CLASSIFICATION OF ACTIONS OF DISCRETE AMENABLE GROUPS ON AMENABLE SUBFACTORS OF TYPE II

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Abstract. We prove a classification result for properly outer actions σ of discrete amenable groups G on strongly amenable subfactors of type II, $N \subset M$, a class of subfactors that were shown to be completely classified by their standard invariant $\mathcal{G}_{N,M}$, in ([Po7]). The result shows that the action σ is completely classified in terms of the action it induces on $\mathcal{G}_{N,M}$. As a an application of this, we obtain that inclusions of type III_λ factors, $0 < \lambda < 1$, having discrete decomposition and strongly amenable graph, are completely classified by their standard invariant.

0. Introduction. In ([C1]) A.Connes classified the amenable semifinite factors showing that, up to isomorphism, there is only one of type II_1, the unique approximately finite dimensional II_1 factor R of Murray and von Neumann, also called the hyperfinite factor, and one of type II_∞ ($R \otimes \mathcal{B}(\mathcal{H})$). Then, motivated by the problem of classifying infinite amenable factors of type III, automorphisms of amenable factors of type II were classified in ([C2,5]). Further classification results were proved for actions of finite and general amenable groups on R and $R \otimes \mathcal{B}(\mathcal{H})$ in ([J1]) and respectively ([Oc]).

For inclusions of factors $N \subset M$ of finite Jones index $[M : N] < \infty$, the suitable notion of amenability was introduced in ([Po7]). Also, it was proved in ([Po7]) that the amenable subfactors coincide with the subfactors that can be approximated by the finite dimensional subalgebras of their higher relative commutants. In the case of a trivial inclusion $N = M \subset M$ this corresponds to the uniqueness of the amenable type II_1 factor. In general, this shows that amenable inclusions are completely classified by their *standard invariant* $\mathcal{G}_{N,M}$, the graph type combinatorial object that encodes the lattice of higher relative commutants in the Jones tower ([Po7]). $\mathcal{G}_{N,M}$ consists of a pair of weighted, pointed, bipartite graphs $(\Gamma_{N,M}, \vec{s})$, $(\Gamma'_{N,M}, \vec{s'})$ called the *standard* (or the *principal*) *graphs* of $N \subset M$ ([J2]), with some additional structure. The invariant gives rise to a *canonical model* $N^{st} \subset M^{st}$ and in fact the theorem in ([Po7]) states that $N \subset M$ is *strongly amenable*, i.e., it is amenable and its standard graph is ergodic, if and only if $N \subset M$ is isomorphic to its canonical model.

We will prove in this paper a classification result for properly outer actions of amenable groups on strongly amenable inclusions of type II_1 and II_∞ factors. The result can be regarded as an equivariant version of ([Po7]). The main motivation for studying this problem is, as in the single von Neumann algebra case, the classification of inclusions of type III_λ factors, $0 < \lambda < 1$, for which a similar Connes type discrete decomposition holds as shown in ([Lo1]). Thus, by our results it follows that the classification of type III_λ inclusions, $0 < \lambda < 1$, reduces to the classification of trace scaling actions of $\mathbb{Z}$ on inclusions of type II_∞ factors.

More precisely, our first result (see Theorem 2.1) shows that if an inclusion $N^\infty \subset M^\infty$ of hyperfinite II_∞ factors is extremal and strongly amenable, then a trace scaling automorphism σ on it splits into the tensor product between the action σ^{st}, implemented by σ on the model inclusion of II_1 factors $N^{st} \subset M^{st}$, and a model action σ_0 on a commonly splitted II_∞ factor. Thus, $(N^\infty \subset M^\infty, \sigma) = ((N^{st} \subset M^{st}) \otimes R^\infty, \sigma^{st} \otimes \sigma_0)$. More generally, we prove that for a trace scaling action of $\mathbb{Z}^n$ on $N^\infty \subset M^\infty$ which is diagonalizable (in the obvious sense) a similar splitting result holds true.

An important application of the above result is the classification of inclusions of hyperfinite type III_λ factors with discrete decomposition and strongly amenable graph, i.e. of the form $(\mathcal{N} \subset \mathcal{M}) = (N^\infty \rtimes \sigma \subset M^\infty \rtimes \sigma)$, with $N^\infty \subset M^\infty$ a strongly amenable inclusion of type II_∞ factors and σ a λ-scaling automorphism of M^∞ leaving N^∞ globally invariant. Thus, our theorem implies that $\mathcal{N} \subset \mathcal{M}$ is isomorphic to the inclusion $(N^{st} \otimes R^\infty \rtimes \sigma^{st} \otimes \sigma_0 \subset M^{st} \otimes R^\infty \rtimes \sigma^{st} \otimes \sigma_0)$, where $N^{st} \subset M^{st}$ is the canonical model associated with $N^\infty \subset M^\infty$, σ^{st} is the action implemented on it by σ, and σ_0 is a model λ-scaling automorphism on the hyperfinite II_∞ factor R^∞.

Although it is not needed for the classification of type III_λ inclusions, we also prove a classification result for properly outer actions θ of arbitrary discrete amenable groups G on strongly amenable inclusions of type II_1 factors $N \subset M$ as well (see Theorem 3.1). It shows that θ is cocycle conjugate to an action of the form $\theta^{\mathrm{st}} \otimes \sigma_0$ on $(N^{\mathrm{st}} \subset M^{\mathrm{st}})\overline{\otimes}R$, with θ^{st} the "standard" part of θ and σ_0 a properly outer action of G on the hyperfinite II_1 factor R.

Thus, in all these cases the actions are completely classified (up to outer conjugacy) by the actions they implement on the standard invariant $\mathcal{G}_{N,M}$. Due to its rigid combinatorial structure $\mathcal{G}_{N,M}$ generally admits only a few (finitely many) actions, oftenly just the trivial action. In the case of the index ≤ 4 all such actions were listed ([Lo1,2]).

The proofs of both theorems rely on *non-commutative ergodic theory techniques*, much in the spirit of ([Po1,2,4,7]). The idea we use is to build "local Rohlin towers" for larger and larger finite parts of the acting group $\mathbb{Z}^n, G$, indexed by corresponding Følner sets. We then "glue them together" by using maximality arguments, very much the same way we did in [P07], inspired by similar arguments in [C3]. As a result of this argument, we obtain that σ splits into a tensor product of σ^{st} and a trace scaling action σ_0. In case $n = 1$, we can then further apply Connes Theorem in [C2] to derive the final result. In turn, for the proof of the classification of (trace preserving) actions of arbitrary amenable groups G on $N \subset M \simeq R$, we ultimately use Ocneanu's uniqueness (up to cocycle conjugacy) of cocycle actions of G on the hyperfinite II_1 factor [Oc]. However, as pointed out in [Po7,8], note that our classification of strongly amenable subfactors in [Po7] does imply both the uniqueness of the trace scaling automorphisms of R^∞ in [C2] and the case G strongly amenable of the results in [Oc], for which it thus provides alternative proofs.

Part of the results in this paper have been presented by the author in a number of lectures during 1991-1992 and in a C. R. Acad. Sci. Paris note ([Po8]). A preliminary form of the paper has been circulated by the author since the fall of

1991 and in its final form as an IHES preprint no. 46/1992.

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Added in the proof, July 2009. While the work presented in this paper has been completed some 18 years ago, its typing proved to be an agonizing process. Thus, my traveling schedule during 1991-1992 resulted into several people being involved with this task. They had to cope with a manuscript full of complicated formulas and a handwriting unfamiliar to them, under limited time-frame. The outcome was, alas, a 1992 preprint with hundreds and hundreds of typos, a nightmare to correct. As the results in the paper were fully accepted by the mathematical community and amply used and cited in subsequent papers, I felt little incentive to go through the necessary proof reading ordeal, for several years. Then other mathematical interests prevailed, all the way until the Summer of 2009, when I needed some of the results and techniques in this paper, to study rigidity properties of certain inductive limits of II_1 factors. This gave me the energy to carefully go through the paper and make the corrections. Other than that, I have chosen to leave the original 1992 preprint essentially unchanged, although I certainly would have written the paper quite differently today...

While initially the main interest in this paper was due to the application to the classification of type III_λ subfactors, a hot topic at the time, the techniques used here seem to be of a broader interest in von Neumann algebras, including deformation/rigidity theory. Hence my decision to revive the paper and seek its publication in a refereed journal, despite so many years of neglect. Some of the ideas used in this approach to the classification of actions of amenable groups on inclusions of hyperfinite type II factors, are quite novel. Thus, unlike the usual strategy for classifying trace scaling automorphisms of the hyperfinite II_∞ factor R^∞ due to Connes ([C2]), which consists in reducing to the trace preserving case (resulting into a "type II_1 treatment"), one works here directly in the II_∞ setting. Actually, one relies heavily on the trace scaling property of the automorphism, to prove it coincides with the model. Also, in the classification (up to cocycle conjugacy) of actions θ of amenable groups G on strongly amenable II_1 subfactor $N \subset M \simeq R$, the model considered is a cocycle action θ_0 on R, rather than a genuine action (see Sec. 3.2)!

It is worth mentioning that the techniques and ideas in this paper have later inspired me in isolating the concept of *central freeness* and *approximate innerness* for subfactors in [P9,10] and in the proof of the classification of approximately inner, centrally free subfactors with amenable graph in [P9,10,11] (note that these are the only papers I have added to the reference list of the initial 1992 version). In particular, [Po10] provided a new proof to the classification of hyperfinite III_λ subfactors with strongly amenable graph, in case the standard part of the action of the trace scaling automorphism in the common discrete decomposition is trivial (i.e., with the above notations, $\sigma^{st} = id$).

1. The standard invariant of an equivariant inclusion. We begin by explaining the standard invariant and model associated to a triple $(N \subset M,\ \theta)$, where $\theta \in \mathrm{Aut}(M,\ N) \overset{\text{def}}{=} \{\sigma \in \mathrm{Aut}M | \sigma(N) = N\}$, as considered by Loi in [Lo1], for inclusions of type II_1 or II_∞ factors $N \subset M$ of finite index.

1.1. *Inclusions of type* II_∞ *factors.* If $N^\infty \subset M^\infty$ are type II_∞ factors then there exists a normal conditional expectation of M^∞ onto N^∞ if and only if $N^\infty \vee (N^{\infty\prime} \cap M^\infty)$ contains finite projections of M^∞ (see e.g. [Po6]). If N^∞ contains finite projections of M^∞ then there exists a unique trace preserving conditional expectation of M^∞ onto N^∞, E_{N^∞} and, in fact, if $N^\infty = N\overline{\otimes}\mathcal{B}(\mathcal{H})$ is any splitting of N^∞ with N a type II_1 factor and $M = \mathcal{B}(\mathcal{H})' \cap M^\infty$, then $(N^\infty \subset M^\infty) = (N\overline{\otimes}\mathcal{B}(\mathcal{H}) \subset M\overline{\otimes}\mathcal{B}(\mathcal{H}))$ and $E_{N^\infty} = E_N \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})}$.

The index of N^∞ in M^∞, $[M^\infty : N^\infty]$, is defined to be ∞ if there is no normal conditional expectation of M^∞ onto N^∞ or, more generally, if N^∞ doesn't contain finite projections of M^∞. It is defined by

$$[M^\infty : N^\infty] = (\max\{\lambda \geq 0 | E_{N^\infty}(x) \geq \lambda x,\ x \in M^\infty_+\})^{-1}$$

in case N^∞ contains finite projections of M^∞. Alternatively, with the above notations, we can define the index of $N^\infty \subset M^\infty$ by $[M^\infty : N^\infty] \overset{\text{def}}{=} [M : N]$. If the index is finite then we can associate to $N^\infty \subset M^\infty$ the tower of embeddings ([Po7]):

$$\begin{array}{ccccccc} N^\infty & \subset & M^\infty & \overset{e_1}{\subset} & M_1^\infty & \overset{e_2}{\subset} & \cdots \\ \cup & & \cup & & \cup & & \\ N & \subset & M & \overset{e_1}{\subset} & M_1 & \overset{e_2}{\subset} & \cdots \end{array}$$

where $M_k^\infty = M_k\overline{\otimes}\mathcal{B}(\mathcal{H})$ and $e_k \in M_k \simeq M_k \otimes 1 \subset M_k \otimes \mathcal{B}(\mathcal{H})$ are the usual Jones projections for the Jones' tower $N \subset M \overset{e_1}{\subset} M_1 \overset{e_2}{\subset} M_2 \subset \cdots$.

Moreover, due to the splitting of $N^\infty \subset M^\infty$ into a type II_1 inclusion $\otimes\mathcal{B}(\mathcal{H})$, there exist projections $e_0 \in N^\infty \subset M^\infty$ so that $E_{N^\infty}(e_0) = \lambda 1 = [M^\infty : N^\infty]^{-1}1$. The proof of [PiPo1] then shows that any two such projections are conjugate by a unitary element of N^∞. If e_0, $e_0^0 \in M^\infty$ are as above, with $E_{N^\infty}(e_0) = E_{N^\infty}(e_0^0) = \lambda 1$, and $N_1^\infty = \{e_0\}' \cap N^\infty$, $N_1^{0,\infty} = \{e_0^0\}' \cap N^\infty$, with $e_0 \in N \otimes 1 \subset N^\infty$ and $N_1 = \{e_0\}' \cap N$, then the unitary element $u \in N^\infty$, $ue_0u^* = e_0^0$, also satisfies $uN_1^\infty u^* = N_1^{0,\infty}$. We have $N_1^\infty = N_1 \otimes \mathcal{B}(\mathcal{H})$ and $N_1^{0,\infty} = N_1^\infty \otimes \mathcal{B}(\mathcal{H}^0)$ where $N_1^0 = uN_1u^*$, $\mathcal{B}(\mathcal{H}^0) = u\mathcal{B}(\mathcal{H})u^*$. Note that the conjugation by u changes the splitting of $N^\infty \subset M^\infty \overset{e_1}{\subset} M_1^\infty \subset \cdots$ by $\mathcal{B}(\mathcal{H})$ into a splitting by $\mathcal{B}(\mathcal{H}^0)$.

It thus follows that we may choose recursively a tunnel of type II_∞ factors $M_\infty^\infty \overset{e_0^0}{\supset} N^\infty \overset{e_{-1}^0}{\supset} N_1^{0,\infty} \overset{e_{-2}^0}{\supset} N_2^{0,\infty} \supset \cdots$ so that for each k there exists $\mathcal{B}(\mathcal{H}_k^0) \subset N_k^{0,\infty}$ with the property that $N_i^0 = \mathcal{B}(\mathcal{H}_k^0)' \cap N_i^{0,\infty}$ are type II_1 factors and $M \overset{e_0^0}{\supset} N \overset{e_{-1}^0}{\supset} N_1^0 \supset \cdots \overset{e_{-k+1}^0}{\supset} N_{k-1}^0 \supset N_k^0$ is a tunnel of type II_1 factors for $M \supset N$.

It thus follows that one can define for a type II_∞ inclusion $N^\infty \subset M^\infty$ its *standard invariant*, in the same way one does for type II_1 inclusion, as the sequence of commuting squares of higher relative commutants $\{N_k^{\infty\prime} \cap N^\infty \subset N_k^{\infty\prime} \cap M^\infty\}_k$, on which one has a canonical finite trace, and which doesn't depend on the choice of the tunnel. Moreover this invariant, that we denote $\mathcal{G}_{N^\infty, M^\infty}$, coincides with the standard invariant $\mathcal{G}_{N,M}$ of the corresponding inclusion of type II_1 factors $N \subset M$, independently on the choice of $N \subset M$ (e.g., by [Po7]). Also, we denote by $M_\infty^\infty = \overline{\cup M_k^\infty}$ with the closure being taken in the strong topology of the common trace.

Note that M^∞_∞ coincides with $M^\infty \otimes B(\mathcal{H})$. It is called the *enveloping algebra* of $N^\infty \subset M^\infty$.

Finally, let us define the notion of amenability for inclusions of type II_∞ factors. In order to avoid lengthy discussions and too abstract statements (which, unlike in the II_1 case where they were quite necessary, are here practically useless) we will adopt the simple minded point of view of reducing to the type II_1 case. We thus put:

Definition. $N^\infty \subset M^\infty$ is *amenable* if $N \subset M$ is amenable. It is *strongly amenable* (respectively, has *ergodic care*) if $N \subset M$ is strongly amenable (respectively, has ergodic core).

1.2. *The standard part of an automorphism.* Let $N \subset M$ be both type II_1 factors and $(N^\infty \subset M^\infty) = N \otimes \mathcal{B}(\mathcal{H}) \subset M \otimes \mathcal{B}(\mathcal{H})$, the corresponding type II_∞ amplification. Assume $[M : N] = [M^\infty : N^\infty] < \infty$. Denote by $\mathrm{Aut}(M^\alpha,\ N^\alpha) = \{\theta \in \mathrm{Aut}M^\alpha | \theta(N^\alpha) = N^\alpha\}$ where $\alpha = 1$ or $\alpha = \infty$.

Let us first point out that the automorphisms in $\mathrm{Aut}(M^\infty, N^\infty)$ always commute with the trace-preserving expectation.

LEMMA. *If* $\theta \in \mathrm{A}ut(M^\infty, N^\infty)$ *then* $\theta E_{N^\infty} = E_{N^\infty}\theta$ *and the action of* θ *on* $N^{\infty\prime} \cap M^\infty$, *equipped with its canonical trace* (*see* 1.1), *is trace preserving.*

Proof. Since an expectation of M^∞ onto N^∞ is uniquely determined by its values on $N^{\infty\prime} \cap M^\infty$, by comparing E_{N^∞} and $\theta E_{N^\infty}\theta^{-1}$ we see that it is sufficient to prove that they agree on $N^{\infty\prime} \cap M^\infty$. For this it is sufficient to show that $\theta|_{N^{\infty\prime}\cap M^\infty}$ is trace preserving. This is clear if $\mathrm{mod}\theta = 1$. If $\mathrm{mod}\theta = \lambda \neq 1$ then let R^∞ be a hyperfinite type II_∞ factor and σ an automorphism of R^∞ with $\mathrm{mod}\sigma = \lambda^{-1}$. Then clearly $(N^\infty \overline{\otimes} R^\infty)' \cap M^\infty \overline{\otimes} R^\infty = N^{\infty\prime} \cap M^\infty \otimes 1$ and the action of $\theta \otimes \sigma$ on $(N^{\infty\prime} \cap M^\infty) \otimes 1$ coincides with that of $\theta|_{N^{\infty\prime}\cap M^\infty} \otimes 1$. But by the above case $\theta \otimes \sigma$ acts trace preservingly on $(N^\infty \otimes R^\infty)' \cap M^\infty \otimes R^\infty$. Q.E.D.

We can now describe Loi's construction ([Lo2]), based on ([PiPo1]), of the action implemented by θ on the higher relative commutants, for arbitrary actions θ on inclusions of type II factors.

(i) Given any choice of the tunnel

$$M^\alpha \overset{e_0}{\supset} N^\alpha \overset{e_{-1}}{\supset} N_1^\alpha \supset \cdots$$

there exist unitary elements $u_k \in N_k^\alpha,\ k \geq 0$, such that $\mathrm{Ad}(u_k \cdots u_0)\theta(e_{-k}) = e_{-k}$ (cf.[PiPo1]).

(ii) For $x \in \bigcup\limits_k (N_k^{\alpha\prime} \cap M^\alpha)$ define

$$\theta^{\mathrm{st}}(x) \overset{\mathrm{def}}{=} \mathrm{Ad}(\cdots u_k \cdots u_0)\theta(x).$$

Then $\theta^{\mathrm{st}}(N_k^{\alpha\prime} \cap M^\alpha) = N_k^{\alpha\prime} \cap M^\alpha,\ \theta^{\mathrm{st}}(N_k^{\alpha\prime} \cap N^\alpha) = N_k^{\alpha\prime} \cap N^\alpha,\ \theta^{\mathrm{st}}(e_{-k}) = e_{-k}$, for all $k \geq 0$. Moreover, in case $\alpha = \infty$, if $N_k^{\infty\prime} \cap M^\infty$ are interpreted as higher relative commutants of the corresponding type II_1 inclusions obtained by splitting all N_k^∞, $i \geq k \geq -1$, by a common $\mathcal{B}(\mathcal{H}) \subset N_k^\infty$, then $\theta^{\rm st}$ as defined above is trace preserving (with respect to the corresponding type II_1 trace).

Thus, both in the II_1 and II_∞ case, $\theta^{\rm st}$ implements a trace preserving automorphism on the union algebra $\cup(N_k^{\alpha\prime} \cap M^\infty)$, leaving $\underset{k}{\cup}(N_k^{\alpha\prime} \cap M_k^\alpha)$, $N_k^{\alpha\prime} \cap M^\alpha$, $N_k^{\alpha\prime} \cap N^\alpha$, globally invariant and all the Jones' projections fixed.

(iii) $\theta^{\rm st}$ implements a trace preserving automorphism on the standard invariant $\mathcal{G}_{N^\alpha, M^\alpha} = \{N_k^{\alpha\prime} \cap N^\alpha \subset N_k^{\alpha\prime} \cap M^\infty\}_{k \geq 0}$, which is independent on the choice of the tunnel N_k^α, on the choice of the unitaries $u_k \in N_k^\alpha$ and on perturbations of θ by inner automorphisms $\mathrm{Ad}u$, with $u \in \mathcal{U}(N^\alpha)$.

(iv) If one also denotes by θ^{st} the trace preserving automorphism of $M^{\rm st} = \overline{\cup(N_k^{\alpha\prime} \cap M^\alpha)}$ leaving $N^{\rm st} = \overline{\underset{k}{\cup}(N_k^{\alpha\prime} \cap N^\alpha)}$ globally fixed, then $\theta \mapsto \theta^{\rm st}$ implements a group homomorphism, from $\mathrm{Aut}(M^\alpha,\ N^\alpha)/\mathrm{Int}N^\alpha$ to $\mathrm{Aut}(M^{\rm st},\ N^{\rm st})$, leaving all Jones projections e_{-k} fixed and all finite dimensional algebras $N_k^{\alpha\prime} \cap N_j^\alpha$ globally invariant.

(v) Define recursively $\theta_{k+1} : M_{k+1}^\alpha \to M_{k+1}^\alpha$ by

$$\theta_{k+1}(\Sigma x_i e_{k+1} y_i) = \Sigma_i \theta_k(x_i) e_{k+1} \theta_k(y_i), k \geq 0, x_i, y_i \in M_k^\alpha.$$

Then $\theta_{k+1|M_k^\alpha} = \theta_k$ and $\theta_{k+1} \in \mathrm{Aut}(M_{k+1}^\alpha, M_k^\alpha)$. Also denote $\theta_\infty : M_\infty^\alpha \to M_\infty^\alpha$ the unique automorphism which on the dense set $\cup M_k^\alpha$ acts by $\theta_{\infty|M_k^\alpha} = \theta_k$ and by $\theta^{st,0} : M^{\alpha\prime} \cap M_\infty^\alpha \to M^{\alpha\prime} \cap M_\infty^\alpha$ its restriction to $M^{\alpha\prime} \cap M_\infty^\alpha$. Then $\theta \mapsto \theta^{st,0}$ is a group morphism from $\mathrm{Aut}(M^\alpha, N^\alpha)$ to $\mathrm{Aut}(M^{\alpha\prime} \cap M_\infty^\alpha, {M^{\alpha}}_1^\prime \cap M_\infty^\alpha)$, with $\theta^{st,0}$ leaving each Jones projection e_k fixed and $M_1^{\alpha\prime} \cap M_k^\alpha \subset M^{\alpha\prime} \cap M_k^\alpha$ globally invariant. Note that if $N^\alpha \subset M^\alpha$ is extremal ([P7]) and ^{op} denotes the canonical antiisomorphism of $N_{k-1}' \cap M$ onto $M' \cap M_k$ ([Po7]), then ^{op} intertwines θ^{st} and $\theta^{st,0}$.

Definition. The automorphism θ^{st} on $N^{st} \subset M^{st}$ is called the *standard part* of the automorphism θ (on $N \subset M$). The automorphism $\theta^{st,0}$ on $M_1' \cap M_\infty \subset M' \cap M_\infty$ is called the *opposite standard part* of θ. The action implemented by θ^{st} on $\mathcal{G}_{N^\alpha, M^\alpha}$ (i.e., on the lattice of higher relative commutants) will be denoted by γ_θ and $(\mathcal{G}_{N,M}, \gamma_\theta)$ is called the *standard invariant* of the equivariant inclusion $(N^\alpha \subset M^\alpha, \theta)$.

We will now summarize some of the properties of θ^{st} that are more or less implicit in the above considerations.

1.3. PROPOSITION. (i) *The application* st *factors to a group morphism from* $\mathrm{Aut}(M^\alpha, N^\alpha)/\mathrm{Int}N^\alpha$ *into* $\mathrm{Aut}(M^{st}, N^{st})$*.*

(ii) *The group morphism* st *is continuous from* $\mathrm{Aut}(M^\alpha, N^\alpha)/\mathrm{Int}N^\alpha$ *with the quotient topology into* $\mathrm{Aut}(M^{st}, N^{st})$ *with its usual topology.*

(iii) *Assume* $N^\alpha \subset M^\alpha$ *is strongly amenable standard invariant (equivalently,* N^{st}*,* M^{st} *are factors and have same higher relative commutants as* $N^\alpha \subset M^\alpha$*, see* [Po7]*). If* $\sigma \in \mathrm{Aut}(M^\alpha, N^\alpha)$ *then* $(\sigma^{st})^{st} = \sigma^{st}$*.*

Proof. We already proved (i) above. Then (ii), and (iii) are trivial by the definitions. Q.E.D.

1.4. *Actions with trivial standard part.* As it turns out in certain situations the standard part of an action follows automatically trivial. Since the standard part of a cocycle action on an inclusion $(N^\alpha \subset M^\alpha, \theta)$ coincides with the standard part of the cocycle action it implements on the associated II_1 inclusion, $N \subset M$, we can reduce our discussion to the case of II_1 inclusions, i.e. when $\alpha = 1$.

PROPOSITION. (i) *If σ is inner on all relative commutants $M' \cap M_k$ and $\Gamma_{N,M}$ is a tree (equivalently $a_{k\ell} \in \{0,1\}, \forall k, \ell$, and $\Gamma_{N,M}$ has no cycles), then $\sigma^{st} = id$.*

(ii) *If $\Gamma_{N,M}$ is a tree and the canonical weights $(s_k)_{k\in K}$, resp. $(t_l)_{l\in L}$, are distinct then, $\sigma^{st,0} = id$ $\forall \sigma \in \mathrm{Aut}(M, N)$.*

(iii) *If $\sup \dim \mathcal{Z}((M' \cap M_k)^\sigma) < \infty$ then σ^{st} is periodic, $\forall \sigma \in \mathrm{Aut}(M^\alpha, N^\alpha)$, i.e., $(\sigma^{st})^n = id$ for some $n < \infty, (\sigma^{st})^k$ properly outer for $0 < k < n$.*

Proof (i) If $\sigma^{st,0}$ is inner on $M' \cap M_k$ (equivalently, σ^{st} inner on all $N'_{k-1} \cap M$) then it acts trivially on its center. If $\Gamma_{N,M}$ is a tree then $M' \cap M_{k+1} = \mathrm{sp}(M' \cap M_k)e_{k+1}(M' \cap M_k) \oplus B_{k+1}$, with B_{k+1} abelian. Since $\sigma(e_{k+1}) = e_{k+1}, B_{k+1} \subset \mathcal{Z}(M' \cap M_{k+1})$ and $\sigma_{|\mathcal{Z}(M'\cap M_{k+1})} = id$, we obtain that $\sigma_{|M'\cap M_k} = id$ implies $\sigma_{|M'\cap M_{k+1}} = id$.

Then (ii) is trivial by (i), since for distinct $(s_k)_k$, $(t_l)_l$, σ is forced to act trivially on the center of $M' \cap M_k$, $\forall k$.

To prove (iii) note first that if $N^{st} \subset M^{st}$ are factors and if for some k_0 we have $ax = \sigma^{st}(x)a, x \in N_k^{st}, a \in M^{st}$, then given any $\varepsilon > 0$ there exists m such that $a \underset{\varepsilon}{\in} N_m^{st\,\prime} \cap M^{st}$. Thus, if $a \neq 0$ then for $x \in N_m^{st}$ we have $\|[x,a]\|_2 \leq 2\varepsilon\|x\|$ and $\|x - \sigma^{st}(x)\|_2 < f(\varepsilon)\|x\|$ with $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus σ^{st} is inner on N_m^{st} for m large enough (cf. e.g. [Ch]), say $\sigma^{st}_{N_m^{st}} = \mathrm{Ad}u, u \in N_m^{st}$. But $\sigma^{st} = (\mathrm{Ad}u^*\sigma)^{st}$ for $u \in N_m^{st} \subset N^{st}$, so that $\sigma^{st} = id$ on N_m^{st}. Since $\sigma^{st}(e_{-i}) = e_{-i}, \forall i \geq 0$, it follows that $\sigma^{st} = id$, unless $a = 0$.

Now, if $\dim \mathcal{Z}((M' \cap M_k)^\sigma)$ is uniformly bounded then by (1.1 in [Po7]), $(M' \cap M_{k-1})^\sigma \subset (M' \cap M_k)^\sigma \overset{e_{k+1}}{\subset} (M' \cap M_{k+1})^\sigma$ is a basic construction for k large enough and the support of e_{k+1} in $(M' \cap M_{k+1})^\sigma$ is 1. Thus the support of e_{k+1} in $M' \cap M_{k+1}$ is also 1 and $N \subset M$ has finite depth (cf. e.g. 1.1 in [Po7]), in particular it is extremal and $N^{st} \subset M^{st}$ are factors (e.g. by [Po4]). Also, we have a sequence of commuting squares

$$\begin{array}{ccccc} M' \cap M_{k-1} & \subset & M' \cap M_k & \overset{e_{k+1}}{\subset} & M' \cap M_{k+1} \\ \cup & & \cup & & \cup \\ (M' \cap M_{k-1})^\sigma & \subset & (M' \cap M_k)^\sigma & \overset{e_{k+1}}{\subset} & (M' \cap M_{k+1})^\sigma \end{array}$$

in which e_{k+1} implements a basic construction on both rows (for k large enough). Also $(M' \cap M_\infty)^\sigma = \overline{(\underset{k}{\cup}(M' \cap M_k)^\sigma)}$ is a factor (e.g., by [Po4, 5], [We]) and by ([GHJ]), if T is the inclusion matrix of $(M' \cap M_{k-1})^\sigma \subset M' \cap M_k$ then $[M' \cap M_\infty : (M' \cap M_\infty)^\sigma] = \|T\|^2$.

By the arguments above, we see that if $(\sigma^{st})^n$ is inner for some n then $(\sigma^{st})^n = id$ on N^{st} and thus on all M^{st}. Thus, either $(\sigma^{st})^n = id$ and $(\sigma^{st})^k$ is outer for $0 < k < n$, where n is the period of σ^{st}, or σ^{st} acts freely on M^{st}. In the latter case though the fixed point algebra $(M^{st})^{\sigma^{st}}$ would have infinite index in M^{st} (cf. e.g. [Lo1]), a contradiction. Q.E.D.

COROLLARY. *(i) If* $\Gamma_{N,M}$ *is a tree then* $\ker({}^{st})$ *contains the connected component of the identity in* $\mathrm{Aut}(M,N)$.

(ii) If $\sigma : G \to \mathrm{Aut}(M,N)$ *is a continuous action of a connected group* G *and either* $\Gamma_{N,M}$ *is a tree, or if* $\sup \dim \mathcal{Z}((M' \cap M_k)^\sigma) < \infty$*, then* $\sigma^{st} = id$.

Proof. (i) If σ is in the connected component of the identity in $\mathrm{Aut}(M,N)$ then $\sigma^{st}_{|N_k' \cap M}$ is inner $\forall k$ and part (i) of the previous proposition applies.

(ii) If $\Gamma_{N,M}$ is a tree then part (i) applies. Also, since G is connected, if for some k $\sigma^{st}_{|N_k' \cap M} \neq id$ then there exists $g \in G$ such that $\sigma^{st}(g)^n \neq id \; \forall n$ (even on $N_k' \cap M$), which contradicts (iii) in the previous proposition. Q.E.D.

1.5. *Proper outerness for actions on inclusions.* In order to be able to prove that the standard invariant of an equivariant inclusion is a complete invariant, we need the actions to satisfy some proper outerness condition.

Definition 1. An automorphism σ of a II_1 inclusion $N \subset M$ is called properly outer if for any k and any choice of the tunnel $M \supset N \supset \cdots \supset N_k$ we have the implication:

(*) If $a \in M$ is such that $ax = \sigma(x)a$, $\forall x \in N_k$, then $a = 0$.

An automorphism σ of a II_∞ inclusion $N^\infty \subset M^\infty$ is properly outer if the corresponding automorphism of its associated II_1 inclusion is properly outer.

Remarks. 1°. If $N^\alpha = M^\alpha$ then the above condition coincides with the usual definition of proper outerness of an action on the single algebra M^α.

2°. If σ is a properly outer action on $N^\alpha \subset M^\alpha$ then it is properly outer on both N^α and M^α, more generally, if $M^\alpha \supset N^\alpha \supset \cdots \supset N_k^\alpha$ is a tunnel and $u \in \mathcal{U}(N^\alpha)$ is so that $\mathrm{Ad}u N_j^\alpha = N_j^\alpha, j \leq k$, then $\mathrm{Ad}u\sigma$ is properly outer on each $N_j^\alpha, j \leq k$.

3°. The above definition doesn't depend on the outer conjugacy class of σ, i.e., if σ satisfies the above condition and $u \in \mathcal{U}(M^\alpha)$ is so that $\mathrm{Ad}u N^\alpha = N^\alpha$ (e.g., if $u \in N^\alpha$ or if $u \in N^{\alpha\prime} \cap M^\alpha$) then $\mathrm{Ad}u\sigma$ satisfies it. Also if $p \in N^\alpha$ is a nonzero projection then σ is properly outer iff σ_p is properly outer.

4°. If the implication (*) in the above definition holds true for some k and some choice of the tunnel up to k then it holds true for any $j \leq k$ and any choice of the tunnel up to j. Thus, in order for σ to be properly outer it is sufficient that there exists a tunnel $M^\alpha \supset N^\alpha \supset \cdots$ such that if for some $k \geq 0$ and $a \in M$ we have $ax = \sigma(x)a, \forall x \in N_k$, then $a = 0$.

PROPOSITION. *Let* θ *be a cocycle action of a discrete group* G *on* $N^\alpha \subset M^\alpha$. *The following conditions are equivalent:*

(i) $\theta(g)$ *is properly outer on* $N^\alpha \subset M^\alpha, \forall g \neq e$.

(ii) *There exists a tunnel* $M^\alpha \supset N^\alpha \supset \cdots$ *such that* $N_k^{\alpha\prime} \cap (M^\alpha \rtimes_\theta G) = N_k^{\alpha\prime} \cap M^\alpha$ *for all* $k \geq 0$.

(ii′) *For any tunnel* $M^\alpha \supset N^\alpha \supset \cdots$ *and any* k, $N_k^{\alpha\prime} \cap (M^\alpha \rtimes G) = N_k^{\alpha\prime} \cap M^\alpha$.

Moreover, if $N^\alpha \subset M^\alpha$ *is extremal, then these conditions are also equivalent to the following:*

(iii) $M^{\alpha\prime} \cap (M_k^\alpha \rtimes G) = M^{\alpha\prime} \cap M_k^\alpha, \forall k$.

(iii′) $M^{\alpha\prime} \cap (M_\infty^\alpha \rtimes G) = M^{\alpha\prime} \cap M_\infty^\alpha$.

Proof. (i) $\Rightarrow$ (ii). If $X = \Sigma x_g u_g \in M^\alpha \rtimes G$ is so that $[X, N_k^\alpha] = 0$ for some k and if $x_g \neq 0$ for some $g \neq e$ then $x_g\theta(g)(x) = xx_g$ for all $x \in N_k$. Thus $\theta(g)$ for that g will not be properly outer.

(ii) $\Rightarrow$ (i). If $\theta(g)$ is not properly outer then for some k and $a \in M^\alpha, a \neq 0$, we have $a\sigma(x) = xa, \forall x \in N_k^\alpha$. Thus $X = au_g \in N_k^{\alpha\prime} \cap (M^\alpha \rtimes_\theta G)$ but $X \notin N_k^{\alpha\prime} \cap M^\alpha$.

(ii) $\Leftrightarrow$ (ii′) is trivial.

(ii) $\Leftrightarrow$ (iii). By 1.2(v) there exists an antiisomorphism of $N_{2k-1}^{\alpha\prime} \cap (M^\alpha \rtimes G)$ onto $M^{\alpha\prime} \cap (M_{2k}^\alpha \rtimes G)$ carrying $N_{2k-1}^{\alpha\prime} \cap M^\alpha$ onto $M^{\alpha\prime} \cap M_{2k}^\alpha$.

(iii) $\Leftrightarrow$ (iii′). If $M^{\alpha\prime} \cap (M_\infty^\alpha \rtimes G) = M^{\alpha\prime} \cap M_\infty^\alpha$ then by the commuting square relation we have $M^{\alpha'} \cap (M_k^\alpha \rtimes G) = E_{M_k^\alpha \rtimes G}(M^{\alpha\prime} \cap (M_\infty^\alpha \rtimes G)) = E_{M_k^\alpha \rtimes G}(M^{\alpha\prime} \cap M_\infty^\alpha) = M^{\alpha\prime} \cap M_k^\alpha$. Q.E.D.

Definition 2. A cocycle action θ of a discrete group G on $N^\alpha \subset M^\alpha$ is properly outer if the equivalent conditions in the above proposition are satisfied. A faithful action θ of a locally compact group G on an extremal inclusion $N^\alpha \subset M^\alpha$ is properly outer if $M^{\alpha\prime} \cap (M_k^\alpha \rtimes G) = M^{\alpha\prime} \cap M_k^\alpha, \forall k, \forall$. (It has been pointed out to us by Y. Kawahigashi that a similar property has been independently considered by M.Choda and H. Kosaki in [ChK]).

In ([EvKa]) there are examples of periodic automorphisms on $N \subset M$ that are properly outer on both N and M but not on $N \subset M$ in the sense of the above 2 definitions. There in fact do exist aperiodic ones as well:

Example. Let P^α be a type II_1 or II_∞ factor and $\sigma = (\sigma_1, \dots, \sigma_n)$ some n automorphisms acting on P^α. Let $M^\alpha = M_{(n+1)\times(n+1)}(P^\alpha)$ and $N_\alpha = \{\Sigma_j \sigma_j(x) e_{jj} \mid x \in P^\alpha\}$, where $\sigma_0 = id$ and $\{e_{ij}\}_{0 \le i,j \le n}$ is a matrix unit for $M_{(n+1)\times(n+1)}(\sigma)$ (see [Po5]). An automorphism θ of M^α fixing $\{e_{ij}\}$ will leave N^α globally invariant iff θ commutes with all σ_i. It is easy to see that such a θ acts properly outer on $N^\alpha \subset M^\alpha$ iff θ doesn't belong to the group generated by the σ_i's and $\mathrm{Int}P^\alpha$ in $\mathrm{Aut}P^\alpha$. In particular, if we take $n = 1$ and $\sigma_1 = \theta$ aperiodic, one obtains an example of an automorphism which is aperiodic on M^α but is not properly outer on the inclusion $N^\alpha \subset M^\alpha$ that was first pointed out by Y. Kawahigashi (private communication).

1.6. *Sufficient conditions for proper outerness.* We will now show that in certain situations an action on an inclusion follows automatically properly outer once it is properly outer on one of the algebras. But first, we will introduce an invariant that measures the "distance" from proper outerness.

LEMMA. *Let θ be a cocycle action of a discrete group on the inclusion $N^\alpha \subset M^\alpha$ and assume that the action is properly outer on each of the algebras N^α, M^α (in the usual sense). If $\{N_k^\alpha\}_{k\ge1}$ is a choice of a tunnel then the algebras $\{N_k^{\alpha'} \cap (M^\alpha \rtimes G)\}_{k \ge -1}$ are finite dimensional and there exists a unique normalized trace τ_0 on $\bigcup_k (N_k^{\alpha'} \cap (M^\alpha \rtimes_\theta \mathbb{Z}))$ such that $E_{N_k^\alpha}^{M^\alpha \rtimes G}(x) = \tau_0(x)1$, for all $k \ge -1$ and all $x \in N_k^{\alpha'} \cap (M^\alpha \rtimes G)$. Moreover, the sequence of inclusions $\mathbb{C} = M^{\alpha'} \cap (M^\alpha \rtimes G) \subset N^{\alpha'} \cap (M^\alpha \rtimes G) \subset \cdots$ and respectively the trace τ_0 are described by a pointed matrix (or pointed bipartite graph) Γ_θ and its transpose and respectively a positive vector $\vec{s}_\theta$ satisfying $\Gamma_\theta \Gamma_\theta^t \vec{s}_\theta = [M^\alpha : N^\alpha] \vec{s}_\theta$. Also, up to trace preserving isomorphism this sequence of inclusions (and thus Γ_θ, s_θ) doesn't depend on the choice of the tunnel.*

Proof. Let R^∞ be a copy of the hyperfinite type II_∞ factor and let θ' be an action of the group $G_0 = G/\ker(\mathrm{mod}\theta)$ on R^∞ such that $\mathrm{mod}\theta'(\pi(g)) = \mathrm{mod}\theta(g)^{-1}, g \in G$, where $\pi : G \to G_0$ is the quotient map. Let $\theta_0 : G \to \mathrm{Aut}R^\infty, \theta_0(g) \in \theta'(\pi(g))$. Then $\theta \otimes \theta_0 : G \to \mathrm{Aut}(M^\alpha \otimes R^\infty, N^\alpha \otimes R^\infty)$ is a cocycle action that will still be properly outer on each algebra and $\mathrm{mod}\theta \otimes \theta_0(g) = 1$ for all g. Also the higher relative commutants and the state τ_0 do not change if we replace θ by $\theta \otimes \theta_0$. So we may assume from the beginning that θ is trace preserving and then, by splitting off some $\mathcal{B}(H)$, that $(N^\alpha \subset M^\alpha) = (N \subset M)$ are type II_1 factors. The trace τ_0 is then simply the restriction of the unique trace on the type II_1 factor $M \rtimes G$. Since $M' \cap (M \rtimes G) = \mathbb{C}$ and since $N'_k \cap (M \rtimes G) \subset N'_{k+1} \cap (M \rtimes G)$ has [PiPo1] index $\leq [M : N]$ (cf. e.g., [Po3]), the algebras are indeed finite dimensional. The fact that the inclusions are determined by a unique pointed matrix and an eigenvector follows then by ([Po7], §1.2). Obviously, all this is independent on the tunnel. Q.E.D.

Definition . The weighted graph $(\Gamma_\theta, s_\theta)$ of the above lemma is called the standard graph of the action θ. Note that, by ([Po7]), $\|\Gamma_\theta\|^2 \leq [M : N]$ and that if Γ_θ is finite we have equality, by the Peron-Frobenius theorem. Note also that θ is properly outer iff $\Gamma_\theta = \Gamma_{N,M}$. In general, this may not be the case though (see [Ka] for examples). However we have:

THEOREM. *Let θ be a cocycle action of a discrete group G on an extremal inclusion of type II_α factors $N^\alpha \subset M^\alpha$. If one of the following conditions $1° - 4°$ holds true, then the action θ is properly outer on $N^\alpha \subset M^\alpha$.*

$1°$ *$\alpha = \infty$ and θ is trace scaling, i.e., $\mathrm{Tr}\theta(g) \neq Tr$, $g \neq e$ (so that necessarily $G \subset \mathbb{R}$).*

$2°$ *$N^\alpha \subset M^\alpha$ has finite depth, G is torsion free and the action θ is properly outer on either M^α or on N^α.*

$3°$ *Γ_θ is a tree (i.e., it has only multiplicities 0 and 1 and it has no cycles), G is torsion free and the action θ is properly outer on M^α.*

$4°$ *The standard vectors $(s_k)_k$, resp. $(t_l)_l$, have distinct entries and θ is properly outer on M^α.*

$5°$. *$\alpha = 1$ and $(N \subset M) = (N^{st} \subset M^{st})$ is a standard inclusion of type II_1 factors and $\theta = \theta^{st}$ is a nontrivial standard action on it such that θ is properly outer on M^{st}.*

Moreover, in the case $\alpha = 1$, we have:

$6°$. *If $M' \cap N^\omega$ is nontrivial, and if $\sigma \in Aut(M, N)$ is properly outer on $M' \cap N^\omega$, then σ is properly outer on $N \subset M$. If in addition $N \subset M$ is strongly amenable then, conversely, if σ is properly outer on $N \subset M$ then it is properly outer on $M' \cap N^\omega$.*

Proof. To prove $1°$, it is clearly sufficient to treat the case θ is a single automorphism. Let $\lambda = \mathrm{mod}\theta$, i.e. $\mathrm{Tr}\theta = \lambda$ Tr, $\lambda \neq 1$. Assume there exists $k \geq 0$ and $0 \neq a \in M^\infty$ such that $\theta(x)a = ax, x \in N^\infty_k$. By conjugating if necessary with a unitary $u \in N^\infty$ we may suppose $\theta N^\infty_k = N^\infty_k$. Thus $\theta(N^{\infty\prime}_k \cap M^\infty) = N^{\infty\prime}_k \cap M^\infty$. Taking polar decomposition of a, it follows that there exists a partial isometry $0 \neq v \subset M^\infty$ such that $\theta(x)v = vx, x \in N^\infty_k$. Also, $v^*v, vv^* \in N^{\infty\prime}_k \cap M^\infty$. If $p \leq v^*v, q \leq vv^*$ are minimal projections in $N^{\infty\prime}_k \cap M^\infty$ such that $qvp \neq 0$, then, by multiplying the equation $\theta(x)y = vx$ by q on the left and p on the right, we may suppose $q = v^*v, p = vv^*$ are minimal projections in $N^{\infty\prime}_k \cap M^\infty$.

Since $N^\infty \subset M^\infty$ is extremal, $N_k^\infty \subset M^\infty$ is extremal (cf. [PiPo2]) so that by [PiPo1], $[qM^\infty q : N_k^\infty q]E_{N_k^\infty}(p) = [pM^\infty p : N_k^\infty p]E_{N_k^\infty}(q)$. But since $\theta(x)v = vx, x \in N_k$, we have $N_k v = vN_k$, so that $(N_k^\infty q \subset qM^\infty q) = v(N_k^\infty p \subset pM^\infty p)v^*$. Thus $[qM^\infty q : N_k^\infty q] = [pM^\infty p : N_k^\infty p]$ and we get $E_{N_k^\infty}(p) = E_{N_k^\infty}(q)$.

It follows that if $f \in N_k^\infty, Trf < \infty$, then $Tr(pf) = Tr(qf) = (Trf)E_{N_k^\infty}(q) = (Trf)E_{N_k^\infty}(p)$. Thus we get for $\lambda = \mathrm{mod}\theta$

$$\begin{aligned}\lambda Tr(fp) &= \lambda Tr(f)E_{N_k^\infty}(p) = Tr(\sigma(f))E_{N_k^\infty}(q)\\ &= Tr(\sigma(f)q) = Tr(\sigma(f)vv^*)\\ &= Tr(vfv^*) = Tr(fv^*v) = Tr(pf).\end{aligned}$$

This contradiction shows that a must be zero and thus θ acts properly outer on $N^\infty \subset M^\infty$.

To prove 2°, it is sufficient to prove the case $G = \mathbb{Z}$. To this end, denote $P_k = N_k^{\alpha\prime} \cap (M^\alpha \rtimes \mathbb{Z})$ and $Q_k = N_k^{\alpha\prime} \cap M^\alpha$. Reasoning as in the proofs of the previous lemma and of Lemma 1.2, it is sufficient to prove the case $(N^\alpha \subset M^\alpha) = (N \subset M)$ is of type II_1.

Let $\hat\theta$ be the action of $\mathbb{T} = \hat{\mathbb{Z}}$ on $M \rtimes_\theta \mathbb{Z}$ dual to θ and note that $\hat\theta(P_k) = P_k, \hat\theta|_{Q_k} = id$. Since $M = (M \rtimes_\theta \mathbb{Z})^{\hat\theta}$ we also have $Q_k = P_k^{\hat\theta}$, by commuting squares. Since $\mathbb{T}$ is simply connected, $\hat\theta$ acts innerly on each P_k and thus trivially on $\mathcal{Z}(P_k)$. Thus, $\sup_k \dim\mathcal{Z}(Q_k) \geq \sup_k \dim\mathcal{Z}(P_k)$, so that if $N \subset M$ has finite depth then Γ_θ is finite. Thus, for k large enough, the commuting square

$$\begin{array}{ccc} P_k & \subset & P_{k+1}\\ \cup & & \cup\\ Q_k & \subset & Q_{k+1}\end{array}$$

is just the basic construction of

$$\begin{array}{ccc} P_{k-1} & \subset & P_k\\ \cup & & \cup\\ Q_{k-1} & \subset & Q_k\end{array}$$

so that by [GHJ] or [We1], we have $[P_\infty : Q_\infty] < \infty$, where $P_\infty = \overline{\bigcup_k P_k}, Q_\infty = \overline{\bigcup_k Q_k}$. Since $P_\infty^{\hat\theta} = Q_\infty$, if we can show that $Q_\infty' \cap P_\infty = \mathbb{C}$ we will get a contradiction, unless $Q_\infty = P_\infty$. In turn, this equality implies θ is properly outer, by Proposition 1.5.

If $x \in Q_\infty' \cap P_\infty$ then let $\varepsilon > 0, k_0 \leq 0$ and $x_{k_0} \in P_{k_0} = N_{k_0}' \cap (M \rtimes \mathbb{Z})$ be so that $\|x - x_{k_0}\|_2 < \varepsilon$. Thus $\|ux_{k_0}u^* - x\|_2 < \varepsilon, u \in \mathcal{U}(Q_\infty)$. Since $\sup_j \dim\mathcal{Z}(N_j' \cap (M \rtimes \mathbb{Z})) < \infty$ we have $\sup_{i,j} \dim\mathcal{Z}(N_j' \cap (M_i \rtimes \mathbb{Z})) < \infty$ (see e.g. the proof of the previous Proposition). Thus, there exists ℓ large enough such that $N_{j+1}' \cap (M_\ell \rtimes \mathbb{Z})$ is the basic construction of $N_j' \cap (M_\ell \rtimes \mathbb{Z})$ by $N_{j-1}' \cap (M_\ell \rtimes \mathbb{Z})$ with Jones projection e_j, for all j. Thus $x_{k_0} \in N_{k_0}' \cap (M \rtimes \mathbb{Z}) \subset N_{k_0}' \cap (M_\ell \rtimes \mathbb{Z}) = sp(N_{k_0-1}' \cap (M_\ell \rtimes \mathbb{Z}))e_{-k_0+1}(N_{k_0-1}' \cap (M_\ell \rtimes \mathbb{Z}))$. But since $N \subset M$ has finite depth $N_{k_0+1} \subset N_{k_0+2}$ has finite depth, so that if $Q_{j,\infty} = \overline{\underset{i}{\cup}(N_i' \cap N_j)}$ then $E_{Q_{j,\infty}' \cap Q_\infty}(e_{-j}) \in \mathbb{C}$. Averaging over $u \in Q_{j,\infty}, j = k_0+1, k_0+2, \dots$ in the relation $\|ux_{k_0}u^* - x\|_2 < \varepsilon$, we obtain recursively that there exists $x_0 \in \underset{j \leq k_0-1}{\cap}(N_j' \cap (M_\ell \rtimes \mathbb{Z})) \cap M = M' \cap M = \mathbb{C}1$ such that $\|x_0 - x\|_2 < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $x \in \mathbb{C}1$.

To prove 3°, it is again sufficient to consider the case $G = \mathbb{Z}$ and we can use the same notations as above. Also, note that if Γ_θ is a tree for the G action, then it is a tree for the $\mathbb{Z}$ action as well. In this case, P_{k+1} is obtained from P_k by the basic construction (obtained with the Jones' projection which is in M and thus on which $\hat{\theta}$ acts trivially) adding in direct sum an abelian algebra. By induction, since $P_0 = \mathbb{C}$, we get by the triviality of $\widehat{\theta}$ on P_k and $\mathcal{Z}(P_{k+1})$ that $\widehat{\theta}$ acts trivially on P_{k+1}, thus $Q_j = P_j$ for all j and θ follows properly outer by Proposition 1.5.

To prove 4°, we may assume θ is a single automorphism, which is outer on M^α. We may also clearly assume $\alpha = 1$. If for some k and $0 \neq a \in M_k$ we have $\theta(x)a = ax$ for all $x \in M$, then taking the polar decomposition of a we may assume $a = v$ is a partial isometry with the right and left supports being minimal projections in $M' \cap M_k$. By the hypothesis, it follows that these supports are in the same direct summand of $M' \cap M_k$. This implies that the normalizer of Mp in pM_kp is non-trivial, where p is a minimal projection in that direct summand of $M' \cap M_k$. But this implies that (s_k) has an entry $s_k = 1$ for some $k \neq *$, a contradiction.

Part 5° has already been proved in the first part of the proof of *(iii)* in Proposition 1.4.

To prove 6°, let $N \subset M$ be an inclusion of type II_1 factors such that $M' \cap N^\omega \neq \mathbb{C}1$, i.e., N contains nontrivial central consequences of M. Note that by arguing as in [McD], it follows that $M' \cap N^\omega$ has no atoms. Let $\theta \in \text{Aut}(M, N)$. Then $\theta(M' \cap N^\omega) = M' \cap N^\omega$. Since $M' \cap N^\omega$ has no atoms, if θ is properly outer on $M' \cap N^\omega$ then by Connes' local Rohlin lemma ([C2], see also [P2]), given any $\varepsilon > 0$ there exists a partition $\{p_i\}$ of 1 with projections in $M' \cap N^\omega$ such that $\|\Sigma p_i \theta(p_i)\|_2 < \varepsilon$. If $\theta(x)a = ax$ for all $x \in N_k$, for some $k \geq 0$ and $a \in M$, then, since $[p_i, a] = 0$ and $p_i \in N_k^\omega$ (because $p_i \in N^\omega$ and $[p_i, e_j] = 0, j \geq 0$), we get $\underset{i}{\Sigma}\theta(p_i)p_i a = \Sigma a p_i = a$, while $\|\underset{i}{\Sigma}\theta(p_i)p_i a\|_2 \leq \varepsilon\|a\|$. Since $\varepsilon > 0$ was arbitrary $a = 0$. Thus θ is properly outer on $N \subset M$.

Conversely, if we assume $N \subset M$ is a strongly amenable inclusion and if θ is properly outer on $N \subset M$ then let $M \overset{e_0}{\supset} N \overset{e_{-1}}{\supset} N_1 \supset \cdots$ be a tunnel such that $N_k' \cap M \nearrow M$ and let $p \in \mathcal{P}(M' \cap N^\omega), p \neq 0$. Since $p \in N^\omega$ and $[p, e_k] = 0, k \geq 0, p \subset \underset{k}{\cap} N_k^\omega$. Thus we may assume p is represented by a sequence $p = (p_n)_n$, with $p_n \in \mathcal{P}(N_k)_n$ and $k_n \to \infty$. Note that, since $N_{k_n}' \cap M \nearrow M$, any sequence $x = (x_n)$ with $x_n \in N_{k_n}$ is in $M' \cap N^\omega$. Since θ is properly outer on $N \subset M$, the proof of the Rohlin lemma in [Po2] shows that given any $\varepsilon > 0$ and any n there exists $\delta = \delta(\varepsilon)$ (independent on n!) and $q_n = \mathcal{P}(N_{k_n}), q_n \leq p_n, \tau(q_n) \geq \delta\tau(p_n)$, such that $\|\theta(q_n)q_n\|_2 < \varepsilon\|q_n\|_2$. But then $q = (q_n)$ is in $M' \cap N^\omega, 0 \neq q \leq p$ and $\|\theta(q)q\|_2 < \varepsilon\|q\|_2$. Thus θ is properly outer on $M' \cap N^\omega$. This proves the converse implication in 6°. Q.E.D.

We mention that the finite depth case of part 6° in the above theorem was also shown independently by Y. Kamahigashi ([Ka2]). Thus, all the examples in [EvKa] of actions of finite groups on $N \subset M$ that act trivially on $M' \cap N^\omega$ (i.e., which are "centrally trivial") are also examples of non-outer automorphisms of $N \subset M$.

2. Classification of trace scaling actions on type II_∞ subfactors. We will prove in this section that a trace-scaling action of $\mathbb{Z}^n$ on a strongly amenable inclusion of type II_∞ factors splits into the tensor product between its standard part (as defined in 1.2) and a trace-scaling action on a common type II_∞ factor. More precisely we will prove:

2.1. THEOREM. *Let $N^\infty \subset M^\infty$ be a strongly amenable inclusion of type II_∞ factors. Let θ be a properly outer action of $\mathbb{Z}^n$ on $N^\infty \subset M^\infty$ such that $\mathrm{mod}\theta(g) \neq 1$, if $g \neq e = (0, \dots, 0)$. Assume there exists a partition of the unity $\{e_g\}_g \in \mathbb{Z}^n$ with finite projections in N^∞ such that $\theta(h)e_g = e_{hg}$, $h, g \in \mathbb{Z}^n$. Then there exists an isomorphism α from M^∞ onto $M^{\mathrm{st}} \otimes R^\infty$, with $\alpha(N^\infty) = N^{\mathrm{st}}\overline{\otimes}R^\infty$, such that $\alpha\theta\alpha^{-1} = \theta^{\mathrm{st}} \otimes \sigma$, where R^∞ is a copy of the hyperfinite type II_∞ factor and σ is an action of $\mathbb{Z}^n$ on R^∞ with $\mathrm{mod}\sigma(g) = \mathrm{mod}\theta(g)$, $g \in \mathbb{Z}^n$.*

When applied to the case $n = 1$, i.e., for actions by one automorphism that scale the trace, by using Connes' theorem showing that all automorphisms acting on the hyperfinite type II_∞ factor and scaling the trace by the same number are conjugate, we get:

2.2. COROLLARY . *Let $N^\infty \subset M^\infty$ be a strongly amenable inclusion of type II_∞ factors and θ a properly outer automorphism on $N^\infty \subset M^\infty$, scaling the trace (i.e., $\theta N^\infty = N^\infty$, $\mathrm{mod}\theta \neq 1$). Then there exists an isomorphism α of M^∞ onto $M^{\mathrm{st}} \otimes R^\infty$, with $\alpha(N^\infty) = N^{\mathrm{st}} \otimes R^\infty$, such that $\alpha\theta\alpha^{-1} = \theta^{\mathrm{st}} \otimes \sigma_0$, where R^∞ is a copy of the hyperfinite type II_∞ factor and σ_0 is the model action on R^∞, with $\mathrm{mod}\sigma_0 = \mathrm{mod}\theta$. Moreover, if $N \subset M$ is extremal then the proper outerness condition is automatically satisfied.*

2.3. COROLLARY . *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of hyperfinite III_λ factors, $0 < \lambda < 1$. Assume there exists a conditional expectation of finite index from $\mathcal{M}$ onto $\mathcal{N}$ which has discrete decomposition, i.e., if N^∞ is the II_∞ core of $\mathcal{N}$ and ϕ is a normal semifinite weight on $\mathcal{N}$ whose centralizer is N^∞, then the centralizer of $\phi \circ \mathcal{E}$ is the II_∞ core of $\mathcal{M}$, M^∞. Let $(\mathcal{N} \subset \mathcal{M}) = (N^\infty \rtimes \sigma \subset M^\infty \rtimes \sigma)$ be the associated discrete decomposition. Assume also that $N^\infty \subset M^\infty$ has strongly amenable graph. Then $(\mathcal{N} \subset \mathcal{M}) \simeq (N^{st} \otimes R^\infty \rtimes \sigma^{st} \otimes \sigma_0 \subset M^{st} \otimes R^\infty \rtimes \sigma^{st} \otimes \sigma_0)$, where σ^{st} is the action implemented by σ on the model II_1 inclusion $N^{st} \subset M^{st}$ associated with $N^\infty \subset M^\infty$ and σ_0 is a λ-scaling automorphism of the hyperfinite II_∞ factor R^∞.*

We will prove 2.1 by building a tunnel for $M^\infty \supset N^\infty$ that is invariant to all $\theta(g)$, $g \in \mathbb{Z}^n$, and so that the algebra of higher relative commutants splits its commutant in N^∞, being a hyperfinite type II_∞ factor containing the "diagonal" $\{e_g\}_{g\in\mathbb{Z}^n}$.

To do this we need a technical lemma, which uses the noncommutative local Rohlin theorem ([Po1,2,7]) and some maximality arguments inspired from ([C3]). We will consistently denote by multiplication the operation in a discrete group, including $\mathbb{Z}^n$.

2.4. LEMMA. *Assume $\mathrm{mod}\theta(g) > 1$ if $g \neq e = (0, \dots, 0)$ and g has only nonnegative entries. Let v_i, $1 \le i \le n$, be partial isometries in N^∞ such that*

$v_iv_i^* = e_e$, $v_i^*v_i \leq e_{\delta_i}$, $1 \leq i \leq n$, where $\delta_i = (0, \dots, 1, 0, \dots, 0)$, 1 appearing only on the i'th entry. Let $m \geq 0$ and denote by $K_m = \{g \in \mathbb{Z}^n \mid 0 \leq g_i \leq m\}$. Let $\varepsilon > 0$ and $X \subset e_eM^\infty e_e$ be a finite set. Then there exist unitary elements $w_i^0 \in e_{\delta_i}N^\infty e_{\delta_i}$, and a partition of the unity $\{q_g^0\}_{g\in K_m}$ with projections in $e_eN^\infty e_e$ of trace $1/_{|K_m|}$ such that

1. $\|w_i^0 - e_{\delta_i}\|_{1,\mathrm{Tr}} \leq 3/_m, 1 \leq i \leq n$.

2. $\mathrm{Ad}(v_iw_i^0)\theta(\delta_i)(q_g^0) = q_{s_i(g)}^0$, $1 \leq i \leq n$, $g \in K_m$, where $s_i(g) = \delta_i g$, if the i'th entry g_i of g is less than m and $s_i(g) = (g_1, \dots, g_{i-1}, 0, g_{i+1}, \dots, g_n)$ if $g_i = m$.

3. $\|x - \sum_g q_g^0 x q_g^0\|_2 < \varepsilon, x \in X$.

Proof. Let $\mathcal{F}$ be the set of all $(n + |K_m|)$-tuples $((w_i)_{1\leq i\leq n}, (q_g)_{g\in K_m})$ in which $(q_g)_{g\in K_m}$ are mutually orthogonal, mutually equivalent projections in $e_eN^\infty e_e$ and w_i are unitary elements in $e_{\delta_i}N^\infty e_{\delta_i}$ such that

(i) $\|w_i - e_{\delta_i}\|_{1,\mathrm{Tr}} \leq 3/_m \mathrm{Tr}(\theta(\delta_i)(\sum_g q_g))$, $1 \leq i \leq n$.

(ii) $\mathrm{Ad}(v_iw_i)\theta(\delta_i)(q_g) = q_{s_i(g)}$, $1 \leq i \leq n$, $g \in K_m$.

(iii) $\|(x - (1 - \sum_g q_g)x(1 - \sum_g q_g)) - \sum_g q_g x q_g\|_{2,\mathrm{Tr}}^2 \leq \varepsilon \mathrm{Tr}(\sum_g q_g), x \in X$.

We define on $\mathcal{F}$ a (strict) order $<$ by letting $((w_i)_i, (q_g)_g) < ((w_i')_i, (q_g')_g)$ if

$$q_g \leq q_g', \forall g \in K_m,$$

$$\sum q_g \neq \sum q_g',$$

$$w_i = w_i'\theta(\delta_i)(\sum_g q_g)$$

$$\|w_i' - w_i\|_{1,\mathrm{Tr}} \leq 3/_m \mathrm{Tr}(\theta(\delta_i)(\sum_g (q_g' - q_g))), \forall 1 \leq i \leq n.$$

Then $(\mathcal{F}, <)$ is clearly inductively ordered. Let $((w_i^0)_i, (q_g^0)_g)$ be a maximal element in $\mathcal{F}$. Assume $\sum_g q_g \neq e_e$ and let $s = e_e - \sum_g q_g^0$. Note that $\mathrm{Ad}(v_iw_i^0)\theta(\delta_i)(s) = s$. To prove the lemma we only need to show that the assumption $s \neq 0$ leads to a contradiction.

For each $g \in K_m$ let v_g be a partial isometry in N^∞ with $v_gv_g^* = s$, $v_g^*v_g \leq \theta(g)(s) \leq e_g$.

Let $\varepsilon_0 > 0$. Since $sN^\infty s$ is hyperfinite, there exists a finite dimensional subfactor $B_0 \subset sN^\infty s$ such that:

a) $\theta(g)^{-1}(v_g^*v_g) \underset{\varepsilon_0}{\in} B_0$, $g \in K_m$.

b) $\theta(\delta_i g)^{-1}(v_{\delta_i g}^* v_i w_i^0 \theta(\delta_i)(v_g)) \underset{\varepsilon_0}{\in} B_0$, $g \in K_m$, $1 \leq i \leq n$.

Since $B_0' \cap sN^\infty s \subset B_0' \cap sM^\infty s$ is strongly amenable, given any $\varepsilon_0' > 0$ there exists a tunnel $M^\infty \supset N^\infty \supset N_1^\infty \supset \cdots \supset N_k^\infty$ such that $s \in N_k$, $B_0 \subset sN_ks$, and such that one has the estimates

c) $\theta(g)^{-1}(v_g x v_g^*) \subset_{\varepsilon_0'} B_0 \vee ((sN_k^\infty s)' \cap sM^\infty s)$, $g \in K_m$.

Next let $\varepsilon_1 > 0$. By the noncommutative local Rohlin theorem ([Po1,2,7]) from the proper outerness condition 1.3 on θ it follows that there exists a partition of the unity $\{r_j\}_j$ with projections in $B_0' \cap sN_k^\infty s$ such that

d) $\sum_{g\neq g'\in K_m}\left\|\sum_j \theta(g)(r_j)v_g^*v_{g'}\theta(g')(r_j)\right\|_{2,\mathrm{Tr}}^2<\varepsilon_1^4\left\|\sum_j r_j\right\|_{2,\mathrm{Tr}}^2$.

It follows by d) that the set J_1 of all the j's for which

$$\sum_{g\neq g'}\|v_g\theta(g)(r_j)v_g^*v_{g'}\theta(g')(r_j)v_{g'}^*\|_{2,\mathrm{Tr}}^2<\varepsilon_1^2\|r_j\|_{2,\mathrm{Tr}}^2 \tag{1}$$

satisfies $\mathrm{Tr}(\sum_{j\in J_1}r_j)\geq(1-\varepsilon_1^2)\mathrm{Tr}(s)$.

Moreover, for each $g\in K_m$, by a) we have:

$$\sum_j \mathrm{Tr}\left(v_g\theta(g)(r_j)v_g^*-v_g\theta(g)(r_j)v_g^*v_g\theta(g)(r_j)v_g^*\right) \tag{2}$$

$$=\mathrm{Tr}\left(\theta(g)(s)v_g^*v_g\right)-\|\sum_j\theta(g)(r_j)v_g^*v_g\theta(g)(r_j)\|_{2,\mathrm{Tr}}^2$$

$$\leq \mathrm{Tr}(v_g^*v_g)-\|\theta(g)(E_{B_0'\cap sP^\infty s}(\theta(g)^{-1}(v_g^*v_g))\|_{2,\mathrm{Tr}}^2$$

$$\leq \mathrm{Tr}(s)-\|\theta(g)(\theta(g)^{-1}(v_g^*v_g))\|_{2,\mathrm{Tr}}^2+\|\theta(g)(\theta(g)^{-1}(v_g^*v_g)-E_{B_0'\cap sP^\infty s}(\theta(g)^{-1}(v_g^*v_g))\|_{2,\mathrm{Tr}}^2$$

$$=\|\theta(g)(\theta(g)^{-1}(v_g^*v_g)-E_{B_0'\cap sP^\infty s}(\theta(g)^{-1}(v_g^*v_g))\|_{2,\mathrm{Tr}}^2$$

$$\leq \varepsilon_0^2\mathrm{mod}\theta(g)\mathrm{Tr}(\Sigma_j r_j).$$

This shows that if J_2 denotes the set of all j's for which

$$\sum_g\|v_g\theta(g)(r_j)v_g^*-(v_g\theta(g)(r_j)v_g^*)^2\|_{1,\mathrm{Tr}}<\varepsilon_0\mathrm{Tr}r_j \tag{3}$$

then $\Sigma_{j\in J_2}\mathrm{Tr}r_j\geq(1-\varepsilon_0\sum_g\mathrm{mod}\theta(g))\mathrm{Tr}s$. Indeed, for if not then $\Sigma_{j\in J_2}\mathrm{Tr}r_j\geq(1-\varepsilon_0\sum_g\mathrm{mod}\theta(g))\mathrm{Tr}s$, so

$$\Sigma_{j\notin J_2}\mathrm{Tr}r_j>\varepsilon_0(\sum_g\mathrm{mod}\theta(g))\mathrm{Tr}s$$

which together with the inequality

$$\sum_{j\notin J_2}\sum_g\|v_g\theta(g)(r_j)v_g^*-(v_g\theta(g)(r_j)v_g^*)^2\|_{1,\mathrm{Tr}}\geq\varepsilon_0\sum_{j\notin J_2}\mathrm{Tr}r_j$$

implies

$$\sum_{j\notin J_2}\sum_g\|v_g\theta(g)(r_j)v_g^*-(v_g\theta(g)(r_j)v_g^*)^2\|_{1,\mathrm{Tr}}$$

$$\geq\varepsilon_0\Sigma_{j\notin J_2}\mathrm{Tr}r_j>\varepsilon_0^2(\sum_g\mathrm{mod}\theta(g))\mathrm{Tr}s$$

contradicting the inequality (2).

Also by first applying the Cauchy-Schwartz inequality and then a), we get:

$$\begin{aligned}
&\sum_j |\mathrm{Tr}(v_g\theta(g)(r_j)v_g^*) - \mathrm{Tr}(r_j)| \\
&\quad = \sum_j |\mathrm{Tr}(\theta(g)(r_j)(v_g^*v_g - E_{\theta(g)(B_0)}(v_g^*v_g))\theta(g)(r_j))| \\
&\quad \leq \| \sum \theta(g)(r_j)(v_g^*v_g - E_{\theta(g)(B_0)}(v_g^*v_g))\theta(g)(r_j)\|_{1,\mathrm{Tr}} \\
&\quad \leq \| \sum_j \theta(g)(r_j)(v_g^*v_g - E_{\theta(g)(B_0)}(v_g^*v_g))\theta(g)(r_j)\|_{2,\mathrm{Tr}} \| \sum_j \theta(g)(r_j)\|_{2,\mathrm{Tr}} \\
&\quad \leq \|v_g^*v_g - E_{\theta(g)(B_0)}(v_g^*v_g)\|_{2,\mathrm{Tr}} \|\theta(g)(s)\|_{2,\mathrm{Tr}} \leq \varepsilon_0 \mathrm{mod}\theta(g) \sum_j \mathrm{Tr} r_j.
\end{aligned}$$

Thus, reasoning exactly as above, it follows that if J_3 denotes the set of all j's for which

$$(4) \qquad \sum_g \left|\mathrm{Tr}(v_g\theta(g)(r_j)v_g^*) - \mathrm{Tr}(r_j)\right| < \varepsilon_0^{1/2}\mathrm{Tr} r_j$$

then $\sum_{j\in J_3} \mathrm{Tr} r_j \geq (1 - \varepsilon_0^{1/2} \sum_g \mathrm{mod}\theta(g))\mathrm{Tr}s$.

Further, we have for all $g \in K_m$ for which $\delta_i g \in K_m$:

$$\begin{aligned}
&\sum_j \left\|\mathrm{Ad}(v_i w_i^0)\theta(\delta_i)(v_g\theta(g)(r_j)v_g^*) - v_{\delta_i g}\theta(\delta_i g)(r_j)v_{\delta_i g}^*\right\|_{2,\mathrm{Tr}}^2 = \\
&= \sum_j \mathrm{Tr}((v_i w_i^0\theta(\delta_i)(v_g)\theta(\delta_i g)(r_j)\theta(\delta_i)(v_g^*)w_i^{0*}v_i^*)^2) + \sum_j \mathrm{Tr}((v_{\delta_i g}\theta(\delta_i g)(r_j)v_{\delta_i g}^*)^2) \\
&- 2\mathrm{Tr}(\theta(\delta_i g)(r_j)(v_{\delta_i g}^* v_i w_i^0\theta(\delta_i)(v_g))\theta(\delta_i g)(r_j)(\theta(\delta_i)(v_g^*)w_i^{0*}v_i^*v_{\delta_i g})) \\
&= \left\|E_{A_0'}(\theta(\delta_i)(v_g^*)w_i^{0*}v_i^*v_i w_i^0\theta(\delta_i)(v_g))\right\|_{2,\mathrm{Tr}}^2 \\
&+ \left\|E_{A_0'}(v_{\delta_i g}^*v_{\delta_i g})\right\|_{2,\mathrm{Tr}}^2 - 2\left\|E_{A_0'}(v_{\delta_i g}^*v_i w_i^0\theta(\delta_i)(v_g))\right\|_{2,\mathrm{Tr}}^2
\end{aligned}$$

where we denoted by $A_0 \subset \theta(\delta_i g)(B_0' \cap sN_k^\infty s)$ the algebra generated by the partition $\{\theta(\delta_i g)(r_j)\}_j$. But then b) shows that this last term is majorized by:

$$\begin{aligned}
&\|\theta(\delta_i)(v_g^*)w_i^{0*}v_i^*v_i w_i^0\theta(\delta_i)(v_g)\|_{2,\mathrm{Tr}}^2 \\
&+ \|v_{\delta_i g}^*v_{\delta_i g}\|_{2,\mathrm{Tr}}^2 - 2\|v_{\delta_i g}^*v_i w_i^0\theta(\delta_i)(v_g)\|_{2,\mathrm{Tr}}^2 + 4\varepsilon_0(\mathrm{mod}\theta(\delta_i)\mathrm{Tr}(s) \\
&= \|v_{\delta_i g}v_{\delta_i g}^* - v_i w_i^0\theta(\delta_i)(v_g v_g^*)w_i^{0*}v_i^*\|_{2,\mathrm{Tr}}^2 + 4\varepsilon_0(\mathrm{mod}\theta(\delta_i) \\
&= \|s - s\|_{2,\mathrm{Tr}}^2 + 4\varepsilon_0\mathrm{mod}\theta(\delta_i)\mathrm{Tr}(s) = 4\varepsilon_0\mathrm{mod}\theta(\delta_i)\sum_j \mathrm{Tr} r_j.
\end{aligned}$$

Thus, if we denote by J_4 the set of all the j's for which

$$(4) \qquad \sum_g \|\mathrm{Ad}(v_i w_i^0)\theta(\delta_i)(v_g\delta(g)(r_j)v_g^*) - v_{\delta_i g}\theta(\delta_i g)(r_j)v_{\delta_i g}^*\|_{2,\mathrm{Tr}}^2 < \varepsilon_0^{1/2}\mathrm{Tr} r_j$$

then $\sum_{j\in J_2}\mathrm{Tr}(r_j)\geq(1-4\varepsilon_0^{1/2}\mathrm{mod}\theta(\delta_i))\mathrm{Tr}s$.

Finally, c) shows that if ε_0' is sufficiently small then

$$\sum_{x\in X}\sum_{g\in K_m}\|v_g^*xv_g-\sum_j\theta(g)(r_j)v_g^*xv_g\theta(g)(r_j)\|_{2,\mathrm{Tr}}^2<\varepsilon_0^2\mathrm{Tr}\left(\sum r_j\right).$$

Thus, if we denote by J_5 the set of all j's for which

$$\sum_x\sum_g\|(\theta(g)(s)-\theta(g)(r_j))v_g^*xv_g\theta(g)(r_j)\|_{2,\mathrm{Tr}}^2<\varepsilon_0\mathrm{Tr}r_j \tag{5}$$

then $\underset{j\in J_5}{\mathrm{Tr}}\, r_j\geq(1-\varepsilon_0)\mathrm{Tr}s$.

From all this, we see that if $\varepsilon_0,\varepsilon_1$ are chosen sufficiently small then $\bigcap_{i=1}^{5}J_i\neq\emptyset$. Let $j\in\bigcap_i J_i$ be fixed. Let $a_g=v_g\theta(g)(r_j)v_g^*$, $g\in K_m$. If $\alpha=\varepsilon_1+2\varepsilon_0^{1/4}$ then by (1) – (5) we get:

(1′) $\|a_ga_{g'}\|_{2,\mathrm{Tr}}<\alpha\|a_g\|_{2,\mathrm{Tr}}$, $\forall g,g'\in K_m$, $g\neq g'$.

(2′) $\|a_g^2-a_g\|_{1,\mathrm{Tr}}<\alpha\|a_g\|_{1,\mathrm{Tr}}$, $\forall g\in K_m$.

(3′) $|\mathrm{Tr}(a_g)-\mathrm{Tr}(a_e)|<\alpha\mathrm{Tr}a_e$, $\forall g\in K_m$, $e=(0,\dots,0)$.

(4′) $\|\mathrm{Ad}v_iw_i^0\theta(\delta_i)(a_g)-a_{\delta_ig}\|_{2,\mathrm{Tr}}<\alpha\|\alpha_g\|_{2,\mathrm{Tr}}$, $g\in K_m$, $1\leq i\leq n$, $\delta_ig\in K_m$.

(5′) $\sum_g\|[sxs,a_g]\|_2^2<\alpha\|a_g\|_{2,\mathrm{Tr}}$, $\forall x\in X$.

But then a standard perturbation argument shows that (1′) – (3′) imply the existence of mutually orthogonal projections $\{p_g\}_{g\in K_m}$ in $sN^\infty s$ such that $\|p_g-a_g\|_{2,\mathrm{Tr}}<f_0(\alpha)\|p_g\|_{2,\mathrm{Tr}}$, $\mathrm{Tr}p_g=\mathrm{Tr}p_e$, $g\in K_m$, where $f_0(\alpha)\to0$ as $\alpha\to0$ (m is fixed!). By (5′) we will then have for any $x\in X$ the estimate:

$$\left\|\left(sxs-\left(s-\sum p_g\right)x\left(s-\sum p_g\right)\right)-\sum p_gxp_g\right\|_{2,\mathrm{Tr}}^2\leq f_1(\alpha)\left\|\sum p_g\right\|_{2,\mathrm{Tr}}^2,$$

with $f_1(\alpha)\to0$ as $\alpha\to0$.

By (4′) it follows the existence of unitary elements w_i' in $\theta(\delta_i)(s)N^\infty\theta(\delta_i)(s)=\theta(\delta_i)(sN^\infty s)$ such that

$$\mathrm{Ad}(v_iw_i^0w_i')\theta(\delta_i)(p_g)=p_{\delta_i}g\ ,\ g\in K_m\text{with }\delta_ig\in K_m$$

$$\|w_i'-\theta(\delta_i)(s)\|_{1,\mathrm{Tr}}<f_2(\alpha)\mathrm{Tr}\left(\sum_g\theta(\delta_i)(p_g)\right)$$

where $f_2(\alpha)\to0$ as $\alpha\to0$.

Since the set $\{g\in K_m\mid\delta_ig\notin K_m\}$ has cardinality $|K_m|/_m$ it follows that there exist unitary elements v_i' in $sN^\infty s$ such that $\|v_i'-s\|_{1,\mathrm{Tr}}<2/_m\mathrm{Tr}(\sum p_g)$ and such that $\mathrm{Ad}(v_i'v_iw_i^0w_i')\theta(\delta_i)(p_g)=p_{s_i(g)}$ (with $s_i(g)$ the bijection on K_m defined in the statement of the Lemma). But then, $w_i'^*w_i^{0*}v_i^*v_i'v_iw_i^0w_i'$ can be completed to a unitary w_i'' in $\theta(\delta_i)(sN^\infty s)$, by defining it to be the identity on the complement of its support. This unitary will still satisfy $\|w'\prime_i - \theta(\delta_i)(s)\|_{1,\mathrm{Tr}} \le 2/_m \mathrm{Tr}(\sum p_g) < 2/_m \mathrm{Tr}\left(\theta(\delta_i)\sum p_g\right)$. If we thus define $w_i = w_i' w_i'' + (e_{\delta_i} - \theta(\delta_i)(s))$ then we have:

$$\mathrm{Ad}v_i w_i^0 w_i \theta(\delta_i)(p_g) = p_{s_i(g)},\ g \in K_m;$$

$$\|w_i - e_{\delta_i}\|_{1,\mathrm{Tr}} < (f_2(\alpha) + 2/_m)\mathrm{Tr}(\sum_g \theta(\delta_i)(p_g));$$

where the inequality follows from the estimate

$$\|w_i' w_i'' - e_{\delta_i}\|_{1,\mathrm{Tr}} \le \|w_i' w_i'' - w_i'\|_{1,\mathrm{Tr}} + \|w_i' - e_{\delta_i}\|_{1,\mathrm{Tr}}$$

$$= \|w_i'' - e_{\delta_i}\|_{1,\mathrm{Tr}} + \|w_i' - e_{\delta_i}\|_{1,\mathrm{Tr}} < (f_2(\alpha) + 2/_m)\mathrm{Tr}(\sum_g \theta(\delta_i)(p_g)).$$

Thus, if α is sufficiently small then $\|w_i - e_{\delta_i}\|_{1,\mathrm{Tr}} < 3/_m \mathrm{Tr}(\sum_g \theta(\delta_i)(p_g))$.

Define $w_i^1 = w_i^0 w_i$ and $q_g^1 = q_g^0 + p_g$, $1 \le i \le n$, $g \in K_m$. Then by the definitions we have:

(i) $\mathrm{Ad}v_i w_i^1 \theta(\delta_i)(q_g^1) = q_{s_i(g)}^1$, $g \in K_m$, $1 \le i \le n$.

(ii) $\|w_i - e_{\delta_i}\|_{1,\mathrm{Tr}} \le \|w_i^0 w_i - w_i\|_{1,\mathrm{Tr}} + \|w_i - e_{\delta_i}\|_{1,\mathrm{Tr}} \le 3/_m \mathrm{Tr}(\theta(\delta_i)(\sum q_g^1))$.

(iii) $w_i^1 \sum_g q_g^0 = w_i^1(1-s) = w_i^0$.

(iv) $\|w_i^1 - w_i^0\|_{1,\mathrm{Tr}} = \|w_i - e_{\delta_i}\|_{1,\mathrm{Tr}} \le 3/_m \mathrm{Tr}(\theta(\delta_i)(\sum q_g^1 - \sum q_g^0))$.

(v)

$$\begin{aligned}
&\left\|\left(x - \left(e_e - \sum q_g^1\right) x \left(e_e - \sum q_g^1\right)\right) - \sum q_g^1 x q_g^1\right\|_{2,\mathrm{Tr}}^2 \\
\le &\left\|\left(x - \left(e_e - \sum q_g^0\right) x \left(e_e - \sum q_g^0\right)\right) - \sum q_g^0 x q_g^0\right\|_{2,\mathrm{Tr}}^2 + \\
+ &\|(sxs \ - \left(s - \sum p_g\right) x \left(s - \sum p_g\right)\right) - \sum p_g x p_g \ \|_{2,\mathrm{Tr}} \le \varepsilon \mathrm{Tr}(\sum_g q_g^1).
\end{aligned}$$

But this shows that $((w_i^1)_i, (q_g^1)_g)$ is in $\mathcal{F}$ and majorizes $((w_i^0)_i, (q_g^0)_g)$, thus contradicting the maximality of the latter. Q.E.D.

We can now obtain the existence of equivariant tunnels for which the higher relative commutant approximate well a given finite set of elements.

2.5. LEMMA. *Let $N^\infty \subset M^\infty$ be a strongly amenable inclusion of type II_∞ factors, $\theta : \mathbb{Z}^n \to \mathrm{Aut}(M^\infty, N^\infty)$ an action scaling the trace and $\{e_g\}_{g\in\mathbb{Z}^n} \subset N^\infty$ a partition of the unity like in the hypothesis of 2.2. Let $X \subset M = e_e M^\infty e_e$ be a finite set, $A_0 \subset e_e N^\infty e_e = N$ be a finite dimensional factor and $w_i \in N^\infty$ be partial isometries such that $w_i w_i^* = e_e$, $w_i^* w_i \le e_{\delta_i}$. Given any $\delta > 0$ there exist a choice of the tunnel $M \overset{e_0^0}{\supset} N \supset \cdots \overset{e_{-\ell+1}^0}{\supset} N_{\ell-1} \supset N_\ell$, up to some ℓ, for $M \supset N$, and a finite dimensional subfactor $B_0 \subset N_\ell$ containing A_0 such that if $e_{-j} = \sum_g \theta(g)(e_{-j}^0)$ then*

(i) $x \underset{\delta}{\in} B_0 \vee (N_\ell' \cap M)$, $x \in X$.

(ii) $w_i \underset{\delta}{\in} N_\ell^\infty$, $1 \le i \le n$, *where* $N_\ell^\infty = \{e_0, e_{-1}, \dots, e_{-\ell+1}\}' \cap N^\infty$.

Proof. By Lemma 2.4, if $\varepsilon > 0$ then there exist $m > 3\varepsilon^{-1}$, unitary elements $\omega_i \in e_{\delta_i} N^\infty e_{\delta_i}$, $1 \leq i \leq n$, and a partition of the identity with mutually orthogonal, mutually equivalent projections $(q_g)_{g \in K_m}$ in $e_e N^\infty e_e$ such that

(1) $\mathrm{Ad}(w_i \omega_i)\theta(\delta_i)(q_g) = q_{s_i(g)}$, $g \in K_m$, $1 \leq i \leq n$.

(2) $\|\omega_i - e_{\delta_i}\|_{2,\mathrm{Tr}} \leq \varepsilon$, $1 \leq i \leq n$.

(3) $\|x - \sum_g q_g x q_g\|_{2,\mathrm{Tr}} \leq \varepsilon$, $x \in X$ or $x = e_{ij} \in A_0$, with $\{e_{ij}\}$ a fixed matrix unit of A_0.

Note that (1) and the fact that $\{q_g\}_g$ have equal traces implicitly means that $[\theta(\delta_i)(q_g),\ \omega_i^* w_i^* w_i \omega_i] = 0$ and $\mathrm{Tr}(\theta(\delta_i)(q_g)\omega_i^* w_i^* w_i \omega_i) = \mathrm{Tr}(q_g) = \mathrm{Tr}(q_e)$, for all $g \in K_m$, $1 \leq i \leq n$.

For each $g \in K_m$ let $w_g \in N^\infty$ with $w_g w_g^* = q_g$, $w_g^* w_g \leq \theta(g)(q_e)$ and $w_e = e$. Like in the proof of the previous lemma, given any $\varepsilon_0 > 0$ there exists a finite dimensional subfactor $B \subset q_e N^\infty q_e$ such that:

(a) $\theta(g)^{-1}(w_g^* w_g) \subset_{\varepsilon_0} B$, $g \in K_m$.

(b) $\theta(\delta_i g)^{-1}(w_{\delta_i g}^* w_i \omega_i \theta(\delta_i)(w_g)) \underset{\varepsilon_0}{\in} B$, $g \in K_m \cap \delta_i^{-1} K_m$.

By a small perturbation depending only on ε_0 it follows that there are partial isometries w'_g in N^∞ such that:

(a′) $w_g'^* w'_g \in \theta(g)(B)$, $w'_g w_g'^* \leq q_g$, $\|w'_g - w_g\|_{2,\mathrm{Tr}} \leq f'_0(\varepsilon_0)$, $g \in K_m$.
Moreover, by enlarging if necessary B, we will also have:

(a″) $q_g e_{ij} q_g \underset{f'_0(\varepsilon_0)}{\in} \mathrm{Ad}w'_g\theta(g)(B)$, $g \in K_m$, $e_{ij} \in A_0$.

It then follows that there are partial isometries $\omega'_{i,g} \in q_{\delta_i g} N^\infty q_{\delta_i g}$ such that:

(b′) $b_{i,g} = w'^*_{\delta_i g}\omega'_{i,g} w_i \omega_i \theta(\delta_i)(w'_g)$ is a partial isometry in $\theta(\delta_i g)(B)$ and we have the estimates

$$\omega'_{i,g}\omega'^*_{i,g} \leq \mathrm{Ad}(w_i\omega_i)\theta(\delta_i)(w'_g w'^*_g), omega'_{i,g}\omega'^*_{i,g} \leq w'_{\delta_i g} w'^*_{\delta_i g}$$

$$\|\omega'_{i,g} - q_{\delta_i g}\|_{2,\mathrm{Tr}} \leq f''_0(\varepsilon_0)\ ,\ g \in K_m \cap \delta_i^{-1} K_m$$

where $f'(\varepsilon_0), f''(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$.

Now, since $N^e = B' \cap q_e N^\infty q_e \subset B' \cap q_e M^\infty q_e = M^e$, and more generally $(w'_g\theta(g)(B)w'^*_g)' \cap (w'_g w'^*_g N^\infty w'_g w'^*_g) \subset (w'_g\theta(g)(B)w'^*_g)' \cap (w'_g w'^*_g M^\infty w'_g w'^*_g)$ are strongly amenable and also since

$$(w'_g\theta(g)(B)w'^*_g)' \cap (w'_g w'^*_g N^\infty w'_g w'^*_g) = \mathrm{Ad}w'_g\theta(g)(N^e),$$
$$(w'_g\theta(g)(B)w'^*_g)' \cap (w'_g w'^*_g M^\infty w'_g w'^*_g) = \mathrm{Ad}w'_g\theta(g)(M^e)$$

it follows that given any $\varepsilon_1 > 0$ there exists a choice of the tunnel up to some ℓ, $M^e \overset{e_0^e}{\supset} N^e \overset{e_{-1}^e}{\supset} N_1^e \supset \cdots \overset{e_{-\ell+1}^e}{\supset} N_{\ell-1}^e \supset N_\ell^e$, for $M^e \supset N^e$, so that to have:

(c′) $w'_g w'^*_g X w'_g w'^*_g \underset{\varepsilon_1}{\subset} \mathrm{Ad}w'_g\theta(g)(B) \vee \mathrm{Ad}w'_g\theta(g)(N_\ell^{e'} \cap M^e)$.

Put $p_e = e_e - \sum_g w'_g w'^*_g$ and let $\{e'_{-j}\}_{0 \leq j < \ell}$ be a set of Jones projections in $p_e M^\infty p_e$, with e'_0 projecting on the scalar $[M^\infty : N^\infty]^{-1} p_e$ when expected onto $p_e N^\infty p_e$ and with $e'_1, \ldots, e'_{\ell-1} \in p_e N^\infty p_e$.

Then define $e^0_{-j} \overset{\text{def}}{=} \sum_{g\in K_m} \mathrm{Ad}w'_g\theta(g)(e^e_{-j}) + e'_{-j}$ and $e_{-j} \overset{\text{def}}{=} \sum_{g\in\mathbb{Z}^n} \theta(g)(e^0_{-j})$, $N^\infty_j \overset{\text{def}}{=} \{e_0, \dots, e_{-\ell+1}\}' \cap N^\infty$. Since $\sum_{g\in K_m} \mathrm{Ad}w'_g\theta(g)(B) \subset e_e N^\infty_\ell e_e$ it follows that given any $\varepsilon_2 > 0$ there exists a finite dimensional subfactor $B_0 \subset e_e N^\infty_\ell e_e$ such that $\{q_g\}_{g\in K_m} \subset B_0$, $B \subset q_e B_0 q_e$, $\mathrm{Ad}w'_g\theta(g)(B) \underset{\varepsilon_2}{\subset} B_0$, $\forall g \in K_m$.

Since by (a′) we have $\|e_e - \sum_g w'_g w'^*_g\|_{2,\mathrm{Tr}} \leq |K_m| f'(\varepsilon_0)$, with $|K_m|f'(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$, and since by (3) we have $X \underset{\varepsilon}{\subset} \sum_g q_g X q_g$, it follows by (c′) and by the definition of B_0 that if we let ε, ε_0, ε_1, ε_2 sufficiently small then condition (i) of the statement is satisfied.

To show that (ii) is also satisfied note that if we let

$$q' \overset{\text{def}}{=} \sum_{g\in K_m\cap\delta_i^{-1}K_m} w'_g w'^*_g \in e_e N^\infty e_e,$$

$$q'' \overset{\text{def}}{=} \sum_{g\in K_m\cap\delta_i^{-1}K_m} \omega'_{i,g}\omega'^*_{i,g} = \sum w_{\delta_i g} b_{i,g} b^*_{i,g} w^*_{\delta_i g},$$

$$\omega'_i = \sum_{g\in K_m\cap\delta_i^{-1}K_m} \omega'_{i,g},$$

then $[q', e_{-j}] = 0$, $[q'', e_{-j}] = 0$. By (b′) we then have:

$$\begin{aligned}
\mathrm{Ad}(\omega'_i w_i \omega_i)\theta(\delta_i)(q' e_{-j}) &= \sum_{g\in K_m\cap\delta_i^{-1}K_m} \mathrm{Ad}(\omega'_i w_i\omega_i\theta(\delta_i)(w'_g))\theta(\delta_i g)(e^e_{-j}) \\
&= \sum_{g\in K_m\cap\delta_i^{-1}K_m} \mathrm{Ad}(w'_{\delta_i g} b_{i,g})\theta(\delta_i g)(e^e_{-j}) \\
&= \sum_{g\in K_m\cap\delta_i^{-1}K_m} \mathrm{Ad}w'_{\delta_i g}(\theta(\delta_i g)(e^e_{-j}) b_{i,g} b^*_{i,g}) \\
&= e_{-j} q''.
\end{aligned}$$

Thus, since $\theta(\delta_i)(e_{-j}) = e_{-j}$, it follows that

$$(q''\omega'_i w_i\omega_i\theta(\delta_i)(q'))e_{-j} = e_{-j}(q''\omega'_i w_i\omega_i\theta(\delta_i)(q')).$$

Since q'' is close to e_e, $\theta(\delta_i)(q')$ is close to e_{δ_i}, ω'_i close to e_e and ω_i close to e_{δ_i}, it follows that there exist $w'_i (= q''\omega'_i w_i\omega_i\theta(\delta_i)(q'))$ in N^∞ that is arbitrarily close to w_i, say $\|w'_i - w_i\|_{2,\mathrm{Tr}} < \delta$, so that $[w'_i, e_{-j}] = 0$, $0 \leq j < \ell$, i.e., so that $w'_i \in N^\infty_\ell$. This shows that for appropriately small ε, ε_0, ε_1, ε_2 condition (ii) of the statement will also be satisfied.

Still, the condition $A_0 \subset B_0$ is not yet achieved. But by the way B_0 was defined, condition (a″) shows that $A_0 \underset{\alpha}{\subset} B_0$ with α as small as we please. Since both A_0, $B_0 \subset e_e N^\infty e_e$ it follows by [Ch] that there exists a unitary element $u_0 \in e_e N^\infty e_e$ close to e_e so that $\mathrm{Ad}u_0(B_0) \supset A_0$. But then, if we conjugate spatially all the previous choices, $\{e_{-j}\}_{0\leq j\leq \ell-1}$, $\{N^\infty_j\}_{1\leq j\leq \ell}$, by $\mathrm{Ad}u$, where $u = \sum_{g\in\mathbb{Z}^n} \theta(g)(u_0)$, then for α small enough the estimates (i), (ii) will still hold true. Q.E.D.

Proof of Theorem 2.1. By changing if necessary the positive cone of $\mathbb{Z}^n$ we may assume $\mathrm{Tr}\theta(g) \geq \mathrm{Tr}$ for all the g having only nonnegative coordinates.

Let $\{x_n\}_n$ be a sequence of elements in $e_e M^\infty e_e$, dense in the strong operator topology in $e_e M^\infty e_e$. We construct recursively an increasing sequence of integers, $i_1 < i_2 < ....$, continuations of the tunnel $M^\infty \overset{e_0}{\supset} N^\infty \overset{e_{-1}}{\supset} N_1^\infty \supset \cdots \overset{e_{-i_k}}{\supset} N_{i_k}^\infty \cdots$ and an finite dimensional factors $B_k \in N_{i_k}^\infty = \{e_0, e_{-1}, ..., e_{-i_k}\}' \cap N^\infty$, such that:

(i) $x_j \underset{2^{-k}}{\in} B_k \vee e_e(N_{i_k}^{\infty'} \cap M^\infty)e_e$, $1 \leq j \leq k$.

(ii) $\|w_k^i - w_{k-1}^i\|_{2,\mathrm{Tr}} \leq 2^{-k}$, $w_k^i w_k^{i*} = e_e$, $w_k^{i*} w_k^i \leq e_{\delta_i}$, $1 \leq i \leq n$.

Assume we made this construction up to some k.

By [C3] there exists $\delta' > 0$ such that if $P^\infty \subset N_k^\infty$ is a subfactor containing the projections $\{e_g\}_{g\in\mathbb{Z}^n}$ and if $b_i \in P^\infty$ are so that $\|b_i\| \leq 1$, $b_i = e_e b_i e_{\delta_i}$, $\|w_k^i - b_i\|_{2,\mathrm{Tr}} < \delta$, then there exists partial isometries $w_i' \in P^\infty$ such that $w_i' w_i'^* = e_e$, $w_i'^* w_i' \leq e_{\delta_i}$, $\|w_k^i - w_i'\|_{2,\mathrm{Tr}} \leq 2^{-k-1}$.

Also, there exist a finite subset $X \subset e_e N_{i_k-1}^\infty e_e$ and a $\delta'' > 0$ such that if $N_{i_k-1}^\infty \overset{e_{-i_k+1}}{\supset} N_{i_k}^\infty \supset \cdots \supset N_{i_{k+1}}^\infty$ is a continuation of the tunnel up to some i_{k+1} such that $B_k \subset N_{i_{k+1}}^\infty$, $\{e_g\}_{g\in\mathbb{Z}^n} \subset N_{i_{k+1}}^\infty$ and if $X \underset{\delta''}{\subset} B_{k+1} \vee e_e(N_{i_{k+1}}^{\infty'} \cap N_{i_k-1}^\infty)e_e$, for some finite dimensional subfactor $B_{k+1} \subset e_e N_{i_{k+1}}^\infty e_e$ with $B_{k+1} \supset B_k$, then $x_j \underset{2^{-k-1}}{\in} B_{k+1} \vee e_e(N_{i_{k+1}}^{\infty'} \cap M^\infty)e_e$, $1 \leq j \leq k+1$. Indeed if one takes $\{m_j\}_j$ to be an orthonormal basis of $e_e N_{i_k-1} e_e$ over $e_e N_{i_k}^\infty e_e$ then one may construct an orthonormal basis $\{m_i^k\}_i$ of $e_e M^\infty e_e$ over $e_e N_{i_k}^\infty e_e$ as words in the m_j's and in $e_0, e_{-1}, \ldots, e_{-i_k+1}$ (see [PiPo1,2]). Writing $x_1, \ldots, x_{k+1}$ in the basis $\{m_i^k\}_i$ of $e_e M^\infty e_e$ over $e_e N_{i_k}^\infty e_e$ we obtain a finite set $X \subset e_e N_{i_k}^\infty e_e$ which for suitably small δ'' will yield the above estimates.

If we now take $\delta = \min\{\delta', \delta''\}$ and apply Lemma 2.5 for $N_{i_k}^\infty \subset N_{i_k-1}^\infty$ (as $N^\infty \subset M^\infty$), $B_k \subset e_e N_{i_k}^\infty e_e$ (as A_0), the restriction of θ to $N_{i_k-1}^\infty$ and the above X and δ then we get the i_{k+1} the θ-equivariant continuation of the tunnel up to i_{k+1}, the algebra B_{k+1} and the partial isometry w_{k+1} satisfying (i), (ii) for $k+1$.

Now, since all $N_{i_k}^\infty$ contain the projections $\{e_g\}_{g\in\mathbb{Z}^n}$, we have $\{e_g\}_g \subset \underset{j}{\cap} N_j^\infty$. By condition (ii) we have $\{w_k^i\}_k$ is Cauchy in the norm $\| \ \|_{2,\mathrm{Tr}}$ and if $w_i = \lim w_i^k$ then w_i are also contained in $\underset{j}{\cap} N_j^\infty$. Moreover $\underset{k}{\cup} B_k \subset \underset{j}{\cap} e_e N_j^\infty e_e$ and if $P = \overline{\underset{k}{\cup} B_k}$ then by (i) we have $e_e M^\infty e_e = P \vee e_e(N_k^{\infty'} \cap M^\infty)e_e$. Thus $e_e(\underset{j}{\cap} N_j^\infty)e_e = P$ so that $e_e(\underset{j}{\cap} N_j^\infty)e_e$ and all $\theta(g)(e_e \cap N_j^\infty e_e) = e_g(\cap N_j^\infty)e_g$ are type II_1 factors. But then the fact that the w_i (and thus all $\theta(g)(w_i)$) are in $\cap N_k^\infty$ shows that the projections $\{e_g\}_{g\in\mathbb{Z}^n} \subset \underset{j}{\cap} N_j^\infty$ are comparable (in the Murray-von Neumann sense) in $\underset{j}{\cap} N_j^\infty$. Thus $\underset{j}{\cap} N_j^\infty$ is a type II_∞ factor.

It follows that if we denote by $R^\infty = \underset{j}{\cap} N_j^\infty$, $M^{st} = \overline{\underset{j}{\cup}(N_j^{\infty'} \cap M)}$, $N^{st} = \overline{\underset{j}{\cup}(N_j^{\infty'} \cap N^\infty)}$ then all the conditions in the theorem are satisfied. Q.E.D.

3. Classification of actions on type II_1 subfactors. In this section we will use ([Po7]) and noncommutative ergodic theory techniques to prove that a properly outer cocycle action θ of a discrete amenable group G on a strongly amenable inclusion of type II_1 factors $N \subset M$ is (cocycle) conjugate to the tensor product of the canonical action θ^{st} on $N^{st} \subset M^{st}$ and a commonly splitted properly outer *model cocycle action* σ of G on a single hyperfinite type II_1 factor R_0. When applied to the case $N \subset M = M_{2\times 2}(\mathbb{C})$, for which the standard part of any action is trivial, this shows that any properly outer cocycle action of G on a hyperfinite II_1 factor R_0 is cocycle conjugate to a cocycle action of the form $id \otimes \sigma$. Altogether, this gives:

3.1. THEOREM. *Let $N \subset M$ be a strongly amenable inclusion of type II_1 factors and G a countable discrete amenable group. Let $\theta : G \to \mathrm{Aut}(M,\ N)$ be such that $\theta(g)$ is properly outer on $N \subset M$ for each $g \neq e$ and such that $\theta(g)\theta(h) = \mathrm{Ad}(u(g,\ h))\theta(gh)$, for $g,\ h \in G$, where $u(g,\ h) \in \mathcal{U}(N)$ satisfy*

$$\begin{aligned} &u(g,\ h)u(gh,\ h') = \theta(g)(u(h,\ h'))u(g,\ hh') \\ &u(e,\ g) = u(g,\ e) = 1. \end{aligned}$$

If R_0 is a copy of the hyperfinite type II_1 factor and σ_0 is a properly outer action of G on R_0, then $(N \subset M,\ \theta)$ is cocycle conjugate to $(N^{\mathrm{st}}\otimes R_0 \subset M^{\mathrm{st}}\otimes R_0,\ \theta^{\mathrm{st}}\otimes\sigma_0)$, i.e., there exists an isomorphism $\alpha : M \to M^{\mathrm{st}}\otimes R_0$ such that $\alpha(N^{\mathrm{st}}\otimes R_0) = N$ and such that $\mathrm{Ad}v(g)\theta(g) = \alpha^{-1}(\theta^{\mathrm{st}}(g) \otimes \sigma_0(g))\alpha$, for all $g \in G$ and for some unitaries $v(g) \in \mathcal{U}(N)$.

The rest of this section is devoted to the proof of this theorem. The idea is to construct a tunnel of subfactors $M \overset{e_0}{\supset} N \overset{e_{-1}}{\supset} N_1 \supset \cdots$ with unitaries $v_k(g) \in \mathcal{U}(N_k)$ such that $\|v_k(g) - 1\|_2 < 2^{-k}$, for $g \in F_k \subset G$, with F_k finite sets satisfying $F_k \subset F_{k-1}$, $\cup F_k = G$, and $\mathrm{Ad}(v_k(g)...v_0(g))\theta(g)(e_{-k}) = e_{-k}$, for all $g \in G$, and such that $S = \overline{\cup(N_k' \cap N)}$, $\underset{k}{\cup}(N_k' \cap M) = R$ splits $N \subset M$, i.e., $S' \cap N = R' \cap N = R' \cap M$ and $(N \subset M) = (S \vee (R' \cap M) \subset R \vee (R' \cap M))$. Then $v(g) = \lim_k v_k(g) \cdots v_0(g)$, $(N^{st} \subset M^{st}) \simeq (S \subset R)$, $\theta^{st}(g) = \mathrm{Ad}v(g)\theta(g)|R$, $R_0 = R' \cap M$, $\sigma(g) = \mathrm{Ad}v(g)\theta(g)_{|R_0}$ will be of the form $\theta^{st} \otimes \sigma_0$ on $(N^{st} \subset M^{st}) \otimes R_0$, with σ_0 an action of G on R_0. Moreover, we will construct this split off of θ so that $\mathrm{Ad}v\theta_{R_0}$ is a prescribed *model* properly outer cocycle action of G on the hyperfinite II_1 factor R_0.

3.2. *Some model cocycle actions.* To construct the above perturbations $v_k(g)$ and prove the Theorem, we first need to introduce some notations. We will then construct a "model" for properly outer cocycle actions of amenable groups on the hyperfinite II_1 factor.
Notations.

(1) Let G be a countable discrete amenable group. For each finite subset $F \subset G$ and $\varepsilon > 0$ we choose a finite (Folner) set $K \subset G$, $K = K(F,\varepsilon)$, with the property that $|FK \triangle K| < \varepsilon|K|$. Moreover, for each $h \in G$ we choose once for all a permutation $s_K(h)$ of the set K, such that for $g \in K$ with $hg \in K$ we have $s_K(h)(g) = hg$.
(2) Let $\{F_n\}_{n\geq 1}$ be an increasing sequence of finite dimensional subsets of G with $G = \cup F_n$ and $K_n = K(F_n, 2^{-n})$. Denote by $w_n(h)$ the unitary element on $\ell^2(K_n)$ defined on the orthonormal basis $\{\delta_g\}_{g\in K_n}$ of $\ell^2(K_n)$ by $w_n(h)\delta_g = \delta_{s_{K_n}(h)(g)}$, $h \in G$, $g \in K_n$.

(3) Let $R = \overline{\bigotimes}_{n\geq 1}\mathcal{B}(\ell^2(K_n), \tau_n)$ be the hyperfinite type II_1 factor realized as an infinite tensor product with respect to the unique normalized traces τ_n of the finite dimensional factors $B(\ell^2(K_n))$. For each $k \geq 1$ and $h \in G$ let $\theta_k(h) \in \operatorname{Aut} R$ be defined as the product type automorphism $\theta_k(h) = \operatorname{Ad}(1 \otimes \cdots \otimes 1 \otimes w_k(h) \otimes w_{k+1}(h) \otimes \cdots$ where $w_k(h)$ appears exactly at the k'th position.

LEMMA. *For each $k \geq 1$ and $h, g \in G$, the element*

$$u_k(h,g) \overset{\text{def}}{=} 1 \otimes \cdots \otimes 1 \otimes w_k(g)w_k(h)w_k(gh)^{-1} \otimes w_{k+1}(g)w_{k+1}(h)w_{k+1}(gh)^{-1} \otimes \cdots$$

is a unitary in R and $\theta_k \colon G \to \operatorname{Aut} R$ is a properly outer cocycle action of G with 2-cocycle u_k.

Proof. Due to the condition $|gK_n \backslash K_n| < 2^{-n}|K_n|$, $|hK_n \backslash K_n| < 2^{-n}|K_n|$ for $h,\ g \in F_n$, it follows by the definition of $s_{K_n}(g),\ s_{K_n}(h),\ s_{K_n}(gh)$ that $s_{K_n}(gh)$ coincides with $s_{K_n}(g)s_{K_n}(h)$ on "most" of K_n, more precisely $\|w_k(g)w_k(h)w_k(gh)^{-1} - 1\|_2^2 < 2^{-n+2}$. This shows that $u_k(g,\ h)$ are all unitaries, $k \geq 1,\ g,\ h \in G$, and that $\lim_{k\to\infty} \|u_k(g,\ h) - 1\|_2 = 0,\ g,\ h \in G$.

Clearly we have $\operatorname{Ad} u_k(g,\ h) = \theta_k(g)\theta_k(h)\theta_k(gh)^{-1}$ and $u_k(g,\ e) = u_k(e,\ g) = 1$ and the identity:

$$
\begin{aligned}
&u_k(g,\ h)u_k(gh,\ h') \\
&= 1 \otimes \cdots \otimes 1 \otimes w_k(g)w_k(h)w_k(gh)^{-1}w_k(gh)w_k(h')w_k(ghh')^{-1} \\
&= 1 \otimes \cdots \otimes 1 \otimes w_k(g)w_k(h)w_k(h')w_k(ghh')^{-1} \otimes 1 \otimes \cdots \\
&= \cdots \otimes 1 \otimes w_k(g)(w_k(h)w_k(h')w_k(hh')^{-1})w_k(g)^{-1}(w_k(g)w_k(hh')w_k(ghh')^{-1}) \otimes \cdots
\end{aligned}
$$

shows that $u_k(g,\ h)u_k(gh, h') = \theta_k(g)(u_k(h,\ h'))u_k(g,\ hh')$, and thus $u_k(g,h)$ is a 2-cocycle for θ_k. Q.E.D.

COROLLARY. *Let $\theta \colon G \to R$ be a cocycle action of the discrete amenable group G on the hyperfinite type II_1 factor R. Given any $\varepsilon > 0$, $x_1, \ldots, x_n \in R$, $F \subset G$ a finite subset and $B_0 \subset R$ a finite dimensional factor, there exist a finite dimensional subfactor $B \subset R$ containing B_0 and unitary elements $v(h) \in R$, $h \in F$, such that*

(1) $\|E_B(x_i) - x_i\|_2 < \varepsilon$, $1 \leq i \leq n$.
(2) $\|v(h) - 1\|_2 < \varepsilon$, $h \in F$.
(3) $\operatorname{Adv}(h)\theta(h)(B) = B$, $h \in F$.

Proof. By the previous Lemma and [Oc], there are unitary elements $v_0(h) \in R$, $h \in G$, such that $\operatorname{Ad} v_0(h)\theta(h)$ is identical to the cocycle model action $\theta_1(h)$, $h \in G$, for some splitting of R in an infinite tensor product, as in (3) above. But then there exists m large enough so that if $B = \mathcal{B}(\ell^2(K_1)) \otimes \cdots \otimes \mathcal{B}(\ell^2(K_m))$, then $\|E_B(x_i) - x_i\|_2 < \delta$, $1 \leq i \leq n$, $\|E_B(v_0(h)) - v_0(h)\|_2 < \delta$, $h \in F$, with δ to be chosen later. Since $\theta_1(h)B = B$, from the last set of inequalities it follows that for $x \in B$, $\|x\| \geq 1$, we have:

$$
\begin{aligned}
\|E_B(\theta(h)(x)) - \theta(h)(x)\|_2 = \|&E_B(v_0(h)^*\theta_1(h)(x)v_0(h)) \\
&- v_0(h)^*\theta_1(h)(x)v_0(h)\|_2 \leq 3\delta.
\end{aligned}
$$

By [Ch] it follows that if δ is small enough then there exist unitary elements $v(h) \in R$ such that $\mathrm{Ad}v(h)\theta(h)B = B$ and $\|v(h) - 1\|_2 < \varepsilon$, $h \in F$. Q.E.D.

3.3. *Construction of Rohlin towers.* We will first construct partitions of the unity in N on which small perturbations of a given finite set of automorphisms θ_g act by permutations, as in the model action 3.2, i.e., like on a Rohlin tower.

LEMMA. *Under the hypothesis of* 3.1 *and with the notations in* 3.2, *let* $\varepsilon > 0$, $X = X^* \subset M$, $F \subset G$ *be finite sets and* $K = K(F, \varepsilon/2)$. *Then, there exist projections* $\{e_g^0\}_{g \in K} \subset N$, *with* $\Sigma e_g^0 = 1$, *and unitary elements* $\{v_0(h)\}_{h \in F} \subset N$ *such that*

$$\|x - \sum_{g \in K} e_g^0 x e_g^0\|_2 \leq \varepsilon, \quad x \in X$$

$$\|v_0(h) - 1\|_1 \leq \varepsilon, \quad h \in F$$

$$\mathrm{Ad}v_0(h)\theta_h(e_g^0) = e_{s_k(h)(g)}^0, \quad h \in F, \quad g \in K$$

Proof. Let $\mathcal{F}$ be the set of all $|F| + |K|$ tuples $((v(h))_{h \in F},\ (e_g)_{g \in K})$ in which $(e_g)_{g \in K}$ are mutually orthogonal projections in N and $(v(h))_{h \in F}$ are unitary elements in N such that

$$\|(x - (1 - \sum_K e_{g'})x(1 - \sum_K e_{g''})) - \sum_K e_g x e_g\|_2^2 \leq \varepsilon^2 \|\sum_K e_g\|_2^2$$

$$\|v(h) - 1\|_1 \leq \varepsilon \sum \tau(e_g)$$

$$\mathrm{Adv}(h)\theta_h(e_g) = e_{s_k(h)(g)}, \quad g \in K, \quad h \in F.$$

We define on $\mathcal{F}$ the (strict) order $<$ given by

$$((v(h))_{h \in F},\ (e_g)_{g \in K}) < ((v'(h))_{h \in F},\ (e'_g)_{g \in K})$$

if $e_g \leq e'_g$, $g \in K, \Sigma e_g \neq \Sigma e'_g$, $v(h) = v'(h) \sum_{g \in K} e_g$ and $\|v'(h) - v(h)\|_n \leq \varepsilon\tau(\Sigma e'_g - \Sigma e_g)$. Then $(\mathcal{F}, <)$ is clearly inductively ordered. Let $((v_0(h))_{h \in F},\ (e_g^0)_{g \in K})$ be a maximal element in $\mathcal{F}$. Assume $\Sigma e_g^0 \neq 1$ and let $s = 1 - \sum_{g \in K} e_g^0$. Note that $\mathrm{Ad}v_0(h)\theta_h(s) = s$.

Now denote $sNs = Q$, $sMs = P$ and $\sigma : G \to \mathrm{Aut}(P,\ Q)$ the cocycle action defined by $\sigma_g = \mathrm{Ad}v_0(g)\theta_{g|P}$ for each $g \in G$ and note that $\sigma_h \sigma_g = \mathrm{Ad}w(h,\ g)\sigma_{hg}$ for some unitary elements $w(h,\ g) \in Q$, $g,\ h \in G$. By 1.3 σ is properly outer on $Q \subset P$. Let $Y = \{\sigma_g^{-1}(sxs) \mid x \in X,\ g \in K\} \cup \{\sigma_{g'}^{-1}(w(h,\ g))^{\pm 1} \mid h \in F,\ g,\ g' \in K\}$ and let $\delta_0 > 0$. Since $Q \subset P$ has the generating property, there exists a choice of the tunnel up to some i, $P \supset Q \supset Q_1 \supset Q_2 \supset \cdots \supset Q_i$ such that $\|E_{Q'_i \cap P}(y) - y\|_2 < \delta_0$, $y \in Y$.

By 1.2 there are unitary elements $w'(g) \in Q$ such that $\mathrm{Ad}w'(g)\sigma_g(Q_j) = Q_j$, $j \leq i$. It follows by the proper outerness of σ that if we denote by L the crossed product algebra $P \rtimes_\sigma G$ and by $u_g \in L$ the unitary elements implementing the action σ_g on P, then $E_{Q'_i \cap L}(u_g^{-1}u_{g'}) = 0$, $g \neq g'$.

Given any $\delta > 0$ there exists by (A.1.4 in [Po7]) a projection $q \in Q_i$ such that

$$\|qy - E_{Q'_i \cap P}(y)q\|_2 < \delta_0/2\|q\|_2, \quad y \in Y$$

$$\|qu_g^{-1}u_{g'}q\|_2 < \delta\|q\|_2, \quad g,\ g' \in K, \quad g \neq g'.$$

Since $Y = Y^*$ the first relation implies

$$\|[q,\ y]\|_2 < \delta_0 \|q\|_2, \quad y \in Y$$

so that one has:

(1) $\|[\sigma_g(q),\ x]\|_2 < \delta_0 \|q\|_2,\ x \in X \cup \{w(h,\ g) \mid h \in F,\ g \in K\}, g \in K$.

The second relation implies $\sigma(g)(q),\ g \in K$, are δ-mutually orthogonal projections.

(2) $\|\sigma_g(q)\sigma_{g'}(q)\|_2 < \delta\|q\|_2,\ g,\ g' \in K,\ g \neq g'$.

When applied to $x = w(h,\ g)$, (1) implies that

(3) $\|\sigma_h\sigma_g(q) - \sigma_{hg}(q)\|_2 < \delta_0\|q\|_2,\ h \in F,\ g \in K \cap h^{-1}K$.

Since $|K \backslash FK| < \varepsilon/2|K|$ and since $\delta,\ \delta_0$ are arbitrarily small, it follows that there exist mutually orthogonal projections $e_g \in Q,\ g \in K$, such that

(4) $\|[e_g,\ x]\|_2 < f_1(\delta_0)\|e_g\|_2,\ \tau(e_g) = \tau(q),\ g \in K$;

(5) $\|e_g - \sigma_g(q)\|_2 < f_2(\delta)\|e_g\|_2,\ g \in K$;

(6) $\|\sigma_h(e_g) - e_{hg}\|_2 < f_3(\delta)\|e_g\|_2,\ h \in F,\ g \in K \cap h^{-1}K$;

where $f_i(\varepsilon) \to 0$ as $\varepsilon \to 0,\ i = 1, 2, 3$.

Thus, if $w(h) = \sum_{g \in K \cap h^{-1}K} e_{hg}\sigma_h(e_g)$, then

$$\|w(h) - \sum_{g \in K \cap h^{-1}K} e_g\|_2^2 < f_3(\delta)^2 \|\sum_{g \in K \cap h^{-1}K} e_g\|_2^2.$$

Since $\sum_{g \in K \backslash hK} \tau(e_g) < \varepsilon/2 \sum_{g \in K} \tau(e_g)$, it follows by letting $\delta,\ \delta_0$ very small and by taking the polar decomposition of $w(h)$ and suitably extending it to a partial isometry $w_1(h)$ from $\sum_{g \in K} \sigma_h(e_g)$ to $\sum_{g \in K} e_g$ carrying $\sigma_h(e_g)$ onto e_{hg} when $g \in K \cap h^{-1}K$ and more generally $\sigma_h(e_g)$ onto $e_{s(h)(g)}$, that we have $w_1(h) \in Q = sNs, h \in F$ and

$$\|w_1(h) - \sum_{g \in K} e_g\|_1 \leq (\varepsilon/2 + f(\delta_0,\ \delta))\|\sum_{g \in K} e_g\|_1.$$

But then, by ([C3]), there exists a suitable extension of $w_1(h)$ to a unitary $w_0(h)$ in Q which will satisfy

$$\|w_0(h) - \|_1 \leq \varepsilon \|\sum_{g \in K} e_g\|_1.$$

Thus, if we take now the $|F| + |K|$ tuple $((v_1(h))_{h \in F}, (e_g^1)_{g \in K})$, with $v_1(h) = ((1 - s) + w_0(h))v_0(h)$ and $e_g^1 = e_g^0 + e_g$, then it is easy to see that it satisfies the necessary conditions to be contained in $\mathcal{F}$. But $((v_1(h))_h, (e_g^1)_g) > ((v_0(h))_h, (e_g^0)_g)$, which contradicts the maximality of $((v_0(h))_{h \in F},\ (e_g^0)_{g \in K})$.

This shows that $\sum_{g \in K} e_g^0 = 1$ and that $v_0(h),\ h \in F$, are unitary elements satisfying the requirements. Q.E.D.

3.4. *Existence of equivariant Jones projections.* We now prove that given any finitely many automorphisms of a cocycle action of an amenable group, there exist Jones projections that are almost invariant to all of them.

LEMMA. *Assume the conditions in the hypothesis of* 3.1 *are satisfied. Let* $X \subset M$, $F \subset G$ *be finite sets and* $\varepsilon > 0$. *Denote* $K = K(F, \varepsilon|F|^{-1}/4)$. *There exist a choice of the tunnel up to some* i, $M \overset{e_0}{\supset} N \overset{e_{-1}}{\supset} N_1 \supset \cdots \supset N_i$, *a partition of the unity* $\{e^0_g\}_{g \in K}$ *in* N_i *and unitary elements* $\{v(h)\}_{h \in F}$ *in* N *such that*

(1) $\|v(h) - 1\|_2 < \varepsilon, h \in F$
(2) $Adv(h)\theta(h)(e^0_g) = e^0_{s_K(h)(g)}, h \in F, g \in K$
(3) $Adv(h)\theta(h)(e_{-j}) = e_{-j}, 0 \le j \le i-1, h \in G$
(4) $\|E_{N_i \vee (N_i' \cap M)}(x) - x\|_2 < \varepsilon, x \in X$.

Proof: By Lemma 3.3, there exist some unitary elements $\{v_0(h)\}_{h \in F} \subset N$ and a partition of the unity by projections $\{e^0_g\}_{g \in K} \subset N$ such that

$$\begin{aligned} \|v_0(h) - 1\|_2 < \varepsilon/2, &\quad h \in F \\ \mathrm{Ad}v_0(h)\theta(h)(e^0_g) = e^0_{s_K(h)(g)}, &\quad h \in F, g \in K \\ \|x - \sum_g e^0_g x e^0_g\|_2 < \varepsilon/2, &\quad x \in X. \end{aligned}$$

Let $\{w_g\}_{g \in K}$ be partial isometries such that $w_g\theta(g)(e^0_e)w_g^* = e^0_g, g \in K$. For any $\delta_0 > 0$, there exists a finite dimensional factor $B_0 \subset e^0_e N e^0_e$ such that if $h \in F, g \in K$, and $hg \in K$ then $\|(\mathrm{Ad}w_{hg}\theta(hg))^{-1}\mathrm{Ad}v_0(h)\theta(h)\mathrm{Ad}w_g\theta(g)(x) - x\|_2 < \delta_0/2$ for all $x \in B_0' \cap e^0_e N e^0_e, \|x\| \le 1$. This is possible because θ is a 2-cocycle action, so that any product $\theta(h)\theta(g)$ differs from $\theta(hg)$ by an inner automorphism and also because N (and thus $e^0_e N e^0_e$) can be approximated by finite dimensional subfactors. Let $Y = \{(\mathrm{Ad}w_g\theta(g))^{-1}(e^0_g x e^0_g) | x \in X, g \in K\} \subset e^0_e M e^0_e$. Since the inclusion $B_0' \cap e^0_e N e^0_e \subset B_0' \cap e^0_e M e^0_e$ has the generating property, it follows that there exist some i and some choice of the tunnel up to i for this inclusion, coming from some tunnel $M \overset{e^0_0}{\supset} N \overset{e^0_{-1}}{\supset} N^0_1 \supset \cdots \supset N^0_i$, with $e^0_e \in N^0_i, B_0 \subset e^0_e N^0_i e^0_e$, such that $\sum \|E_{B_0 \vee (e^0_e N' e^0_e \cap e^0_e M e^0_e)}(y) - y\|_2^2 < 3\varepsilon^2/4$, the sum being taken over all y in Y.

Finally, let $e_{-j} \overset{\text{def}}{=} \sum_{g \in K} w_g\theta(g)(e^0_{-j}e^0_e)w_g^*, 0 \le j \le i-1$. If $B \overset{\text{def}}{=} \Sigma_g \mathrm{Ad}w_g\theta(g)(B_0)$ then $e^0_g \in N_i, B \subset N_i$ and for each $x \in X$ we have

$$\begin{aligned} \|E_{N_i \vee (N_i' \cap M)}(x) - x\|_2^2 &\le \|E_{\Sigma_g B e^0_g \vee N_i' \cap M}(x) - x\|_2^2 \\ &= \sum_g \|E_{Be^0_g \vee (N_i' \cap M)e^0_g}(e^0_g x e^0_g) - e^0_g x e^0_g\|_2^2 + \|x - \sum_g e^0_g x e^0_g\|_2^2 \\ &\le \sum_g \|E_{B_0 \vee (N^0_i \cap M)e^0_e}((\mathrm{Ad}w_g\theta(g))^{-1}(e^0_g x e^0_g)) \\ &\quad - (\mathrm{Ad}w_g\theta(g))^{-1}(e^0_g x e^0_g)\|_2^2 + (\varepsilon/2)^2 < \varepsilon^2. \qquad \blacksquare \end{aligned}$$

This proves that the tunnel $M^{e_0} \supset N^{e_{-1}} \supset N_1 \supset \cdots \overset{e_{-i+1}}{\supset} N_{i-1} \supset N_i$ satisfies (4).

Next, due to the choice of B_0, we have that $\mathrm{Ad}v_0(h)\theta(h)\mathrm{Ad}w_g\theta(g)|_{e^0_e B_0' \cap M e^0_e}$ differs from $\mathrm{Ad}w_{hg}\theta(hg)|_{e^0_e B_0' \cap M e^0_e}$ by an inner automorphism $\mathrm{Ad}u'_{h,g}$ with $u'_{h,g} \in e^0_{hg}\mathrm{Ad}w_{hg}\theta(hg)(B_0' \cap N)e^0_{hg}, \|u'_{h,g} - 1\|_2 < f(\delta_0)$ and $f(\delta_0) \to 0$ as $\delta_0 \to 0$.

We now let $u'_h = \Sigma_g u'_{h,g}$, with the sum taken over $g \in K' = \cap_{h \in F} h^{-1}K$ and define $u_h = u'_h + u''_h$, where $u''_h = \Sigma u''_{h,g}$ the sum being taken over all $g \in K \setminus K'$, with $u''_{h,g}$ satisfying $\mathrm{Ad}w_{hg}\theta(hg)(e^0_{-j}(e^0_e) = \mathrm{Ad}u''_{h,g}\mathrm{Ad}w_g\theta(g)(e^0_{-j}e^0_e)$. Then $v(h) =$

$u_h v_0(h)$ is a unitary element in N that satisfies (1) and (3). Since u_h lies in $\{e^0_g\}'_{g\in K} \cap N$, condition (2) will still be satisfied, since $v_0(h)$ satisfies it. Q.E.D.

3.5. *Proof of Theorem 3.1.* Let $\{x_n\}_n$ be a dense sequence (in the norm $\| \ \|_2$) in M. Let F_n be an increasing sequence of finite subsets of G such that $\cup F_n = G$.

We construct recursively an increasing sequence of integers $0 = i_0 < i_1 < i_2 < \cdots < i_k < \cdots$, for each k a choice of the tunnel $\overset{e_{-i_{k-1}}}{\supset} N_{i_{k-1}} \supset \cdots \overset{e_{-i_k+1}}{\supset} N_{i_k-1} \supset N_{i_k}$, finite dimensional subfactors $\{B_k\}_{k\geq 0}$, with $B_k \subset \cap_j N_j$, $[B_k, B_{k'}] = 0, k \neq k'$, and unitary elements $\{v_k(h)\}_{k\geq 0}, h \in G$, with $v_k(h) \in (B_0 \vee \cdots \vee B_{k-1})' \cap N_{i_k}$, and satisfying:

(1) $\|x_j - E_{B_0\vee B_1\vee\cdots\vee B_k\vee(N'_{i_k}\cap M)}(x_j)\|_2 < 2^{-k}, j \leq k$.

(2) $\mathrm{Ad}(v_k(h)\ldots v_0(h))\theta(h)(e_{-j}) = e_{-j}, 0 \leq i_k, h \in G$.

(3) $\mathrm{Ad}(v_k(h)\ldots v_0(h))\theta(h)(B_k) = B_k, h \in G$ with $B_k \simeq \mathcal{B}(\ell^2(K_k))\otimes M_{n_k\times n_k}(\mathbb{C})$, where $K_k = K(F_k, 2^{-k-2}|F_k|^{-1})$ and $\mathrm{Ad}(v_k(h)\ldots v_0(h))\theta(h)e^k_g = e^k_{s_{K_k}(h)(g)}$, for $h \in F_k, \{e^k_g\}_{g\in K_k}$ being the diagonal of $\mathcal{B}(\ell^2(K_k))$.

(4) $\|v_k(h) - 1\|_2 < 2^{-k}, h \in F_k$.

Assume we made this construction up to some $k \geq 0$. We claim that there exist $\varepsilon > 0$ and a finite subset $X \subset (B_0 \vee \cdots \vee B_k)' \cap N_{i_k-1}$ such that if $N_{i_k-1} \overset{e_{-i_k}}{\supset} N_{i_k} \supset \cdots \overset{e_{-m+1}}{\supset} N_{m-1} \supset N_m$ is a continuation of the tunnel satisfying $B = B_0\vee\cdots\vee B_k \subset N_m$ and $X \underset{\varepsilon}{\subset} N'_m \cap N_{i_k-1}$, then $x_j \underset{2^{-k-2}}{\in} N_m \vee (N'_m \cap M), \forall j \leq k+1$. To see this, we argue exactly as in the proof of Theorem 2.1. Thus, we take $\{m_j\}_j$ to be an orthonormal basis of N_{i_k-1} over N_{i_k-1} and note that one can construct an orthonormal basis $\{m^k_i\}_i$ of M over N_{i_k} as words in the m_j's and in $e_0, e_{-1}, \ldots, e_{-i_k+1}$ (see [PiPo1,2]). Writing $x_1, \ldots, x_{k+1}$ in the basis $\{m^k_i\}_i$ of M over N_{i_k} we obtain a finite set $X \subset N_{i_k}$ which for suitably small ε will yield the estimates.

We can thus apply Lemma 3.4 to the inclusion $B'\cap N_{i_k} \subset B'\cap N_{i_k-1}$, to the above $\varepsilon > 0$ and $X \subset B'\cap N_{i_k-1}$, to the automorphisms $\{\mathrm{Ad}(v_k(h)\ldots v_0(h))\theta(h)|_{B'\cap N_{i_k-1}}\}$ and to the finite set $F_{k+1} \subset G$, to get a continuation of the tunnel up to some $i_{k+1}, \ldots N_{i_k-1} \overset{e_{-i_k}}{\supset} N_{i_k} \supset \ldots \overset{e_{-i_{k+1}+1}}{\supset} N_{i_{k+1}-1} \supset N_{i_{k+1}}$, with $B \subset N_{i_{k+1}}$, unitary elements $v^0_{k+1}(h) \in B'\cap N_{i_k}$ and a partition of the unity $\{e^{k+1}_g\}_{g\in K_{k+1}}$ in $B'\cap N_{i_{k+1}}$ indexed by the Folner set $K_{k+1} = K(F_{k+1}, |F_{k+1}|^{-1}2^{-k-3})$, such that

(1′) $\|E_{((B_0\vee\ldots\vee B_k)'\cap N_{i_{k+1}}\vee(N'_{i_{k+1}}\cap N_{k_k-1})}(x) - x\|_2 < \varepsilon, x \in X$.

(2′) $\mathrm{Ad}v^0_{k+1}(h)v_k(h)\ldots v_0(h)\theta(h)(e_{-j}) = e_{-j}, i_k \leq j < i_{k+1}, h \in G$.

(3′) $\mathrm{Ad}v^0_{k+1}(h)v_k(h)\ldots v_0(h)\theta(h)(e^{k+1}_g) = e_{s_{K_{k+1}}(h)(g)}, h \in F_{k+1}, g \in K_{k+1}$.

(4′) $\|v'_{k+1}(h) - 1\|_2 < 2^{-1}2^{-k-1}, h \in F_{k+1}$.

Indeed, in order to be able to apply Lemma 3.4 we only need to show that $\mathrm{Ad}(v_k(h)\ldots v_0(h))$
$\theta(h), h \notin e$, are all properly outer when restricted to both N_{i_k-1} and N_{i_k}. To see this we only need to show that if $\sigma \in \mathrm{Aut}(M, N), \sigma, \sigma|_N$ are properly outer, $M \overset{e_0}{\supset} N \supset N_1$ is a downward basic construction and $v \in N$ is a unitary such that $\mathrm{Ad}v\sigma(e_0) = e_0, \mathrm{Ad}v\sigma(N_1) = N_1$, then $\mathrm{Ad}v\sigma|_{N_1}$ is also properly outer. But $\mathrm{Ad}v\sigma$ properly outer on M and $\mathrm{Ad}v\sigma(e_0) = e_0$ implies $\mathrm{Ad}v\sigma|_{e_0Me_0}$ properly outer and since $e_0Me_0 = N_1e_0 \simeq N_1$, it follows that $\mathrm{Ad}v\sigma|_{N_1}$ is also properly outer.

Now, since $N_{i_{k+1}}$ is hyperfinite, there exists a finite dimensional factor $B^0_{k+1} \subset (B_0 \vee \ldots \vee B_k)' \cap N_{i_{k+1}}$ containing the projections $\{e^{k+1}_g\}_{g \in K_{k+1}}$ such that

(1″) $\|E_{B^0_{k+1} \vee (N'_{i_{k+1}} \cap N_{i_k - 1})}(x) - x\|_2 < \varepsilon, x \in X$.

But then, by Corollary 3.2, there exist a finite dimensional subfactor $B_{k+1} \subset B' \cap N_{i_{k+1}}$ containing B^0_{k+1} and unitary elements $v^1_{k+1}(h) \in B' \cap N_{i_{k+1}}, h \in G$, such that

(3″) $\mathrm{Ad}v^1_{k+1}(h)v^0_{k+1}(h)\theta(h)(B_{k+1}) = B_{k+1}, h \in G$.

(4″) $\|v^1_{k+1}(h) - 1\|_2 < 2^{-k-2}, h \in F_{k+1}$.

Thus, if we define $v_{k+1}(h) = v^1_{k+1}(h)v^0_{k+1}(h)$ then, by the above properties of X, ε, it follows that all the conditions $(1) - (4)$ are satisfied at step $k+1$.

We now let $R = \overline{\cup_k (N'_k \cap M)}$, $S = \overline{\cup_k (N'_k \cap N)}$, $R_0 = B_0 \vee B_1 \vee \ldots$, $v(h) = \lim_k (v_k(h) \ldots v_0(h))$, $\sigma_0(h) = \theta(h)|_{R_0}$, $h \in G$.

Condition (1) implies that $M = R \vee R_0 \simeq R \otimes R_0$. Condition (4) shows that $v(h)$ are all unitary elements in N. Then (2) shows that $\mathrm{Ad}v(h)\theta(h)$ restricted to $S \subset R$ (which is isomorphic to $N^{st} \subset M^{st}$) is nothing but $\theta^{st}(h)$. Finally, by (3), we have that $\mathrm{Ad}v(h)\theta(h)(R_0) = R_0$ and also that $\mathrm{Ad}v(h)\theta(h)$ is of the form $\theta_0 \otimes \theta_1$, for some splitting $R_0 = R \otimes R_1$, with θ_0 a model properly outer cocycle action of G on the hyperfinite II_1 factor R, as in Section 3.2. Thus, $\mathrm{Ad}v(h)\theta(h)_{R_0}, h \in G$, is a cocycle action of G on R_0. But then by Ocneanu's theorem in [Oc], $\mathrm{Ad}v(h)\theta(h)|_{R_0}$ is the same as the chosen action σ_0 on R_0, after perturbing if necessary with unitaries in R_0. Q.E.D.

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