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Sorin Popa

Mathematical Sciences Research Institute
1000 Centennial Drive Berkeley, California 94720–5070

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SORIN POPA

University of CaliforniaΓ LosAngeles

0. Introduction.

The first examples of type II_1 factors M having countable fundamental group $\mathcal{F}(M)$ have been constructed by Connes in ([C1])Γ as group algebras$M = L(G)$ for G discrete ICC groups with the property T of Kazhdan. Since thenΓ many more examples of such factors have been constructed (see e.g.Γ [Po1Γ3] or[Gol]). But all such constructions are using property T groupsΓ in a way or another. For instanceΓ it is proved in ([Po1]) that if a type II_1 factor M contains a group von Neumann algebra $L(G)$ for some ICC property T group G then $\mathcal{F}(M)$ is countable.

All the factors M having $\mathcal{F}(M)$ countable that have been constructed so far do not have non-trivial asymptotically central sequencesΓ i.e.Γ they do not have the property Γ of Murray and von Neumann (equivalentlyΓ theyare full factorsΓ in the sense of [C]). In this respectΓ note that if a factor M has non-commuting such central sequences then by a result of McDuff ([McD]) they split-off the hyperfinite type II_1 factorΓ thus having fundamental group equal to $\mathbb{R}_+^*$. ThusΓ if it is for a property Γ factor M to have fundamental group $\neq \mathbb{R}_+^*$ Γ then its central sequences must commute.

In this paper we construct a class of examples of factors which do have the property Γ yet have countable fundamental group. The construction does not use property T groupsΓ but instead uses the rigidity properties of the inclusion $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$$\Gamma$ also due to Kazhdan ([Kaz]). We will also use perturbation resultsΓ separability arguments and the recent striking results of Gaboriau on the cost of equivalence relations.

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The paper is organised as follows: In Section 1 we prove some perturbation result for Cartan subalgebras in type II_1 factorΓin the spirit of some well known results of Eric Christensen. In Section 2 we prove the main result of the paper (Theorem 2.3)Γ providing some classes of factors with countable fundamental groupΓ both with and without property Γ. These factors do not contain type II_1 factors with the property T of ([CJ1])Γnor can be embedded into free group factorsΓyet they seem to be closer to the latter class. In Section 3 we make some commentsΓ and sketch another construction of property Γ factors with countable fundamental groupΓthis time by using property T groups (Remark 3.1). We also construct a class of examples of property Γ factors with fundamental group $\mathbb{R}_+^*$ which however are not McDuff (Remark 3.2).

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1. Some perturbation results.

In this Section we prove some perturbation results for subalgebras in type II_1 factors.

The first such perturbation result concerns maximal abelian $*$-subalgebras (abreviated as m.a.s.a. hereafter) of type II_1 factors. Besides the technique from [Ch]Γ the proof uses the "pull down" lemma in the basic construction ([PiPo]) and some considerations on the geometry of projectionsΓnotably a result of Kadison ([K]).

1.1. Theorem. *Let N be a type II_1 factor and $B_0, B_1 \subset N$ be m.a.s.a.'s of N such that* $\sup\{\|u - E_{B_1}(u)\|_2 \mid u \in \mathcal{U}_0\} < 1$, *for some unitary subgroup $\mathcal{U}_0 \subset B_0$ satisfying $\mathcal{U}_0'' = B_0$. Then there exists a non-zero partial isometry $v \in N$ such that $v^*v \in B_0, vv^* \in B_1$ and $vB_0v^* = B_1vv^*$.*

Proof. Consider first the basic construction for the inclusion $B_1 \subset N$: ThusΓ we let e be the orthogonal projection of $L^2(N, \tau)$ onto $L^2(B_1, \tau)$Γand which is known to satisfy $exe = E_{B_1}(x)e, \forall x \in N$. Then we let $N_1 = \langle N, e\rangle$ be the von Neumann algebra generated inside $\mathcal{B}(L^2(N, \tau))$ by N and e. Note that $eN_1e = B_1e$. We endow N_1 with the unique normal semifinite faithful trace Tr which satisfies $Tr(xey) = \tau(xy), \forall x, y \in N$. Note that there exists a unique $N - N$ bilinear map Φ from $\mathrm{sp}NeN \subset N_1$ into N satisfying $\Phi(xey) = xy, \forall x, y \in N$. This bilinear map satisfies the "pull down" identity $eX = e\Phi(eX), \forall X \in N_1$ from ([PiPo]).

Let now $K_e = \overline{\mathrm{co}}^{\mathrm{w}}\{u_0eu_0^* \mid u_0 \in \mathcal{U}_0\}$. We clearly have $0 \leq a \leq 1$ and $Tr(a) \leq 1, \forall a \in K_e$. MoreoverΓ$K_e$ is contained in the Hilbert space $L^2(N_1, Tr)$Γ whereit is still weakly closed. Let $h \in K_e$ be the unique element of minimal norm $\| \|_{2,Tr}$ in

K_e. Since $uhu^* \in K_e$ and $\|uhu^*\|_{2,Tr} = \|h\|_{2,Tr}, \forall u \in \mathcal{U}_0$Γby the uniqueness of h it follows that $uhu^* = h, \forall u \in \mathcal{U}_0$. Thus $h \in \mathcal{U}_0' \cap N_1 = B_0' \cap N_1$.

On the other handΓsince $Tr(eueu^*) = \|E_{B_1}(u)\|_2^2 = 1 - \|u - E_{B_1}(u)\|_2^2$Γif we denote $\delta = 1 - \sup\{\|u - E_{B_1}(u)\|_2 \mid u \in \mathcal{U}_0\}$Γ it follows that $Tr(eueu^*) \geq \delta, \forall u \in \mathcal{U}_0$. Taking appropriate linear combinations and weak limitsΓ it follows that $Tr(eh) \geq \delta$. Since by hypothesis we have $\delta > 0$Γ itfollows that $h \neq 0$.

Let e_0 be a non-zero spectral projection of h. ThusΓe_0 is a finite projection in N_1 and e_0 commutes with B_0 (since h does). Since $B_0 e_0$ is abelianΓ it is contained in a maximal abelian subalgebra B of $e_0 N_1 e_0$. (Note that any element in B commutes with B_0.) By a result of Kadison ([K])ΓB contains a non-zero abelian projection e_1 of N_1 (i.e.Γ$e_1 N_1 e_1$ is abelian). Since e has central valued (semifinite) trace equal to 1Γ itfollows that e majorizes e_1.

Let $V \in N_1$ be a partial isometry such that $V^*V = e_1 \leq e_0$ and $VV^* \leq e$. Moreover VBe_1V^* is a subalgebra of $eN_1e = B_1e$. Since e_1 commutes with B_0Γit follows that if we denote by f' the maximal projection in B_0 such that $f'e_1 = 0$ and let $f_0 = 1 - f'$Γthen there exists a unique isomorphism φ from $B_0 f_0$ into B_1 such that $\varphi(b)e = VbV^*, \forall b \in B_0 f$. Let $f_1 = \varphi(f_0) \in B_1$.

It follows that $\varphi(b)eV = eVb, \forall b \in B_0 f_0$. By applying Φ to both sides and denoting $a = \Phi(eV) \in N$Γit follows that $\varphi(b)a = ab, \forall b \in B_0$. Since $ea = eV = V$Γ it follows that $a \neq 0$.

By the usual trickΓif we denote by $v_0 \in N$ the unique partial isometry in the polar decomposition of a such that the right supports of a and v_0 coincideΓthen $p_0 = v_0^* v_0 \in B_0' \cap N = B_0$Γ$p_1 = v_0 v_0^* \in \varphi(B_0)f_1' \cap f_1 N f_1$ and $\varphi(b)v_0 = v_0 b, \forall b \in B_0 f_0$.

But $B_0 f_0$ maximal abelian in $f_0 N f_0$ implies that $vB_0v^* = \varphi(B_0)p_1$ is maximal abelian in $v_0 v_0^* N v_0 v_0^*$. Since any element in $p_1 B_1 p_1$ commutes with $\varphi(B_0)p_1$Γ which is maximal abelianΓ it follows that $p_1 B_1 p_1 = \varphi(B_0)p_1$. ThusΓif P_1 denotes the von Neumann algebra generated by p_1 and $B_1 f_1$ inside $f_1 N f_1$Γthen P_1 is like a basic construction for the inclusion $\varphi(B_0)f_1 \subset B_1 f_1$Γwith p_1 playing the role of the "Jones projection". In particualarΓ$p_1 P_1 p_1 = \varphi(B_0)p_1$ is abelian.

ThusΓp_1 is an abelian projection in P_1. Since P_1 is a finite von Neumann algebraΓ there exists a central projection z_1 of P_1Γunder the central support of p_1 in P_1Γ such that $p_1 z_1 \neq 0$ and such that $P_1 z_1$ is homogeneous of type nΓfor some $n \geq 1$. Note that the center of P_1 is included in $B_1 f_1$ the latter being maximal abelian in $f_1 N f_1$Γthus in $P_1 \subset f_1 N f_1$. ThusΓ$z_1 \in B_1 f_1$.

NowΓsince $p_1 z_1$ has central trace equal to $1/n$ in $P_1 z_1$ and $B_1 z_1$ is maximal abelian in $P_1 z_1$Γ it follows that there exists a projection $f_{11} \in B_1 f_1$ such that f_{11} is equivalent to $p_1 z_1$ in $P_1 z_1$ (see [K]). Let $v_1 \in P_1 z_1$ be such that $v_1 v_1^* = f_{11}, v_1^* v_1 = p_1 z_1$. Since $p_1 z_1$ is abelian in P_1Γf_{11} is also abelianΓthus $f_{11} P_1 f_{11} = B_1 f_{11}$. This implies that $v_1(\varphi(B_0)p_1z_1)v_1^* = B_1f_{11}$.

FinallyΓ since$B_0f_0 \ni b \mapsto \varphi(b) \in \varphi(B_0)f_1$ and p_1z_1 belongs to $\varphi(B_0)p_1$Γ it follows that there exists a projection f_{00} in B_0f_0 such that $\varphi(f_{00})p_1 = z_1p_1$. In particularΓ $v_1^*v_1 = v_0f_{00}v_0^*$.

AltogetherΓ this shows that if we denote $v = v_1v_0f_{00}$Γ then$v$ is a partial isometry satsifying $v^*v = f_{00} \in B_0, vv^* = f_{11} \in B_1$ and $vB_0v^* = B_1vv^*$. Q.E.D.

In Section 2 we will in fact need a consequence of Theorem 1.1. To state itΓ recall from ([D1]) that a m.a.s.a. A of a von Neumann factor N is called *semiregular* if the set of unitaries of N that normalize A generate a factor. AlsoΓA is called *regular* if this normalizer generetes all the ambient factor N. Such regular m.a.s.a.'s were later called *Cartan subalgebras* in ([FM])Γ a terminology that seems to have prevailed and which we will therefore adopt.

We will also use the following notations from ([Ch]):

1.2. Notation. Let $\mathcal{B}_0, \mathcal{B}_1$ be von Neumann subalgebras of a type II_1 factor N. If $\sup\{\|x_0 - E_{\mathcal{B}_1}(x_0)\|_2 \mid x_0 \in \mathcal{B}_0, \|x_0\| \leq 1\} \leq \varepsilon$ then we write $\mathcal{B}_0 \subset_\varepsilon \mathcal{B}_1$. AlsoΓ we denote by $d(\mathcal{B}_0, \mathcal{B}_1)$ the maximum between $\sup\{\|x_0 - E_{\mathcal{B}_1}(x_0)\|_2 \mid x_0 \in \mathcal{B}_0, \|x_0\| \leq 1\}$ and $\sup\{\|x_0 - E_{\mathcal{B}_1}(x_0)\|_2 \mid x_0 \in \mathcal{B}_0, \|x_0\| \leq 1\}$).

1.3. Corollary. *Let N be an arbitrary type II_1 factor and $A_0, A_1 \subset N$ be semiregular m.a.s.a.'s of N. If $A_0 \subset_{1-\delta} A_1$ for some $\delta > 0$ then there exists a unitary element $u \in N$ such that $uA_0u^* = A_1$. In particular, this is the case if A_0, A_1 are Cartan subalgebras of N.*

Proof. The condition implies that $\sup\{\|u - E_{A_1}(u)\|_2 \mid u \in \mathcal{U}(A_0)\} < 1$. ThusΓ by Theorem 1.1 there exists a non-zero partial isometry $v \in N$ such that $v^*v \in A_0, vv^* \in A_1, vA_0v^* = A_1vv^*$. MoreoverΓ by cutting v from the right with a smaller projection in A_0Γwe may clearly assume $\tau(vv^*) = 1/n$ for some integer n.

Since A_0, A_1 are semiregularΓthere exist partial isometries $v_1, v_2, ..., v_n$Γrespectively $w_1, w_2, ..., w_n$ in the normalizing groupoids of A_0 respectively A_1 such that $\Sigma_i v_iv_i^* = 1, \Sigma_j w_jw_j^* = 1$ and $v_i^*v_i = v^*v, w_j^*w_j = vv^*, \forall i, j$. But then $u = \Sigma_i w_ivv_i^*$ is a unitary element and $vA_0v^* = A_1$. Q.E.D.

Let us also mention another application of Theorem 1.1 whichΓalthough not needed later in this paperΓ has some independent interest. ThusΓ recall from ([D1]) that a m.a.s.a. A of a von Neumann algebra N is *singular* if the only unitaries in N that normalize A are the unitaries of A. When N is a type II_1 factorΓ in ([P3]) a numerical invariant $\delta(A)$ was associated to m.a.s.a.'s A of NΓas a "measure of singularity"Γ asfollows:

For each non-zero partial isometry $v \in N$ with vv^*, v^*v mutually orthogonal projections in AΓdenote $\delta(vAv^*, A) = \sup\{\|x - E_A(x)\|_2 \mid x \in vAv^*, \|x\| \leq 1\}$.

Then define $\delta(A)$ to be the infimum of $\delta(vAv^*, A)/\|vv^*\|_2$Γas v runs over the set of all such partial isometries. In ([Po3]) it was noted that $\delta(A) > 0$ implies A is singular and examples of m.a.s.a.'s with $\delta(A) > 0$ were constructed in any type II_1 factor with separable predual. But it was left as an open question to calculate the possible values the constants $\delta(A)$ can take. This problem was recently revived in ([SiSm]). From 1.1 we get:

1.4. Corollary. *Let N be an arbitrary type II_1 factor and $A \subset N$ a singular m.a.s.a. Then $\delta(A) = 1$.*

Proof. If for some partial isometry $v \in N$ with $vv^*, v^*v \in A$Γ$v \neq 0, v^2 = 0$ we have $\sup\{\|x - E_{Avv^*}(x)\|_2/\|vv^*\|_2 \mid x \in vAv^*, \|x\| \leq 1\} < 1$Γthen it follows that the m.a.s.a.'s $B_0 = vAv^*, B_1 = Avv^*$ in the factor vv^*Nvv^* (with its normalized trace) verify the condition in Theorem 1.1. ThusΓ there exists a non-zero partial isometry $v_0 \in vv^*Nvv^*$ such that $v_0^*v_0 \in vAv^*, v_0v_0^* \in Avv^*$ and $v_0vAv^*v_0^* \subset A$. But this contradicts the singularity of A. Q.E.D.

We end this section by mentioning two well known perturbation results from ([Ch])Γ needed in the sequel. We have included a proof for the sake of completeness.

1.5. Lemma. [Ch]. *Let N be a type II_1 factor and $B_0, B_1 \subset N$ be von Neumann subalgebras of N. Assume there exists a subgroup $\mathcal{U}_0$ of the unitary group of B_0 such that $\mathcal{U}_0'' = B_0$ and $\|u_0 - E_{B_1}(u_0)\|_2 \leq \varepsilon, \forall u_0 \in \mathcal{U}_0$. Then $B_1' \cap N \subset_{2\varepsilon} B_0' \cap N$.*

Proof. Let $x \in B_1' \cap N, \|x\| \leq 1$. Since $E_{B_0' \cap N}(x)$ is the element of minimal norm-2 in the weakly compact convex set $\overline{\mathrm{co}}^{\mathrm{w}}\{u_0xu_0^* \mid u_0 \in \mathcal{U}_0\}$Γit follows that $\forall \delta > 0, \exists u_1, u_2, ..., u_n \in \mathcal{U}_0$ such that $\|1/n\Sigma_i u_ixu_i^* - E_{B_0' \cap N}(x)\|_2 < \delta$.

But by hypothesisΓfor all $i = 1, 2, .., n$ we have the estimates:

$$\|x - u_ixu_i^*\|_2 \leq \|x - E_{B_1}(u_i)xu_i^*\|_2 + \|u_i - E_{B_1}(u_i)\|_2$$
$$= \|x - xE_{B_1}(u_i)u_i^*\|_2 + \|u_i - E_{B_1}(u_i)\|_2 \leq 2\|u_i - E_{B_1}(u_i)\|_2 \leq 2\varepsilon.$$

AltogetherΓ this implies that $\|x - E_{B_0' \cap N}(x)\|_2 < 2\varepsilon + \delta$Γ with δ arbitraryΓ showing that $x \in_{2\varepsilon} B_0' \cap N$. Since x was taken arbitrary in the unit ball of $B_1' \cap N$Γ we have shown that $B_1' \cap N \subset_{2\varepsilon} B_0' \cap N$. Q.E.D.

1.6. Lemma. [Ch]. *Let N be a type II_1. Assume there exists a subgroup $\mathcal{U}_0$ of the unitary group of N and a group morphism $\rho : \mathcal{U}_0 \to \mathcal{U}(N)$ such that $\|\rho(u_0) - u_0\|_2 \leq \varepsilon, \forall u_0 \in \mathcal{U}_0$. Then there exists a partial isometry $v \in N$ such that $v^*v \in \mathcal{U}_0' \cap N$, $vv^* \in \rho(\mathcal{U}_0)' \cap N$, $\|1 - v\|_2 \leq 2\varepsilon$ and $\mathrm{Ad}(v)(u_0) = \rho(u_0)vv^*, \forall u_0 \in \mathcal{U}_0$.*

Proof. Let $K_\rho = \overline{\mathrm{co}}^{\mathrm{w}}\{\rho(u_0)u_0^* \mid u_0 \in \mathcal{U}_0\}$. Let $k \in K_\rho$ be the unique element of minimal norm-2. Since $\rho(u_0)K_\rho u_0^* \subset K_\rho$ and $\|\rho(u_0)k'u_0^*\|_2 = \|k'\|_2$Γforall $u_0 \in$

$\mathcal{U}_0, k' \in K_\rho$Γit follows that $\rho(u_0)ku_0^* = k, \forall u_0 \in \mathcal{U}_0$. ThusΓ$\rho(u_0)k = ku_0, \forall u_0 \in \mathcal{U}_0$. By the usual trickΓit follows that if v denotes the partial isometry in the polar decomposition of k with right support equal to the right support of k then v is an intertwiner between ρ and id on $\mathcal{U}_0$. AlsoΓsince any element k' in K_ρ satisfies $\|k'-1\|_2 \leq \varepsilon$Γ by standard estimates (see e.g.Γ [C2] or [Ch]) one gets$\|v-1\|_2 \leq 2\varepsilon$. Q.E.D.

2. Examples of factors M with countable $\mathcal{F}(M)$.

In this section we construct a class of examples of type II_1 factors with countable fundamental group. Some of them have the property Γ of Murray and von Neumann while others don't. The construction relies on the rigidity properties of the embedding of the group $G_0 = \mathbb{Z}^2$ inside the group $G = \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}^2)$Γdiscovered by Kazhdan in the late 60's ([Kaz]). HoweverΓ as we will prove in the next sectionΓ the factors that we construct in this section do not contain any subfactor with the property TΓother than the finite dimensional ones.

2.1. The construction. Let (X_0, μ_0) be the 2-dimensional thorusΓregarded as the dual group of $\mathbb{Z}^2$Γ endowed with the Haar measure μ_0. Note that μ_0 is the same as the Lebesgue measure on $X_0 = \mathbb{T}^2$. Let σ_0 be the action of $SL(2, \mathbb{Z})$ on X_0 implemented by the action of $SL(2, \mathbb{Z})$ on $\mathbb{Z}^2$. Let (X_1, μ_1) be a probability space with a measure preserving ergodic transformation σ_1 of $SL(2, \mathbb{Z})$ on it. Let $\sigma = \sigma_0 \times \sigma_1$ be the product action on the probability space $(X, \mu) = (X_0 \times X_1, \mu_0 \times \mu_1)$. Denote $A = L^\infty(X, \mu)$ and $M = A \rtimes_\sigma SL(2, \mathbb{Z})$. AlsoΓdenote $A_0 = L^\infty(X_0, \mu_0)$ and $M_0 = A_0 \rtimes_{\sigma_0} SL(2, \mathbb{Z})$.

Note that we can regard A_0 as a subalgebra AΓin which case the canonical unitaries $u_g, g \in SL(2, \mathbb{Z}) \subset M$ implementing the action σ on A also implement the action σ_0 on A_0. ThusΓM_0 can be viewed as a subfactor of M. MoreoverΓwe can view A_0 as $L(\mathbb{Z}^2)$Γin which case M_0 is identified with $L(\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}))$.

Given any arbitrary ergodic action σ_1 as in 2.1Γthe action σ and the algebras defined above have the following properties:

2.2. Lemma. 1°. *The action σ is a free, ergodic action of $SL(2, \mathbb{Z})$ on the probability space (X, μ).*

2°. *The action σ_0 is strongly ergodic. The action $\sigma = \sigma_0 \times \sigma_1$ is strongly ergodic if and only if σ_1 is strongly ergodic.*

3°. *There exist ergodic actions σ_1 of $SL(2, \mathbb{Z})$ that are not strongly ergodic.*

4°. *M_0 is a non-Γ type II_1 factor and $A_0 \subset M_0$ is a Cartan subalgebra of M_0. More generally, M is a type II_1 factor, $A \subset M$ is a Cartan subalgebra in M and M is non-Γ if and only if σ_1 is strongly ergodic.*

5°. *When regarded as a subalgebra in M, A_0 satisfies $A_0' \cap M = A$.*

Proof. 1°. It is well known that a non-inner automorphism of a group G implements a properly outer automorphism on $L(G)$ that preserves the canonical trace of $L(G)$. ThusΓ since each non-trivial element g in the group $SL(2, \mathbb{Z})$ implements a non-inner automorphism of $\mathbb{Z}^2$ it follows that $\sigma_0(g)$ is properly outer $\forall g \in SL(2, \mathbb{Z}), g \neq e$.

Since the tensor product of any properly outer automorphism with an arbitrary automorphism is properly outerΓit follows that $\sigma(g)$ is properly outer $\forall g \in SL(2, \mathbb{Z}), g \neq e$Γas well.

FurthermoreΓ the actionσ_0 is well known to be mixing. More preciselyΓ it is easy to see that given any finite set $F \subset SL(2, \mathbb{Z}), e \notin F$Γthere exists $g \in SL(2, \mathbb{Z})$ such that $gF \cap F = \emptyset$. But this implies not only that σ_0 is ergodicΓbut also that its tensor product with any ergodic action is still ergodic.

2°. The first part is a well known result of Klaus Schmidt ([S1]). The second part is an immediate consequence of the proof of this result in ([S1]).

3°. This is a consequence of a theorem of Connes and Weiss ([CW])Γshowing that any discrete group G_1 which doesn't have the property T has a freeΓergodic but not strongly ergodic action σ_1 on a probability space. ThusΓ one simply applies this result to $G_1 = SL(2, \mathbb{Z})$Γ which doesn't have the property T.

One can in fact avoid using the general result in ([CW])Γby noticing that since $SL(2, \mathbb{Z})$ has an infinite amenable group H as a quotient (see e.g.Γ [dHV])Γany ergodic action of H on a non-atomic probability space (e.g.Γ a Bernoulli shift action of H) composed with the quotient map $G_1 \to H$ gives an ergodic but not strongly ergodic action of $SL(2, \mathbb{Z})$ (note that the resulting action of $SL(2, \mathbb{Z})$ is not free thoughΓin fact not even faithfulΓbut freeness is not necessary in the construction 1.1).

4°. Since $SL(2, \mathbb{Z})$ is close to be a free group (see e.g.Γ[dHV])Γit is easy to see that any central sequences in a type II_1 factor of the form $B \rtimes_\sigma SL(2, \mathbb{Z})$ obtained as the cross product of a finite von Neumann algebra (B, τ) by a freeΓτ-preserving action σ of $SL(2, \mathbb{Z})$ on it must be supported on B.

5°. Note that if $b = \Sigma_g a_g u_g \in M = A \rtimes_\sigma SL(2, \mathbb{Z})$ commutes with all $a \in A_0$ and $a_g \neq 0$ for some $g \neq e$ then $a a_g u_g = a_g u_g a, \forall a \in A_0$Γimplying that $a_g a = a_g \sigma_g(a), \forall a \in A_0$. This in turn contradicts the fact that σ_g is properly outer on A_0. ThusΓall a_g with $g \neq 0$ must be equal to 0Γ implying that$b$ lies in A. Q.E.D.

2.3. Theorem. *Given any ergodic action σ_1 of $SL(2, \mathbb{Z}^2)$ on a probability space (X_1, μ_1), the type II_1 factor M constructed in 2.1 has countable fundamental group.*

To prove the TheoremΓ we'll use first a "separability argument"Γ thenthe result of Kazhdan on the rigidity of the embedding of the group $\mathbb{Z}^2$ inside $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ ([Kaz])Γthen the perturbation results from the previous sectionΓand finally the recent rigidity result of the free ergodic actions of the free groups on probability spaces of Gaboriau ([G]).

We split the corresponding arguments in a series of lemmas.

2.4. Lemma. *Let N be a separable type II_1 factor. If $\mathcal{F}(N)$ is uncountable then there exist projections $p_n \in \mathcal{P}(N)$, with $p_n \neq 1, p_n \nearrow 1$, and isomorphisms $\theta_n : N \simeq p_nNp_n$, such that $\lim_{n\to\infty} \|\theta_n(x) - x\|_2 = 0, \forall x \in N$. Moreover, the projections $\{p_n\}_n$ can be taken to lie in any given diffuse abelian von Neumann subalgebra A_0 of N.*

Proof. Let $\{q_t \mid 0 \leq t \leq 1\} \subset N$ be a totally ordered set of projections with $\tau(q_t) = t, \forall 0 \leq t \leq 1$. For each $t \in S = (1/2, 1) \cap \mathcal{F}(N)$ choose an isomorphism $\theta'_t : N \simeq q_tNq_t$ and denote $\mathcal{I} = \{\theta'_t \mid t \in S\}$. Note that$\Gamma$ since$\mathcal{F}(N)$ is uncountableΓ S is uncountable.

Let $\{x_n\}_{n\geq 1}$ be a sequence of elements in the unit ball N_1 of $N\Gamma$such that as a set it is dense in N_1 in the norm $\| \|_2$ and such that each element appears infinitely many times in the sequence.

Since $\{\theta'_t(x_1)\}_{t\in S}$ is a subset of $N_1\Gamma$by covering N_1 with countably many open balls of $\| \|_2$-radius $1/2$ and using the separability of $(N_1, \| \|_2)$ it follows that there exists an uncountable subset $S_1 \subset S$ such that $\|\theta_t(x_1) - \theta'_{t'}(x_1)\|_2 < 1, \forall t, t' \in S_1$. Similarily$\Gamma$ by repeatedly using the separability of N_1 to cover it by countably many balls of radius $1/2n\Gamma$ we construct recursively a decreasing sequence of uncountable sets $S_n \subset S, n \geq 1\Gamma$such that

$$\|\theta'_t(x_n) - \theta'_{t'}(x_n)\|_2 < 1/n, \forall t, t' \in S_n.$$

For each $n \geq 1$ choose now two distinct $t_n, t'_n \in S_n\Gamma$say with $t_n > t'_n$. Since $\theta'_{t_n}(1) \geq \theta'_{t'_n}(1)\Gamma$

$$\theta_n(x) = \theta'^{-1}_{t_n}(\theta'_{t'_n}(x)), x \in N,$$

gives a well defined isomorphism of N onto $p_nNp_n\Gamma$where

$$p_n = \theta'^{-1}_{t_n}(\theta'_{t'_n}(1)) = \theta^{-1}_{t_n}(q_{t'_n}) \in \mathcal{P}(N).$$

MoreoverΓsince $t_n, t'_n \in S_j, \forall j \leq n\Gamma$we have for each $n \geq 1$ and $1 \leq j \leq n$ the estimates:

$$\|\theta_n(x_j) - x_j\|_2^2 = \tau(q_{t_n})^{-1}\|\theta'_{t'_n}(x_j) - \theta'_{t_n}(x_j)\|_2^2 \leq 2/j.$$

Since each x_j appears infinitely many timesΓit follows that $\lim_{n\to\infty} \|\theta_n(x_j) - x_j\|_2 = 0, \forall j \geq 1$. By the density of $\{x_n\}_n$ in $N_1\Gamma$ it follows that $\lim_{n\to\infty} \|\theta_n(x) - x\|_2 = 0, \forall x \in N_1\Gamma$and thus for all $x \in N$.

FinallyΓin case we want the projections $\{p_n\}_n$ to lie in a given diffuse abelian von Neumann subalgebra A_0Γthen we first choose projections $\{p'_n\}_n \subset A_0$ with $\tau(p'_n) = \tau(p_n), \forall n$Γ andnote that $\|p'_n - p_n\|_2 \to 0$ (because both sequences tend to 1). ThusΓthere exist partial isometries $v_n \in N$ such that $v_n v_n^* = p_n, v_n^* v_n = p'_n, \forall n$ and $\|v_n - 1\|_2 \to 0$ (see e.g.Γ [C2]or [Ch]). But thenΓby replacing n p'_n for p_n and $\mathrm{Ad}(v_n)\theta_n$ for θ_nΓ$\forall n \geq 1$Γ thelast condition will also be satisfied. Q.E.D.

We are now going to use the rigidity of the embedding of the group $\mathbb{Z}^2$ inside $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ (cf. [Kaz]). We recall that if G_0 is a subgroup of a discrete group G then we say that G_0 *has the property* T *inside* G if there exist $g_1, g_2, ..., g_n \in G$ and $\varepsilon > 0$ such that if π is a unitary representation of G on a Hilbert space $\mathcal{H}$ such that for some unit vector $\xi \in \mathcal{H}$ one has $\|\pi(g_i)\xi - \xi\| \leq \varepsilon, \forall i$ then there exists a non-zero vector $\xi_0 \in \mathcal{H}$ such that $\pi(h)\xi_0 = \xi_0, \forall h \in G_0$. Note that if this is the caseΓthen by a well known argument it follows that $\exists K \geq 1$ such that $\forall \delta < \varepsilon$Γif $\|\pi(g_i)\xi - \xi\| \leq \delta, \forall i$ then $\|\pi(h)\xi - \xi\| \leq K\delta, \forall h \in G_0$ (see e.g.Γ [De-Ki]).

2.5. Lemma. *Let G be a discrete group and $G_0 \subset G$ a property T embedding of a group G_0 into G (e.g., $G_0 = \mathbb{Z}^2, G = \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}^2)$). Let $(A_0 \subset M_0) = (L(G_0) \subset L(G))$. Denote by $u_g, g \in G$, the canonical unitaries in M_0. Let N be a type* II_1 *factor that contains M_0. Assume $\{\theta_n\}_n$ is a sequence of non-necessarily unital endomorphisms of N such that $\lim_{t\to 0}\|\theta_n(u_g) - u_g\|_2 = 0, \forall g \in G$. Then for any $\varepsilon > 0$ there exists n_ε such that if $n \geq n_\varepsilon$ then $\|\theta_n(u_h) - u_h\|_2 \leq \varepsilon$, $\forall h \in G_0$.*

Proof. Let $\pi_n : \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}^2) \to \mathcal{U}(\mathcal{H}_n)$ be the unitary representation on the Hilbert space $\mathcal{H}_n = \theta_n(1)L^2(M,\tau))$ defined by $\pi_n(g)(\xi) = \theta_n(u_g)\xi u_g^*, g \in \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}^2)$. If we let $\xi_n \in \mathcal{H}_n$ be the projection $p_n \in M$ regarded as a vector in $\mathcal{H}_n$ then

$$\|\pi_n(g)\xi_n - \xi_n\| = \|\theta_n(u_g)u_g^* - p_n\|_2$$

$$= \|\theta_n(u_g) - p_n u_g\|_2 \leq \|\theta_n(u_g) - u_g\|_2 + \|1 - p_n\|_2, \forall n \geq 1, g \in G.$$

SimilarilyΓ

$$\|\pi_n(g)\xi_n - \xi_n\| \geq \|\theta_n(u_g) - u_g\|_2 - \|1 - p_n\|_2, \forall n \geq 1, g \in G.$$

By the hypothesis and the first set of estimates it follows that $\lim_{n\to\infty} \|\pi_n(g)\xi_n - \xi_n\| = 0, \forall g \in \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}^2)$. Since $\|\xi_n\|^2 = \tau(p_n) \to 1$ as $n \to \infty$Γ by the property T of $\mathbb{Z}^2$ inside $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}^2)$ and the second set of estimatesΓ itfollows that

$$\lim_{n\to\infty} \sup_{h\in\mathbb{Z}^2} \|\theta_n(u_h) - u_h\|_2 = 0.$$

Q.E.D.

2.6. Lemma. *With the same notations as in the statement of* Lemma 2.5, *if in addition we assume the projections* $p_n = \theta_n(1)$ *lie in* $A_0, \forall n$, *then for any* $n \geq n_\varepsilon$ *we have* $d(\theta_n(A_0)' \cap N, A_0 p_n' \cap N) \leq 2\varepsilon(1-\varepsilon)^{-1}$.

Proof. By using the fact that $p_n \in A_0 = \{u_h\}''_{h\in\mathbb{Z}^2}$ and applying Lemma 1.5Γfirst to $\mathcal{U}_0 = \theta_n(\{u_h\}_{h\in\mathbb{Z}^2})$ then to $\mathcal{U}_0 = \{u_h p_n\}_{h\in\mathbb{Z}^2}$Γregarded as unitary subgroups in the type II_1 factor $p_n N p_n$Γ endowed with the normalized trace $\tau(p_n)^{-1}\tau$Γ it follows that

$$d(A_0 p_n' \cap p_n N p_n, \theta_n(A_0)' \cap p_n N p_n) \leq 2\varepsilon\tau(p_n)^{-1/2} \leq 2\varepsilon(1-\varepsilon)^{-1}.$$

Q.E.D.

Proof of Theorem 2.3. If $\mathcal{F}(M)$ is uncountable then by Lemma 2.4 there exists a sequence of projections $p_n \in A_0, p_n \neq 1$Γ and isomorphismsθ_n of M onto $p_n M p_n$ such that $\lim_{n\to\infty} \|\theta_n(x) - x\|_2 = 0, \forall x \in M$. In particularΓaplying this to $x = u_g \in M_0 \subset M, g \in \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$Γ we have

$$\lim_{n\to\infty} \|\theta_n(u_g) - u_g\|_2 = 0, \forall g \in \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}).$$

By 2.5 and 2.6Γit follows that for n large enoughΓif we denote $p = p_n$ and $\theta = \theta_n$ then we have $d(A_0 p' \cap pMp, \theta(A_0)p' \cap pMp) < 1$. But by Lemma 2.1Γ $A_0 p' \cap pMp = Ap$ is a Cartan subalgebra of pMp. SimilarilyΓ

$$\theta(A_0)p' \cap pMp = \theta(A_0' \cap M) = \theta(A)$$

is a Cartan subalgebra of pMp as well. By Corollary 1.4 it follows that the Cartan subalgebras Ap and $\theta(A)$ of pMp are conjugate by a unitary element u of pMp. ThusΓ$\theta' = \mathrm{Ad}(u) \circ \theta$ is an isomorphism of M onto pMp carying A onto Ap. But by the results of Gaboriau the cost $c(A \subset M)$ of the Cartan subalgebra A of M is equal to 13/12 while the cost $c(Ap \subset pMp)$ of its restriction to p is equal to $1/12\tau(p) + 1$Γ which is larger than 13/12. Since the cost is invariant to the isomorphism θ'Γwe get a contradiction. ThusΓ$\mathcal{F}(M)$ is uncountable. Q.E.D.

2.7. Corollary. 1°. *The type* II_1 *factor* $M_0 = L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ *is non*-Γ *and has countable fundamental group* $\mathcal{F}(M_0)$.

2°. *If* σ_1 *is chosen to be ergodic but not strongly ergodic* (*cf* $2.2.3^\circ$), *then the type* II_1 *factor* $M = A \rtimes_\sigma SL(2,\mathbb{Z})$, *where* A *and* σ *are defined as in 2.1, then* M *has the property* Γ *and countable fundamental group* $\mathcal{F}(M)$.

Proof. This is now an immediate consequence of Lemma 2.2 and Theorem 2.3. Q.E.D.

3. Further remarks.

3.1. More examples of factors M with countable $\mathcal{F}(M)$. Recall that the first examples of factors M with countable fundamental group were constructed by Connes in ([C1])Γas group von Neumann factors associated to dicrete ICC groups G having the property T of Kazhdan. More examples of factors with countable fundamental group were constructed in ([Po1Γ3Γ5][Gol]). All these examples had relied in a way or another on constructions involving property T groups and they were non-Γ factors. The examples in the previous section are still using some rigidity phenomenaΓbut the milder one involving the inclusion $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$. Although these examples seem "closer" to free group factorsΓthey cannot be embedded in any free group factorΓby the same argument as the one in [CJ1] (see [Po]). But they cannot contain property T groups eitherΓbecause of the "relative Haagerup property" that the inclusion $L(\mathbb{Z}^2) \subset L(\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}))$ has (see [Po]). .

One can actually construct another class of examples of factors having the property Γ but countable fundamental groupΓ by using property T groups and a strategy reminiscent of ([C1]Γ [CJ1]) and([Po1Γ3])Γ as follows:

Let G be an ICC group with the property T of Kazhdan and satisfying the following condition:

(3.1.1). Given any finite set $F \subset G - \{e\}$ there exists $g \in G$ such that $gFg^{-1}F \cap FgFg^{-1} = \emptyset$.

Note that this implies in particular $gFg^{-1} \cap F = \emptyset$. (We are greatful to George Skandalis for confirming to us that the groups $SL(n, \mathbb{Z})$ with n oddΓ$n \geq 3$ do satisfy this conditionΓbesides having the property T by Kazhdan's classical result).

Then we let $(\mathbb{T}_g, \mu_g), g \in G$, be copies of the thorus $\mathbb{T}$ with its Lebesgue measure and (X, μ) be the product of these spacesΓ with the corresponding product measure. We let σ be the action of G on (X, μ) by Bernoulli shifts. We let M_0 be the corresponding type II_1 factor obtained by the group-measure space constructionΓi.e.Γ $M_0 = L^\infty(X, \mu) \rtimes_\sigma G$. FinallyΓ we consider the action α of $\mathbb{Z}$ on M_0Γ implemented by the "transversal" action of $\mathbb{Z}$ on (X, μ) given by rotation by the same irrational number on each copy of the thorus $(\mathbb{T}_g, \mu_g), \forall g \in G$. We denote $M = M_0 \rtimes_{\sigma_0} \mathbb{Z}$.

It is easy to see that M has the property Γ. Using Lemmas 2.4 and 2.5 (the latter for $G_0 = G$) and adding some more workΓone can show that if $\mathcal{F}(M)$ is uncountable then there exists a projection $p \in A_0 = \{u_h\}''_{h \in \mathbb{Z}} \subset M$Γwith $p \neq 1$Γ and an isomorphism $\theta : M \to pMp$Γwith $\theta(u_g) = u_g p, \forall g \in G$. This implies $\theta(A_0) = A_0 p$ and then the condition (3.1.1) easily implies that $\theta(L^\infty(X, \mu) \rtimes \mathbb{Z}) = p(L^\infty(X, \mu) \rtimes \mathbb{Z})p$. This forces each Haar unitary v_h generating $L^\infty(\mathbb{T}_h)$ to be so that $\theta(v_h)$ has a certain specific form. But thenΓby using again (3.1.1)Γit follows that for appropriate $g \in G$ the unitaries $\theta(v_h)$ and $\sigma_g(\theta(v_h))$ are "supported" on almost disjoint finite segments of the product $\otimes_g L^\infty(\mathbb{T}_g)$ cross product with $\mathbb{Z}$. But then it is easy to see that $\theta(v_h)$ and $\sigma_g(\theta(v_h))$ cannot commuteΓ acontradiction.

3.2. Examples of factors M with $\mathcal{F}(M) = \mathbb{R}_+^*$. Recall that in ([V]Γ [Ra]) the free group factor $L(\mathbb{F}_\infty)$ was proved to have fundamental group equal to $\mathbb{R}_+^*$. This is the most striking exampleΓso farΓof a non-Γ type II_1 factor with fundamental group same as the hyperfinite factor.

The following construction provides a class of examples of type II_1 factors which have the property ΓΓare not McDuffΓand yet have fundamental group $\mathbb{R}_+^*$:

Let Q be an arbitrary non-atomic finite von Neumann algebra. Let $A \subset R$ be the hyperfinite type II_1 factor with its Cartan subalgebra. Let $M = Q\overline{\otimes}A *_A R$ be the amalgamated free product with respect to the trace preserving conditional expectationsΓas in ([Po3]). By the proof of (4.1 and 7.1 in [Po3])Γ we see that any central sequence of M must lie in A. On the other handΓ any central sequence of R that lies in A is also a central sequence of M. ThusΓM has the property ΓΓ without being McDuff.

Note that M can be alternatively described as follows: Denote by σ the irrational rotation on the thorusΓviewed as an automorphism of $L(\mathbb{Z})$. We still denote by σ the action $id * \sigma$ of $\mathbb{Z}$ on $Q * L(\mathbb{Z})$. Then $M \simeq Q * L(\mathbb{Z}) \rtimes_\sigma \mathbb{Z}$Γ with$Q$ (respectively $A \subset R$) in the first construction corresponding to Q (resp. $\lambda(\mathbb{Z})'' \subset L(\mathbb{Z}) \rtimes_\sigma \mathbb{Z}$) in this construction. The fact that M has Γ but is not McDuff can then also be deduced from ([Po4]).

Using the first representation of MΓ we see that if p is a non-zero projection in A then $p(Q\overline{\otimes}A *_A R)p$ is trivially isomorphic to $Q\overline{\otimes}Ap *_{Ap} pRp$. But by Dye's theorem $(Ap \subset pRp)$ is isomorphic to $(A \subset R)$Γ while$Q\overline{\otimes}Ap$ is trivially isomorphic to $Q\overline{\otimes}A$. ThusΓ$pMp \simeq M$ for any non-zero projection in M. Thus $\mathcal{F}(M) = \mathbb{R}_+^*$.

Due to the resemblence of the above construction of M with the amalgamated free product construction in ([Po3])Γ it is likely that in the case $Q = L(\mathbb{F}_\infty)$ the factor M has the same universality property ([PoSh]) as the free group factor $L(\mathbb{F}_\infty)$Γ namely that any standard lattice $\mathcal{G}$ can "act" on it.

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MATH.DEPT., UCLA, LA, CA 90095-155505
E-mail address: popa@math.ucla.edu