ n \geq 3\} \cup [4,\infty]$.

1.4.2. PROPOSITION. (i) If $\mathcal{H}_0 \subset \mathcal{H}$, $\dim_{N,M} \mathcal{H}_0 \leq \dim_{N,M} \mathcal{H}$.

(ii) $\dim_{N,M} \mathcal{H} = \dim_{M,N} \bar{\mathcal{H}}$.

(iii) $\dim_{N,M}(\mathcal{H} \circ \mathcal{K}) = (\dim_{N,P} \mathcal{H})(\dim_{P,M} \mathcal{K})$, where $\mathcal{H} \in \mathrm{Corr}(N,P)$, $\mathcal{K} \in \mathrm{Corr}$

(iv) If $N_0 \subset N$, $M_0 \subset M$ are subfactors and $\mathcal{H}_0$ is a correspondence between N_0 and M_0 then $\dim_{N,M}({}^N\mathcal{H}_0^M) = [N:N_0][M:M_0] \dim_{N_0,M_0} \mathcal{H}_0$. If $\mathcal{H}$ is a correspondence between N and M and we regard it, by restriction, as a correspondence between N_0 and M_0 then $\dim_{N_0,M_0} \mathcal{H} = [M:M_0][N:N_0] \dim_{N,M} \mathcal{H}$.

Proof. (i) is trivial.

(ii) follows by 1.3.4 and the properties of the index of subfactors in [ ].

(iii) is a consequence of (ii). Q.E.D.

1.4.3. STABLE EQUIVALENCE. Let N and M be finite factors. Then N and M are stable equivalent, $N \sim M$, if there exists a correspondence of index one between them. This is the same as to say that there exists an isomorphism of N onto same finite amplification M_α of M (with $0<\alpha<\infty$), or equivalently, an isomorphism between fNf and eMe for some projections $e \in M$, $f \in N$.

Then we say that N and M are w-stable equivalent and write $N \underset{w}{\sim} M$ if there exists a correspondence of finite index between them. These are clearly equivalence relations.

Since stable equivalence cannot distinguish between the amplifications of a factor and since by [ ] there exist II_1 factors with small fundamental group, it follows that this equivlaence relation is strictly weaker than the usual isomorphism of factors. Moreover, since by [ ] there exists a II_1 factor M with a period 2 automorphism σ so that M is not isomorphic to $M \rtimes \sigma$ and so that $M \cong M \otimes R$, it follows that w-stable equivalence is strictly weaker than stable equivalence of factors.

We note that by 1.4.1 to show that two factors N and M are w-stable equivalent amounts to find an embedding with finite index of N into some finite amplification of M or vice-versa, an embedding with finite index of M is some amplification of N.

In connection with a well known classical problem in type II_1 factors asking whether $L(\mathbb{F}_n)$ are mutually isomorphic or not (cf. [ ], [ ], [ ]), let us mention here the following exemple of w-stable equivalent II_1 factors.

1.4.4. PROPOSITION. If $\mathbb{F}_n$ is the free group on n generators then $L(\mathbb{F}_n)$ are mutually w-stable equivalent for all $2 \leq n \leq \infty$.

Proof. It is sufficient to embed $L(\mathbb{F}_\infty)$ into $L(\mathbb{F}_n)$ with finite index, for each $n \geq 2$. Let $s: \mathbb{F}_n \to \mathbb{Z}/2\mathbb{Z}$ be a surjective group morphism. Then ker s is easily seen to be isomorphic to $\mathbb{F}_\infty$ (see e.g. [ ]) and thus we have $\mathbb{F}_\infty \subset \mathbb{F}_n$ with $[\mathbb{F}_n : \mathbb{F}_\infty] = 2$. Thus $[L(\mathbb{F}_n) : L(\mathbb{F}_\infty)] = 2$.
Q.E.D.

Note in connection with the preceding proposition that we can go further and define the modulus of w-stable equivalence to be the infimum $\dim_{N,M} \mathcal{H}$ over all correspondences $\mathcal{H}$ between N and M. It would be very interesting to show that this infimum is in fact a minimum. Since if $\dim_{N,M} \mathcal{H} < 4$ implies $\dim_{N,M} \mathcal{H} \in \{4\cos^2 \frac{\pi}{n} \mid n \geq 3\}$ this is indeed the case if we can find a bimodule of such small index. By (the proof of) 1.4.4 (and by [ ]) it follows that either the modulus of stable equivalence between $L(\mathbb{F}_n)$ and $L(\mathbb{F}_\infty)$ is 2 or $L(\mathbb{F}_n)$ is isomorphic to some finite amplification of $L(\mathbb{F}_\infty)$ (i.e. $L(\mathbb{F}_n) \sim L(\mathbb{F}_\infty)$)

1.4.5. INDEX AND ENTROPY OF ϕ. Given a normal completely positive map ϕ we call the number $\dim \mathcal{H}_\phi$ the index of the completely positive map ϕ.

Moreover, in the same line, we call the relative entropy of the associated embedding $\rho(N) \subset M$ (cf. 1.1.4) as defined in [ ], the entropy of ϕ. (To use the term entropy in this second definition is in fact quite improprious since in case $\phi = \theta \in \mathrm{Aut}\, M$ it doesn't coincide with the usual notion of entropy [ ]). There are at least two aspects to be considered about this index : to use it in the study of completely positive maps and to use completely positive maps (of finite index) in the study of subfactors of finite index of a given factor.

One can easily formulate a lot of problems about this notion. The reader may do this by himself and we mention here is one problem that however seems of most interest: find necessary and/or sufficient conditions for ϕ to have finite index, or even more, to have a certain number as index. We make here a guess on this problem: if $\phi: M \to M$ are such that $\phi(x) \geq cx$ for some $c>0$ and any $x \geq 0$ then the index of ϕ is finite.

1.4.6. TYPES OF CORRESPONDENCES. As we have seen, a correspondence $\mathcal{H}$ between N and M is in fact a unital *-representation π of $N \otimes M^\circ$ on $\mathcal{H}$. So we may speak about the type of this correspondence as being the type of π. Thus $\mathcal{H}$ can be irreducible, factorial, of type I, II, III etc. Such considerations seem of interest if we want to find out how much the two algebras N and M are related. For instance if there exists an irreducible correspondence between them this may be an indication that in some sense N and M are close one to the other. Note however that this is the same as having an embedding of N in some M_α so that $N' \cap M_\alpha = \mathbb{C}$. It then follows by [ ] that there exists irreducible correspondneces between the hyperfinite II_1 factor R and any other separable II_1 factor. Note also problem in [ ] .

§1.5 Comments

1.5.1. The notions, terminology, properties and results presented in §1.1, and §1.3 are due to A. Connes ([ ],[ ]).

Moreover the construction of a correspondence from a normal completely positive map and, vice-versa, of a completely positive map from a correspondence are also due to Connes (i.e. §1.2.1).

Another notion of index for a correspondence of the form $\mathcal{H}_N$ for $N \subset M$ has been considered in [ ]. That index of $\mathcal{H}_N$ coincides with the square root of our index of $\mathcal{H}_N$.

In the case $\rho: M \to M$ is a unital *-isomorphism our index of $\mathcal{H}_\rho = L(\rho)$ coincides with the index of ρ as defined in [ ].

1.5.2. People which are familiar with Hilbert C*-bimodules will note that the construction of the composition product of correspondences follows step by step the construction of the tensor product in that theory (see e.g. [ ],[ ],[ ]), since in fact on $\mathcal{H}_o^o$ we do have an N-M Hilbert C*-bimodule structure, as it explicitely appears in the proofs of §1.3. However in the theory of von Neumann algebras the completion of $\mathcal{H}_o^o$ relative to its Hilbert norm (i.e. $\mathcal{H}$ itself) will play a crucial role and the existence of $\mathcal{H}_o^o \subset \mathcal{H}$ will be carried in mind only for technical reasons.

1.5.3. The possibility for a normal completely positive map Φ to be "included" in $\mathcal{H}_{co}$ when Φ have certain nice properties (e.g. Φ with finite range or, more generally, a Hilbert-Schmidt operator on $L^2(M,\tau)$) was suggested to us by Connes' unpublished work on correspondences.

CH.2 TOPOLOGY ON CORRESPONDENCES

§2.1 The definitions

Like for groups and algebra representations one can define a suitable topology on Corr(N,M): it is given by the topology of the corresponding representations of $N \otimes M^o$ as in [ ] Ch. (cf. [ ],[ ],[ ]). To describe this topology we define its neighbourhoods.

2.1.1. FIRST DEFINITION. Let $\hat{\mathcal{H}}_o \in \mathrm{Corr}(N,M)$, $\epsilon>0$, and $F \subset N$, $E \subset M$, $S=\{\xi_1,\dots,\xi_p\} \subset \mathcal{H}_o$ some finite sets of elements. We denote by $U(\hat{\mathcal{H}}_o;F;E;S) \subset \mathrm{Corr}(N,M)$ to be the set of classes of correspondences $\hat{\mathcal{H}}$ such that there exists $\{\eta_1,\eta_2,\dots,\eta_n\} \subset \mathcal{H}$ with $|\langle y\xi_i x,\xi_j\rangle - \langle y\eta_i x,\eta_j$ $<\epsilon$ for all $y \in F$, $x \in E$, $1 \le i,j \le p$. We consider on Corr (N,M) the topology for which these sets U are basis of neighbourhoods. Note that if we regard correspondences as *-representations of $N \otimes M^o$ then this topology is the usual topology on classes of representations of $N \otimes M^o$.

2.1.2. SECOND DEFINITION. As for representations of groups and algebras the topology on Corr(N,M) may also be characterized by strong operator convergence. As before we describe this topology by its neighborhoods. For simplicity we assume that N and M have separable preduals and only consider for them correspondences with separable infinite dimensional underlying Hilbert space. We denote $\mathrm{Corr}_o(N,M)$ the set of such classes of correspondences. Let $\hat{\mathcal{H}}_o \in \mathrm{Corr}_o(M,M)$, $\epsilon>0$, and $F \subset N$, $E \subset M$, $S \subset \mathcal{H}_o$ be finite sets of elements. We denote by $V(\hat{\mathcal{H}}_o;\epsilon;F;E;S)$ the set of all classes of correspondences $\hat{\mathcal{H}} \in \mathrm{Corr}_o(N,M)$ having the property: there exists a correspondence $\mathcal{H}$ in the class $\hat{\mathcal{H}}$ such that $\mathcal{H}$ coincides with $\mathcal{H}_o$ as a

Hilbert space and such that if $y\cdot\xi\cdot x$ denotes the bimodule structure on $\mathcal{H}$ (with $\xi\in\mathcal{H}=\mathcal{H}_o$) and $y\xi x$ the one in $\mathcal{H}_o$ then $||y\cdot\xi\cdot x-y\xi x||<\varepsilon$ for all $y\in F$, $x\in E$, $\xi\in S$.

2.1.3. PROPOSITION. The two topologies given by 2.1.1 and 2.1.2 coincide. More precisely if ε, F, E, S are as before and if we denote $F'=\{1_N\}\cup F\cup F^*F$, $E'=\{1_M\}\cup E\cup E^*E$ and $S'=S\cup\{y\xi x \mid y\in F,\ x\in E,\ \xi\in S\}$ then there exists $\varepsilon'>0$ such that $U(\hat{\mathcal{H}}_o;\varepsilon';F';E';S')\subset V(\hat{\mathcal{H}}_o;\varepsilon;F;E;S)$.

Proof. The proof is just a translation to this context of the proof of in [ ] Q.E.D.

2.1.4. REMARKS. 1°. If $\xi_o\in\mathcal{H}_o$ is a cyclic vector for $\mathcal{H}_o$, i.e. $\overline{\text{span}}\, N\xi_o M=\mathcal{H}_o$, then it is easy to see that the neighborhoods of the form $U(\hat{\mathcal{H}}_o;\varepsilon;F;E;\{\xi_o\})$ (or $V(\hat{\mathcal{H}}_o;\varepsilon;F;E;\{\xi_o\})$) give a basis of neighborhoods for the topology on Corr(N,M). Moreover F and E can be chosen finite sets in given subsets spaning linearly N and M (e.g. in the sets of unitary or selfadjoint elements).

2°. For correspondences of the form $\mathcal{H}_\phi$ we have a nice sufficient condition for convergence, as follows: let $\phi_i:N\to M$ be a net of normal completely positive maps with $\sup_i||\phi_i||<\infty$ and let $\phi:N\to M$ be also normal and completely positive. If $\phi_i(y)$ tend weakly to $\phi(y)$ for all $y\in N$ then $\mathcal{H}_{\phi_i}\to\mathcal{H}_\phi$.

3°. We have by a contravariant equivalence between $\{\mathcal{H}_\theta \mid \theta\in \text{Aut } M\}$ with composition product 1.3.1 and the above topology and Aut/Int M with its usual structure of polish group.

2.1.5. A NOTATION. Let $\mathcal{H}_o$, $\mathcal{H}$ be two correspondence between N and M. We say that $\mathcal{H}_o$ is <u>weakly subequivalent</u> to $\mathcal{H}$ (or that $\mathcal{H}_o$ is <u>weakly contained</u> in $\mathcal{H}$) if $\mathcal{H}$ is in the closure of $\mathcal{H}_\mathcal{J}$. We write this $\mathcal{H}_o\lesssim\mathcal{H}$ (or $\hat{\mathcal{H}}_o\lesssim\hat{\mathcal{H}}$). We say that $\mathcal{H}_o$ is <u>w-equivalent</u> to $\mathcal{H}$ if $\mathcal{H}_o\lesssim\mathcal{H}$, $\mathcal{H}\lesssim\mathcal{H}_o$.

§2.2. Continuity of operations

As a direct consequence of the definitions we obtain that all the operations that we introduced are continuous in the above topology. We summarize this in the next:

2.2.1. PROPOSITION. (i) The composition product $\text{Corr}(N,P)\times\text{Corr}(P,M)\ni(\mathcal{H},\mathcal{K})\to\mathcal{H}\circ\mathcal{K}\in\text{Corr}(N,M)$ is separately continuous in each variable .

(ii) The adjoint operation $\mathcal{H}\to\overline{\mathcal{H}}$ is continuous.

(iii) The restriction operation is continuous.

(iv) If $N_o\subset N$, $M_o\subset M$ then $\text{Corr}(N_o,M_o)\ni\mathcal{H}\mapsto {}^N\mathcal{H}^M$ Corr(N,M) is continuous.

Proof. (ii) and (iii) are trivial and clearly (i) => (iv). (i) follows by the definition of $\circ$ in 1.3.1 and the observation at the end of that paragraph.

Q.E.D.

Note in connection with 2.2.1 that it is easy to construct exemples showing that the composition product is not continuous simultaneously in the two variables.

We mention that the index is not a continuous function on Corr(N,M) (exercise!). This will follow implicitely from the results of Ch.3.

§2.3. Neighborhoods of $\mathcal{H}_{id}$

Let M be a finite factor and $\mathcal{H}_{id}$ its identity correspondence. For futher purposes it is important to have a better understanding of the neighborhoods of $\mathcal{H}_{id}$. A more suitable description to work with is given bellow (cf.[ ]).

Let $\varepsilon>0$ and $E\subset M$ a finite set. We denote by $W(\varepsilon,E)$ the set of classes of correspondences $\mathcal{H}$ of M such that there exists $\xi\in\mathcal{H}$, $||\xi||=1$, with $||x\xi-\xi x||<\varepsilon$ for $x\in E$.

2.3.1. LEMMA. The sets W form a basis of neighborhoods of $\mathcal{H}_{id}$.

Proof. If $E>1$ and if we denote $\xi_o \in L^2(M,\tau)$ the trace vector then we clearly have $V(\mathcal{H}_{id};\varepsilon/2;F;F;\xi_o) \subset W(\varepsilon;F)$. Let's show that given $\varepsilon>0$, $F\subset M$ there exist $\varepsilon'>0$, $F'\subset M$ such that $W(\varepsilon',F') \subset U(\mathcal{H}_{id};\varepsilon;F;F;\xi_o)$.

By Dixmier's theorem there exist unitary elements $u_1,\dots,u_m \in M$ such that $\|\frac{1}{m}\sum_i u_i^* y_1 y_2 u_i - \tau(y_1 y_2) 1_M\| < \varepsilon'$ for all $y_{1,2} \in F$. Let then $F' = F \cup F^* \cup \{u_i^* y_1 y_2 \mid y_1, y_2 \in F\} \cup \{u_i\}_i$. If $\mathcal{H} \in W(\varepsilon';F')$ then there exists $\xi \in \mathcal{H}$, $\|\xi\|=1$ such that $\|x\xi - \xi x\| < \varepsilon'$ for all $x \in F'$. It follows that if $x_1, x_2 \in F'$ then:

$$\text{(i)} \quad |\langle x_1 \xi x_2, \xi\rangle - \langle x_1 \xi_o x_2, \xi_o\rangle| = |\langle x_1 \xi x_2, \xi\rangle - \tau(x_1 x_2))| \le$$

$$\le |\langle x_1 x_2 \xi, \xi\rangle - \tau(x_1 x_2)| + \varepsilon' \|x_1\|;$$

$$\text{(ii)} \quad |\langle x_1 x_2 \xi, \xi\rangle - \langle x_2 x_1 \xi, \xi\rangle| \le |\langle x_1 x_2 \xi, \xi\rangle - \langle x_2 \xi x_1, \xi\rangle| + \varepsilon' \|x_2\| =$$

$$= |\langle x_1 x_2 \xi, \xi\rangle - \langle x_2 \xi, \xi x_1^*\rangle| + \varepsilon' \|x_2\| \le$$

$$\le |\langle x_1 x_2 \xi, \xi\rangle - \langle x_2 \xi, x_1^* \xi\rangle| + 2\varepsilon' \|x_2\| = 2\varepsilon' \|x_2\|.$$

Taking then $x_1 = u_i^* y_1 y_2 \in F'$ and $x_2 = u_i \in F'$ it follows that

$$|\langle u_i^* y_1 y_2 u_i \xi, \xi\rangle - \langle y_1 y_2 \xi, \xi\rangle| \le 2\varepsilon' \sup_{x \in F'} \|x\| = 2\varepsilon' c$$

so that $|\langle \frac{1}{m}\sum_i u_i^* y_1 y_2 u_i \xi, \xi\rangle - \langle y_1 y_2 \xi, \xi\rangle| \le 2\varepsilon' c$ and thus $|\tau(y_1 y_2) - \langle y_1 y_2 \xi, \xi\rangle| \le$ $\le 3\varepsilon' c$ which together with (i), (ii) give for all $y_1, y_2 \in F \subset F'$:

$$|\langle y_1 \xi y_2, \xi\rangle - \langle y_1 \xi_o y_2, \xi_o\rangle| \le 3\varepsilon' c \; .$$

Taking $\varepsilon' = \varepsilon/3c$ the lemma follows.

Q.E.D.

We end this section with a result showing that in some sense, the coarse correspondence is the smallest one.

2.3.2. PROPOSITION. If M is a separable type II_1 factor then $\mathcal{H}_{co} \le \mathcal{H}$ for any correspondence $\mathcal{H}$ of M.

This result is a consequence of the noncommutative ergodic phenomena specific for type II_1 factors proved in [ ] . It will follow easily by the next:

2.3.3. PROPOSITION. Let M is a separable type II_1 factor and $\varepsilon > 0$, $x_1,\dots,x_n \in M$. Then there exists a maximal abelian $*$-subalgebra A of M such that $\|E_A(x_i) - \tau(x_i) 1_M\|_2 < \varepsilon$ for all i. Moreover, there exists a nonzero projection $e \in M$ such that $\|e x_i e - \tau(x_i) e\| < \varepsilon$ for all i.

Proof. By [ ] there exists a hyperfinite type II_1 subfactor $R \subset M$ such that $R' \cap M = \mathbb{C}$. Moreover there exists in R a sequence of subfactor R_n such that $R_n' \cap R = \mathbb{C}$ and $\|E_{R_n}(x) - \tau(x) 1_M\|_2 \to 0$ for all $x \in R$. This can be easily seen using the techniques in [ ] and is also a consequence of [ ] . Now let n be sufficiently large so that $\sum_i \|E_{R_n}(x_i) - \tau(x_i) 1\|_2^2 < (\varepsilon/4)^2$. Since $R_n' \cap R = \mathbb{C}$ by [ ], there exists a finite dimensional abelian von Neumann subalgebra A_o in R_n so that $\sum_i \|E_{A_o' \cap R}(E_R(x_i)) - \tau(x_i) 1\|_2^2 < (\varepsilon/4)^2$. Moreover, by [ ] there exists a finite dimensional abelian refinement A_1 of A_o in R such that if $x_i' = x_i - E_R(x_i)$ then $\sum_i \|E_{A_1' \cap M}(x_i')\|_2^2 <$ $< (\varepsilon/2)^2$. We thus get $\sum_i \|E_{A_1' \cap M}(x_i) - \tau(x_i) 1_M\|_2^2 < 2\sum_i \|E_{A_1' \cap M}(x_i')\|_2^2 +$ $+ 2\sum_i \|E_{A_1' \cap M}(E_R(x_i)) - \tau(x_i) 1_M\|_2^2 < \varepsilon^2/4 + 2\sum_i \|E_{A_1' \cap R}(E_R(x_i)) - \tau(x_i) 1\|_2^2 < \varepsilon^2$.

Thus any maximal abelian $A \subset M$ with $A \supset A_1$ will do.

Moreover, if $e_1, \dots, e_n$ are the minimal projections of A_1 then the above inequality can be written

$$\sum_j \sum_i \| e_j x_i e_j - \tau(x_i) e_j \|_2^2 < \varepsilon^2 \sum_j \| e_j \|_2^2 .$$

Thus, for some j we have $\sum_i \| e_j x_i e_j - \tau(x_i) e_j \|_2^2 < \varepsilon^2 \| e_j \|_2^2$.

It follows by arguing as in [ ] or by 1.2.1 in [ ] that for some projection $0 \neq e \leq e_j$ we have $\| e x_i e - \tau(x_i) e \| < \delta(\varepsilon)$, with $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Q.E.D.

The proof of 2.3.2 is now quite easy. Let $\xi \in \mathcal{H}_0^0$ be as in 1.2.2 . Let $\varepsilon > 0$, $x_1, \dots, x_m \in M$. Let $0 \neq e \in M$ be a projection such that $\| e x_i e - \tau(x_i) e \| < \varepsilon$. There exist unitary elements $u, v \in M$ such that $e u^* \xi v e \neq 0$. We define $\xi_0 = e u^* \xi v e / \| e u^* \xi v e \| \in \mathcal{H}_0^0$. Then we have for $K = \max_j \| x_j \|$,

$$| \langle x_i \xi_0 x_j , \xi_0 \rangle - \tau(x_i) \tau(x_j) |$$

$$\leq K\varepsilon + | \langle \tau(x_i) \xi_0 x_j , \xi_0 \rangle - \tau(x_i) \tau(x_j) | \leq 2K\varepsilon .$$

This shows that $\mathcal{H}_{co} \subseteq \mathcal{H}$.

Q.E.D.

Let us finally note that we do not have in general $\mathcal{H}_\Phi \subseteq \mathcal{H}_{id}$ for normal completely positive maps Φ . In fact if M is a rigid factor, as it will be defined in Ch. 4, with a non inner automorphism θ then $\mathcal{H}_{\theta^{-1}} \not\subseteq \mathcal{H}_{id}$. Indeed because otherwise $\mathcal{H}_{id} = \mathcal{H}_{\theta^{-1}} \circ \mathcal{H}_\theta \subseteq \mathcal{H}_\theta$ which would imply $\mathcal{H}_{id} \subset \mathcal{H}_\theta$ contradicting the outerness of θ . However it will be shown in Ch.3 that if $M = R$

is the hyperfinite II_1 factor then the closure of any correspondence $\mathcal{H}$ equals Corr(M).

§2.4. Comments

The topology on Corr(N,M) was considered by Connes when he first defined correspondences and the property T for type II_1 factors (see §4.1). For general von Neuamnn algebras it was defined in [ ] . The lemma 2.3.1 details a remark in [ ] .

Like in the theories of group or C*-algebras representations the topology on Corr(M) will help us get some informations about the algebra M knowing certain topological properties of Corr(M). For instance in the next two chapters we will exploit the privileged position of $\mathcal{H}_{id}$ and $\mathcal{H}_{co}$ in the topological space Corr(M) to define amenability and property T for M. Following ideas from C*-algebra representation theory it seems to us that other topological properties of Corr(M) may be helpful to classification or structure properties of M.

CH.3 AMENABILITY

§3.1 Amenable factors

It is well known ([ ]) that the amenability of a discrete group G can be characterized by a condition involving only the representation theory of that group: by a result of Hulanicki G is amenable iff the trivial representation of G is weakly contained in its regular representation. For Kac algebras, which may be viewed as generalized groups (see e.g. [ ]), the amenability may also be defined starting from a similar property (cf. [ ]). For von Neumann algebras however, using other considerations as a starting point, the amenability is defined in terms of cohomological conditions (see [ ], [ ]).

It turns out that in the framework of correspondences we may consider a definition of amenability which is exactly the translation in this representation theory of the above characterisation of amenability for groups. To do this we only need to find a good analogue of the regular representation: this will be here the coarse correspondence.

3.1.1. DEFINITION. A finite von Neumann algebra M (or more generaly an arbitrary von Neumann algebra) is called <u>amenable</u> if the identity correspondence of M, $\mathcal{H}_{id}$, is weakly contained in the coarse correspondence of M, $\mathcal{H}_{co}$, i.e. if $\mathcal{H}_{id} \leq \mathcal{H}_{co}$.

It is easy to observe that if M comes from a discrete group G then M is amenable iff G is amenable. In fact we have:

3.1.2. THEOREM. A finite von Neumann algebra M is amenable iff it is injective (i.e. there exists a conditional expectation of $\mathcal{B}(L^2(M,\tau))$ onto M, [ ]).

Proof. The proof of injectivity implies amenability is just the interpretation in this context of the proof of Connes' Folner type condition for injective algebras. Indeed by his result given any finite set $F \subset M$ there exist Hilbert-Schmidt operators on $L^2(M)$ (in fact finite rank projections) η_n such that $\|x\eta_n - \eta_n x\|_{HS} \to 0$ for any $x \in F$. The operators η_n in turn are obtained by just applying Day's trick to the hypertrace on M (which is the composition of the trace with the conditional expectation of $\mathcal{B}(L^2(M,\tau))$ onto M, see [ ] for all this). Since η_n may be viewed as vectors in $\mathcal{H}_{co}$, this shows that $\mathcal{H}_{id} \leq \mathcal{H}_{co}$.

Conversely, assume M is amenable and denote by I the directed set of finite subsets of M. Let $\{\eta_i\}_{i \in I} \subset \mathcal{H}_{co} = L^2(M) \bar{\otimes} L^2(M)$ be a net of unit vectors such that $\|x\eta_i - \eta_i x\| \underset{i}{\to} 0$ for all $x \in M$. Let then $\phi(T) = \lim_I \langle T\eta_i, \eta_i \rangle$ for $T \in \mathcal{B}(L^2(M,\tau_M))$ denote a Banach limit over I (see e.g. [ ]). It follows that ϕ is a state on $\mathcal{B}(L^2(M,\tau))$ and that for any unitary element $u \in M$ we have

$$\phi(uTu^*) = \lim_I \langle uTu^*\eta_i, \eta_i \rangle = \lim_I \langle Tu^*\eta_i, u^*\eta_i \rangle =$$

$$= \lim_I \langle T\eta_i u^*, \eta_i u^* \rangle = \lim_I \langle T\eta_i, \eta_i \rangle = \phi(T) .$$

Thus ϕ is a hypertrace on $M \subset \mathcal{B}(L^2(M))$ which means that M is injective. Q.E.D.

3.1.3. THE EQUIVALENCE OF AMENABILITY AND HYPERFINITENESS. If M is hyperfinite and $B_n \subset M$ is an increasing sequence of finite dimensional *-subalgebras with $\overline{\cup B_n} = M$ then by the remark 2.2.3, 3° $\mathcal{H}_{B_n} \to \mathcal{H}_{id}$. But by 1.2.5 we have $\mathcal{H}_{B_n} \subset \mathcal{H}_{co}$. This shows that $\mathcal{H}_{id} \leq \mathcal{H}_{co}$

and thus that M is amenable. The converse implication is the hard part of Connes' celebrated theorem [ ] . One may ask whether we can get any real use of this new setting to get a simplified and more conceptual proof of this part of Connes' theorem. The proof given in [ ] may be viewed as giving a partly positive answer to this question. Indeed we used there implicitely the context of correspondence as follows: let M be amenable, then $\mathcal{H}_{id} \leq \mathcal{H}_A$ for any maximal abelian *-subalgebra. Using the bimodule structure of $L^2(M)$ over A this approximate inclusion can be translated in a number of norm two inequaliteis only involving elements in M. Using then the local Rohlin lemma of [ ] we can construct from the given elements of M a local matrix unit which "approximate" M on a small "corner" (i.e. under a projection). By a maximality argument we obtain a finite dimensional subalgebra of M with the same unit as M and "approximating" M. This proves the hyperfiniteness of M.

We now prove a general property about the correspondences of the hyperfinite type II_1 factor R.

3.1.4. PROPOSITION. Any two correspondences $\mathcal{H}$, $\mathcal{H}'$ of the hyperfinite type II_1 factor R are w-equivalent (i.e. $\mathcal{H} \leq \mathcal{H}' \leq \mathcal{H}$).

Proof. By 2.3.2 we only have to show that $\mathcal{H} \leq \mathcal{H}_{co}$ for any correspondence $\mathcal{H}$ of R. Since $\mathcal{H}_{id} \oplus \mathcal{H}_{id} \leq \mathcal{H}_{id}$ (see §3.3 below) and $\mathcal{H}_{id} \leq \mathcal{H}_{co}$ (by 3.1.3 and 2.3.2) and since any $\mathcal{H}$ is a direct sum of correspondences of the form $\mathcal{H}_\Phi$, for some normal completely positive maps Φ, it follows that it is sufficient to prove that $\mathcal{H}_\Phi \leq \mathcal{H}_{co}$. Since $R = \overline{\bigcup_n M_n}^w$ for an increasing sequence of matrix algebras M_n, by taking $\Phi_n = E_{M_n} \circ \Phi \circ E_{M_n}$ we have $\|\Phi_n(x) - \Phi(x)\|_2 \to 0$ for all $x \in R$ so that $\mathcal{H}_{\Phi_n} \to \mathcal{H}_\Phi$ and thus if is sufficient to show that $\mathcal{H}_{\Phi_n} \leq \mathcal{H}_{co}$. But this follows by 1.2.6.

Q.E.D.

Note that, Hagerup's proof [ ] to Connes' fundamental theorem depends on a careful interpratation of the fact that if M is an injective type II_1 factor then given any normal completely positive map with finite dimensional range Φ, $\mathcal{H}_\Phi \leq \mathcal{H}_{id}$.

§3.2 Relative amenability

As the preceding comments show, the condition $\mathcal{H}_{id} \leq \mathcal{H}_B$ for subalgebras $B \subset M$ seems natural to consider. A closer look to this condition will show that it implies the existence of certain amenability properties of M relative to B, so it is natural to consider the following:

3.2.1. DEFINITION. Let $B \subset M$ be finite von Neumann algebras. We say that M is <u>amenable relative to</u> B (<u>or over</u> B, or that <u>the inclusion</u> $B \subset M$ <u>is amenable</u>) if $\mathcal{H}_{id} \leq \mathcal{H}_B$.

Note that M is amenable if and only if it is amenable over $\mathbb{C}1_M$.

The same way amenability can be proved to be equivalent to other conditions (such as injectivity), we now show that the relative amenability can be characterized by the corresponding "relative type" conditions.

To state the last condition we need some preliminaries. We use all the way the notations and terminology in [ ]. So, if M is as usual a finite von Neumann algebra with a faithful normal finite trace τ_M, we put $\mathrm{Bil}_B^\sigma(M,M)$ to be the set of bounded normal bilinear forms F on M satisfying $F(xb,y)=F(x,by)$ for all $b\in B$, $x,y\in M$, with the usual Banach norm, and we let $M \otimes^\sigma_B M = \mathrm{Bil}_B^\sigma(M,M)^*$ with its obvious dual M-M bimodule structure. Also we let $\pi : R\otimes^\sigma_B M \to M$ be the w*-continuous extension of $x\otimes y \mapsto xy$ like in [ ]. Then we say that $V\in M\otimes^\sigma_B M$ is a normal virtual B-diagonal if $xV=Vx$, $x\in M$, and $\pi(V)=1$.

M be a finite factor and

3.23. THEOREM. Let $B\subset M$ be a finite von Neumann sub algebra~~s~~. The following conditions are equivalent.

(i) $B\subset M$ is amenable.

(ii) If M_1 is the extension of M by B (relative to some trace τ_M on M) then there exists a conditional expectation of M_1 onto M.

(iii) If M_1 is as in (ii) then M_1 has a state that contains M in its centralizer.

(iv) $H_B^1(M,X)=0$ for any dual M bimodule X, i.e. any w-continuous derivation $\delta : M\to X$ with $\delta_{|B}\equiv 0$ is inner.

(v) M has a normal virtual B-diagonal

Proof. (i) $\Rightarrow$ (iii). Let I be the set of finite subsets i of the unit ball of M ordered by inclusion and let $|i|$ be the number of elements in M. For each $i\in I$ let $\eta_i\in \mathcal{X}_B$, $\|\eta_i\|=1$, with $\|x_k\eta_i-\eta_i x_k\|<|i|^{-1}$ for all $k\in i$. For each $T\in M_1$ (=the extension of M by B) we put $\Phi(T)=\lim_I \langle T\eta_i,\eta_i\rangle$, where the right hand side represents a Banach limit after I (see e.g. [ ], Ch.10). Since the usual limit of $\|u\eta_i-\eta_i u\|$ is zero for all unitary elements $u\in M$, we have $\Phi(uTu^*)=\lim_I\langle uTu^*\eta_i,\eta_i\rangle=\lim_I\langle Tu^*\eta_i,u^*\eta_i\rangle=\lim_I\langle T\eta_i u^*,\eta_i u^*\rangle=\Phi(T)$. Since clearly Φ is a state on M_1 this proves (iii).

(iii) $\Rightarrow$ (i). Using Day's trick it follows the existence of a net of elements $\eta_i'\in L^1(M_1, {}_{M_1})_+$ with $\tau_{M_1}(\eta_i')=1$ for all $i\in I$ where I is as in the proof of (i) $\Rightarrow$ (iii) and τ_{M_1} is the unique normal semifinte faithful trace on M_1 as in 1.3, such that $\|u\eta_i'u^*-\eta_i'\|_1\to 0$ for all $u\in\mathcal{U}(M)$. By the Powers-Stormer inequality it follows that $\eta_i=(\eta_i')^{1/2}\in L^2(M_1,\tau_{M_1})$ satisfies $\|\eta_i\|_2=1$ and $\|u\eta_i u^*-\eta_i\|_2\to 0$ for all $u\in\mathcal{U}(M)$. This proves (i).

(ii) $\Rightarrow$ (iii). If $E:M_1\to M$ is a conditional expectation then $\tau_M\circ E$ is a state on M_1 containing M in its centralizer.

(iii) $\Rightarrow$ (ii). Let $E:M_1\to M$ be defined by $\tau(E(T)x)=\Phi(Tx)$ for $x\in M$, where Φ is a state on M_1 having M in its centralizer. (that $x\mapsto\Phi(Tx)$ is normal on M follows easily from the fact that M is a factor). Then it is easy to verify that E is a conditional expectation of M_1 onto M.

(iv) $\Rightarrow$ (iii). The proof of this implication follows step by step Connes' proof of the case $B=\mathbb{C}$ in sec. 2.3 of [ ]. So let $X=\{\varphi\in M_1^* \mid |\varphi(xTy)|\le K\|x\|_2\|T\|\,\|y\|_2$ for some $K>0$ and any $T\in M_1$, $x,y\in M$ and $\varphi(x)=0$ for all $x\in M\}$.

It is easy to check that $\varphi \in X$ and $x, y \in M$ implies $x\varphi y \in X$ and that these actions are norm continuous, so X becomes a Banach M-bimodule and in fact it is a normal dual bimodule by the same argument as in [ ]. For $x \in M$ we define $\delta(x) = \omega x - x\omega$ where $\omega(T) = \langle Te_B, e_B\rangle = \tau_{M_1}(e_B T e_B)$ for $T \in M_1$ where $e_B \in L^2(M_1, \tau_{M_1})$ is as defined in 1.3 and T is regarded as an operator acting by left multiplication on $L^2(M_1, \tau_{M_1})$. Then $\delta(x) \in X$ (and $X \neq \{0\}$), δ is a normal derivation and for any $b \in B$ we have $\delta(b) = 0$. Thus by (iv) there exists $\psi \in X$ such that $x\psi - \psi x = x\omega - \omega x$ for all $x \in M$. Since clearly $\omega \notin X$, $\varphi = \omega - \psi \neq 0$ and $x\varphi = \varphi x$ for all $x \in M$. To get a positive φ we do like in [ ].

(v) $\Rightarrow$ (iv). The proof of this implication is the same as the first part of the proof of Theorem 3.1 in [ ]: given a normal derivation $\delta : M \to X$ with $\delta(B) = 0$ in the normal dual M-bimodule X we let $F(x,y) = \delta(x)y$ which is bilinear and normal in each variable and satisfies $F(xb, y) = F(x, by)$ for $b \in B$. Then, if V is a normal virtual B-diagonal of M, we let

$$f_o = \int F(x,y)\,dV(x,y) = \int \delta(x)y\,dV(x,y) \in X$$

as in [ ], and we have for $a \in M$

$$af_o = \int a\delta(x)y\,dV(x,y) = \int \delta(ax)y\,dV(x,y) - \int \delta(a)xy\,dV(x,v) =$$

$$= \int \delta(x)ya\,dV(x,y) - \delta(a)\int xy\,dV(x,y) = f_o a - \delta(a).$$

(i) $\Rightarrow$ (V). If $B \subset M$ is amenable then for each $i \in I$ (defined as in the proof of (i) $\Rightarrow$ (iii)) we let $\eta_i \in \mathcal{H}_B$ be a scalar multiple of a finite projection in M_1 (when $\mathcal{H}_B$ is interpreted as a Hilbert subalgebra of M_1) such that $\|\eta_i\| = 1$, $\|\eta_i x_k - x_k \eta_i\| < |i|^{-1}$ for all $x_k \in i$. Then

suitable normalizations of η_i may be viewed as norm one elements φ_i in $\mathrm{Bil}_B^{\sigma}(M,M)^*$, via the inclusion $M_* \underset{B}{\otimes} M_* \subset M \underset{B}{\otimes}^{\sigma} M = \mathrm{Bil}_B^{\sigma}(M,M)^*$ (defined like in [ ]). Then the above estimates imply that $x\varphi_i - \varphi_i x$ tend σ-weakly to zero for all $x \in M$ and $\pi(\varphi_i)$ tend weakly to 1. Thus, if V is a weak limit of φ_i then $xV = Vx$ for all $x \in M$ and $\pi(V) = 1$.

Q.E.D.

The next theorem lists the main properties of the relative amenability and provides new motivations for considering this notion and for calling it like this.

3.2.4. THEOREM. 1°. If $B_o \subset B \subset M$ then $B_o \subset M$ is amenable iff both $B_o \subset B$ and $B \subset M$ are amenable.

2°. If $M = N \otimes N_o$ then $N \subset M$ is amenable iff N_o is amenable.

3°. Suppose M is a cocycle crossed product of the finite von Neumann algebra B by a cocycle action of a discrete group G, with measure preserving transformations. Then $B \subset M$ is amenable iff G is an amenable group.

4°. If $N \subset M$ are finite factors, $[M:N] < \infty$, then $N \subset M$ is amenable.

5°. If $B \subset M$ and if $M_n \uparrow M$, with $B \subset M_n$, then $B \subset M_n$ amenable for all n implies $B \subset M$ amenable.

Proof. If $B_o \subset B$ and $B \subset M$ are amenable then $L^2(B, \tau_B) \lessapprox {}^{B}L^2(B_o, \tau_{B_o})^{B}$ and $L^2(M,\tau) \lessapprox {}^{M}L^2(B, \tau_B)^{M}$ so that by 2.2 we have $L^2(M,\tau) \lessapprox {}^{M}L^2(B,\tau_B)^{M} \lessapprox {}^{M}({}^{B}L^2(B_o,\tau)^{B})^{M} = {}^{M}L^2(B_o,\tau)^{M}$ which shows that $B_o \subset M$ is amenable.

If $B_o \subset M$ is amenable then $B_o \subset B$, $B \subset M$ follow amenable by 3.2.3 (iii) respectively 3.2.3 (iv).

2°. " $\Leftarrow$ " follows by 3°, taking $N_o = L(G)$ for some infinite amenable discrete ICC group G.

2° " $\Rightarrow$ " follows by 3.2.3 (ii) and 3.1.2.

3° " $\Leftarrow$ " If $e_B \in \mathcal{H}_B$ is defined as usual and $K_n \uparrow G$ are finite Følner subsets of G then normalizations of the vectors $\eta_n = \sum_{g \in K_n} u_g e_B u_g^*$ satisfy $\| \eta_n x - x \eta_n \| \to 0$ for $x \in M$.

3° " $\Rightarrow$ " is the some as in the case $B = \mathbb{C}$ in [ ].

4°. Is clear, since $\mathcal{H}_{id} \subset \mathcal{H}_N$.

5°. We have $\mathcal{H}_{M_n} \to \mathcal{H}_{id}$ and $L^2(M_n,\tau) \lessapprox {}^{M_n}L^2(B,\tau)^{M_n}$. Thus $\mathcal{H}_{M_n} = {}^{M}L^2(M_n,\tau)^{M} \lessapprox {}^{M}({}^{M_n}L^2(B,\tau)^{M_n})^{M} = \mathcal{H}_B$.

Q.E.D.

§3.3. Asimptotic commutativity

We note in this section that the property Γ of Murray and von Neumann (see [ ]) can be characterized in terms of correspondences.

3.3.1. THEOREM. Let M be a type II_1 factor. Then $L^2(M,\tau) \oplus L^2(M,\tau) \lessapprox L^2(M,\tau)$ iff M has the property Γ.

Proof. The implication " $\Leftarrow$ " is clear. The converse implication follows by [ ].

Q.E.D.

3.3.2. PROBLEM. In [ ] it is shown that if N is a property Γ type II_1 factor and σ is a free action of $\mathbb{Z}$ on N then the corresponding crossed product $M = N \rtimes_\sigma \mathbb{Z}$ also has property Γ. By [ ] we may expect that the sâme result holds true if $\mathbb{Z}$ is replaced by an arbitrary amenable group G. Moreover in [ ] it is proved that if $N \subset M$ are type II_1 factors and $[M:N] < \infty$ then again M has property Γ. It is therefore natural to ask whether the following question has an affirmative answer:

Let $N \subset M$ be separable type II_1 factors. Suppose N has property Γ and $N \subset M$ is amenable. Does this imply that M has property Γ

§3.4. Comments

3.4.1. There are several equivalent descriptions of the amenability for type II_1 factors for which we couldn't find good analogue notions equivalent to the relative amenability. These are semi-discreteness, the isomorphism of $C^*(R, R')$ and $R \bar{\otimes} R$, innerness of the flip automorphism on $R \bar{\otimes} R$ and existence of normal finite range completely positive maps tending to the identity. We believe it would be important to find such notions. For the last of these conditions we do have a candidate as follows: We say that M is aproximately finite dimensional over $B \subset M$ if there exists a sequence of normal completely positive maps $\Phi_n : M \to M$ so that $\Phi_n(b) = b$ for $b \in B$, $\|\Phi_n(x) - x\|_2 \to 0$ and $E_B \Phi_n(x) \geq \lambda_n \Phi_n(x)$ for all $x \in M_+$ where $\lambda_n > 0$. Then the theorem would be that this condition is equivalent to the amenability of $B \subset M$. If $\Phi_n = E_{M_n}$ where $B \subset M_n \subset M$ then this condition implies that B has finite index in M_n (cf. [ ]) but, of course, the condition of the existence of such M_n's is too strong to hold true in general (e.g. when $M = B \rtimes \mathbb{Z}$ and B is a factor).

We mention that our main purpose for considering the notion of relative amenability was to provide a tool for the approach to the problem of vanishing (or nonvanishing) of the second cohomology for cocycle actions of $\mathbb{Z}^2$ on arbitrary type II_1 factors (see [ ]).

3.4.2. In [ ] Zimmer considered a notion of amenable actions of arbitrary groups. This notion has been generalized in [ ] . When the algbra B on which the group acts is finite

and the action is measure preserving the amenability of the action is equivalent to the amenability of the group ([ ]). In general if the group is discrete then there is a normal conditional expectation of $M = B \rtimes G$ onto B. Then the constructio of the M-correspondence $\mathcal{H}_B$ is the same as the one described in §1.2 so that the condition $L^2(M) \subset \mathcal{H}_N$ makes sense in this context. More generally we may consider arbitrary von Neumann algebras $B \subset M$ with the condition of the existence of a normal conditional expectation of M onto B and define the amenabilit of M relative to B by $L^2(M) \subseteq \mathcal{H}_B$. It worth verifying whether this definition coincides with the one in [ ] and [ ] for the particular case $M = B \rtimes G$.

CH.4 RIGIDITY IN TYPE II_1 FACTORS

Connes introduced the notion of correspondences to have an appropriate framework to define his property T as an intrinsic property of a von Neumann algebra. Moreover correspondences provided the natural setting to obtain rigidity results about such algebras.

In this chapter we continue the study of type II_1 factors with property T and we prove various rigidity results about arbitrary type II_1 factors.

§4.1 Definitions and basic properties

4.1.1. DEFINITION. We say that a finite factor M has <u>property</u> T (or is <u>rigid</u>) if there is a neighborhood U of the identity correspondence $\mathcal{H}_{id}$ such that any correspondence in U contains $\mathcal{H}_{id}$.

This definition is formally very similar to the definition of the property T for groups. We'll see that in fact a type II_1 factor coming from a discrete group G has property T iff the group G has it.

From the description of the neighborhoods of $\mathcal{H}_{id}$ given in 2.2 we readily get the following reformulation of the property T.

4.1.2. LEMMA. M has property T iff there exist $\varepsilon>0$, $x_1,\dots,x_n\in M$ such that if $\mathcal{H}$ is a correspondence of M with a vector

ξ, $||\xi||=1$, satisfying $||x_i\xi-\xi x_i||<\varepsilon$, $1\le i\le n$, then $\mathcal{H}$ has a nonzero central vector for M.

4.1.3. DEFINITION. Let M be a finite factor and $B\subset M$ a von Neumann subalgebra of it. We say that M has <u>property T relative to B</u> (or that <u>the inclusion $B\subset M$ is rigid</u>) if there exist $\varepsilon>0$, $x_1,\dots,x_n\in M$ such that if $\mathcal{H}$ is a correspondence of M with a central vector for B $\xi\in\mathcal{H}$, $||\xi||=1$, satisfying $||x_i\xi-\xi x_i||<\varepsilon$, $1\le i\le n$, then $\mathcal{H}$ has a nonzero central vector for M, i.e. $\mathcal{H}\supset\mathcal{H}_{id}$. We then say that $\varepsilon>0, x_1,\dots,x_n\in M$ give a <u>critical neighborhood</u> of $\mathcal{H}_{id}$.

Note that this definition is different from Moore's relative property T in [ ] . In the case B=A is a Cartan subalgebra of M then our definition agrees with Zimmer's property T of the corresponding equivalence relation. We'll discuss all this in the final section of this chapter.

We now prove some results that will justify the preceding definitions (4.1.1, 4.1.3) and will show that they are good. A basic technical device needed in what follows is that given an almost central vector for M one can find a central vector for M close to it. To prove it we use a characterization of a kind of relative property Γ in terms of the automorphism group of the factor. This is A.1 in the Appendix. Moreover we use the next theorem which is a generalization of the first rigidity result in II_1 factors ([ ]).

4.1.4. THEOREM. Let $B\subset M$ be a von Neumann subalgebra and denote $\mathrm{Aut}_B M=\{\theta\in\mathrm{Aut}\, M \mid \theta_{|B}=\mathrm{id}_B\}$ and $\mathrm{Int}_B M=\mathrm{Aut}_B M\cap \mathrm{Int}\, M$. If $B\subset M$ is rigid then $\mathrm{Int}_B M$ is open and closed in $\mathrm{Aut}_B M$.

Proof. Since $\mathrm{Aut}_B M$ is a topological group, it is sufficient to prove $\mathrm{Int}_B M$ is open in $\mathrm{Aut}_B M$. Let $\varepsilon>0$, $x_1,\dots,x_n$ M give a critical neighborhood of $\mathcal{H}_{id}$. Suppose $\theta\in\mathrm{Aut}_B M$ is so that $||\theta(x_i)-x_i||_2<\varepsilon$, $1\le i\le n$. Then the vector $\hat{1}\in\mathcal{H}_\theta$ satisfies $b\cdot 1=1\cdot b$ (because $\theta(b)=b$) for all $b\in B$ and $||x_i\cdot 1-1\cdot x_i||_2=||\theta(x_i)1-1x_i||_2=||\theta(x_i)-x_i||_2<\varepsilon$. Thus $\mathcal{H}_\theta$ enters in the critical neighborhood of $\mathcal{H}_{id}$, so that there exists $\eta\in\mathcal{H}_\theta$ $(=L^2(M,\tau)$ as a Hilbert space) with $\theta(x)\eta=x\cdot\eta=\eta\cdot x=\eta x$. Regarding η as a square sumable operator affiliated with M, it follows that the partial isometry v in the polar decomposition of η is in M and satisfies $\theta(x)v=vx$, which implies that θ is inner (see [ ] or [ ]) and thus $\theta\in\mathrm{Int}_B M$. Q.E.D.

4.1.5. LEMMA. Let $B\subset M$ be a rigid inclusion. There exist $K>0$, $\varepsilon_0>0$, $x_1,\dots,x_n\in M$ such that given any $\delta\le\varepsilon_0$ and any correspondence $\mathcal{H}$ of M with a vector $\xi\in\mathcal{H}$, $||\xi||=1$, central for B satisfying $||x_i\xi-\xi x_i||<\delta$, there exists a vector $\eta\in\mathcal{H}$ central for M with $||\eta-\xi||<K\delta$.

Proof. Since $\mathrm{Int}_B M$ is closed in $\mathrm{Aut}_B M$ it follows by A.1 that there exists a finite set $\{u_1,\dots,u_p\}$ of unitary elemetns in M and a constant $c>0$ such that $||\xi||^2\le c\sum||u_i\xi-\xi u_i||^2$ for any $\xi\in L^2(B'\cap M,\tau)$, with $<\xi,1>=0$. Indeed, because otherwise for any finite set of elements $v_1,v_2,\dots,v_q\in\mathcal{U}(M)$ there would exist a sequence $\{\xi_n\}\subset L^2(B'\cap M,\tau)$, $||\xi_n||=1$, $<\xi_n,1>=0$ and $||[\xi_n,v_i]||_2\to 0$ for all i., which by A.1 is a contradiction. Moreover the inequality $||\xi||^2\le c\sum_i||[u_i,\xi]||^2$ is true if instead of $\xi\in L^2(B'\cap M,\tau)$ we let ξ be an arbitrary B-central vector of a correspondence $\mathcal{H}$ which is a direct sum of copies of $L^2(M,\tau)$. Now let $\varepsilon>0$, $y_1,\dots,y_n\in M$ give a critical neighborhood of $\mathcal{H}_{id}$ and denote by $\{x_i\}_i=\{y_i\}_j\cup\{u_k\}_k$.

Let $\mathcal{H}$ be a correspondence with a vector ξ central for B with $||[x_i,\xi]||<\delta\le\varepsilon$. Let ξ_0 be the projection of ξ on the subcorrespondence of $\mathcal{H}$ which is a direct sum of copies of $L^2(M,\tau)$ and denote $\xi_1=\xi-\xi_0$. Let $\xi_0=\xi_0'+\xi_0''$ with ξ_0' central and ξ_0'' orthogonal to central vectors. Then $b\xi_0''=\xi_0''b$ for $b\in B$ and $\|\xi_0''\|^2\le pc\delta^2$. Indeed, because otherwise $pc\delta^2<\|\xi_0''\|^2\le c\sum_i\|u_i\xi_0''-\xi_0''u_i\|^2\le pc\delta^2$, a contradiction. Moreover we have $\|\xi_1\|\le\delta/\varepsilon$, because otherwise $\|\frac{\varepsilon}{\delta}\xi_1\|\ge 1$ and $\|x_i\xi_1-\xi_1x_i\|<\delta$, so that $\|x_i(\frac{\varepsilon}{\delta}\xi_1)-(\frac{\varepsilon}{\delta}\xi_1)x_i\|<\varepsilon$. But then, since $b\xi_1=\xi_1 b$ for all $b\in B$ it follows that $\mathcal{H}_1=\overline{\mathrm{span}}M\xi_1M$ contains a nonzero central vector, again a contradiction.

Thus ξ_0' is central for M and we have

$$\|\xi-\xi_0'\|^2=\|\xi_0''+\xi_1\|^2<\left(\left(\tfrac{1}{\varepsilon}\right)^2+pc\right)\delta^2. \qquad \text{Q.E.D.}$$

4.1.6. REMARK. Let $B\subset M$ be a rigid inclusion and $M_0\subset M$ a weakly dense *-subalgebra of M. Let $u_1,\dots,u_p$ be some unitary elements in M so that $\sum_i\|u_i\eta u_i^*-\eta\|_2^2\ge c\|\eta\|^2$ for any $\eta\in L^2(B'\cap M,\tau)$, $<\eta,1>=0$ as in A.1 (cf. 4.1.4). Then there are $y_1^0,\dots,y_m^0\in M_0$, $\|y_i^0\|\le 1$, such that $\varepsilon_0=\varepsilon^2/2^{12}pc$ and $\{x_i\}_i=\{y_j^0\}_j\cup\{u_k\}_k$ give a critical neighborhood of $L^2(M,\tau)$ (where ε is the one giving a critical neighborhood of $L^2(M,\tau)$). In other words, up to the unitaries $u_1,\dots,u_p$, the elements given the critical neighborhood of $L^2(M,\tau)$ may be chosen in M_0.

To see this, let $\mathcal{H}$ be an M-correspondence and $\xi\in\mathcal{H}$ with $\|\xi x_i-x_i\xi\|\le\varepsilon_0$. We claim that from the condition $\|\xi u_k-u_k\xi\|\le\varepsilon_0$ it follows that there is $\xi_0\in\mathcal{H}$, $\|\xi_0-\xi\|\le 2(2pc\varepsilon_0)^{1/2}\le\varepsilon/16$ such that ξ_0 is bounded on the left and right by 1, i.e. $M\ni x\longmapsto<x\xi_0,\xi_0>$ and $<\xi_0x,\xi_0>$ are majorized by τ_M. If we show this and if $\varepsilon>0$, $y_1,\dots,y_n\in M$, $\|y_i\|\le 1$ give the critical neighborhood of $L^2(M,\tau)$, then $y_1^0,\dots,y_n^0 \in M_0$ with $\|y_i^0\| \le 1$, $\|y_i^0 - y_i\|_2 \le \varepsilon/4$ will do. Indeed, we have $\|y_i\xi - \xi y_i\| \le 4\|\xi - \xi_0\| + \|(y_i^0 - y_i)\xi_0\| + \|\xi_0(y_i - y_i^0)\|_2 + \|\xi y_i^0 - y_i^0\xi\| \le \varepsilon$ which shows that if $\|u_k\xi - \xi u_k\| \le \varepsilon_0$ and $\|y_j^0\xi - \xi y_j^0\| \le \varepsilon_0$ then $\|y_i\xi - \xi y_i\| \le \varepsilon$ so that $\mathcal{H}$ has an M central vector.

Now if $\|\xi u_k - u_k\xi\| \le \varepsilon_0$ then let $X_{1,2} \in L^1(M,\tau)_+$ be the Radon--Nykodim derivatives of $\langle x\xi,\xi\rangle = \langle x\xi u_k, \xi u_k\rangle$ respectively $\langle \xi x, \xi\rangle = \langle u_k\xi x, u_k\xi\rangle$ and let $\eta_i = X_i^{1/2} \in L^2(M,\tau)$. Note that in fact $\eta_i \in L^2(B'\cap M,\tau)$ and let α_i be the orthogonal projection of η_i onto $\mathbb{C}.1 \subset L^2(B'\cap M,\tau)$. Then by A.1 and the Powers-Stormer inequality (see e.g.[ ], 10.24) we have $\|\eta_i - \alpha_i\|_2^2 \le c\sum_k \|u_k\eta_i - \eta_i u_k\|_2^2 \le c\sum_k \|u_k X_i u_k^* - X_i\|_1 = c\sum_k \|\omega_{u_k\xi, u_k\xi} - \omega_{\xi u_k, \xi u_k}\| \le 2c\sum_k \|u_k\xi - \xi u_k\| \le 2pc\varepsilon_0$. It follows that $z_i = f(\eta_i)$ where $f:[0,\infty) \to [0,1]$, $f(t) = \begin{cases} t, & \text{if } t \le 1 \\ t^{-1}, & \text{if } t > 1 \end{cases}$, satisfy $\|\eta_1 - z_1\eta_1\|_2 \le \|\eta_1 - \alpha_1\|_2 \le (2pc\varepsilon_0)^{1/2}$ and $\|\eta_2 - \eta_2 z_2\|_2 \le \|\eta_2 - \alpha_2\|_2 \le (2pc\varepsilon_0)^{1/2}$. Then an easy computation shows that $\xi_0 = z_1\xi z_2$ will satisfy the above requirements.

This remark may be of further use. In this paper we will need it only to show how property T behaves to tensor products. For this purpose note that if $M = N \bar{\otimes} N_0$ and the inclusion $N \subset M$ is nonΓ (i.e. $\mathrm{Int}_N M$ is closed in $\mathrm{Aut}_N M$, see A.1) then the unitaries u_k as above can be taken in $N_0 = 1 \otimes N_0 = N' \cap M$.

We can now prove the main properties of rigid inclusions.

4.1.7. THEOREM. Let M be a type II_1 factor.

(i) Suppose M is the cocycle crossed product of a finite von Neumann algebra B by the cocycle action of a discrete group G by measure preserving transformations. Then the inclusion $B \subset M$ is rigid iff G has the property T of Kazhdon

(ii) Let $B \subset M$ be a von Neumann subalgebra and suppose $\mathcal{N}(B)'' = M$. Suppose G is a discrete group with property T of Kazhdan and that $\pi: G \to \mathcal{N}(B)$ is a $*$-representation of G such that $(\pi(G) \cup B)'' = M$. Then $B \subset M$ is a rigid inclusion.

(iii) If $M = N \otimes N_0$ then $N \subset M$ is a rigid inclusion iff N_0 is a rigid factor. Moreover M is rigid iff both N and N_0 are rigid factors.

Proof. (i) Suppose G has property T and let $\varepsilon > 0$, $g_1,\dots,g_n$ give a critical neighborhood of the trivial representation of G. Let $\mathcal{H}$ be a correspondence of M and $\xi_0 \in \mathcal{H}$, $\|\xi_0\| = 1$, a central vector for B with $\|u_{g_i}\xi_0 - \xi_0 u_{g_i}\| < \varepsilon$, where $\{u_g\}_{g\in G} \subset M$ is the family of unitary elements implementing the given action σ of G on B with 2-cocycle $\mu: G \times G \to \mathcal{U}(B)$, i.e. $u_g b u_g^* = \sigma_g(b)$ and $u_g u_h = \mu(g,h)u_{gh}$. Let $0 \ne \mathcal{H}_0$ be the Hilbert subspace of all B-central vectors in $\mathcal{H}$ Then $\pi: G \to \mathcal{U}(\mathcal{H}_0)$, $\pi(g)\xi = u_g\xi u_g^*$, $g \in G$ is a well defined unitary representation of G on $\mathcal{H}_0$ and $\|\pi(g_i)\xi_0 - \xi_0\| < \varepsilon$. Thus $(\pi, \mathcal{H}_0)$ has a nonzero fixed vector η_0. Since η_0 is also central for B, it is central for M.

Suppose $B \subset M$ is a rigid inclusion. Let $\varepsilon > 0$, $x_1,\dots,x_n \in M$ give a critical neighborhood of $\mathcal{H}_{id}$ as in 4.1.5. We may clearly suppose $\|x_i\|_2 = 1$ for each i. If $x_i = \sum_g b_g^i u_g$ is the expresion of x_i in $M = B \rtimes_{\sigma,\mu} G$ then let $F \subset G$ be a finite set such that $\sum_{g\notin F} \|b_g^i\|_2^2 < \delta^2$

where $\delta \leq \varepsilon$ as in 4.1.5.

Let $\pi: G \to \mathcal{U}(\mathcal{H}_0)$ be a unitary representation of G and suppose $\xi_0 \in \mathcal{H}_0$ is a unit vector with $\|\pi(g)\xi_0 - \xi_0\| < \delta$ for $g \in F$. Let $\mathcal{H} = \ell^2(G, \mathcal{H}_0) = \ell^2(G) \bar{\otimes} \mathcal{H}_0$ and $\tilde{\xi}_0 \in \mathcal{H}$, $\tilde{\xi}_0(e) = \xi_0$, $\tilde{\xi}_0(g) = 0, g \neq e$. Let M act on the left by $b \cdot \tilde{\xi} = (b \otimes 1)\tilde{\xi}$, $u_g \cdot \tilde{\xi} = (u_g \otimes \pi(g))\tilde{\xi}$, for $b \in B$, $g \in G$ and on the right by $\tilde{\xi} \cdot x = \tilde{\xi}(x \otimes 1)$.

Then we have

$$\|x_i \cdot \tilde{\xi}_0 - \tilde{\xi}_0 \cdot x_i\|^2 = \|\sum_g ((b_g^i u_g \otimes \pi(g))\tilde{\xi}_0 - \tilde{\xi}_0 (b_g^i u_g \otimes 1))\|^2 =$$

$$= \|\sum_g b_g^i u_g \otimes (\pi(g)\xi_0 - \xi_0)\|_2^2 = \sum_{g \in G} \|b_g^i\|_2^2 \|\pi(g)\xi_0 - \xi_0\|^2 \leq$$

$$\leq 4\delta^2 + \sum_{g \in F} \|b_g^i\|_2^2 \|\pi(g)\xi_0 - \xi_0\|^2 \leq 5\delta^2 .$$

By 4.1.5 there is a vector $\tilde{\eta} \in \mathcal{H}$ central for M and close to $\tilde{\xi}_0$, $\|\tilde{\eta} - \tilde{\xi}\| < \sqrt{5}K\delta$. Since $\tilde{\xi}_0(e) = \xi_0$, it follows that for δ small enough, $\eta = \tilde{\eta}(e) \neq 0$. But $\eta \in \mathcal{H}_0$ and by the definition it is fixed by $\pi(g)$. Thus $\mathcal{H}_0$ contains the trivial representation of G. This shows that G has property T.

(ii) The proof is the same as the first implication of (i).

(iii) If N_0 is rigid then let $\varepsilon > 0$, $y_1, \dots, y_n \in N_0$ give a critical neighborhood of $L^2(N_0)$. Let $\mathcal{H}$ be a correspondence of $M = N \bar{\otimes} N_0$ with a vector $\xi \in \mathcal{H}$, $\|\xi_0\| = 1$, ξ_0 central for N and $\|\xi_0 y_i - y_i \xi_0\| < \varepsilon$, $\forall i$. Let $0 \neq \mathcal{H}_0 \subset \mathcal{H}$ be the set of N-central vectors of $\mathcal{H}$. Then $N_0 \mathcal{H}_0 N_0 \subset \mathcal{H}_0$ so by restriction $\mathcal{H}_0$ becomes an N_0-N_0 bimodule and since $\xi_0 \in \mathcal{H}_0$, by the definition of rigidity $\mathcal{H}_0$ has an N_0-central vector η_0. This η_0 will also be central for N and thus for M.

Conversely if $N \subset M$ is a rigid inclusion then let $\varepsilon > 0$, $x_1, \dots, x_n \in M$ give a critical neighborhood of $L^2(M)$. By 4.1.6 we

may suppose $x_i = \sum_j y_{ij} \otimes y_{ij}^0$ for some $y_{ij} \in N$, $y_{ij}^0 \in N_0$ and moreover we may take $\{y_{ij}\}_j$ to be mutually ortogonal (with respect to the trace) for each i and of norm two equal to one. Let $\mathcal{H}_0$ be a correspondence of N_0 with $\xi_0 \in \mathcal{H}_0$, $\|\xi_0\| = 1$, $\|[\xi_0, y_{ij}^0]\| < \varepsilon/K$. Let $\mathcal{H} = L^2(N) \bar{\otimes} \mathcal{H}_0$ which is an M correspondence in the obvious way and for which $\xi = 1 \otimes \xi_0$ is an N-central vector with $\|[\xi, \sum_j y_{ij} \otimes y_{ij}^0]\|^2 = \sum_j \|y_{ij}\|_2^2 \|[y_{ij}^0, \xi_0]\|_2^2 \leq \varepsilon^2$. Thus by the definition of relative rigidity $\mathcal{H}$ has a nonzero M central vector η close to ξ. Thus the projection η_0 of η onto $\mathcal{H}_0 = \hat{1} \otimes \mathcal{H}_0 \subset \mathcal{H}$ is close to ξ_0 (= the projection of ξ on $\mathcal{H}_0$). So, $\eta \neq 0$, and N_0 has property T.

The proof of the rest of (iii) is similar to the above so we omit further details. Q.E.D.

4.1.8. THEOREM. (i) If $B \subset N \subset M$ are von Neumann subalgebras with N and M factors and $B \subset N$, $N \subset M$ are rigid inclusions then $B \subset M$ is rigid. If $B \subset M$ is rigid then $N \subset M$ is rigid.

(ii) Let $B \subset N \subset M$ and suppose $[M:N] < \infty$. Then $B \subset N$ is rigid iff $B \subset M$ is rigid.

(iii) If $N_0 \subset N \subset M$ are finite factors and $[N:N_0] < \infty$ then $N_0 \subset M$ is rigid iff $N \subset M$ is rigid.

(iv) If $N \subset M$ are factors and $[M:N] < \infty$ then $N \subset M$ is rigid. If $N' \cap M$ is finite dimensional and $N \subset M$ is both rigid and amenable then $[M:N] < \infty$.

Proof. (i) Suppose $B \subset N$, $N \subset M$ are rigid inclusions and let $\mathcal{H}$ be a correspondence of M with a B-central vector ξ, $\|\xi\| = 1$, $\|[\xi, y_i]\| < \varepsilon$, $\|[\xi, x_i]\| < \varepsilon'$, where $\varepsilon > 0$, $y_1, \dots, y_n \in N$ give the critical neighborhood of $L^2(N)$ (for the rigid inclusion $B \subset N$) and $\varepsilon' > 0$, $x_1, \dots, x_m \in M$ the one of $L^2(M)$ (for $N \subset M$).

By regarding $\mathcal{H}$ as an N correspondence (by restriction) it follows that there exists in $\mathcal{H}$ an N-central vector η close to ξ. Moreover if ε is small enough we can obtain η so that $\|\xi-\eta\|<\varepsilon'(\sum\|x_i\|+1)^{-1}$. But then $\|[\eta,x_i]\|<\varepsilon'$ and since $N\subset M$ is rigid, $\mathcal{H}$ contains an M central vector.

The other affirmation in (i) is trivial.

(ii) Since $[M:N]<\infty$ by [ ] there exists an orthonormal basis of M over N, i.e. $m_o,\dots,m_n\in M$ with $E_N(m_i^*m_j)=\delta_{ij}$ for $i\neq 0$ or $j\neq 0$ and $E_N(m_o^*m_o)=f$ for some projection $f\in N$, and so that $x=\sum_j m_jE_N(m_j^*x)$ for all $x\in M$. Suppose $B\subset M$ is rigid and let $\varepsilon>0$, $x_1,\dots,x_n\in M$ give a critical neighborhood of $L^2(M)$. Let $x_im_j=\sum_k m_ky_{ij}^k$ for $y_{ij}^k=E_N(m_k^*x_im_j)$. Then $m_j^*x_i=\sum_k y_{ik}^j m_k^*$. We infer that there is a $\delta>0$ which together with $(y_{jk}^i)_{i,j,k}$ give a critical neighborhood of $L^2(N)$ thus showing that $B\subset N$ is rigid. Indeed if $\mathcal{H}_o$ is an N-correspondence with $\xi_o\in\mathcal{H}_o$, $\|\xi_o\|=1$, ξ_o central for B and $\|[y_{ij}^k,\xi_o]\|<\delta$ then let $\mathcal{H}={}^M\mathcal{H}_o^M$ be the induced of $\mathcal{H}_o$ to M, $\mathcal{H}=L^2(M)\underset{N}{\otimes}\mathcal{H}_o\underset{N}{\otimes}L^2(M)$ and denote $\xi=\sum_j m_j\otimes\xi_o\otimes m_j^*$. Then $\|\xi\|\geq 1$ and $x_i\xi-\xi x_i=\sum_{j,k}m_j\otimes([y_{jk}^i,\xi_o])\otimes m_k^*$ so that $\|[x_i,\xi]\|^2\leq$ $\leq\delta(\sum_j\|m_j\|_2^2)^2=\delta[M:N]^2$. It follows that if $\delta[M:N]^2<\varepsilon$ then $\mathcal{H}$ has an M central vector η close to ξ so that the projection η_o of η on $\mathcal{H}_o=\hat{1}\otimes\mathcal{H}_o\otimes\hat{1}$ is close to the projection ξ_o of ξ on $\mathcal{H}_o$. Thus $\mathcal{H}_o\ni\eta_o\neq 0$ and η_o is central for N.

Conversely if $B\subset N$ is rigid and $\varepsilon>0$, $y_1,\dots,y_n\in N$ give the critical neighborhood of $L^2(N)$ then let $(x_i)=(y_j)\cup(m_k)$ and $\delta=$ Suppose $\mathcal{H}$ is an M-correspondence with a B-central unit vector $\xi\in\mathcal{H}$ such that $\|[x_i,\xi]\|<\varepsilon$. Then in particular $\|[y_i,\xi]\|<\varepsilon$ so that if we regard $\mathcal{H}$ as N-correspondence (by res-

triction), it follows that $\mathcal{H}$ has an N-central vector η close to ξ. In particular $\|\eta\|\geq 1/2$ and thus $\eta'=\sum m_j\eta m_j^*$ is central for M (by the same computations as above) and we have $\|\eta'-[M:N]\eta\|=\|\sum m_j\eta m_j^*-$ $-[M:N]\eta\|=\|\sum(m_j\eta m_j^*-\eta m_jm_j^*)\|\leq\sum_j\|m_j\eta-\eta m_j\|\|m_j^*\|\leq([M:N]+1)^2\delta$. Thus if δ is small enough, since $\|\eta\|\geq 1/2$, it follows that $\eta'\neq 0$.

(iii) If $N_o\subset M$ is rigid then by (i), $N\subset M$ is rigid. Conversely let $N\subset M$ be a rigid inclusion, $(m_j)_j$ be an orthonormal basis of N over N_o and $\varepsilon\geq 0$, $y_1,\dots,y_n$ M give the critical neighborhood for the rigid inclusion $N\subset M$. Let $(x_i)=(y_j)\cup(m_k)$ and let $\mathcal{H}$ be an M-correspondence with an N_o-central vector ξ, $\|\xi\|=1$, and $\|[x_i,\xi]\|<\varepsilon$, $1\leq i\leq n$. Let $\xi'=\sum m_j\xi m_j^*$. It follows that ξ' is close to $\xi\sum m_jm_j^*=[M:N]\xi$ and that ξ' is N-central. By the rigidity of the inclusion $N\subset M$ it follows that $\mathcal{H}$ contains a nonzero central vector for M and thus $N_o\subset M$ is also rigid.

(iv) If $N\subset M$ is amenable then $\mathcal{H}_{id}\leq\mathcal{H}_N$ and if $N\subset M$ is also rigid then by definition it follows that $\mathcal{H}_{id}\subset\mathcal{H}_N$. But then 2.4 implies $[M:N]<\infty$. If $[M:N]<\infty$ then $\mathcal{H}_N$ coincides with the Hilbert space $L^2(M_1)$ where M_1 is the finite factor obtained as the extension of M by N. Thus $1\in M_1\subset L^2(M_1)=\mathcal{H}_N$ is a central vector for M so that $L^2(M)\subset\mathcal{H}_N$ which shows that $N\subset M$ is amenable. Since the inclusion $M\subset M$ is clearly rigid, by (iii) it follows that $N\subset M$ is also rigid.

Q.E.D.

Let us also note that the rigidity properties are inherited by inducing or reducing von Neumann algebras by projections.

4.1.9. THEOREM. Let B be a von Neumann subalgebra of M.

1°. If $e \in B \cup (B' \cap M)$ is a nonzero projection and $eBe \subset eMe$ is rigid then $B \subset M$ is rigid.

2°. If $e \in B$ (respectively $e \in B' \cap M$) is a nonzero projection and if we assume the normalizer of B (respectively $B' \cap M$) in M acts ergodically on the center of B (respectively the center of $B' \cap M$) then $B \subset M$ rigid implies $eBe \subset eMe$ is rigid.

Proof. 1°. Suppose $e \in B \cup (B' \cap M)$ and $eBe \subset eMe$ is rigid and let $\varepsilon > 0$, $y_0 = e$, $y_1, \dots, y_m \in eMe$, $\|y_j\| \leq 1$ give the critical neighborhood of $L^2(eMe, \tau)$. Since M is a factor there exists partial isometries $e_{1i} \in M$, $0 \leq i \leq n$, so that $e_{1i} e_{1i}^* = e$ for $1 \leq i \leq n$, $e_{10} e_{10}^* \leq e$ and $\sum_{i=0}^{n} e_{1i}^* e_{1i} = 1_M$. Let then $\varepsilon' > 0$ and $\{x_i\}_i = \{y_j\}_j \cup \{e_{1p}, e_{1p}^*\}_p$.

Assume $\mathcal{H}$ is a correspondence of M and $\xi \in \mathcal{H}$, $\|\xi\| = 1$, $\|x_i \xi - \xi x_i\| < \varepsilon'$ for all i, $b\xi = \xi b$ for $b \in B$. Then in particular $\xi_0 = e \xi e$ will satisfy $b_0 \xi_0 = \xi_0 b_0$ for $b_0 \in eBe$, $\|y_j \xi_0 - \xi_0 y_j\| = \|y_j e \xi e - e \xi e y_j\| \leq 2\|e\xi - \xi e\| + \|y_j \xi - \xi y_j\| \leq 3\varepsilon'$. Moreover

$$\|\xi\|^2 = \Big\|\sum_{i=0}^{n} e_{1i}^* e_{1i} \xi\Big\|^2 = \sum_i \|e_{1i}^* e_{1i} \xi\|^2 \leq \sum_i \|e_{1i}^* \xi e_{1i}\|^2 +$$

$+ (n+1)\varepsilon'^2 \leq (n+1)(\|\xi_0\|^2 + \varepsilon'^2)$, so that $\|\xi_0\|^2 \geq 1/n+1 - \varepsilon'^2$. Thus if $3\varepsilon'(1/n+1 - \varepsilon'^2)^{-1} \leq \varepsilon$ where $n+1 \geq \tau(e)^{-1}$, then the eMe correspondence $e\mathcal{H}e$ enters in the critical neighborhood of $L^2(eMe, \tau)$ and will thus contain an eMe central vector $0 \neq \eta_0 \in e\mathcal{H}e$. But then a trivial computation shows that $\sum_{i=0}^{n} e_{1i}^* \eta_0 e_{1i}$ is central for M.

2°. Suppose $B \subset M$ is rigid, $e \in B$ and the normalizer of B, $\mathcal{N}(B)$, acts ergodically on $Z(B)$. By 1° to show that $eBe \subset eMe$ is rigid it is sufficient to prove that $e_0 B e_0 \subset e_0 M e_0$ is rigid

for some $0 \neq e_0 \in B$, $e_0 \leq e$. Since B is finite there exists a projection $f_0 \in B$, $0 \neq f_0 \leq e$, which divides a central projection of B, i.e. there are $f_0, f_1, \dots, f_n \in B$ equivalent in B with $\sum_i f_i = z \in Z(B)$. Since $\mathcal{N}(B)$ acts ergodically on $Z(B)$ it follows that for some $z_0 \leq z$, $0 \neq z_0 \in Z(B)$ there exists projections $z_1, \dots, z_m \in Z(B)$ and $1 = u_0, u_1, \dots, u_m \in \mathcal{N}(B)$ such that $u_i z_0 u_i^* = z_i$ and $\sum_{i=0}^{m} z_i = 1$. Let $e_i = f_i z_0$ and denote by $v_0, v_1, \dots, v_n \in B$ some partial isometries satisfying $v_i^* v_i = e_0$, $v_i v_i^* = e_i$. Let now $\varepsilon > 0$, $x_1, \dots, x_p \in M$ give the critical neighborhood of $L^2(M, \tau)$. We define $\{y_i\}_i = \{v_s^* u_t^* x_k u_j v_i \mid 0 \leq j, t \leq m,\ 0 \leq i, s \leq n,\ 1 \leq k \leq p\}$ and put $\varepsilon' = \tau(e_0)\varepsilon$, $\alpha = (m+1)(n+1) = \tau(e_0)^{-1}$. Suppose $\mathcal{H}_0$ is an $e_0 M e_0$-correspondence with $e_0 B e_0$-central vector ξ_0 satisfying $\|\xi_0\| = 1$, $\|\xi_0 y_i - y_i \xi_0\| \leq \varepsilon$ for all i. Let $\mathcal{H}$ denote the α-amplification of $\mathcal{H}_0$, i.e. $\mathcal{H} = L^2(M_{\alpha \times \alpha}, \mathrm{Tr}) \otimes \mathcal{H}_0$, regarded as an M-correspondence like in 1.3.10. Let $\xi = \sum_{i,j} u_t v_s \xi_0 v_s^* u_t^* \in \mathcal{H}$. It is easy to verify that ξ is B central and clearly $\|\xi\|^2 = \alpha$. Moreover if we denote $e_{st} = u_t v_s v_s^* u_t^*$ then

$$\|\xi x_k - x_k \xi\|_2^2 = \Big\|\sum_{s,t} u_t v_s \xi_0 v_s^* u_t^* x_k - x_k u_t v_s \xi_0 v_s^* u_t^*\Big\|^2 =$$

$$= \sum_{s,t,i,j} \|u_t v_s \xi_0 v_s^* u_t^* x_k e_{ij} - e_{st} x_k u_j v_i \xi_0 v_i^* u_j^*\|^2 =$$

$$= \sum_{s,t,i,j} \|\xi_0 v_s^* u_t^* x_k u_j v_i - v_s^* u_t^* x_k u_j v_i \xi_0\|^2 \leq \alpha^2 \varepsilon'^2 = \varepsilon^2.$$

Thus $\mathcal{H}$ has an M central vector $0 \neq \eta \in \mathcal{H}$. Then $e_0 \eta e_0 \neq 0$, $e_0 \eta e_0 \in e_0 \mathcal{H} e_0 = \mathcal{H}_0$ is central for $e_0 M e_0$ in $\mathcal{H}_0$.

The proof of the case $e \in B' \cap M$ and $\mathcal{N}(B' \cap M)$ acts ergodically on $Z(B' \cap M)$ is exactly the same.

Q.E.D.

4.1.10. EXAMPLES. 1°. Since $G=SL(3,\mathbb{Z})$, has the property T and is an ICC group it follows that $M=L(G)$ has the property T. But we may construct free ergodic actions of G on a non-atomic probability measure space A so that the crossed product type II_1 factor $A \rtimes G$ has the property T or not (the inclusion $A \subset A \rtimes G$ is allways rigid!). Indeed if $A=\bigotimes_{g\in G} A_g$ and

$A_g=\bigotimes_{n\geq 1} (L^{\infty}([0,1]))_n$ for each $g\in G$ then the Bernoulli shift action σ on A has the property that tere exist distinct abelian subalgebras $A_n \subset A$, with $A_n \subset A_{n+1}$, $\overline{\bigcup_{n\geq 1} A_n}^w = A$, $\sigma_g(A_n)=A_n$ for all n.

Thus $M_n=A_n \rtimes G \subset A \rtimes_\sigma G$ is an increasing sequence of subfactors in $A \rtimes G$ with $\overline{\bigcup_n M_n}=A \rtimes G$ but $M_n \neq A \rtimes G$ and $M_n' \cap (A \rtimes_\sigma G)=\mathbb{C}$ for all n. If $A \rtimes_\sigma G$ would be rigid this would contradict 4.4.1 below. Thus $A \rtimes_\sigma G$ is not rigid.

On the other hand $\mathbb{Z}^3 \rtimes SL(3,\mathbb{Z})$ has the property T by [ ], so that if $A=L(\mathbb{Z}^3)$ then the corresponding cross product $A \rtimes SL(3,\mathbb{Z})=L(\mathbb{Z}^3 \rtimes SL(3,\mathbb{Z}))$ has the property T.

2°. Free products of von Neumann algebras are not rigid in general. In fact if M_o, M_1 are finite von Neumann algebras with normal finite faithful traces τ_o, τ_1 and if both M_o, M_1 have dimension ≥ 3 then $M=(M_o,\tau_o)*(M_1,\tau_1)$ is a type II_1 factor, but if we assume either M_o or M_1 has a nondiscrete automorphism group Aut M_o then M will also have nondiscrete automorphism group and by 4.1.4 this implies that M is not rigid. Note however that if M_o is a rigid type II_1 factor then $M_o' \cap M=\mathbb{C}$. Thus the discreteness of the automorphisms of a factor M does not follow by only assuming the existence of a rigid subfactor $M_o \subset M$ with trivial relative

commutant $M_o' \cap M=\mathbb{C}$. We'll see however that other rigidity properties of such factors M hold (cf. 4.6.1).

In particular from the preceding considerations it follows that even if M_o, M_1 are rigid II_1 factors, $M=M_o * M_1$ is not rigid (because Int M_o is not discrete!). Thus if a type II_1 factor M has two rigid subfactors that generate it then this doesn't imply that M itself is rigid. The best positive result in this direction that we could get is the following:

4.1.11. PROPOSITION. Let $B\subset M$ be a von Neumann subalgebra, M_o, $M_1 \subset M$ type II_1 factors that contain B and generate M as a von Neumann algebra. Suppose $B\subset M_o$, $B\subset M_1$ are rigid inclusions. Moreover suppose the group $\mathcal{U}_o=\{u_o \in \mathcal{U}(M_o) \mid u_o M_1 u_o^*=M_1\}$ generate M_o as a von Neumann algebra. Then $B\subset M$ is rigid.

Proof. Let $\mathcal{H}$ be an M-M correspondence and denote by p_i the orthogonal projection onto the subspace $\mathcal{H}_i$ of all central vectors for M_i, $i=0,1$. Then p_i may be realized as follows: if $\xi \in \mathcal{H}$ let $K_\xi^i=\overline{co}^w\{u_i \xi u_i^* \mid u_i \in \mathcal{U}(M_i)\}$ and $\eta_i(\xi) \in K_\xi^i$ the unique vector of minimal norm in K_ξ^i. Then $\eta_i(\xi) \in \mathcal{H}_i$ and in fact $\eta_i(\xi)=p_i(\xi)$. Indeed, if $\xi \in \mathcal{H}_i$ this is clear and if $\xi \perp \mathcal{H}_i$ then $K_\xi^i \perp \mathcal{H}_i$ so that $\eta_i(\xi)=0$.

But by hypothesis we also have $u_o \mathcal{H}_1 u_o^*=\mathcal{H}_1$. Thus, by the above construction of p_o it follows that $p_o(\mathcal{H}_1)\subset \mathcal{H}_1$. Since we also have $p_o(\mathcal{H}_1)\subset \mathcal{H}_o$ it follows that $p_o(\mathcal{H}_1)\subset \mathcal{H}_o \cap \mathcal{H}_1$. Now, if $B\subset M_o$, $B\subset M_1$ are rigid and $\varepsilon>0$, $x_1^o,\dots,x_n^o \in M_o$,

$x_1^1, \dots, x_m^1 \in M_1$ give the critical neighborhoods of $L^2(M_0, \tau)$ respectively $L^2(M_1, \tau)$ then we let $\{x_i\}_i = \{x_i^j \mid i, j\}$. If $\mathcal{H}$ is an M-correspondence and ξ is invariant to B and ε'-invariant to all x_i then it follows that there is $\xi_1 \in \mathcal{H}$ invariant to M_1 and close to ξ. In particular ξ_1 is almost invariant to M_0 so that by 4.1.5 its projection ξ_0 onto the invariant vectors for M_0 is close to ξ. Thus $\xi_0 \neq 0$ and by the preceding remarks ξ_0 is central for $M_0 \cdot M_1$ and thus for M.

Q.E.D.

§4.2 Rigidity and completely positive maps

In this section we prove a rigidity result about completely positive maps defined on factors with property T. It generalises the main argument in the proof of Theorem 3 in [ ], which shows that if M has property T and $\phi: M \to M$ is a normal completely positive map close to the identity in certain finitely many points then ϕ is uniformly close to the identity. Our generalization consists in leting ϕ take values in an arbitrary algebra and replacing the identity by a *-morphism and of course, as usual, assuming a relative property T instead of the full property T. We'll get many applications of this technical result in the next sections. It is fearly possible that other rigidity results will come out from it.

4.2.1. THEOREM. Let $\underbrace{B \subset M}_{\text{be a } W^*\text{-subalgebra of } M \text{ such that } B_0 = B + \mathbb{C}(1_M - 1_B) \subset M}$ be a rigid inclusion. Let $k>0$. There exist $\varepsilon>0$ and $x_1, \dots, x_n \in M$ such that $\overbrace{\text{if}}^{\delta \le \varepsilon,}$ $\phi: M \to M_0$ is a normal completely position map into a finite von Neumann algebra (M_0, τ_0), with $\|\phi\| \le k$ and $\rho: M \to M_0$ is a *-isomorphism (not necessarily unital!) with $\tau_0(\rho(1)) \ge k$ [?], $\phi|_B = \rho|_B$ and $\|\phi(x_i) - \rho(x_i)\|_2 < \delta^2/200k_0^2$, $1 \le i \le n$, then $\|\phi(x) - \rho(x)\|_2 \le 3k_0^{1/4}\delta^{1/4} + k$ [?] for any $x \in M$, $\|x\| \le 1$, where k_0 is the constant appearing in 4.1.5 and depends only on the inclusion $B \subset M$.

Proof. Note first that since the fixed normal finite faithful trace τ_0 on M_0 satisfies $\tau_0(1_{M_0}) = 1$ we have $k \ge 1$. Since M is a factor, by the uniqueness of the trace on M it follows that $\tau_0 \circ \rho$ is a scalar multiple of τ, so that ρ is normal and $N = \rho(M)$ is a subfactor of M_0 but with $1_N = \rho(1_M)$ not necesarily equal to 1_{M_0}.

We first show that we may assume $\phi(1) \le \rho(1)$. Indeed, if we let 1_M belong to the set $(x_i)_i$, we have $\|\phi(1) - \rho(1)\|_2 < \delta^2/200k^{1/2}$ and $\|\phi(1)\| = \|\phi\| \le k$. Let $g: [0, \infty) \to [0, \infty)$, $g(t) = \begin{cases} 0, & 0 \le t \le 1-\delta' \text{ or } t > 1+\delta' \\ t^{-1/2}, & 1+\delta' \ge t > 1-\delta' \end{cases}$

Let $a=\chi(\phi(1))$ and note that $e_o=a\phi(1)a$ is a projection and $\|e_o-a\|$, $\|(1-e_o)\phi(1)\|_2$ are small. Thus $\|a\phi(x)a-\phi(x)\|_2\le$ $\le\|a\phi(x)a-e_o\phi(x)e_o\|_2+\|e_o\phi(x)e_o-\phi(x)\|_2=\mathcal{O}(\delta')+2\|e_o\phi(x)-\phi(x)\|_2$.

But for $\|x\|\le 1$, we have $\|e_o\phi(x)-\phi(x)\|_2^2=\tau(\phi(x^*)\phi(x)-\tau(\phi(x^*)e_o\phi(x))=$ $=\tau(\phi(x)\phi(x^*)(1-e_o))\le\|\phi\|\tau(\phi(xx^*)(1-e_o))\le\|\phi\|\tau(\phi(1)(1-e_o))=\mathcal{O}(\delta')$.

Thus $\|a\phi(x)a-\phi(x)\|_2=\mathcal{O}(\delta')$ is small uniformly in $x\in M$, $\|x\|\le 1$. Since the projection $e_o=a\phi(1)a$ satisfies $\|e_o-\rho(1)\|_2\le\mathcal{O}(\delta')+\delta^2/10k^{1/2}$ we can find a projection $f\le e_o$ majorized by $\rho(1)$ (in the sense that f is equivalent to a subprojection of $\rho(1)$) and such that $\|f-e_o\|_2\le\mathcal{O}(\delta')+3\delta^2/10k^{1/2}$ and $f\ge\Phi(1_B)\cdot\rho(1_B)\cdot\rho(1_M)$. By 1.4 in [ ] there exists a partial isometry $v\in M$ with $v^*v=f$, $vv^*\le\rho(1_M)$, $v\rho(1_B)=\rho(1_B)$, $\|v-f\|_2=\mathcal{O}(\delta')+9\delta^2/10k^{1/2}$. It follows that for δ' small enough the completely positive map $\phi_o(x)=va\phi(x)av^*$ satisfies $\phi_o(1)\le\rho(1)$, $\Phi_o(b)=\rho(b),\ b\in B_1$, $\|\phi_o(x)-\Phi(x)\|_2\le\|va\phi(x)av^*-$ $-\phi(x)\|_2$ $\le 2\|v-f\|_2\|a\phi(x)a\|+\|a\phi(x)a-\phi(x)\|_2$ $\le 20k\delta^2/10^2k^{1/2}=1/5\,k^{1/2}\delta^2$, uniformly in $x\in M$, $\|x\|\le 1$. This shows that by replacing if necessary ϕ by ϕ_o we can assume $\phi(1)\le\rho(1)$.

We denote by E_N the unique normal conditional expectation of M_o onto N which preserve the trace on $\rho(1)M_o\rho(1)$. We also denote by ρ^* the adjoint of ρ in the sense of 1.3. Note that $\rho^*=\rho^{-1}\circ E_N$ where ρ^{-1} is the inverse of $\rho:M\to N$. Put $\psi:M\to M$, $\psi(x)=\rho^*(\phi(x))$. Then $\|\psi(x)-x\|_2=\|\rho^*(\phi(x))-x\|_2=\|\rho(\rho^*(\phi(x)))-\rho(x)\|_2\cdot\|\rho(1_M)\|_2^{-1}\le$ $\le k^{1/2}\|E_N(\phi(x))-\rho(x)\|_2\overset{k^{1/2}}{=}\|E_N(\phi(x)-\rho(x))\|_2\overset{k^{1/2}}{\le}\|\phi(x)-\rho(x)\|_2$. Moreover $\psi(b)=b$ for $b\in B$. Let $\mathcal{H}_\psi$ denote the correspondence associated to ψ and $\xi=1\otimes 1\in\mathcal{H}_\psi$. Then $\|\xi\|^2=\tau(\psi(1))$, $b\xi=\xi b$ for $b\in B$ and $\|y\xi-\xi y\|^2=$ $=\tau(\Psi(y^*y))+\tau(\Psi(1)yy^*)-2\mathrm{Re}\,\tau(\psi(y^*)y)$ for $y\in M$. Since if $z\in M$ we have $|\tau(z)|\le$ $\le\|z\|_2$, it follows that $|\tau(\psi(y^*y)-y^*y)|\le\|\psi(y^*y)-y^*y\|_2$, $\tau(\psi(1))\ge$ $\ge 1-\|1-\psi(1)\|_2$ and $\|y\xi-\xi y\|^2\le\|\Psi(y^*y)-y^*y\|_2+\|1-\Psi(1)\|_2\|y\|^2+2\|\Psi(y)-y\|_2\|y\|$.

Let now $\varepsilon>0$, $y_1,\dots,y_n\in M$, $\|y_i\|\le 1$, give a critical neighborhood of $\mathcal{H}_{id}$ (for the rigid inclusion $B\subset M$) as in 4.1.5. By the above computations,

if we put $\{x_i\}_i=\{1\}\cup\{y_j\}_j\cup\{y_k^*y_k\}$, $\delta\le\varepsilon$ and if ψ is defined as before $\psi=\rho^*\circ\phi$ with $\Phi\,(=\phi_o)$ satisfying $\|\phi(x_i)-\rho(x_i)\|_2<(4k^{1/2})^{-1}\delta$ for all i, then it follows that $\mathcal{H}_\psi$ enters in the critical neighborhood of $\mathcal{H}_{id}$ given by $\delta\le\varepsilon$ and $\{y_j\}$. Thus there exists by 4.1.5 a vector $\eta\in\mathcal{H}_\psi$ central for M and close to ξ, $\|\eta-\xi\|\le k_o\delta$, for some k_o only depending on the inclusion $B\subset M$. We may clearly assume $\|\eta\|=1$. Then we have the following estimates for arbitrary $x\in M$, $\|x\|\le 1$:

$$\|\psi(x)-x\|_2^2=\tau(\psi(x^*)\psi(x))+\tau(x^*x)-2\mathrm{Re}\,\tau(\psi(x^*)x)=$$

$$=(\langle x\xi,\xi\psi(x)\rangle-\mathrm{Re}\langle x\eta,\eta\psi(x)\rangle)+(\langle x\eta,\eta x\rangle-\mathrm{Re}\langle x\xi,\xi\quad\rangle)$$

$$\le 2\|\xi-\eta\|\,\|\psi\|+2\|\xi-\eta\|\le\quad 4k_o\delta\ .$$

Since $\phi(1)\le\rho(1)$ we have $\tau(\phi(x))=\tau(E_N(\phi(x)))=\tau(\rho^*\phi(x))$ so that by Kadison's inequality, $\|\phi(x)-\rho(x)\|_2^2=$ $\tau_o(\phi(x^*)\phi(x))+\tau_o\rho(x^*x)-2\mathrm{Re}\,\tau_o(\rho(x^*)\phi(x))\le\tau_o(\phi(x^*x)+\tau_o\rho(x^*x)-$ $-2\mathrm{Re}\,\tau_o(\rho(x^*)\phi(x)))=\tau(\Psi(x^*x))+\tau(x^*x)-2\mathrm{Re}\,\tau(x^*\Psi(x))$ and by the preceding estimate the last term is small uniformly in $x\in M$, $\|x\|\le 1$. More precisely we have:

$$\|\Phi(x)-\rho(x)\|_2^2\le 6k_o^{1/2}\delta^{1/2},\ \text{for } x\in M,\ \|x\|\le 1.$$

By the first part of the proof this shows that for general Φ satisfying the conditions in the statement we have:

$$\|\Phi(x)-\rho(x)\|_2\le 3k_o^{1/4}\delta^{1/4}+1/5\,k^{-1/2}\delta^2,\ \text{for } x\in M,\ \|x\|\le 1.$$

Q.E.D.

§4.3. Embedding rigid factors

We present in this section the result of Connes and Jones in [ ] showing that a rigid factor cannot be embedded in the II_1 factor $L(\mathbb{F}_n)$ coming from the free group on n generators $\mathbb{F}_n$, $n\geq 2$. First we present a version of the original proof in [ ] then we describe another approach to this problem. ~~The proof uses Haagerup's theorem, that the identity in $L(\mathbb{F}_n)$ can be pointwise approximated by compact completely positive maps and the case $M=M_o$, $\rho=id$, $B=\{0\}$ of the preceding theorem.~~

4.3.1. THEOREM. $L(\mathbb{F}_n)$ contains no rigid type II_1 subfactors.

Proof. Since $L(\mathbb{F}_2)\subset L(\mathbb{F}_n)$ for any $n\geq 2$ it is sufficient to prove the statement for $n=2$. Suppose $M\subset L(\mathbb{F}_2)$ is a rigid subfactor (we do not assume $1_M=1$). By [ ] there exist unital normal completely positive trace preserving maps $\psi_n:L(\mathbb{F}_2)\to L(\mathbb{F}_2)$ such that $||\psi_n(x)-x||_2\to 0$ for all $x\in L(\mathbb{F}_2)$ and such that ψ_n send the unit ball of $L(\mathbb{F}_2)$ (in the uniform norm) in a compact set relative to the topology of the norm $||\ ||_2$ (in fact in [ ] ψ_n are so that ψ_n send the unit ball of $L^2(L(\mathbb{F}_2),\tau)$ in the norm $||\ ||_2$ in a compact subset of $L^2(L(\mathbb{F}_2),\tau)$). Then $\phi_n=E_M\psi_n|_M:M\to M$ are also normal completely positive compact maps (in the above sense) and satisfy $\phi_n(1)\leq 1$ and $||\phi_n(x)-x||_2\to 0$ for all $x\in M$. By 4.2.1 it follows that $||\phi_n(x)-x||_2\to 0$ uniformly for $x\in M$, $||x||\leq 1$. Now let $A\subset M$ be a maximal abelian *-subalgebra of M. Since M is of type II_1, A is completely nonatomic so that (A,τ) is isomorphic to $L^\infty(\mathbb{T},\mu)$ where μ is the Lebesgue measure. Hence there is a unitary element $u\in A$ such that $\tau(u^k)=0$ for all $k\neq 0$ (the image of $z\in L^\infty(\mathbb{T},\mu)$). Then $(u^k)_{k\geq 1}$ tends to zero in the w-topology and since ϕ_n are normal, $(\phi_n(u^k))_{k\geq 1}$ also tend to zero in the w-topology, for each n. By the relative compactness of $(\phi_n(u^k))_k$ in the norm $||\ ||_2$ it follows that $||\phi_n(u^k)||_2 \underset{k}{\to} 0$ for each n. On the other hand

$||\phi_n(u^k)||_2\geq ||u^k||_2-||u^k-\phi_n(u^k)||_2=1-||u^k-\phi_n(u^k)||_2$. But as we previously showed, for large n, $||u^k-\phi_n(u^k)||_2$ is uniformly small in k. This gives the contradiction. Q.E.D.

Second approach. This proof is based on a completely different property of the algebras coming from free groups. We use the fact that the automorphisms of $L(\mathbb{F}_n)$ coming from automorphisms of $\mathbb{F}_n$ are connected to the identity automorphism of $L(\mathbb{F}_n)$. It is an open question whether $Aut(L(\mathbb{F}_n))$ is pathwise connected or not but let's not here:

4.3.2. LEMMA. Let $u,v\in L(\mathbb{F}_2)$ be the unitaries corresponding to the generators of $\mathbb{F}_2$ and A_u, A_v the abelian von Neumann algebras generated by u respectively v. The trace preserving automorphisms of A_u, A_v implement in a natural way automorphisms of $L(\mathbb{F}_2)$. Let $G_o\subset Aut(L(\mathbb{F}_2))$ be the group generated by these automorphisms and by those implemented by automorphisms of $\mathbb{F}_2$. Then G_o is pathwise connected in $Aut(L(\mathbb{F}_2))$.

Proof. Let σ be an automorphism of A_u preserving τ and $\{e_t\}_{1\geq t\geq 0}$ a nest of projections generating A_u let σ'_t be the restriction (in the ergodic theory sense, see [ ]) of σ to e_t and $\sigma_t=\sigma'_t+id_{1-e_t}$. Then σ_t is a path (in the point norm-two topology) of automorphisms connecting σ to id. Then σ_t*id on $A_u*A_v=L(\mathbb{F}_2)$ connects $\sigma*id$ to id. Now if $u \xleftrightarrow{\theta} uv$, $v \xleftrightarrow{\theta} v$ is an automorphisms of $\mathbb{F}_2$ then let $v_t\in A_v$ be any path of unitaries relating v to 1. Then $u \xrightarrow{\theta_t} uv_t$,

$v \xleftrightarrow{\theta_t} v$ implement automorphisms of $L(\mathbb{F}_2)$ that relate θ to id. Since any automorphism of $\mathbb{F}_2$ is a composition of automorphisms θ as above and $u \leftrightarrow u^{-1}$, $v \leftrightarrow v$ (cf.[ ]) the proof is complete.

Q.E.D.

Now the idea of this second approach to 4.3.1 is quite simple. We assume $L(\mathbb{F}_2)$ contains a rigid factor M_o so that for some projection $e \in M_o$, $(eM_oe)' \cap eL(\mathbb{F}_2)e = \mathbb{C}$ (the proof of the general case requires a longer argument that we do not detail here). We may assume $\tau(e)=1/n$ so that the algebra M generated by eM_oe and a suitable n by n matrix algebra is also rigid and $M' \cap L(\mathbb{F}_2)=\mathbb{C}$. Let $L(\mathbb{F}_2) \subset L(\mathbb{F}_4)$ in the obvious way and θ the automorphism reversing the first two generators of $\mathbb{F}_4$ (which are the generators of $L(\mathbb{F}_2) \subset L(\mathbb{F}_4)$) with the last two. By 4.3.2 there is a path $\{\theta_t\}_{1 \geq t \geq 0}$ of automorphism with $\theta_o = \mathrm{id}$, $\theta_1 = \theta$. By 4.2.1 this path is continuous in the uniform norm on the unit ball of M and so by A.4 and by $\theta_t(M)' \cap L(\mathbb{F}_4) = \mathbb{C}$ (cf.[ ]), there are unitary elements $u_1, \dots, u_n \in L(\mathbb{F}_4)$ so that $u_i \theta_{t_{i-1}}(x) u_i^* = \theta_{t_i}(x)$, $x \in M$, where $0=t_o < t_1 < \dots < t_n = 1$. Thus M and $\theta(M)$ are inner conjugate in $L(\mathbb{F}_4)$ in contradiction with in [ ] .

We mention that in [ ] Connes and Jones obtained a surprising consequence of the above result: an example of nonvanishing 2-cohomology for a free action of a property T group G on $L(\mathbb{F}_\infty)$.

There construction is as follows. Since G is finitely generated ([ ]) there is a presentation $\mathbb{F}_n \to G \to 0$ of G. The kernel of $\mathbb{F}_n \to G \to 0$ is easily seen to be isomorphic to $\mathbb{F}_\infty$, being a normal proper subgroup of $\mathbb{F}_n$, see e.g. [ ] . By general properties of free groups it follows that given any $e \neq g \in \mathbb{F}_n$ the conjugacy class of g by elements in $\mathbb{F}_\infty \subset \mathbb{F}_n$ is infinite. Thus $L(\mathbb{F}_\infty)' \cap L(\mathbb{F}_n) = \mathbb{C}$. Moreover the normalizer $\mathcal{N}$ of $L(\mathbb{F}_\infty)$ in $L(\mathbb{F}_n)$ generates $L(\mathbb{F}_n)$ and if $\mathcal{U}$ is the unitary group of $L(\mathbb{F}_\infty)$ then $\mathcal{N}/\mathcal{U} \simeq \mathbb{F}_n/\mathbb{F}_\infty = G$. In fact this way $L(\mathbb{F}_n)$ may be viewed as the cocycle crossed product of $L(\mathbb{F}_\infty)$ by G. Now if there would be a lifting from G to $\mathcal{N}$, $u_g \in \mathcal{N}$, $g \in G$, so that $u_g u_h = u_{gh}$, $g,h \in G$, then this would imply that there exists a copy of the left regular representation of G in $L(\mathbb{F}_n)$, thus $L(G) \subset L(\mathbb{F}_n)$, in contradiction with 4.3.1.

Note that the above construction provides an example of a rigid inclusion $L(\mathbb{F}_\infty) \subset L(\mathbb{F}_n)$ with $[L(\mathbb{F}_n) : L(\mathbb{F}_\infty)] = \infty$. In fact this example satisfies the conditions of 4.1.7, (i). It would be interesting to know whether any II_1 factor M has a hyperfinite subfactor R so that $R \subset M$ be rigid. In connection with this problem, note the results in [ ] , [ ], [ ] .

The problem of embedding a factor into another seem quite difficult. SUch problems were first posed by Murray and von Neumann who asked whether a nonhyperfinite II_1 factor, such as

$L(F_2)$, can be embedded into R. The complete answer to this problem was only given by Connes in [ ] : as a consequence of his theorem on the equivalence between injectivity and hyperfiniteness, any subfactor of R is isomorphic to R or finite dimensional. The fact that $L(F_2) \not\subset R$ was noted before, as a consequence of [ ] . On the other hand it should be mentioned here the old problem about whether any nonamenable group contains a copy of F_2 or not. This problem is now solved in the negative ([ ]). However its operator algebra analogue is stil an open question: does any nonhyperfinite II_1 factor contain a copy of $L(F_2)$? Let us mention here that $L(F_2)$ can be approximately embedded in any II_1 factor (cf.[ ]).

§4.4 Rigidity and convergence of conditional expectations

An important rigidity phenomena about property T groups is the following (cf [ ]): if G is a discrete group with property T and $G_n \subset G$ is an increasing sequence of subgroups with $\cup_n G_n = G$ then, for some n_0 , $G_{n_0} = G$. In particular this shows that rigid groups are finitely generated.

Using his approach with correspondences Connes obtained an operator algebra analogue of this result (cf [ ], 6.2). This result was checked independently by Bion-NAdal in [ ] . Moreover Moore proved in [ ] a result of this type for his property T relative to Cartan subalgebras.

The next theorem gives a unifying generalisation of these results in the context of our definition of relative property T.

4.4.1. THEOREM. Let $B \subset M$ be a rigid inclusion and suppose $(M_n)_n$ is an increasing sequence of von Neumann subalgebras of M,

all containing B, so that $\overline{\cup M_n} = M$. Then there exist projections $f_n \in M_n' \cap M$ such that $f_n \uparrow 1$ and $f_n M_n f_n = f_n M f_n$. In particular if $M_n' \cap M = \mathbb{C}$ (or, more generally, if it is finite dimensional) for each n, then $M_n = M$ for n large enough. Thus M is finitely generated over B.

Proof. Since $\overline{\cup M_n} = M$ it follows that $\|E_{M_n}(x) - x\|_2 \to 0$ for each $x \in M$, so that by 4.2.1 $\|E_{M_n}(x) - x\|_2 \to 0$ uniformly in $x \in M$, $\|x\| \leq 1$. By A.2 in the appendix it follows that for n large enough $M_n' \cap M$ has atoms and that if f_n is the atom of maximal trace in $M_n' \cap M$ then $f_n M_n f_n = f_n M f_n$. Q.E.D.

Note that in the preceding proof we used the condition $M_n \uparrow M$ only to get $\|E_{M_n}(x) - x\|_2 \to 0$ for each $x \in M$. So we could directly put this weaker condition as hypothesis in 4.4.1. Using the perturbation results in [ ] (see the appendix) we can actually do much more than that: even if the ambiant factor M is not rigid but we have a sequence of subfactors $N_k \subset M$ "tending pointwise" to a rigid subfactor $N \subset M$ then for k large enough the factors N_k "contain" the rigid factor N in a sense that we make now more precise:

4.4.2. PROPOSITION. Let $N_o \subset M$ be a type II_1 subfactor of M and $B \subset N_o$ a W^*-subalgebra of N_o so that if $B_o = B + \mathbb{C}(1_{N_o} - 1_B)$ then the inclusion $B_o \subset N_o$ is rigid (we allow here $1_B \neq 1_{N_o} \neq 1_M$).

Let 1 $\varepsilon > 0$, $x_1, \ldots, x_n \in N_o$ be given by 4.2.1 for the inclusion $B_o \subset N_o$ and $k_o = k_o(B_o \subset N_o)$ the constant appearing in 4.1.5. If $\delta \leq \varepsilon$ and $N \subset M$ is a subfactor with $B \subset N$ and $\| E_N(x_i) - x_i \|_2 \leq \delta^2 \tau(1_{N_o})^{3/2}/200$, $1 \leq i \leq n$, then $\| E_N(x) - x \|_2 \leq (6k_o^{1/4}+1)^{1/2} \delta^{1/8}$ for all $x \in N_o$, $\| x \| \leq 1$. Moreover, there exist

a) projections $e_o \in N_o$, $e \in N$ and a unital $*$-isomorphism $\theta : e_o N_o e_o \to eNe$;

b) projections $f_o \in (e_o N_o e_o)' \cap e_o M e_o$, $f \in \theta(e_o N_o e_o)' \cap eMe$ and partial isometry $u \in M$ satisfying the following conditions:

1) $u^* u = f_o$, $uu^* = f$ and $ux = \theta(x) u$ for all $x \in e_o N_o e_o$;

2) $\| \theta(e_o x e_o) - e_o x e_o \|_2 \leq$, $x \in N_o$, $\| x \| \leq 1$,

and $\| u - 1_{N_o} \|_2$, $\| u - 1_N \|_2 \leq$

Proof. Let $\Phi_o : N_o \to N_o$, $\Phi_o(x) = E_{N_o} E_N(x)$. Then Φ_o is normal, completely positive, $\Phi_o(b) = b$ for $b \in B$, $\| \Phi_o \| \leq 1$, and $\| \Phi_o(x_i) - x_i \|_2 = \| E_{N_o}(E_N(x_i) - x_i) \|_2 \leq \| E_N(x_i) - x_i \|_2 \leq \delta^2 \tau(1_{N_o})^2/20$

Thus, by 4.2.1 we have $\| \Phi_o(x) - x \|_2 \leq (3k_o^{1/4} + 1/5)\delta^{1/4}$ for all $x \in N_o$, $\| x \| \leq 1$. But then $\| x - E_N(x) \|_2^2 = \| x \|_2^2 - \| E_N(x) \|_2^2 \leq$

$\leq \| x \|_2^2 - \| E_{N_o} E_N(x) \|_2^2 \leq 2 \| x \|_2 (\| x \|_2 - \| E_{N_o} E_N(x) \|_2)$

$\leq 2 \| x - E_{N_o} E_N(x) \|_2 = 2 \| x - \Phi_o(x) \|_2 \leq 2(3k_o^{1/4} + 1/5)\, \delta^{1/4}$ for all

$x \in N_o$, $\| x \| \leq 1$ which proves the first part of the proposition.

The rest of the statement follows now by just applying directly A.3.

Q.E.D.

Note that the first part of 4.4.2 generalizes the technical argument used in [ ]. Moreover, as we noted before, the above proposition generalizes Theorem 4.4.1 (just take $N=M$). We presented first 4.4.1 to underline the simplicity of its proof, which do not use deformation arguments.

§4.5 The set of rigid subfactors of a II_1 factor

Using 4.4.2 and elementary topological arguments we now show that the set of rigid subfactors of an arbitrary separable II_1 factor is in some sense very poor. First we consider subfactors with the same unit as M.

4.5.1. THEOREM. Let M be a separable type II_1 factor, $B\subset M$ a von Neumann subalgebra of M. Consider the set $\mathcal{R}_o=\{N\subset M$ subfactor$\mid$ N contains B and $B\subset N$ is rigid, $N'\cap M$ is finite dimensional$\}$. Consider on the set $\mathcal{R}_o$ the equivalence $\approx$ given by inner conjugacy with unitary elements of M. Then $\mathcal{R}_o/\approx$ is countable.

Proof. For each class in $\mathcal{R}_o/\approx$ we choose a type II_1 factor N in that class. Denote by $\mathcal{R}_o'$ the set of these factors N. For each $N\in\mathcal{R}_o'$ let $\varepsilon_N>0$, $F_N=\{x_1,\dots,x_n\}\subset N$, $\|x_i\|\le 1$ give a critical neighborhood of $L^2(N,\tau)$ as in 4.1.5. Let k_N be the constant associated to $B\subset N$ as in 4.1.5. If we assume $\mathcal{R}_o/\approx$ is uncountable it follows that for some n_o the set $\mathcal{R}_1'=\{N\subset\mathcal{R}_o'\mid$ cardinal $F_N\le n_o$, $\varepsilon_N\ge n_o^{-1}$, $k_N\le n_o$, $\tau(e)\ge n_o^{-1}$ for all $e\in N'\cap M\}$ is uncountable. Now let $\mathcal{H}_N=\operatorname{span} F_N\subset L^2(M,\tau)$ and P_N the corresponding orthogonal projection onto $\mathcal{H}_N$. Then $\mathcal{H}_N\subset N$ and $P_N\le e_N$ where $e_N\in\mathcal{B}(L^2(M,\tau))$ is the extension by continuity of the conditional expectation E_N to the orthogonal projection of $L^2(M,\tau)$ onto $L^2(N,\tau)=\overline{N}\subset L^2(M,\tau)$. Since $\dim\mathcal{H}_N\le n_o$ for all $N\in\mathcal{R}_1'$ it follows that the set $\{p_N\mid N\in\mathcal{R}_1'\}\subset\mathcal{B}(L^2(M,\tau))$ is separable in the uniform norm. Since $\mathcal{R}_1'$ is uncountable it follows that given any $\delta>0$, $\delta\le n_o^{-1}$ there are N_o, $N_1\in\mathcal{R}_1'$, N_o not inner conjugate to N_1

such that $\|p_{N_o}-p_{N_1}\|\le\delta^2/200$. Then we have for $i=0,1$ and $x_j^i\in F_{N_i}$, $\|E_{N_i}(x_j^i)-x_j^i\|_2^2=\|x_j^i\|_2^2-\|E_{N_i}(x_j^i)\|_2^2\le$ $\le\|x_j^i\|_2^2-\|p_{N_i}(x_j^i)\|_2^2=\|p_{N_i}(x_j^i)-x_j^i\|_2^2$ so that $\|E_{N_i}(x_j^i)-x_j^i\|_2$ $\le\delta^2/200$. Thus, by 4.4.2 we have $\|E_{N_i}(x_i)-x_i\|_2\le$ $\le(6n_o^{1/4}+1)^{1/2}\delta^{1/8}$ for all $x_i\in N_i$, $\|x_i\|\le 1$, $i=0,1$. Also by 4.4.2 we have a unital $*$-isomorphism $\theta:p_oN_op_o\to p_1N_1p_1$, for some projections p_i, uniformly close to the identity and with $\|p_i-1_n\|_2$ small $i=0,1$ (depending on δ), more precisely $\|\theta(x_o)-x_o\|_2\le$ for $x_o\in p_oN_op_o$, $\|x_o\|\le 1$ and $\|p_i-1_n\|_2\le$

It follows that for any $x_1\in p_1M_1p_1$ with $\|x_1\|\le 1$ we have $\|E_{\theta(p_oN_op_o)}(x_1)-x_1\|_2\le\|E_{\theta(p_oN_op_o)}\circ E_{p_oN_op_o}(x_1)-$ $-E_{p_oN_op_o}(x_1)\|_2+2\|E_{p_oN_op_o}(x_1)-x_1\|_2$ (here, as allways, when we have a W^*-subalgebra B in M with unit $1_B=e\in M$ we denote by E_B the unique trace preserving conditional expectation of M onto B that preserve the trace on eMe, i.e. $E_B(x)=E_B(exe)$, and it coincides with the restriction to M of the orthogonal projection of $L^2(M,\tau)$ onto the closure of B in $L^2(M,\tau)$). Now we have

$$\|E_{p_oN_op_o}(x_1)-x_1\|_2=\|E_{N_o}(p_ox_1p_o)-x_1\|_2\le 2\|p_o-1\|_2+$$

$$+\|E_{N_o}(x_1)-x_1\|_2 \quad\text{and}\quad \|E_{\theta(p_oN_op_o)}E_{p_oN_op_o}(x_1)-E_{p_oN_op_o}(x_1)\|_2$$

$$\le\|\theta(E_{p_oN_op_o}(x_1))-E_{p_oN_op_o}(x_1)\|_2$$

Thus by A.1 it follows that if p is the atom of maximal trace in $\theta(p_oN_op_o)'\cap p_1N_1p_1$ then $\|p-1\|_2\leq\|p-p_1\|_2+\|p_1-1\|_2\leq$ and $\theta(p_oN_op_o)p=pN_1p$. Hence, if we denote by $\theta':p_oN_op_o\to pN_1p$, $\theta'(x)=\theta(x)p$ then θ' is a surjective unital $*$-isomorphism. By A.3 it follows that there exists a partial isometry $v\in M$ such that $\|v-1\|_2\leq$, $vx_o=\theta'(x_o)v$ for all $x_o\in p_oN_op_o$ and $v^*v\in(p_oN_op_o)'\cap p_oMp_o=p_o(N_o'\cap M)p_o$, $vv^*\in(pN_1p)'\cap pMp=$ $=p(N_1'\cap M)p$. Now if $v^*v\neq p_o$ then since $N_o\in\mathcal{Q}_1'$ it follows that $\tau(p_o-v^*v)\geq\tau(p_o)n_o^{-1}$. Since we also have $\|p_o-v^*v\|_2\leq$, for δ small enough we get a contradiction. Thus $v^*v=p_o\in N_o$. Similary, for δ small enough we also have $vv^*=p\in N_1$. Moreover $vN_ov^*=pN_1p\subset N_1$. Since both N_o and N_1 are type II_1 factors it follows that N_o, N_1 are inner conjugate in M. This final contradiction completes the proof.

Q.E.D.

As a consequence of 4.5.1 we get the "relative version" of Theorem 2 in [ ] :

4.5.2. COROLLARY. Let $B\subset M$ be a rigid inclusion. Then the set $\mathcal{I}_B(M)=\{[M:N]\mid B\subset N\subset M \text{ subfactor}\}$ is countable. In particular, if M is [illegible] then $\mathcal{I}(M)$ and the fundamental group $\mathcal{F}(M)$ of M are countable (i.e. [ ] and [ ]).

Pr-of. By 4.1.7 any subfactor with finite index $N\subset M$ with $B\subset N$ is still rigid relative to B. If the set $\mathcal{I}_B(M)$ is uncountable then for some $N_o\geq 1$, $\mathcal{I}_B(M)\cap[1,N_o]$ is also uncountable and by [ ] if $[M:N]<\infty$ then $N'\cap M$ is finite dimensional. Thus since the index is invariant to conjugacy 4.5.1 applies. Since by [ ] we have an injective map from $\mathcal{F}(M)$ into $\mathcal{I}(M)$ the rest of the statement follows.

Q.E.D.

4.5.3. COROLLARY. There are uncountable many nonisomorphic rigid type II_1 factors.

Theorem 4.5.1 also gives a partial answer to the old standing problem on whether there exists a universal separab II_1 factor, i.e. a separable II_1 factor containing copies of any other separable factor. Indeed by 4.5.1 and 4.5.3 we get

4.5.4. COROLLARY. There exist no separable II_1 factor containing copies of any rigid factor N so that $N'\cap M$ be fini dimensional.

Recall from 1.3.2 that two factors are stable equivale if one of them is isomorphic to a reduced algebra of the oth in other words if there exists a correspondence of index 1 between them.

4.5.5. CONJECTURE. The set of classes of stable equivalent rigid factors is countable. In particular the union $\cup\mathcal{F}(M)$ of the fundamental groups of all rigid factors is countable.

In connection with this problem using the same ideas as in the proof of 4.5.1 we get a result concerning the set of all rigid subfactors of M, not necesarily with the same unit as M.

4.5.6. THEOREM. Let M be a separable type II_1 factor. Then the set of classes of stable equivalence of rigid subfactors N in M (with 1_N not necessarily equal to 1_M)

is countable.

Proof. We proceed the same way we did in the proof of 4.5.1. We denote by $\sim$ the relation of stable equivalence and denote by $\mathcal{R}_o$ the set of all rigid subfactors N of M with 1_N not necessarily equal to 1_M. If $\mathcal{R}_o/\sim$ is uncountable then there is an n_o such that the set $\mathcal{R}_1=\{N\in\mathcal{R}_o \mid \text{cardinal } F_N\leq n_o,\ t_N\geq n_o^{-1},\ k_N\leq n_o,\ \tau(1_N)\geq n_o^{-1}\}$ is uncountable modulo $\sim$, where F_N, t_N, k_N have the same meaning as in the proof of 4.5.1. Like there, it follows that given any $\delta>0$ there are factors $N_o, N_1\in\mathcal{R}_1$, $N_o\not\sim N_1$, so that $\| E_{N_1}(x_j^i)-x_j^i\|_2\leq \delta^2/200\, n_o^2$, $i=0,1$, $x_j^i\in F_{N_i}$. Thus $\| E_{N_o}(x_1)-x_1\|_2\leq$,
$\| E_{N_1}(x_o)-x_o\|_2\leq$ for all $x_i\in N_i$, $\|x_i\|\leq 1$, $i=0,1$.
By 4.4.2 (or directly by A.3) we have a unital $*$-isomorphism $\theta: p_oN_op_o \to p_1N_1p_1$, where $p_i\in N_i$, $\| p_i-1_{N_i}\|_2\leq$
and $\|\theta(x_o)-x_o\|_2\leq$ for all $x_o\in p_oN_op_o$, $\|x_o\|\leq 1$.
Arguing as in the proof of 4.5.1 it follows that $\theta(p_oN_op_o)\subset$ $\subset p_1N_1p_1$ is close to $p_1N_1p_1$ so that by A.1 the projection of maximal trace in $\theta(p_oN_op_o)'\cap p_1N_1p_1$, p, will satisfy $\theta(p_oN_op_o)p=pN_1p$. Thus $\theta':p_oN_op_o\to pN_1p$ defined by $\theta'(x_o)=$ $=\theta(x)p$ is a surjective $*$-isomorphism. Thus N_o is stable equivalent to N_1, a contradiction.

Q.E.D.

Note that by 4.5.5 a negative answer to the conjecture 4.5.4 would imply that there exist no universal separable II_1 factors. However we strongly believe 4.5.4 holds true.

In [ ] Connes posed the rigidity problem for II_1 factors:

§4.6 Rigidity and fundamental groups of factors

In this section we give another approach to the problem of estimating the fundamental group of a type II_1 factor and the set of indices of its subfactors. We show that for these sets to be countable it is sufficient that M contains a rigid subfactor with small relative commutant.

4.6.1. THEOREM. Let M be an arbitrary separable type II_1 factor. If M contains a rigid subfactor N (we allow $1_N\neq 1_M$) so that $N'\cap M$ has a nonzero atomic part then the set $\mathcal{J}(M)$ of indices of the subfactors of M is countable. In particular, the fundamental group of M, $\mathcal{F}(M)$, is countable.

Proof. Note first that for any projection $e\in\mathcal{P}(M)$ we have $\mathcal{J}(M)=\mathcal{J}(pMp)$ (cf. [ ],). Thus in estimating $\mathcal{J}(M)$ we may assume that M contains a rigid subfactor $N\subset M$ with $1_N=1_M$ and $N'\cap M=\mathbb{C}$. For each $k\in\mathcal{J}(M)$ let $M_k\subset M$, $[M:M_k]=k$. If $\mathcal{J}(M)$ is uncountable then for some k_o , $S_o=\mathcal{J}(M)\cap[1,k_o]$ is uncountable. For each $k\in S_o$ let $M_k^o\subset M_k$ be so that M is the extension of M_k by M_k^o (cf. [ ],) and let $e_k\in(M_k^o)'\cap M$ implement the conditional expectation of M_k onto M_k^o as in [ ] , i.e. $\tau(e_k)=k^{-1}$, $e_kxe_k=E_{M_k^o}(x)e_k$ for $x\in M_k$. Since N is a type II_1 factor, for each $k\in S_o$ there is a unitary element $u_k\in M$ so that $u_kNu_k^*\ni e_k$. Then $N_k^o=e_ku_kNu_k^*e_k$ is a rigid subfactor of $M_k^oe_k$ which in turn is isomorphic to M_k^o. We denote by $N_k\subset M_k^o$ the image of N_k^o via this isomorphism. Since $(N_k^o)'\cap M_k^oe_k\subset(N_k^o)'\cap e_kMe_k=\mathbb{C}$ it follows that $N_k'\cap M_k^o=\mathbb{C}$. Thus by A.5 $N_k' \cap M$ is finite dimensional for each $k \in S_o$. Morover we have $e_k \in N_k' \cap M$ by construction of N_k. Now by theorem 4.5.1 the set $\{N_k \mid k \in S_o\}$ is countable modulo conjugacy by unitary elements in M. But $N_k' \cap M$ is an invariant for such conjugacy. Thus the set $\{0<t<1 \mid$ there exists $k \in S_o$ and $e \in N_k' \cap M$ with $\tau(e)=t\}$ is countable. This is a contradiction, because $e_k \in N_k' \cap M$ and the corresponding traces $\tau(e_k)=k^{-1}$ form an uncountable set when k runs over S_o.

Q.E.D.

4.6.2. COROLLARY. There exist separable type II_1 factors without property T having countable fundamental group. In particular there exist uncountable many nonisomorphic non Γ type II_1 factors without property T.

Proof. Let M be a rigid type II_1 factor and (M_o, τ_o) an arbitrary finite von Neumann algebra with a nondiscrete trace preserving automorphisms group Aut M_o (τ_o is as usual a normalized faithful normal trace on M_o). By 4.1.11, $M*M_o$ is not rigid and by [ ] we have $M' \cap (M*M_o)=\mathbb{C}$.

Another example can be obtained as follows. Let N be a rigid factor, ω a free ultrafilter on $\mathbb{N}$, N^ω the corresponding ultrapower II_1 factor ([ ]). Then by [ ], $N' \cap N^\omega = \mathbb{C}$ so that any von Neumann subalgebra M with $N \subset M \subset N^\omega$ is a II_1 factor with $N' \cap M = \mathbb{C}$. Since N^ω is nonseparable there exists an increasing sequence of separable distinct II_1 factors $M_k \subset N^\omega$ all containing N. By 4.4.1, $M=\overline{\cup M_k}$ is not rigid.

Q.E.D.

4.6.3. REMARKS. 1°. Note that rigidity properties of the type 4.1.4 (i.e. discreteness of the automorphisms group) or of the type 4.4.1 don't follow from the existence of rigid subfactors with trivial relative commutant. Indeed the exemple $M \subset M*M_o$, where M_o is completely nonatomic and M is a rigid type II_1 factor, has the property $M' \cap (M*M_o)=\mathbb{C}$ but Aut/Int$(M*M_o)$ contains an injective image of Aut M (and in particular Int M) while the second exemple in the proof of 4.6.2 has an increasing sequence of subfactors which don't stop.

2°. In [ ] Connes proved the existence of non Γ (or full) type II_1 factors with nontrivial fundamental group. On the other hand in the years 1970's it was a feeling that there are only few non Γ II_1 factors. Of course by Connes' result [ ] (see 4.5.3) and by 4.6.2 this is not the case. However the following strengthening of the conjecture 4.5.5 may hold true: there are only countable many classes of stable equivalent non Γ type II_1 factors.

3°. We mention that the methods we used until now as well as the techniques of the next section may give the possibility to construct a type II_1 factor with property Γ (but not isomorphic with its tensor product by R!) with countable fundamental group. We leave this as an open problem.

§4.7 Rigidity and Cartan subalgebras

A particular case of 4.6.1 in the preceding section is as follows: let G be an I.C.C. group with property T acting freely, ergodically and measure preserving on a completely nonatomic probability space $(X, \mathcal{X}, \mu)$. Let M denote the cross product type II_1 factor $M = L^\infty(X,\mu) \rtimes G$ and $N \subset M$ be the rigid subfactor $L(G) \subset L^\infty(X,\mu) \rtimes G$. Let also $A = L^\infty(X,\mu) \subset M$. Then $N' \cap M = \mathbb{C}$ so that by 4.6.1 $\mathcal{F}(M)$ is countable. In particular it follows that if R(G) is the measured equivalence relation given by the action of G on X then, except for a countable set of values t, the restricted equivalence relations $R(G)|_{E_t}$, where $E_t \subset X$ is a subset of measure t, are not orbit equivalent to R(G) (in the sense of [ ]). The reason is that the associated II_1 factors ([ ]) are nonisomorphic, and this is of course a sufficient condition for the corresponding measured equivalence relations not to be orbit equivalent. But Connes and Jones have shown in [ ] that this is not a necessary condition: the II_1 factors may well be isomorphic but not necessarily so that the Cartan subalgebras be carried one onto the other.

The results of this section deal precisely with this kind of problems: given a von Neumann subalgebra $B \subset M$ and a projection $e \in B$ we try to find obstructions for the existence of isomorphisms of M onto eMe carying B onto eBe. Of course, the interesting case is when M is isomorphic to eMe and B to eBe. In these cases the obstructions

for the existence of such isomorphisms will come out from rigidity properties. We first state the theorem that provided the motivation for this study: it shows that if a measured equivalence relation $\mathcal{R}$ contains an ergodic action of a rigid group then most of the restrictions of $\mathcal{R}$ to subsets of positive measure are not orbit equivalent to $\mathcal{R}$. This is a new type of rigidity result, even in ergodic theory.

4.7.1. THEOREM. Let $\mathcal{R}$ be an ergodic countable measured equivalence relation on a nonatomic probability space with $\mathcal{R}$-invariant measure $(X, \mathcal{X}, \mu)$. Suppose $\mathcal{R}$ contains the free ergodic action of an I.C.C. group with property T. Then there is a countable set $S_o \subset [0,1]$ such that whenever $F \in \mathcal{X}$, and $\mu(F) \notin S_o$ the restriction of $\mathcal{R}$ to F is not orbit equivalent to $\mathcal{R}$. In other words, if M is the type II_1 factor with normalized trace τ constructed from $\mathcal{R}$ as in [ ] and if $A = L^\infty(X, \mathcal{X}, \mu) \subset M$ is the corresponding Cartan subalgebra then for any projection $e \in A$ with $\tau(e) \notin S_o$ there are no isomorphisms of M onto eMe carying A onto Ae.

A main interest of the above theorem is related to the following:

4.7.2. COROLLARY. There exist separable type II_1 factors with uncountable many nonconjugate Cartan subalgebras. More precisely there exists a separable type II_1 factor M with a Cartan subalgebra $A \subset M$ so that $M \simeq M \bar{\otimes} R$ (and thus $M \simeq eMe$ for all e), but so that for a certain countable set $S_o \subset [0,1]$, given any projection $e \in A$ with $\tau(e) \notin S_o$ there are no isomorphisms of M onto eMe carying A onto Ae.

Proof of 4.7.2. By [ ], given any I.C.C. group G_o with the property T there is a free measure preserving action σ of $G_o \times H$, where H is a suitable amenable group, on the nonatomic probability space $(X, \mathfrak{X}, \mu)$ so that $\sigma|_{G_o}$ is ergodic and so that the corresponding II_1 factor $M = L^\infty(X,\mu) \rtimes_\sigma (G_o \times H)$ has noncommuting central sequences. Thus $M = \dot{M} \bar{\otimes} R$ (by [ ]) and the rest of the conclusion follows by 4.7.1.

Q.E.D.

Although with a statement of ergodic theory flavor, Theorem 4.7.1 has a purely operator algebra proof. It is in fact the immediate consequence of the following more general:

4.7.3. THEOREM. Let M be a separable type II_1 factor and $B \subset M$ a von Neumann subalgebra ~~of B in M generates a factor~~. Assume there exist type II_1 subfactors $N_o \subset N \subset M$ so that $B \subset N$, $N_o' \cap N = \mathbb{C}$, $N' \cap M = \mathbb{C}$ and so that the factor N_o and the inclusion $B \subset N$ are rigid. Assume further that the normalizer of B in N acts ergodically on the center of B. Then there exists a countable set $S_o \subset [0,1]$ such that for every projection $e \in B$ with $\tau(e) \notin S_o$ there are no isomorphisms of M onto eMe carrying B onto eBe.

Proof of 4.7.3. Suppose the set $S_o = \{0 < t < 1 \mid$ there exists a projection $e \in B$, $\tau(e) = t$, and an isomorphism of M onto eMe carrying B onto eBe$\}$ is uncountable. For each $t \in S_o$ we choose a projection $e_t \in B$, $\tau(e_t) = t$, with an isomorphism θ_t of M onto $e_t M e_t$ such that $\theta_t(B) = e_t B e_t$. We denote by $N_t = \theta_t^{-1}(e_t N e_t)$. Since $e_t B e_t \subset e_t N e_t$ is rigid (cf. 4.(9.1)) it follows that $B = \theta_t^{-1}(e_t B e_t) \subset \theta_t^{-1}(e_t N e_t) = N_t$ is also rigid, for all $t \in S_o$.

By 4.5.1 it follows that there is an uncountable set $S_1 \subset S_o$ so that for all $t, t' \in S_1$, N_t is inner conjugate to $N_{t'}$. So, if we fix $t_1 \in S_1$ and $N^1 = N_{t_1}$, then for each $t \in S_1$ there is a unitary element $u_t \in M$ so that $u_t^* N_t u_t = N^1$. Thus $\theta_t \mathrm{Ad} u_t (N^1) = e_t N e_t$, $t \in S_1$, (but no longer $\theta_t \circ \mathrm{Ad} u_t (B) = e_t B e_t$ – anyway we don't need this condition anymore). Since N is a type II_1 factor, there are unitary elements $v_t \in N$ such that $\{v_t e_t v_t^*\}_{t \in S}$ is a totally ordered set of projections (i.e. $t \leq t'$ implies $v_t e_t v_t^* \leq v_{t'} e_{t'} v_{t'}^*$). Let $\sigma_t = \mathrm{Ad} v_t \circ \theta_t \circ \mathrm{Ad} u_t$. Then for each pair $t, t' \in S_1$, $t < t'$ we have a surjective $*$-isomorphism $\theta_{t,t'} = \sigma_{t'}^{-1} \sigma_t : N^1 \to f N^1 f$ where $f = \sigma_{t'}^{-1} \sigma_t(1)$ and $\tau(f) = t/t'$.

But N^1 contains a rigid subfactor with trivial relative commutant, namely if v is a unitary element in N so that $v N_o v^* \ni e_{t_1}$ (which allways exists, since N_o is a type II_1 factor) th $e_{t_1} v N_o v^* e_{t_1}$ is a rigid subfactor of $e_{t_1} N e_{t_1}$ so that $N^1 = \theta_{t_1}^{-1}(e_{t_1} N e_{t_1})$ will contain $N^o = \theta_{t_1}^{-1}(e_{t_1} v N_o v^* e_{t_1})$ as a rigid subfactor and since $(e_{t_1} v N_o v^* e_{t_1})' \cap e_{t_1} N e_{t_1} = \mathbb{C}$ we also have $(N^o)' \cap N^1 = \mathbb{C}$. Thus by 4.6.1 the fundamental group of N^1 is countable. Thus, with the above notations, it follows that the set $\{t/t' \mid t < t', t, t' \in S_1\}$ is countable. But this is a contradiction, since S_1 itself is uncountable. This contradiction completes the proof.

Q.E.D.

Proof of 4.7.1. If $M \supset A$ are as defined in the statement then it follows by the hypothesis that there is a free ergodic action σ of the I.C.C. property T group G on A. Let the corresponding crossed product type II_1 factor $N=A\rtimes_{\sigma} G$ be embedded in M in the obvious way to contain A. If $N_o=L(G)$ then the inclusion $A\subset N$ and the factor N_o are rigid and $N_o'\cap N=\mathbb{C}$, $N'\cap M=\mathbb{C}$. Thus 4.7.3 applies.

Q.E.D.

Finally note that to prove 4.7.3 (and thus 4.7.1 too) we used the notion and technical results an rigid inclusions of the preceding sections in full generality, thus providing an effective motivation for considering such generalizations.

APPENDIX

We prove here several technical devices that have been use in the paper.

The next result is a generalization of Connes' characterisation of non Γ II_1 factors by a property of their automorphism group.

A.1. PROPOSITION. Let $B\subset M$ be a von Neumann subalgebra of the type II_1 factor M. The following conditions are equivalent:

1°. Given any finite set of elements $F\subset M$ There exists a sequence of unit vectors $\{\xi_n\}_n\subset L^2(B'\cap M,\tau)$ such that $\|\xi_n x-x\xi_n\|_2\to 0$, $x\in F$, $\langle\xi_n,1\rangle=0$, $\forall n$.

2°. Given any finite set of elements $F\subset M$ There exists a sequence of unitary elements $\{u_n\}_n$ in $B'\cap M$ such that $\|[u_n,x]\|_2\to 0$, $x\in F$, and $\tau(u_n)=0$, $n\geq 1$.

3°. $\mathrm{Int}_B M$ is not closed in $\mathrm{Aut}_B M$.

Proof. The proof of the equivalence between 2° and 3° is the same as the proof of the case $B=\mathbb{C}$ in [ ] while the proof of 1° <=> 2° is the same as the one in [ ].

The next result is a strengthening of [ ]. The proof uses [ ].

A.2. PROPOSITION. Let M be a finite factor $M_o \subset M$ a von Neumann subalgebra and $\alpha = \sup\{\|x-E_{M_o}(x)\|_2 \mid x \in M,\ \|x\| \leq 1\}$.

Let f be an arbitrary projection in $M_o' \cap M$ and f_o an atom of $M_o' \cap M$. Then we have:

a) $\alpha \geq 2\min\{\tau(f),\ 1-\tau(f)\}$

b) If t is the index of $f_o M_o f_o$ in $f_o M f_o$ then $\alpha \geq \tau(f_o)\sqrt{1-t^{-1}}$

In particular if $M_o' \cap M = \mathbb{C}$ then $\alpha \geq 1-[M:M_o]^{-1}$.

Proof.

The next two results belong to E. Christensen. For the reader's convenience, we give them full proofs here.

A.3. PROPOSITION. Let M be a type II_1 factor, $N_o, N \subset M$ type II_1 subfactors of M. Suppose $\sup\{\|x-E_N(x)\|_2 \mid x \in N_o, \|x\| \leq 1\} = \delta < 10^{-6}$. Then there exist projections $e_o \in N_o$, $e \in N$, a unital $*$-isomorphism $\theta: e_o N_o e_o \to eNe$, projections $f_o \in N_o' \cap M$, $f \in \theta(e_o N_o e_o)' \cap eMe$ and a partial isometry $u \in M$ such that

1°. $\|1-e_o\|_2 < 2\delta^{1/2}$; $\|1-e\|_2 < 2\delta^{1/2}$;

2°. $\|\theta(e_o x e_o) - e_o x e_o\|_2 \leq 10^2 \delta^{1/2}$;

3°. $\|1-f_o\|_2 < 10^3 \delta^{1/2}$; $\|1-f\|_2 \leq 10^3 \delta^{1/2}$;

4°. $u^*u = e_o f_o$; $uu^* = ef$; $\|1-u\|_2 < 2 \cdot 10^3 \delta^{1/2}$

5°. $u e_o x e_o = \theta(e_o x e_o) u$, $x \in e_o N_o e_o$.

A.4. PROPOSITION. Let M be a finite von Neumann algebra and $N\subset M$ a von Neumann subalgebra (with 1_N possible different from 1_M) and $\theta : N \to M$ a *-isomorphism of N into M so that $\|\theta(x)-x\|_2 \le t$ for all $x\in N$, $\|x\|\le 1$. Then there exist projections $f\in N'\cap 1_N M 1_N$, $p\in\theta(N)'\cap\theta(1)M\theta(1)$ and a partial isometry $v\in M$ such that

1°. $\|1-f\|_2\le 2t$; $\|1-p\|_2\le 2t$; $\|1-v\|_2\le 4t$;

2°. $v^*v=f$; $vv^*=p$;

3°. $vx=\theta(x)v$, $x\in N$.

The final result we present in this appendix estimates the relative commutant of a subfactor when passing to larger factors.

A.5. THEOREM. Let $N\subset M_0\subset M$ be type II_1 factors. Suppose $[M:M_0]<\infty$ and $N'\cap M_0$ is finite dimensional. Then $N'\cap M$ is finite dimensional. if $N'\cap M_0 = \mathbb{C}$ then $\dim N'\cap M \le [M:M_0]$. ~~More precisely dim $N'\cap M\le(\dim N'\cap M_0)([M:M_0]$~~

~~Proof. The fact that $N'\cap M$ is finite dimensional follows trivially by in [ ]. To get the estimate we proceed as follows.~~

Proof. Suppose there exists a sequence of projections $0 \neq f_n < N' \cap M$ such that $\tau(f_n) \to 0$. Let $e_1, \dots, e_m$ be minimal projections in $N' \cap M_o$ such that $\sum e_i = 1$. For each n there is an i so that $\| e_i f_n e_i \| \geq 1/m^2$. Indeed because otherwise $\| f_n e_i \| < 1/m$ so that $\| f \| \leq \sum_i \| f_n e_i \| < 1$ a contradiction. Taking a subsequence if necessary we may assume $\| e_i f_n e_i \| \geq 1/m^2$ for all n. Then $E_{M_o}(e_i f_n e_i) \in N' \cap M_o$ so that $E_{M_o}(e_i f_n e_i)$ is a scalar multiple of e_i say $E_{M_o}(e_i f_n e_i) = \lambda_n e_i$. Since $E_{M_o}(x) \geq \lambda x$, where $x \in M_+$ and $\lambda = [M:M_o]^{-1}$, it follows that $\lambda_n / m^2 \geq \lambda$ for all n. But $\lambda_n \leq \tau(f_n)/\tau(e_i)$ so that $\tau(f_n) \geq \lambda \tau(e_i) m^2$, a contradiction.

The estimate on dim $N' \cap M$ when $N' \cap M_o = \mathbb{C}$ follows easily by 1.3 in[ ], taking an orthonormal basis $m_1, \dots, m_k$ of $N' \cap M$ with respect to the trace τ . Other estimates for the general case follows by 6.1 in [ ].

Q.E.D.

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