An introduction to II_1 factors

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Abstract.

Contents

Part 1

CHAPTER 1

A first approach: examples

This chapter presents some basic constructions of von Neumann algebras arising from measure theory, group theory, group actions and equivalence relations. All these examples are naturally equipped with a faithful trace and are naturally represented on a Hilbert space. This provides a plentiful source of tracial von Neumann algebras to play with. More constructions will be given in Chapter 5.

The most general von Neumann algebras are obtained from simpler building blocks, called factors. These are the von Neumann algebras with a trivial center. We will see that they appear frequently, under usual assumptions. Infinite dimensional tracial factors (II_1 factors) are our main concern. We end this chapter with the most elementary example, the hyperfinite II_1 factor, which is constructed as an appropriate closure of an increasing sequence of matrix algebras.

1.1. Notation and preliminaries

Let $\mathcal{H}$ be a complex Hilbert space with inner-product $\langle \cdot, \cdot \rangle$ (*always assumed to be antilinear in the first variable*), and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. Equipped with the involution $x \mapsto x^*$ (adjoint of x) and with the operator norm, $\mathcal{B}(\mathcal{H})$ is a Banach $*$-algebra with unit $\mathrm{Id}_{\mathcal{H}}$. We will denote by $\|x\|$, or sometimes $\|x\|_\infty$, the operator norm of $x \in \mathcal{B}(\mathcal{H})$. Throughout this text, we will consider the two following weaker topologies on $\mathcal{B}(\mathcal{H})$:

- the *strong operator topology* (s.o. topology), that is, the locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the seminorms
$$p_\xi(x) = \|x\xi\|, \qquad \xi \in \mathcal{H},$$
- the *weak operator topology* (w.o. topology), that is, the locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the seminorms
$$p_{\xi,\eta}(x) = |\omega_{\xi,\eta}(x)|, \quad \xi, \eta \in \mathcal{H},$$

where $\omega_{\xi,\eta}$ is the linear functional $x \mapsto \langle \xi, x\eta \rangle$ on $\mathcal{B}(\mathcal{H})$.

This latter topology is weaker than the s.o. topology. It is strictly weaker when $\mathcal{H}$ is infinite dimensional (see Exercise 1.1). An important observation is that the unit ball of $\mathcal{B}(\mathcal{H})$ is w.o. compact. This is an immediate consequence of Tychonoff's theorem.

This unit ball, endowed with the uniform structure associated with the s.o. topology, is a complete space. In case $\mathcal{H}$ is separable, both w.o. and s.o. topologies on the unit ball are metrizable and second-countable. On the other hand, when $\mathcal{H}$ is infinite dimensional, this unit ball is not separable with respect to the operator norm (Exercise 1.2).

A *von Neumann algebra M on a Hilbert space $\mathcal{H}$* is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ (*i.e.,* a subalgebra invariant under the $*$-operation) which is closed in the s.o. topology and contains the identity operator $\mathrm{Id}_{\mathcal{H}}$.[1] We will sometimes write $(M, \mathcal{H})$ to specify the Hilbert space on which M acts, but $\mathcal{H}$ will often be implicit in the definition of M. The unit $\mathrm{Id}_{\mathcal{H}}$ of M will also be denoted 1_M or simply 1. We use the letters $\mathcal{H}, \mathcal{K}, \mathcal{L}$ to denote complex Hilbert spaces, while the letters M, N, P, Q will typically denote von Neumann algebras.

Given a subset S of $\mathcal{B}(\mathcal{H})$, we denote by S' its *commutant* in $\mathcal{B}(\mathcal{H})$:

$$S' = \{x \in \mathcal{B}(\mathcal{H}) : xy = yx \text{ for all } y \in S\}.$$

The commutant $(S')'$ of S' is denoted S'' and called the *bicommutant* of S. Note that S' is a s.o. closed unital subalgebra of $\mathcal{B}(\mathcal{H})$; if $S = S^*$, then $S' = (S')^*$ and therefore S' is a von Neumann algebra on $\mathcal{H}$. We will see in the next chapter that every von Neumann algebra appears in this way (Theorem 2.1.3).

The first example of von Neumann algebra coming to mind is of course $M = \mathcal{B}(\mathcal{H})$. Then, $M' = \mathbb{C}\,\mathrm{Id}_{\mathcal{H}}$. When $\mathcal{H} = \mathbb{C}^n$, we get the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries, the simplest example of a von Neumann algebra.

We recall that a *C^*-algebra on $\mathcal{H}$* is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the norm topology. Hence a von Neumann algebra is a C^*-algebra, but the converse is not true. For instance the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on an infinite dimensional Hilbert space $\mathcal{H}$ is not a von Neumann algebra on $\mathcal{H}$: its s.o. closure is $\mathcal{B}(\mathcal{H})$.

We assume that the reader has a basic knowledge about C^*-algebras. We have gathered in the appendix, with references, the main facts that we will use. Note that for us, a homomorphism between two C^*-algebras preserves the algebraic operations and the involution[2]. We recall that it is automatically a contraction and a positive map, *i.e.*, it preserves the positive cone (Appendix A).

Remark 1.1.1. A C^*-algebra can be defined abstractly as a Banach $*$-algebra A such that $\|x^*x\| = \|x\|^2$ for every $x \in A$. A celebrated theorem of Gelfand and Naimark states that such an algebra is isometrically isomorphic to a norm closed $*$-algebra of operators on some Hilbert space.

Similarly, for von Neumann algebras, there are two points of view: the concrete and the abstract one (see the notes at the end of this chapter).

[1] We will see in Theorem 2.1.3 that we may require, equivalently, that M is closed in the w.o. topology.

[2] In the literature, very often one says $*$-homomorphism to emphasize the fact that the involution is also preserved.

In this monograph, we have chosen to define von Neumann algebras as concretely represented operator algebras on some Hilbert space (even if this Hilbert space will not always be explicitly mentioned).

1.2. Measure space von Neumann algebras

Every probability measure space (X,μ) gives rise in a natural way to an abelian von Neumann algebra.

PROPOSITION 1.2.1. *Let (X,μ) be a probability measure space. We set $A = L^\infty(X,\mu)$.*

(i) *For $f \in L^\infty(X,\mu)$, we denote by M_f the multiplication operator by f on $L^2(X,\mu)$, that is, $M_f\xi = f\xi$ for $\xi \in L^2(X,\mu)$. Then M_f is a bounded operator and $\|M_f\| = \|f\|_\infty$.*

(ii) *If A is identified with a subalgebra of $\mathcal{B}(L^2(X,\mu))$ via $f \mapsto M_f$, then $A = A'$. In particular, A is a von Neumann algebra on $L^2(X,\mu)$ and a maximal abelian subalgebra of $\mathcal{B}(L^2(X,\mu))$.*

PROOF. (i) Obviously, M_f is a bounded operator with $\|M_f\| \leq \|f\|_\infty$ and it is a classical exercise in measure theory to show that $\|M_f\| = \|f\|_\infty$.

(ii) Since A is abelian, we have $A \subset A'$. Let $T \in A'$ and set $f = T(1)$. Then, for $h \in L^\infty(X,\mu)$, we have $T(h) = TM_h 1 = M_h T(1) = hf$ and $\|fh\|_2 \leq \|T\|\|h\|_2$. It follows that $f \in L^\infty(X,\mu)$ with $\|f\|_\infty \leq \|T\|$ and so $T = M_f$. $\square$

REMARK 1.2.2. Recall that $L^\infty(X,\mu)$ is the dual Banach space of $L^1(X,\mu)$. The *weak* topology* on $L^\infty(X,\mu)$ is defined by the family of seminorms $q_g(f) = \left|\int_X fg \,\mathrm{d}\mu\right|$, $g \in L^1(X,\mu)$. Equivalently, it is defined by the family of seminorms

$$f \mapsto p_{\xi,\eta}(f) = \left|\int_X f\bar{\xi}\eta \,\mathrm{d}\mu\right|$$

with $\xi, \eta \in L^2(X,\mu)$. Therefore, the weak* topology coincides with the w.o. topology on $L^\infty(X,\mu)$ acting on $L^2(X,\mu)$.

1.3. Group von Neumann algebras

Let G be a group[3]. We denote by λ (or λ_G in case of ambiguity) and ρ (or ρ_G) the *left*, and respectively *right*, *regular representation* of G in $\ell^2(G)$, *i.e.*, for all $s, t \in G$,

$$\lambda(s)\delta_t = \delta_{st}, \quad \rho(s)\delta_t = \delta_{ts^{-1}},$$

where $(\delta_t)_{t\in G}$ is the natural orthonormal basis of $\ell^2(G)$.[4]

[3]For us G will be a *discrete* group, unless otherwise stated, and we are mostly interested in infinite countable groups.

[4]Given a set X, we denote by δ_x both the characteristic function of $\{x\}$ and the Dirac measure at $x \in X$.

1.3.1. Definition and first properties. We denote by $L(G)$ the strong operator closure of the linear span of $\lambda(G)$. This von Neumann algebra is called the (left) *group von Neumann algebra of* G. Similarly, one introduces the strong operator closure $R(G)$ of the linear span of $\rho(G)$. Obviously, these two algebras commute: $xy = yx$ for $x \in L(G)$ and $y \in R(G)$ and we will see in Theorem 1.3.6 that each one is the commutant of the other. These von Neumann algebras come equipped with a natural trace, as shown below.

DEFINITION 1.3.1. A linear functional φ on a von Neumann algebra M is *positive* if $\varphi(x^*x) \geq 0$ for every $x \in M$ (*i.e.*, $\varphi(x) \geq 0$ for $x \geq 0$). Whenever, in addition, $\varphi(x^*x) = 0$ implies $x = 0$, we say that φ is *faithful*. If φ is positive with $\varphi(1) = 1$ we say that φ is a *state*[5].

A positive linear functional such $\varphi(xy) = \varphi(yx)$ for every $x, y \in M$ is a *trace*. If moreover it is a state, we call it a *tracial state*.

We recall that a positive linear functional is norm continuous with $\|\varphi\| = \varphi(1)$.

DEFINITION 1.3.2. Given a von Neumann algebra M acting on a Hilbert space $\mathcal{H}$, a vector $\xi \in \mathcal{H}$ is called *cyclic for* M if $M\xi$ is dense in $\mathcal{H}$. It is called *separating for* M if, for $x \in M$, we have $x\xi = 0$ if and only if $x = 0$.

We denote by e the unit of G. One easily checks that δ_e is a cyclic and separating vector for $L(G)$ (and $R(G)$). We define a faithful state on $L(G)$ by

$$\tau(x) = \langle \delta_e, x\delta_e \rangle.$$

For $s_1, s_2 \in G$, we have $\tau(\lambda(s_1)\lambda(s_2)) = 1$ if $s_1 s_2 = e$ and $\tau(\lambda(s_1)\lambda(s_2)) = 0$ otherwise. It follows immediately that τ is a trace. We observe that this trace is continuous with respect to the w.o. topology.

Thus, $L(G)$ and $R(G)$ are examples of tracial von Neumann algebras in the following sense[6].

DEFINITION 1.3.3. A *tracial von Neumann algebra* (M, τ) is a von Neumann algebra $(M, \mathcal{H})$ equipped with a faithful tracial state τ whose restriction to the unit ball is continuous with respect to the w.o. topology (equivalently, equipped with a faithful normal tracial state, see Proposition 2.5.5). In case of ambiguity, the given trace of M will be denoted by τ_M.

Since δ_e is a separating vector for $L(G)$, the map $x \mapsto x\delta_e$ provides a natural identification of $L(G)$ with a dense linear subspace of $\ell^2(G)$ that we are going to characterize.

Recall first that for $f, f_1 \in \ell^2(G)$, the convolution product

$$L_f(f_1) = f * f_1,$$

[5]Recall that $\{x^*x : x \in M\}$ is the cone of all positive elements in M (see Appendix A.2).

[6]$L^\infty(X, \mu)$ equipped with the integral $\tau_\mu : f \mapsto \int_X f \,\mathrm{d}\mu$ is of course another example.

defined by

$$(f * f_1)(t) = \sum_{s \in G} f(s) f_1(s^{-1}t),$$

belongs to $\ell^\infty(G)$. More precisely, using the Cauchy-Schwarz inequality, we see that

$$\|f * f_1\|_\infty \le \|f\|_2 \|f_1\|_2. \tag{1.1}$$

We say that f is a *left convolver for G* if $f * f_1 \in \ell^2(G)$ for every $f_1 \in \ell^2(G)$. Observe that every finitely supported function f is a left convolver and that $L_f = \sum_{s \in G} f(s)\lambda(s)$.

LEMMA 1.3.4. *Let $f \in \ell^2(G)$.*

(i) *If f is a left convolver, L_f is a bounded operator on $\ell^2(G)$.*

(ii) *f is a left convolver if and only if there exists $c > 0$ such that $\|f * k\|_2 \le c\|k\|_2$ for every finitely supported function k on G.*

PROOF. (i) It is sufficient to prove that L_f has a closed graph. Let (f_n) be a sequence in $\ell^2(G)$ such that $\lim_n f_n = 0$ and $\lim_n L_f(f_n) = h$ in $\ell^2(G)$. It follows from the inequality (1.1) that

$$\lim_n \|f * f_n\|_\infty = 0$$

and therefore $h = 0$.

(ii) Assume the existence of a bounded operator T such that $T(k) = f * k$ for every finitely supported function k on G. Let $h \in \ell^2(G)$ and let (h_n) be a sequence of finitely supported functions on G with $\lim_n \|h - h_n\|_2 = 0$. Then we have $\lim_n \|T(h) - f * h_n\|_2 = 0$ and $\lim_n \|f * h - f * h_n\|_\infty = 0$, so that $T = L_f$. □

We denote by $LC(G)$ the space of all left convolvers for G. Note that for $f \in LC(G)$ and $t \in G$, we have

$$L_f \circ \rho(t) = \rho(t) \circ L_f.$$

Since $f \mapsto L_f$ is injective, it follows that we may (and will) view $LC(G)$ as a subspace of $\rho(G)' \subset \mathcal{B}(\ell^2(G))$.

PROPOSITION 1.3.5. *$LC(G)$ is a von Neumann subalgebra of $\rho(G)'$.*

PROOF. Let $f \in LC(G)$. Then $(L_f)^* = L_{f^*}$ where $f^*(t) = \overline{f(t^{-1})}$, so that $LC(G)$ is stable under involution. Let now f_1, f_2 be in $LC(G)$. For $t \in G$, we have

$$\begin{aligned} L_{f_1} \circ L_{f_2}(\delta_t) &= L_{f_1} \circ \rho(t^{-1}) \circ L_{f_2}(\delta_e) = \rho(t^{-1}) \circ L_{f_1}(f_2) \\ &= \rho(t^{-1})(f_1 * f_2) = (f_1 * f_2) * \delta_t, \end{aligned}$$

so that, by Lemma 1.3.4 (ii), $f_1 * f_2 \in LC(G)$ with $L_{f_1} L_{f_2} = L_{f_1 * f_2}$.

Let us show next that $LC(G)$ is s.o. closed. Let $T \in \mathcal{B}(\ell^2(G))$ be such that there exists a net (f_i) of left convolvers with $\lim_i L_{f_i} = T$ in the s.o. topology. We put $h = T\delta_e \in \ell^2(G)$. Since $L_{f_i}\delta_e = f_i$, we get

$$\|h - f_i\|_2 = \|T\delta_e - L_{f_i}\delta_e\|_2 \to 0.$$

To conclude, we show that $T = L_h$. For $f \in \ell^2(G)$, we have

$$\|Tf - f_i * f\|_\infty \le \|Tf - f_i * f\|_2 \to 0$$

and

$$\|h * f - f_i * f\|_\infty \le \|h - f_i\|_2 \|f\|_2 \to 0,$$

and therefore $Tf = h * f$. Hence, $h \in LC(G)$ with $L_h = T$. $\square$

Since $\lambda(G) \subset LC(G)$, it follows from the above proposition that $L(G) \subset LC(G)$. Similarly, we may introduce the von Neumann $RC(G)$ generated by the *right convolvers* R_f *for* G. It commutes with $LC(G)$, that is, $LC(G) \subset RC(G)'$.

We will see that $L(G)$ is exactly the subspace of $\ell^2(G)$ formed by the left convolvers and prove simultaneously that $L(G) = R(G)'$.

Theorem 1.3.6. *We have* $LC(G) = L(G) = R(G)'$ *and* $RC(G) = R(G) = L(G)'$.

Proof. We already know that

$$L(G) \subset LC(G) \subset RC(G)' \subset R(G)'.$$

Let us prove that $R(G)' \subset LC(G)$. To this end, we consider $T \in R(G)'$ and set $f = T\delta_e$. Then for $t \in G$, we have

$$T\delta_t = T\rho(t^{-1})\delta_e = \rho(t^{-1})T\delta_e = f * \delta_t.$$

It follows that $Tk = f * k$ for every finitely supported function k on G. Then, by Lemma 1.3.4 (ii), we see that $f \in LC(G)$ and $T = L_f$.

So, we have proved that $LC(G) = RC(G)' = R(G)'$. Similarly, we have $RC(G) = LC(G)' = L(G)'$. Now, we use one of the fundamental tools of the theory of von Neumann algebras, that will be established in the next chapter, namely the *von Neumann bicommutant theorem*. It tells us that every von Neumann algebra is equal to its bicommutant (see Theorem 2.1.3). It follows that

$$L(G) = L(G)'' = RC(G)' = LC(G) = R(G)'$$

and, similarly,

$$R(G) = RC(G) = L(G)'.$$

$\square$

Remark 1.3.7. Usually, for $g \in G$, we will put $u_g = \lambda(g) \in L(G)$ and this unitary operator will be identified with the vector $\lambda(g)\delta_e = \delta_g \in \ell^2(G)$. Therefore, every $f \in \ell^2(G)$ is written as $f = \sum_{g \in G} f_g u_g$ and, in particular, every $x \in L(G)$ is written as

$$x = \sum_{g \in G} x_g u_g. \tag{1.2}$$

Observe that $\tau(x^* x) = \sum_{g \in G} |x_g|^2$ and that $x_g = \tau(x u_g^*)$. In analogy with developments in Fourier series, the scalars x_g are called the *Fourier coefficients of* x. The unitaries u_g are called the *canonical unitaries of* $L(G)$. We warn the reader that in (1.2) the convergence is in ℓ^2-norm and not with respect to the s.o. or w.o. topology.

Denote, as usually, by $\mathbb{C}[G]$ the *group algebra* of G, that is, the $*$-subalgebra of $L(G)$ formed by the elements $\sum_{g\in G} x_g u_g$ where $x_g = 0$ except for a finite number of indices. Then $L(G)$ is the s.o. closure of $\mathbb{C}[G]$.

Example 1.3.8. Consider the group $G = \mathbb{Z}$ of all integers. Since $\mathbb{Z}$ is abelian, $L(\mathbb{Z})$ is an abelian von Neumann algebra which coincides with $R(\mathbb{Z}) = L(\mathbb{Z})'$.

Let $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ be the Fourier transform, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$, equipped with the Lebesgue probability measure m. Then $\mathcal{F}\delta_n = e_n$ with $e_n(z) = z^n$ and, for $f \in LC(\mathbb{Z})$, we have $\mathcal{F}L_f\mathcal{F}^{-1} = M_{\widehat{f}}$ where $M_{\widehat{f}}$ is the multiplication operator by the Fourier transform $\widehat{f}$ of f. Hence, $\widehat{f}$ is a *multiplier for* $\mathbb{T}$, that is, a function ψ on $\mathbb{T}$ such that $h \mapsto \psi h$ is a bounded operator from $L^2(\mathbb{T})$ into itself. It follows that $\mathcal{F}LC(\mathbb{Z})\mathcal{F}^{-1}$ is the von Neumann subalgebra of $\mathcal{B}(L^2(\mathbb{T}))$ formed by the multiplication operators by these multipliers for $\mathbb{T}$. It can be identified in a natural way with $L^\infty(\mathbb{T})$.

The canonical tracial state τ on $L(\mathbb{Z})$ becomes, after Fourier transform, the integration with respect to the Lebesgue probability measure on $\mathbb{T}$:

$$\tau(L_f) = \int_{\mathbb{T}} \widehat{f}\,\mathrm{d}m.$$

The same observations hold for any abelian group G: the group von Neumann algebra $L(G)$ is abelian and isomorphic to $L^\infty(\widehat{G}, m)$ where $\widehat{G}$ is the dual group and m is the Haar probability measure on this compact group.

However, the most interesting examples for us come from groups such that $L(G)$ has, in sharp contrast, a center reduced to the scalar operators. A von Neumann algebra with such a trivial center is called *a factor*.

Proposition 1.3.9. *Let G be a group. The following conditions are equivalent:*

(i) *$L(G)$ is a factor;*
(ii) *G is an* ICC *(infinite conjugacy classes) group, that is, every non trivial conjugacy class $\{gsg^{-1} : g \in G\}$, $s \neq e$, is infinite.*

Proof. Let x be an element of the center of $L(G)$. For $t \in G$ we have

$$x\delta_e = \lambda(t)x\lambda(t^{-1})\delta_e = \lambda(t)x\rho(t)\delta_e = \lambda(t)\rho(t)(x\delta_e).$$

It follows that $x\delta_e$ is constant on conjugacy classes. Therefore, if G is ICC, since $x\delta_e$ is square summable, we see that $x\delta_e = \alpha\delta_e$ with $\alpha \in \mathbb{C}$, and therefore $x = \alpha\,\mathrm{Id}_{\mathcal{H}}$.

Assume now that G is not ICC and let $C \subset G$ be a finite non-trivial conjugacy class. An easy computation shows that the characteristic function $f = \mathbf{1}_C$ of C defines an element L_f of the center of $L(G)$ which is not a scalar operator. $\square$

There are plenty of countable ICC groups. Among the simplest examples, let us mention:

- $S_\infty = \bigcup_{n=1}^{\infty} S_n$, the group of those permutations of $\mathbb{N}^*$ fixing all but finitely many integers[7] (S_n is the group of all permutations of $\{1, 2, \cdots, n\}$) (Exercise 1.7);
- $\mathbb{F}_n$, $n \geq 2$, the free group on n generators (Exercise 1.8);
- wreath products $G = H \wr \Gamma$ where H is non trivial and Γ is an infinite group, as well as many generalized wreath products.

Let us give some details about the third example. Suppose we are given a non-trivial group H and a group Γ acting on a set I. We denote by $H^{(I)}$ the direct sum of copies of H, indexed by I, that is, $H^{(I)}$ is the group of all maps $\xi : I \to H$ such that $\xi_i = e$ for all but finitely many i. We let Γ act on $H^{(I)}$ by $(\gamma\xi)_i = \xi_{\gamma^{-1}i}$. The *generalized wreath product* $H \wr_I \Gamma$ is the semi-direct product $H^{(I)} \rtimes \Gamma$. The *wreath product* $H \wr \Gamma$ is the particular case where $I = \Gamma$ on which Γ acts by left translations.

PROPOSITION 1.3.10. *Let H, Γ and I be as above. We denote by Γ_f the subgroup of elements in Γ whose conjugacy class is finite. We assume that the orbits of $\Gamma \curvearrowright I$ are infinite and that the restricted action $\Gamma_f \curvearrowright I$ is faithful. Then the group $G = H \wr_I \Gamma$ is* ICC. *In particular every wreath product $H \wr \Gamma$, where H is non trivial and Γ is infinite, is* ICC.

PROOF. We denote by e the unit of $H^{(I)}$ and by ε the unit of Γ. Given $g \in G$ and a subgroup $K \subset G$ we set $g^K = \{kgk^{-1} : k \in K\}$.

Let $g = (\xi, \gamma)$ be an element of G distinct from the unit. Assume first that $\xi \neq e$. Then its support is non-empty and has an infinite orbit under Γ, so g^Γ and *a fortiori* g^G are infinite. Assume now that $g = (e, \gamma)$ with $\gamma \neq \varepsilon$. If $\gamma \notin \Gamma_f$ then of course the conjugacy class of g is infinite. It remains to consider the case where $\gamma \in \Gamma_f \setminus \{\varepsilon\}$. Since Γ_f acts faithfully on I, there is an $i_0 \in I$ such that $\gamma i_0 \neq i_0$. Let ξ_0 be an element in $H^{(I)}$ having all its components trivial except the one of index i_0 and take $g_0 = (\xi_0, \varepsilon)$. Then $g_0^{-1} g g_0 = (\xi_0^{-1}\gamma(\xi_0), \gamma)$ has an infinite conjugacy class since $\xi_0^{-1}\gamma(\xi_0) \neq e$, and we see that g^G is infinite. $\square$

1.3.2. A remark about $L(S_\infty)$. The factor $L(S_\infty)$ has a very important property that we will often meet later, and already in Section 1.6: it is the s.o. closure of the union of an increasing sequence of finite dimensional von Neumann algebras namely the von Neumann algebras $L(S_n)$, $n \geq 1$. Indeed, these algebras are finite dimensional since the groups S_n are finite. Moreover, $L(S_n)$ is naturally isomorphic to the linear span of $\lambda_{S_\infty}(S_n)$ in $L(S_\infty)$, as a consequence of the following proposition.

[7] In this text $\mathbb{N}$ denotes the set of non-negative integers and $\mathbb{N}^*$ is the set of strictly positive integers.

PROPOSITION 1.3.11. *Let H be a subgroup of a countable group G. Then the restriction of λ_G to H is a multiple of the left regular representation of H.*

PROOF. Write G as the disjoint union of its right H-cosets: $G = \cup_{s \in S} Hs$, where S is a set of representatives of $H \setminus G$. Then $\ell^2(G) = \oplus_{s \in S} \ell^2(Hs)$. It is enough to observe that $\ell^2(Hs)$ is invariant under the restriction of λ_G to H, and that this restriction is equivalent to the left regular representation of H. $\square$

1.3.3. II_1 factors and type I factors. Until now, we have met two kinds of factors: the von Neumann algebras $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a finite or infinite dimensional Hilbert space (which are easily seen to be factors) and the factors of the form $L(G)$ where G is an ICC group. Since we are mainly interested in the study of these objects up to isomorphism, let us first specify what we mean by isomorphic von Neumann algebras.

DEFINITION 1.3.12. We say that two von Neumann algebras M_1 and M_2 are *isomorphic*, and we write $M_1 \simeq M_2$, if there exists a bijective homomorphism (*i.e.*, an isomorphism) $\alpha : M_1 \to M_2$.

When $M_1 = M_2 = M$, we denote by $\mathrm{Aut}\,(M)$ the automorphism group of M. If u is a unitary of M (*i.e.*, such that $uu^* = 1_M = u^*u$), then $\mathrm{Ad}\,(u) : x \in M \mapsto uxu^*$ is an automorphism of M called an *inner automorphism.* The set of these inner automorphisms is a normal subgroup of $\mathrm{Aut}\,(M)$, which is denoted by $\mathrm{Inn}\,(M)$. The quotient group $\mathrm{Aut}\,(M)/\mathrm{Inn}\,(M)$ is also of interest. It is denoted by $\mathrm{Out}\,(M)$ and is called the *outer automorphism group of M*.

An isomorphism preserves the algebraic structures as well as the involution. We recall that it is automatically an isometry (see Appendix A). On the other hand it is not necessarily continuous with respect to the w.o. or s.o. topology (see Exercise 1.3) but we will see later (Remark 2.5.10) that its restriction to the unit ball is continuous with respect to these topologies.

Since we have defined von Neumann algebras as acting on specified Hilbert spaces, the following stronger notion of isomorphism is also very natural.

DEFINITION 1.3.13. We say that the von Neumann algebras $(M_1, \mathcal{H}_1)$, $(M_2, \mathcal{H}_2)$, are *spatially isomorphic* if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $x \mapsto UxU^*$ is an isomorphism (called spatial) from M_1 onto M_2.

Two isomorphic von Neumann algebras need not be spatially isomorphic (see Exercise 1.4). Classification, up to spatial isomorphism, involves in addition a notion of multiplicity.

Let us now come back to the examples of factors mentioned at the beginning of this section. A basic result of linear algebra tells us that the (finite dimensional) von Neumann algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices has a unique tracial state τ, namely $\tau = (1/n)\mathrm{Tr}$ where Tr is the usual trace of matrices[8].

On the other hand, it is easily shown that there is no tracial state on the von Neumann algebra $\mathcal{B}(\mathcal{H})$ when $\mathcal{H}$ is infinite dimensional. Indeed, we may write $\mathcal{H}$ as the orthogonal direct sum of two Hilbert subspaces $\mathcal{H}_1, \mathcal{H}_2$ of the same dimension as the dimension of $\mathcal{H}$. If p_1, p_2 are the orthogonal projections on these subspaces, there exist partial isometries u_1, u_2 with $u_i^* u_i = \mathrm{Id}_{\mathcal{H}}$ and $u_i u_i^* = p_i$, $i = 1, 2$. The existence of a tracial state τ on $\mathcal{B}(\mathcal{H})$ leads to the contradiction

$$1 = \tau(p_1) + \tau(p_2) = \tau(u_1 u_1^*) + \tau(u_2 u_2^*) = \tau(u_1^* u_1) + \tau(u_2^* u_2) = 2.$$

Thus, when G is an ICC group, $L(G)$ is not isomorphic to any $\mathcal{B}(\mathcal{H})$: it belongs to the class of II_1 factors that are defined below, whereas $\mathcal{B}(\mathcal{H})$ belongs to the class of type I factors.

Definition 1.3.14. A II_1 *factor* is an infinite dimensional tracial von Neumann algebra M whose center is reduced to the scalar operators[9].

Definition 1.3.15. A factor M is said to be of *type* I if it is isomorphic to some $\mathcal{B}(\mathcal{H})$. If $\dim \mathcal{H} = n$, we say that M (which is isomorphic to $M_n(\mathbb{C})$) is of *type* I_n. If $\dim \mathcal{H} = \infty$, we say that M is of *type* I_∞.

Factors of type I (on a separable Hilbert space) are classified, up to isomorphism, by their dimension. On the other hand, the classification of II_1 factors is out of reach[10]. Already, given two countable ICC groups G_1, G_2, to determine whether the II_1 factors $L(G_1)$ and $L(G_2)$ are isomorphic or not is a very challenging question.

1.4. Group measure space von Neumann algebras

We will describe in this section a fundamental construction, associated with an action of a group G on a probability measure space (X, μ). The previous section was concerned with the case where X is reduced to a point.

1.4.1. Probability measure preserving actions. Recall that two probability measure spaces (X_1, μ_1) and (X_2, μ_2) are *isomorphic* if there exist conull subsets Y_1 and Y_2 of X_1 and X_2, respectively, and a Borel isomorphism $\theta : Y_1 \to Y_2$ such that $\theta_* \mu_{1|_{Y_1}} = \mu_{2|_{Y_2}}$, *i.e.*, $(\theta_* \mu_{1|_{Y_1}})(E) = \mu_1(\theta^{-1}(E)) = \mu_2(E)$ for every Borel subset E of Y_2. Such a map θ is called a *probability measure preserving* (*p.m.p.*) *isomorphism*, and a *p.m.p. automorphism* whenever $(X_1, \mu_1) = (X_2, \mu_2)$. We identify two isomorphisms

[8]By convention, τ will always denote tracial states whereas Tr will denote not necessarily normalized traces.

[9]We will see later (Theorem 6.3.5) that it is enough to require the existence of a tracial state: for factors, such a tracial state is automatically faithful and has the desired continuity property. Moreover, it is unique.

[10]Indeed, II_1 factors on separable Hilbert spaces, up to isomorphism, are not classifiable by countable structures [**ST09**].

that coincide almost everywhere. We denote by $\mathrm{Aut}\,(X,\mu)$ the group of (classes modulo null sets of) p.m.p. automorphisms of a probability measure space (X,μ).

Every element $\theta \in \mathrm{Aut}\,(X,\mu)$ induces an automorphism $f \mapsto f\circ\theta$ of the algebra $L^\infty(X,\mu)$ which preserves the functional $\tau_\mu : f \mapsto \int_X f\,\mathrm{d}\mu$, *i.e.*,

$$\forall f \in L^\infty(X,\mu), \quad \int_X f\circ\theta\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu.$$

We will see later that, for nice probability measure spaces (the so-called standard ones, see Section B.2 in the appendix), every automorphism of $L^\infty(X,\mu)$ comes from an element of $\mathrm{Aut}\,(X,\mu)$ (see Corollary 3.3.3). *The most useful examples of probability measure spaces are the standard ones, where the measure does not concentrate on a point* (see Appendix B). *It will be our implicit assumption in the sequel.*

Definition 1.4.1. A *probability measure preserving (p.m.p.) action* $G \curvearrowright (X,\mu)$ of a group G on a probability measure space (X,μ) is a group homomorphism from G into $\mathrm{Aut}\,(X,\mu)$. The action of $g\in G$ on $w \in X$ will be written gw.

The most classical examples of p.m.p. actions are *Bernoulli actions*. Let (Y,ν) be a probability measure space and let $X = Y^G$ be equipped with the product measure $\mu = \nu^{\otimes G}$. The Bernoulli action $G \curvearrowright (X,\mu)$ is defined by $(gx)_h = x_{g^{-1}h}$ for $x = (x_h)_{h\in G} \in X$ and $g \in G$. As a particular case, we may take $Y = \{0,1\}$ and $\nu(\{0\}) = p, \nu(\{1\}) = 1-p$, for a given $p \in]0,1[$.

1.4.2. Construction of the group measure space algebra. Let $G \curvearrowright (X,\mu)$ be a p.m.p. action of G on a probability measure space (X,μ). Let A be the von Neumann algebra $L^\infty(X,\mu)$, acting by mutiplication on $L^2(X,\mu)$. Let σ be the unitary representation of G on $L^2(X,\mu)$ defined by $(\sigma_g f)(w) = f(g^{-1}w)$. By restriction to $L^\infty(X,\mu) \subset L^2(X,\mu)$, this induces an action of G by automorphisms on $L^\infty(X,\mu)$.

We encode this action of G on A through the involutive algebra $A[G]$ generated by a copy of A and a copy of G, subject to the covariance relation $gag^{-1} = \sigma_g(a)$. More precisely, $A[G]$ is the space of formal sums of the form $\sum_{g\in G} a_g g$ where $a_g \in A$ and where the set of $g\in G$ with $a_g \neq 0$ is finite. The product is defined by

$$(a_1 g)(a_2 h) = a_1\sigma_g(a_2)gh,$$

and the involution by

$$(ag)^* = \sigma_{g^{-1}}(a^*)g^{-1}, \quad \text{where} \quad a^*(w) = \overline{a(w)}.$$

These operations are consistent with the operations on A and G: $a \in A \mapsto ae$ is an injective $*$-homomorphism from A into $A[G]$ and $g\in G \mapsto 1_A g$ is an injective group homomorphism into the unitary group of $A[G]$.[11] Of course,

[11] When X is reduced to a point, $A[G]$ is just the group algebra $\mathbb{C}[G]$.

A will be identified with the corresponding subalgebra of $A[G]$. To avoid confusion, $g \in G$ when viewed as the element $1_A g$ of $A[G]$ will be written u_g. Thus, a generic element of $A[G]$ is written as

$$\sum_{g \in G} a_g u_g. \tag{1.3}$$

For $a = \sum_{g \in G} a_g u_g$ and $b = \sum_{g \in G} b_g u_g$ in $A[G]$, we have

$$\Big(\sum_{g \in G} a_g u_g\Big)\Big(\sum_{g \in G} b_g u_g\Big) = \sum_{g \in G} (a * b)_g u_g$$

where $a * b$ is the *twisted convolution product*

$$(a * b)_g = \sum_{h \in G} a_h \sigma_h(b_{h^{-1}g}). \tag{1.4}$$

We will complete $A[G]$ in order to get a von Neumann algebra. The first step is to represent $A[G]$ as a $*$-algebra of operators acting on the Hilbert space $\mathcal{H} = L^2(X, \mu) \otimes \ell^2(G)$ by sending $a \in A \subset A[G]$ to $L(a) = a \otimes 1$ and u_g to $L(u_g) = \sigma_g \otimes \lambda_g$. Since the algebraic homomorphism rules for A and G are satisfied, as well as the covariance rule $L(u_g)L(a)L(u_g)^* = L(\sigma_g(a))$, this gives a $*$-homomorphism L from $A[G]$ into $\mathcal{B}(\mathcal{H})$.

The *group measure space von Neumann algebra associated with* $G \curvearrowright (X, \mu)$, or *crossed product*, is the von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $L(A) \cup \{L(u_g) : g \in G\}$, that is, the s.o. closure of $L(A[G])$ in $\mathcal{B}(\mathcal{H})$. We denote it by $L(A, G)$, or $A \rtimes G$.

Since $L(\sum_{g \in G} a_g u_g)(1 \otimes \delta_e) = \sum_{g \in G} a_g \otimes \delta_g$ we see that L is injective and we identify $A[G]$ with the corresponding s.o. dense subalgebra of $A \rtimes G$. We also identify it in an obvious way with a dense subspace of the Hilbert space $L^2(X, \mu) \otimes \ell^2(G)$. Note that $L^2(X, \mu) \otimes \ell^2(G)$ is the Hilbert space of all $f = \sum_{g \in G} f_g \otimes \delta_g$ with $f_g \in L^2(X, \mu)$ and

$$\sum_{g \in G} \|f_g\|^2_{L^2(X)} < +\infty.$$

It is convenient to set

$$f_g u_g = f_g \otimes \delta_g$$

and thus to write f as the sum $\sum_{g \in G} f_g u_g$, with coefficients $f_g \in L^2(X, \mu)$. This is consistent with the above identification of $a = \sum_{g \in G} a_g u_g \in A[G]$ with $\sum_{g \in G} a_g \otimes \delta_g \in L^2(X, \mu) \otimes \ell^2(G)$. Then we have

$$\Big(\sum_{g \in G} a_g u_g\Big)\Big(\sum_{g \in G} f_g u_g\Big) = \sum_{g \in G} (a * f)_g u_g$$

where $a * f$ is defined as in Equation (1.4)

Similarly, $A[G]$ acts on $\mathcal{H}$ by right convolution:

$$R(au_g)(fu_h) = (fu_h)(au_g) = f\sigma_h(a)u_{hg},$$

and thus
$$\Big(\sum_{g\in G} f_g u_g\Big)\Big(\sum_{g\in G} a_g u_g\Big) = \sum_{g\in G} (f * a)_g u_g.$$
We denote by $R(A,G)$ the von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ generated by this right action R. Obviously, $L(A,G)$ and $R(A,G)$ commute. The vector $u_e = 1\otimes\delta_e \in \mathcal{H}$ is cyclic for $L(A,G)$ and $R(A,G)$ and therefore is also separating for these two algebras. In particular, the elements of $L(A,G)$ may be identified to elements of $L^2(X,\mu)\otimes\ell^2(G)$ by $x\mapsto xu_e$ and thus are written (in compatibility with (1.3)) as
$$x = \sum_{g\in G} x_g u_g, \tag{1.5}$$
with $\sum_{g\in G}\|x_g\|^2_{L^2(X)} < +\infty$. Observe that A appears as a von Neumann subalgebra of $L(A,G)$.

Let τ be the linear functional on $L(A,G)$ defined by
$$\tau(x) = \langle u_e, xu_e\rangle = \int_X x_e \,\mathrm{d}\mu, \quad \text{for} \quad x = \sum_{g\in G} x_g u_g.$$
Using the invariance of the probability measure μ, it is easily seen that τ is a tracial state (of course w.o. continuous). We also remark that τ is faithful, with
$$\tau(x^*x) = \sum_{g\in G}\int_X |x_g|^2\,\mathrm{d}\mu.$$

Following the lines of the proof of Theorem 1.3.6 (which corresponds to the case $A=\mathbb{C}$), one shows that $L(A,G)$ is the subspace of $f=\sum_{g\in G} f_g u_g \in \mathcal{H} = L^2(X,\mu)\otimes\ell^2(G)$ that are left convolvers in the sense that there exists $c>0$ with $\|f*k\|_{\mathcal{H}} \le c\|k\|_{\mathcal{H}}$ for every finitely supported $k\in\mathcal{H}$.[12] In particular, for every $g\in G$, we have $f_g\in L^\infty(X,\mu)\subset L^2(X,\mu)$ with $\|f_g\|_\infty \le c$. One also gets $L(A,G) = R(A,G)'$ and $R(A,G) = L(A,G)'$.

Thus, the coefficients x_g in (1.5) belong in fact to $L^\infty(X,\mu)$. They are called the *Fourier coefficients* of x. The u_g's are called the *canonical unitaries of the crossed product*. Again, we warn the reader that the convergence of the series in (1.5) does not occur in general with respect to the s.o. topology.

We now introduce conditions on the action, under which $A\rtimes G$ turns out to be a factor, and so a II_1 factor.

Definition 1.4.2. A p.m.p. action $G\curvearrowright(X,\mu)$ is (essentially) *free* if every $g\in G$, $g\neq e$, acts (essentially) freely, *i.e.*, the set $\{w\in X : gw = w\}$ has μ-measure 0.

The action is said to be *ergodic* if every Borel subset E of X such that $\mu(gE\setminus E)=0$ for every $g\neq e$ is either a null set or a conull set.

[12] See Chapter 7, Section 7.1 for a general study of this property.

We give equivalent formulations, which in particular will allow us later to extend these notions to group actions on any von Neumann algebra (see Definition 5.2.2).

LEMMA 1.4.3. *Let $G \curvearrowright (X,\mu)$ be a p.m.p. action. The following conditions are equivalent:*

(i) *the action is ergodic;*
(ii) (*resp.* (ii')) *the only functions $f \in L^\infty(X,\mu)$ (resp. $f \in L^2(X,\mu)$) that are fixed under the G-action (i.e., $\sigma_g(f) = f$ for every $g \in G$) are the constant (a.e.) functions;*
(iii) *the only measurable functions $f : X \to \mathbb{C}$ that are fixed under the G-action are the constant (a.e.) functions.*

PROOF. We only prove that (i) $\Rightarrow$ (iii), from which the whole lemma follows immediately. Let $f : X \to \mathbb{R}$ be a measurable G-invariant function. For every $r \in \mathbb{R}$, the set $E_r = \{w \in X : f(w) < r\}$ is invariant, so has measure 0 or 1. Set $\alpha = \sup\{r : \mu(E_r) = 0\}$. Then for $r_1 < \alpha < r_2$, we have $\mu(E_{r_1}) = 0$ and $\mu(E_{r_2}) = 1$. It follows that $f \equiv \alpha$ (a.e.). $\square$

For the next two results, we will need the following property of standard Borel spaces: they are *countably separated Borel space*. This means the existence of a sequence (E_n) of Borel subsets such that for $w_1 \neq w_2 \in X$ there is some E_n with $w_1 \in E_n$ and $w_2 \notin E_n$.

LEMMA 1.4.4. *Let (X,μ) be a probability measure space, $g \in \mathrm{Aut}\,(X,\mu)$, and let σ_g be the corresponding automorphism of $L^\infty(X,\mu)$. The following conditions are equivalent:*

(i) *g acts freely;*
(ii) *for every Borel subset Y with $\mu(Y) > 0$, there exists a Borel subset Z of Y with $\mu(Z) > 0$ and $Z \cap gZ = \emptyset$;*
(iii) *if $a \in L^\infty(X,\mu)$ is such that $a\sigma_g(x) = xa$ for every $x \in L^\infty(X,\mu)$, then $a = 0$.*

PROOF. (i) $\Rightarrow$ (ii). Let (E_n) be a separating family of Borel subsets as above. Assume that (i) holds and let Y be such that $\mu(Y) > 0$. Since $Y = \bigcup \big(Y \cap (E_n \setminus g^{-1}E_n)\big)$ (up to null sets) there exists n_0 such that

$$\mu(Y \cap (E_{n_0} \setminus g^{-1}E_{n_0})) \neq 0$$

and we take $Z = Y \cap (E_{n_0} \setminus g^{-1}E_{n_0})$.

(ii) $\Rightarrow$ (iii). Let $a \in L^\infty(X,\mu)$ such that $a\sigma_g(x) = xa$ for every $x \in L^\infty(X,\mu)$. If $a \neq 0$, there exists a Borel subset Y of X with $\mu(Y) > 0$ such that, for every $x \in L^\infty(X,\mu)$, we have $x(g^{-1}w) = x(w)$ for almost every $w \in Y$. Taking $x = \mathbf{1}_Z$ with Z as in (ii) leads to a contradiction.

Finally, the easy proof of (iii) $\Rightarrow$ (i) is left to the reader. $\square$

PROPOSITION 1.4.5. *Let $G \curvearrowright (X,\mu)$ be a p.m.p. action and set $A = L^\infty(X,\mu)$.*

(i) *$A' \cap (A \rtimes G) = A$ if and only if the action is free.*

(ii) *Assume that the action is free. Then $A \rtimes G$ is a factor if and only if the action is ergodic.*

Proof. Recall that A is naturally embedded into $A \rtimes G$ by $a \mapsto au_e$. Let $x = \sum_{g \in G} x_g u_g \in A \rtimes G$. Then for $a \in A$ we have

$$ax = \sum_{g \in G} a x_g u_g, \quad \text{and} \quad xa = \sum_{g \in G} x_g \sigma_g(a) u_g.$$

It follows that x belongs to $A' \cap (A \rtimes G)$ if and only if $a x_g = x_g \sigma_g(a)$ for every $g \in G$ and $a \in A$. Assertion (i) is then immediate.

To prove (ii), we remark that x belongs to the center of $A \rtimes G$ if and only if it commutes with A and with the u_g, $g \in G$. Assuming the freeness of the action, we know that the center of $A \rtimes G$ is contained in A. Moreover, an element $a \in A$ commutes with u_g if and only if $\sigma_g(a) = a$. Hence, the only elements of A commuting with u_g for every g are the scalar operators if and only if the action is ergodic. This concludes the proof $\square$

1.4.3. Examples. It follows that when $G \curvearrowright (X, \mu)$ is a free and ergodic p.m.p. action of an infinite group G, then $L^\infty(X, \mu) \rtimes G$ is a II_1 factor. Examples of such free and ergodic p.m.p. actions are plentiful. We mention below the most basic ones.

First, let G be a countable[13] dense subgroup of a compact group X. Denote by μ the Haar probability measure on X. The left action of G onto X by left multiplication is of course measure preserving. It is obviously free. It is ergodic since any function in $L^2(X, \mu)$ which is G-invariant is invariant under the action of the whole group X (using the density of G) and therefore is constant.

The simplest such example is $X = \mathbb{T}$ and $G = \exp(i 2\pi \mathbb{Z} \alpha)$ with α irrational. For one more nice example, consider $X = (\mathbb{Z}/(2\mathbb{Z}))^{\mathbb{N}}$, the group operation being the coordinate-wise addition, and take for G the subgroup of sequences having only finitely many non-zero coordinates.

Secondly, let G be any countable group, (Y, ν) a probability measure space and $X = Y^G$, equipped with the product measure $\mu = \nu^{\otimes G}$. We assume, as always, that ν does not concentrate on a single point.

Proposition 1.4.6. *The Bernoulli action $G \curvearrowright X$ is free and ergodic.*

Proof. We begin by showing that the action is free. Let $g \neq e$ and choose an infinite subset I of G such that $gI \cap I = \emptyset$. Then we have

$$\begin{aligned} \mu(\{x : gx = x\}) &\leq \mu(\big\{x : x_{g^{-1}h} = x_h, \forall h \in I\big\}) \\ &= \prod_{h \in I} \mu(\big\{x : x_{g^{-1}h} = x_h\big\}) = 0, \end{aligned}$$

since the $(\nu \times \nu)$-measure of the diagonal of $Y \times Y$ is strictly smaller than 1.

[13]For us, a countable group will implicitly mean countably infinite, whereas countable sets may be finite

We now prove a stronger property than ergodicity, that is the *mixing property*: for any Borel subsets A, B we have $\lim_{g\to\infty} \mu(A \cap gB) = \mu(A)\mu(B)$. Using basic arguments appealing to monotone classes, it suffices to prove this property when A, B are both of the form $\prod_{g\in G} E_g$ where $E_g = Y$ for all except finitely many g. But then, obviously there is a finite subset $F \subset G$ such that $\mu(A \cap gB) = \mu(A)\mu(B)$ for $g \notin F$. $\square$

REMARK 1.4.7. It is also interesting to deal with *generalized Bernoulli actions*. We let G act on an infinite countable set I and we set $X = Y^I$, endowed with the product measure $\mu = \nu^{\otimes I}$. This gives rise to the following p.m.p. action on (X, μ), called a generalized Bernoulli action:

$$\forall x \in X, \forall g \in G, \quad (gx)_i = x_{g^{-1}i}.$$

Ergodicity and freeness of these actions are studied in Exercise 1.12.

As a last example of a free and ergodic action, let us mention the natural action of $SL(n,\mathbb{Z})$ on $(\mathbb{T}^n, m)$ where m is the Lebesgue probability measure on $\mathbb{T}^n$ (see Exercise 1.13).

1.5. Von Neumann algebras from equivalence relations

We now present a construction that allows one to obtain factors from not necessarily free group actions.

1.5.1. Countable p.m.p. equivalence relations.

DEFINITION 1.5.1. A *countable* or *discrete equivalence relation* is an equivalence relation $\mathcal{R} \subset X \times X$ on a standard Borel space X that is a Borel subset of $X \times X$ and whose equivalence classes are countable.

Let $G \curvearrowright X$ be an action of a countable group G by Borel automorphisms of the Borel standard space X. The corresponding *orbit equivalence relation* is

$$\mathcal{R}_{G\curvearrowright X} = \{(x, gx) : x \in X, g \in G\}.$$

It is an example of a countable equivalence relation, and is in fact the most general one (see Exercise 1.15).

Coming back to the general case of Definition 1.5.1, a *partial isomorphism* $\varphi : A \to B$ between two Borel subsets of X is a Borel isomorphism from A onto B. We denote by $[[\mathcal{R}]]$ the set of such φ whose graph is contained into $\mathcal{R}$, *i.e.*, $(x, \varphi(x)) \in \mathcal{R}$ for every $x \in A$. The *domain* A of φ is written $D(\varphi)$ and its *range* B is written $R(\varphi)$. This family of partial isomorphisms is stable by the natural notions of composition and inverse. It is called the (*full*) *pseudogroup of the equivalence relation*. The pseudogroup $[[\mathcal{R}_{G\curvearrowright X}]]$ is described in Exercise 1.17.

Given a probability measure μ on X, one defines a σ-finite measure ν on $\mathcal{R}$ by

$$\nu(C) = \int_X |C^x| \,\mathrm{d}\mu(x)$$

where $|C^x|$ denotes the cardinality of the set $C^x = \{(x,y) \in C : y\mathcal{R}x\}$. Similarly, we may define the measure $C \mapsto \int_X |C_x| \,\mathrm{d}\mu(x)$ where $|C_x|$ denotes the cardinal of the set $C_x = \{(y,x) \in C : y\mathcal{R}x\}$. When these two measures are the same, we say that $\mathcal{R}$ *preserves the probability measure* μ. In this case, we say that $\mathcal{R}$ is a *countable probability measure preserving* (p.m.p.) *equivalence relation on* (X,μ). We will implicitly endow $\mathcal{R}$ with the measure ν.

LEMMA 1.5.2. *Let $\mathcal{R}$ be a countable equivalence relation on a probability measure space (X,μ). The two following conditions are equivalent:*

(i) *$\mathcal{R}$ preserves the measure μ;*
(ii) *for every $\varphi : A \to B$ in $[[\mathcal{R}]]$, we have $\varphi_*(\mu_{|A}) = \mu_{|B}$.*

When an action $G \curvearrowright X$ of a countable group G is given and $\mathcal{R} = \mathcal{R}_{G\curvearrowright X}$, these conditions are also equivalent to

(iii) *$G \curvearrowright X$ preserves μ.*

PROOF. Obviously (i) implies (ii). Conversely, assume that (ii) holds. Let E be a Borel subset of $\mathcal{R}$. Since the two projections from $\mathcal{R}$ onto X are countable to one, there exists a Borel countable partition $E = \cup E_n$ such that both projections are Borel isomorphisms from E_n onto their respective ranges, as a consequence of a theorem of Lusin-Novikov (see B.5 in the appendix). Each E_n is the graph of an element of $[[\mathcal{R}]]$, and the conclusion (i) follows.

When $\mathcal{R}$ is defined by $G \curvearrowright X$, it suffices to observe that for every $\varphi : A \to B$ in $[[\mathcal{R}]]$, there exists a partition $A = \cup_{g\in G} A_g$ such that $\varphi(x) = gx$ for $x \in A_g$. $\square$

1.5.2. The von Neumann algebras of a countable p.m.p. equivalence relation. To any countable equivalence relation $\mathcal{R}$ on X, we associate an involutive algebra $\mathcal{M}_b(\mathcal{R})$ generalizing matrix algebras, which correspond to trivial equivalence relations on finite sets, where all the elements are equivalent. By definition, $\mathcal{M}_b(\mathcal{R})$ is the set of bounded Borel functions $F : \mathcal{R} \to \mathbb{C}$ such that there exists a constant $C > 0$ with, for every $x, y \in X$,

$$|\{z \in X : F(z,y) \neq 0\}| \leq C, \quad \text{and} \quad |\{z \in X : F(x,z) \neq 0\}| \leq C.$$

It is easy to see that $\mathcal{M}_b(\mathcal{R})$ is an involutive algebra, when the product and the involution are given respectively by the expressions

$$(F_1 * F_2)(x,y) = \sum_{z\mathcal{R}x} F_1(x,z)F_2(z,y),$$
$$F^*(x,y) = \overline{F(y,x)}.$$

Viewing the elements of $\mathcal{M}_b(\mathcal{R})$ as matrices, these operations are respectively the matricial product and adjoint. Note also that $\mathcal{M}_b(\mathcal{R})$ contains the algebra $B_b(X)$ of bounded Borel functions on X: one identifies $f \in B_b(X)$ to the diagonal function $(x,y) \mapsto f(x)\mathbf{1}_\Delta(x,y)$ where $\mathbf{1}_\Delta$ is the characteristic function of the diagonal $\Delta \subset \mathcal{R}$. The algebra $\mathcal{M}_b(\mathcal{R})$ also contains the full pseudo-group $[[\mathcal{R}]]$ when the element $\varphi : A \to B$ of $[[\mathcal{R}]]$ is identified with the characteristic function S_φ of the set $\{(\varphi(x), x) : x \in A\}$.

Every finite sum[14]

$$F(x,y) = \sum f_\varphi(x) S_\varphi(x,y), \tag{1.6}$$

where $\varphi \in [[\mathcal{R}]]$ and $f_\varphi : R(\varphi) \to \mathbb{C}$ is a bounded Borel function, belongs to $\mathcal{M}_b(\mathcal{R})$. Using again the Lusin-Novikov theorem B.5, it can be shown that $\mathcal{M}_b(\mathcal{R})$ is exactly the space of such functions (see Exercise 1.14).

Assume in addition that $\mathcal{R}$ preserves the probability measure μ. We define a representation L of $\mathcal{M}_b(\mathcal{R})$ in $L^2(\mathcal{R}, \nu)$ by the expression

$$L_F(\xi)(x,y) = (F * \xi)(x,y) = \sum_{z \mathcal{R} x} F(x,z)\xi(z,y),$$

for $F \in \mathcal{M}_b(\mathcal{R})$ and $\xi \in L^2(\mathcal{R}, \nu)$. We leave it to the reader to check that $F \mapsto L_F$ is a $*$-homomorphism from the $*$-algebra $\mathcal{M}_b(\mathcal{R})$ into $\mathcal{B}(L^2(\mathcal{R}, \nu))$. Moreover the restriction of L to $B_b(X)$ induces an injective representation of $L^\infty(X, \mu)$, defined by

$$(L_f \xi)(x,y) = f(x)\xi(x,y)$$

for $f \in L^\infty(X, \mu)$ and $\xi \in L^2(\mathcal{R}, \nu)$. Note also that for $\varphi, \psi \in [[\mathcal{R}]]$, we have $L_{S_\varphi} * L_{S_\psi} = L_{S_{\varphi \circ \psi}}$ and $(L_{S_\varphi})^* = L_{S_{\varphi^{-1}}}$. It follows that the element $u_\varphi = L_{S_\varphi}$ is a partial isometry: $u_\varphi^* u_\varphi$ and $u_\varphi u_\varphi^*$ are the projections in $L^\infty(X, \mu) \subset \mathcal{B}(L^2(\mathcal{R}, \nu))$ corresponding to the multiplication by the characteristic functions of the domain $D(\varphi)$ of φ and of its range $R(\varphi)$ respectively. We have $(u_\varphi \xi)(x,y) = \xi(\varphi^{-1}(x), y)$ if $x \in R(\varphi)$ and $(u_\varphi \xi)(x,y) = 0$ otherwise.

The *von Neumann algebra of the countable p.m.p. equivalence relation* $\mathcal{R}$ is the s.o. closure $L(\mathcal{R})$ of $\{L_F : F \in \mathcal{M}_b(\mathcal{R})\}$ in $\mathcal{B}(L^2(\mathcal{R}, \nu))$. Observe that $L^\infty(X, \mu)$ is naturally embedded as a von Neumann subalgebra of $L(\mathcal{R})$. From the expression (1.6) we see that $L(\mathcal{R})$ is the von Neumann algebra generated by the partial isometries u_φ where φ ranges over $[[\mathcal{R}]]$.

Similarly, we may let $\mathcal{M}_b(\mathcal{R})$ act on the right by

$$R_F(\xi)(x,y) = (\xi * F)(x,y) = \sum_{z \mathcal{R} x} \xi(x,z) F(z,y).$$

We denote by $R(\mathcal{R})$ the von Neumann algebra generated by these operators R_F with $F \in \mathcal{M}_b(\mathcal{R})$. We may proceed as in Sections 1.3 and 1.4 to prove the following facts:

- $\mathbf{1}_\Delta$ is a cyclic and separating vector for $L(\mathcal{R})$. In particular, $T \mapsto T\mathbf{1}_\Delta$ identifies $L(\mathcal{R})$ with a subspace of $L^2(\mathcal{R}, \nu)$. Note that $L_F \mathbf{1}_\Delta = F$ for $F \in \mathcal{M}_b(\mathcal{R})$.
- $\tau(L_F) = \langle \mathbf{1}_\Delta, L_F \mathbf{1}_\Delta \rangle = \int_X F(x,x)\, \mathrm{d}\mu(x)$ defines a faithful w.o. continuous tracial state on $L(\mathcal{R})$.

[14]analogous to the expression (1.3)

We might prove, as we did for group von Neumann algebras, that $L(\mathcal{R})' = R(\mathcal{R})$ and that the elements of $L(\mathcal{R})$ (resp. $R(\mathcal{R})$), viewed as functions, are the left (resp. right) convolvers for $\mathcal{R}$ (see Section 7.1 for another proof).

Definition 1.5.3. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation and let A be a Borel subset of X. We denote by $[A]_{\mathcal{R}} = p_1(p_2^{-1}(A)) = p_2(p_1^{-1}(A))$ the $\mathcal{R}$-*saturation* of A, where p_1, p_2 are the left and right projections from $\mathcal{R}$ onto X. We say that A is *invariant* (or *saturated*) if $[A]_{\mathcal{R}} = A$ (up to null sets). The relation $(\mathcal{R}, \mu)$ is called *ergodic* if every invariant Borel subset is either null or co-null.

Remark 1.5.4. The Borel set A is invariant if and only if $\mathbf{1}_A \circ p_1 = \mathbf{1}_A \circ p_2$ ν-a.e. More generally, a Borel function f on X is said to be invariant if $f \circ p_1 = f \circ p_2$ ν-a.e. The equivalence relation is ergodic if and only if the only invariant bounded Borel functions are the constant (up to null sets) ones.

Proposition 1.5.5. *Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X, μ).*

(i) $L^\infty(X,\mu)' \cap L(\mathcal{R}) = L^\infty(X,\mu)$, *that is, $L^\infty(X,\mu)$ is a maximal abelian subalgebra of $L(\mathcal{R})$.*

(ii) *The center of $L(\mathcal{R})$ is the algebra of invariant functions in $L^\infty(X,\mu)$. In particular, $L(\mathcal{R})$ is a factor if and only if the equivalence relation is ergodic.*

Proof. (i) Let $T \in L(\mathcal{R}) \cap L^\infty(X,\mu)'$. We set $F = T\mathbf{1}_\Delta \in L^2(\mathcal{R},\nu)$. For every $f \in L^\infty(X,\mu)$ we have

$$L_f T\mathbf{1}_\Delta = TL_f\mathbf{1}_\Delta = T(\mathbf{1}_\Delta * f),$$

where $(\xi * f)(x,y) = \xi(x,y)f(y)$ for $\xi \in L^2(\mathcal{R},\nu)$. Moreover, T commutes with the right convolution $\xi \mapsto \xi * f$ by f, whence $L_f F = F * f$, that is $f(x)F(x,y) = F(x,y)f(y)$ ν-a.e. It follows that F is supported by the diagonal Δ, and belongs to $L^\infty(X,\mu)$ since T is bounded.

(ii) $f \in L^\infty(X,\mu)$ belongs to the center of $L(\mathcal{R})$ if and only if

$$f(x)F(x,y) = F(x,y)f(y), \quad \nu\text{-}a.e.,$$

for every $F \in \mathcal{M}_b(\mathcal{R})$, therefore if and only if $f \circ p_1 = f \circ p_2$ ν-*a.e.* □

In particular, the von Neumann algebra of an ergodic countable p.m.p. equivalence relation on a Lebesgue probability measure space (X,μ) (*i.e.*, without atoms) is a II_1 factor.

Remark 1.5.6. When $G \curvearrowright (X,\mu)$ is a free p.m.p. action and $\mathcal{R} = \mathcal{R}_{G\curvearrowright X}$, the von Neumann algebras $L(\mathcal{R})$ and $L^\infty(X,\mu) \rtimes G$ coincide. Indeed, the map $\phi : (x,g) \mapsto (x, g^{-1}x)$ induces a unitary operator $V : \xi \mapsto \xi \circ \phi$ from $L^2(\mathcal{R},\nu)$ onto $L^2(X \times G, \mu \otimes \lambda) = L^2(X,\mu) \otimes \ell^2(G)$, where λ is the counting measure on G. This holds because the action is free, and therefore ϕ is an isomorphism from $(X \times G, \mu \otimes \lambda)$ onto $(\mathcal{R}, \nu)$. We immediately see that $V^*(L^\infty(X,\mu) \rtimes G)V \subset L(\mathcal{R})$. In fact $L^\infty(X,\mu)$ is identically preserved, and we have $V^* u_g V = L_{S_g}$ where S_g is the characteristic function of $\{(gx, x) : x \in X\} \subset \mathcal{R}$. Similarly, we see that the commutant of $L^\infty(X,\mu) \rtimes G$ is sent into the commutant $R(\mathcal{R})$ of $L(\mathcal{R})$, whence $V^*(L^\infty(X,\mu) \rtimes G)V = L(\mathcal{R})$ thanks to the von Neumann bicommutant theorem.

1.5.3. Isomorphisms of p.m.p. equivalence relations.

Definition 1.5.7. We say that *two countable p.m.p. equivalence relations* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ on (X_1,μ_1) and (X_2,μ_2) respectively *are isomorphic* (and we write $\mathcal{R}_1 \simeq \mathcal{R}_2$) if there exists an isomorphism $\theta : (X_1,\mu_1) \to (X_2,\mu_2)$ (of probability measure spaces, *i.e.*, $\theta_*\mu_1 = \mu_2$) such that $(\theta \times \theta)(\mathcal{R}_1) = \mathcal{R}_2$, up to null sets, that is, after restriction to conull subsets we have $x \sim_{\mathcal{R}_1} y$ if and only if $\theta(x) \sim_{\mathcal{R}_2} \theta(y)$. Such a θ is said to *induce* or *implement the isomorphism between the equivalence relations*. If this holds when $\mathcal{R}_1 = \mathcal{R}_{G_1}$ and $\mathcal{R}_2 = \mathcal{R}_{G_2}$ we say that the actions $G_1 \curvearrowright (X_1,\mu_1)$ and $G_2 \curvearrowright (X_2,\mu_2)$ are *orbit equivalent*. This means that for *a.e.* $x \in X_1$, we have $\theta(G_1 x) = G_2\theta(x)$.

Let $\theta : (X_1,\mu_1) \to (X_2,\mu_2)$ as above. Then $U : \xi \mapsto \xi \circ (\theta \times \theta)$ is a unitary operator from $L^2(\mathcal{R}_2,\nu_2)$ onto $L^2(\mathcal{R}_1,\nu_1)$ such that $UL(\mathcal{R}_2)U^* = L(\mathcal{R}_1)$. Moreover, this spatial isomorphism sends $L^\infty(X_2,\mu_2)$ onto $L^\infty(X_1,\mu_1)$. More precisely, for $f \in L^\infty(X_2,\mu_2)$, we have $UL_fU^* = L_{f\circ\theta}$. We also observe that this isomorphism preserves the canonical traces on $L(\mathcal{R}_1)$ and $L(\mathcal{R}_2)$.

We deduce from Remark 1.5.6 that when $G_1 \curvearrowright (X_1,\mu_1)$ and $G_2 \curvearrowright (X_2,\mu_2)$ are free p.m.p. actions that are orbit equivalent through $\theta : (X_1,\mu_1) \to (X_2,\mu_2)$, the isomorphism $f \mapsto f \circ \theta$ from $L^\infty(X_2,\mu_2)$ onto $L^\infty(X_1,\mu_1)$ extends to a spatial isomorphism from the crossed product von Neumann algebra $L^\infty(X_2,\mu_2) \rtimes G_2$ onto $L^\infty(X_1,\mu_1) \rtimes G_1$. We will study the converse in Chapter 12 (Corollary 12.2.6).

1.6. Infinite tensor product of matrix algebras

In this section, we describe a way to construct II_1 factors, starting from increasing sequences of matrix algebras.

For any integer n, we embed the matrix algebra $M_n(\mathbb{C})$ into $M_{2n}(\mathbb{C})$ by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

We consider the sequence of inclusions

$$M_2(\mathbb{C}) \hookrightarrow M_{2^2}(\mathbb{C}) \hookrightarrow \cdots M_{2^k}(\mathbb{C}) \hookrightarrow \cdots$$

and we set $\mathcal{M} = \cup_{n\geq 1} M_{2^n}(\mathbb{C})$. Since the inclusions are isometric, we have a natural norm on $\mathcal{M}$: if $x \in \mathcal{M}$, we let $\|x\|$ be $\|x\|_{M_{2^n}(\mathbb{C})}$, where n is any integer such that $x \in M_{2^n}(\mathbb{C})$. There is also a natural trace defined by $\tau(x) = \tau_n(x)$, where again n is such that $x \in M_{2^n}(\mathbb{C})$ and τ_n is the (unique) tracial state on $M_{2^n}(\mathbb{C})$. Obviously, we have $\tau(x^*x) \geq 0$ for every $x \in \mathcal{M}$, and $\tau(x^*x) = 0$ if and only if $x = 0$. We denote by $\mathcal{H}$ the completion of $\mathcal{M}$ equipped with the inner product $\langle x, y\rangle = \tau(x^*y)$ and by $\|\cdot\|_\tau$ the corresponding norm. An element x of $\mathcal{M}$, when viewed as a vector in $\mathcal{H}$, will be written $\hat{x}$. For $x, y \in \mathcal{M}$, we set

$$\pi(x)\hat{y} = \widehat{xy}.$$

Then, we have (for some n),

$$\|\pi(x)\hat{y}\|_\tau^2 = \tau(y^*x^*xy) = \tau_n(y^*x^*xy) \leq \|x\|^2\tau_n(y^*y) = \|x\|^2\|\hat{y}\|_\tau^2.$$

Therefore, $\pi(x)$ extends to an element of $\mathcal{B}(\mathcal{H})$, still denoted by $\pi(x)$. It is easily checked that $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ is an injective $*$-homomorphism and we will write x for $\pi(x)$. Let R be the s.o. closure of $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$.

This construction of R from $(\mathcal{M}, \tau)$ is an example of the GNS construction that we will meet later.

For $x \in \mathcal{M}$, we observe that $\tau(x) = \langle \hat{1}, x\hat{1}\rangle$, and we extend τ to R by the same expression. Using the density of $\mathcal{M}$ into R we see that τ is still a tracial state on R. We also note that this tracial state is continuous on R equipped with the w.o. topology.

Similarly, we may define a $*$-antihomomorphism $\pi^0 : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ by:

$$\forall x, y \in \mathcal{M}, \quad \pi^0(x)\hat{y} = \widehat{yx}.$$

Obviously, $\pi^0(x)$ commutes with R. We deduce from this observation that τ is a faithful state. Indeed, assume that $x \in R$ is such that $\tau(x^*x) = 0$, that is $x\hat{1} = 0$. Then $x\hat{y} = x\pi^0(y)\hat{1} = \pi^0(y)(x\hat{1}) = 0$ for every $y \in \mathcal{M}$, which implies that $x = 0$.

Finally, we show that R is a factor, thus a II_1 factor. Let x be an element of the center of R and let x_i be a net in $\mathcal{M}$ which converges to x in the s.o. topology. In particular, we have

$$\lim_i \left\|x\hat{1} - \widehat{x_i}\right\|_\tau = \lim_i \left\|x\hat{1} - x_i\hat{1}\right\|_\tau = 0.$$

Since τ is a trace, we see that $\left\|uyu^*\hat{1}\right\|_\tau = \left\|y\hat{1}\right\|_\tau$ for every $y \in R$ and every unitary element $u \in R$. Therefore, if n is such that $x_i \in M_{2^n}(\mathbb{C})$ and if u is in the group $\mathcal{U}_{2^n}(\mathbb{C})$ of unitary $2^n \times 2^n$ matrices, we get

$$\left\|x\hat{1} - ux_iu^*\hat{1}\right\|_\tau = \left\|uxu^*\hat{1} - ux_iu^*\hat{1}\right\|_\tau = \left\|x\hat{1} - \widehat{x_i}\right\|_\tau.$$

Let λ be the Haar probability measure on the compact group $\mathcal{U}_{2^n}(\mathbb{C})$. Since $\int_{\mathcal{U}_{2^n}(\mathbb{C})} ux_iu^* \,\mathrm{d}\lambda(u)$ commutes with every element of $\mathcal{U}_{2^n}(\mathbb{C})$, it belongs to the center of $M_{2^n}(\mathbb{C})$, and therefore is a scalar operator $\alpha_i 1$. We have

$$\left\|x\hat{1} - \alpha_i\hat{1}\right\|_\tau = \left\|x\hat{1} - \int_{\mathcal{U}(M_{2^n})} ux_iu^* \,\mathrm{d}\lambda(u)\right\|_\tau \leq \left\|x\hat{1} - \widehat{x_i}\right\|_\tau.$$

It follows that $\lim_i \left\|x\hat{1} - \alpha_i\hat{1}\right\|_\tau = 0$, and therefore x is a scalar operator.

This factor R is called *the hyperfinite* II_1 *factor*. Since $M_{2^k}(\mathbb{C}) = M_2(\mathbb{C})^{\otimes k}$, we write $R = M_2(\mathbb{C})^{\overline{\otimes}\infty}$.

Remark 1.6.1. This construction may be extended to any sequence of inclusions

$$M_{n_1}(\mathbb{C}) \hookrightarrow M_{n_2}(\mathbb{C}) \hookrightarrow \cdots M_{n_k}(\mathbb{C}) \hookrightarrow \cdots$$

where $n_{k+1} = p_k n_k$, and $x \in M_{n_k}(\mathbb{C})$ is embedded into $M_{n_{k+1}}(\mathbb{C})$ by putting diagonally p_k copies of x.[15] Like $L(S_\infty)$, these factors are the s.o. closure of an increasing union of finite dimensional von Neumann algebras (indeed matrix algebras here). We will see in Chapter 11 that all these factors are isomorphic to the above factor R and thus find the explanation for the terminology *"the"* hyperfinite II_1 factor.

Comments. So far, we have now at hand various examples of II_1 factors. In the sequel, we will meet several constructions giving rise to possibly new examples (see for instance Chapter 5).

We leave this chapter with many questions. A first one is, since we have defined von Neumann algebras in a concrete way as operator algebras acting on a given Hilbert space, what are the possible concrete representations for a given von Neumann algebra? This will be studied in Chapter 8.

A much more important and challenging problem is the classification of II_1 factors, up to isomorphism. Those factors are so ubiquitous that there is a serious need to detect whether they are isomorphic or not, hence a serious need of invariants. Among the most useful invariants (up to isomorphism) for a II_1 factor M, we will meet the *fundamental group* $\mathfrak{F}(M)$, the *set* $\mathcal{I}(M)$ *of indices of subfactors* (see respectively Definitions 4.2.4 and 9.4.9) and the *outer automorphism group* $\mathrm{Out}\,(M)$ (Definition 1.3.12). We will also introduce several invariant properties such as amenability (Chapter 10), the Kazhdan property (T) (Chapter 14), and the Haagerup property (H) (Chapter 16).

Exercises

Exercise 1.1. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$. Let x_n be the operator sending e_0 onto e_n, and such that $x_n(\xi) = 0$ whenever ξ is orthogonal to e_0. Check that $\lim_n x_n = 0$ with respect to the w.o. topology but not with respect to the s.o. topology.

Exercise 1.2. Let $\mathcal{H}$ be a separable Hilbert space.

(a) Show that the unit ball $\big(\mathcal{B}(\mathcal{H})\big)_1$ of $\mathcal{B}(\mathcal{H})$ is metrizable and compact (hence second-countable) relative to the w.o. topology.

(b) Show that $\big(\mathcal{B}(\mathcal{H})\big)_1$ is metrizable and second-countable relative to the s.o. topology, and complete for the corresponding uniform structure.

(c) When $\mathcal{H}$ is infinite dimensional, show that $\big(\mathcal{B}(\mathcal{H})\big)_1$ is not separable relative to the operator norm topology (take $\mathcal{H} = L^2([0,1])$ for instance).

[15] This construction will be generalized in section 5.1.2.

Exercise 1.3. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and let α be the isomorphism sending $x \in \mathcal{B}(\mathcal{H})$ onto $\alpha(x) \in \mathcal{B}(\mathcal{H}^{\oplus\infty})$ with $\alpha(x)((\xi_n)_n) = (x\xi_n)_n$ for every $(\xi_n)_n \in \mathcal{H}^{\oplus\infty}$. Show that $\alpha(\mathcal{B}(\mathcal{H}))$ is a von Neumann algebra on $\mathcal{H}^{\oplus\infty}$, but that α is not continuous with respect to the w.o. (or s.o.) topologies[16].

Exercise 1.4. Let $\mathcal{H}$ be a separable Hilbert space. Let $k \in \mathbb{N}^*$ and let α_k be the isomorphism sending $x \in \mathcal{B}(\mathcal{H})$ onto $\alpha_k(x) \in \mathcal{B}(\mathcal{H}^{\oplus k})$ with $\alpha_k(x)((\xi_n)_n) = (x\xi_n)_n$ for every $(\xi_n)_n \in \mathcal{H}^{\oplus k}$. Show that the von Neumann algebras $\alpha_{k_1}(\mathcal{B}(\mathcal{H}))$ and $\alpha_{k_2}(\mathcal{B}(\mathcal{H}))$ are spatially isomorphic if and only if $k_1 = k_2$.

Exercise 1.5. Let (X, μ) be a probability space and $A = L^\infty(X, \mu)$. We view A as a subspace $L^2(X, \mu)$ and as a von Neumann algebra on $L^2(X)$. Show that on the unit ball $(A)_1$ of the von Neumann algebra A, the s.o. topology coincides with the topology defined by $\|\cdot\|_2$. Show that $((A)_1, \|\cdot\|_2)$ is a complete metric space.

Exercise 1.6. Let $(M_i, \mathcal{H}_i)$ be a family of von Neumann algebras. Given (T_i) with $T_i \in M_i$ and $\sup \|T_i\| < +\infty$, let T be the operator acting on the Hilbert space direct sum $\mathcal{H} = \oplus_i \mathcal{H}_i$ by $T\big((\xi_i)_i\big) = (T\xi_i)_i$. We denote by $\sum_i^{\oplus} M_i$, or also by $\prod_i M_i$, the set of such operators T. Show that $\sum_i^{\oplus} M_i$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$.

It is called the *direct sum*, or also the (ℓ^∞-)*product of the von Neumann algebras* M_i (both terminologies and notation are usual in the literature). Note that the projections 1_{M_i} belong to the center of the direct sum.

Exercise 1.7. Let $S_\infty = \cup_{n=1}^{\infty} S_n$ be the group of finite permutations of $\mathbb{N}^*$. Let $\sigma \in S_n$ be a non-trivial permutation and let i be such that $\sigma(i) \neq i$. For $j > n$, denote by s_j the transposition permuting i and j. Show that $\left\{ s_j \sigma s_j^{-1} : j > n \right\}$ is infinite.

Exercise 1.8. Show that the free group $\mathbb{F}_n, n \geq 2$, is ICC.

Exercise 1.9. Let $G \curvearrowright (X, \mu)$ be a p.m.p. action of a countable group G and $A = L^\infty(X, \mu)$. We keep the notation of Section 1.4.2. Let W be the unitary operator of $\mathcal{H} = L^2(X, \mu) \otimes \ell^2(G) = \ell^2(G, L^2(X, \mu))$ defined by $W(f)(s) = \sigma_s(f(s))$ for $f : s \mapsto f(s) \in L^2(X, \mu)$. For $a \in L^\infty(X, \mu)$, we define the operator $\pi(a)$ on $\mathcal{H}$ by $(\pi(a)f)(s) = \sigma_{s^{-1}}(a)f(s)$. Show that $W(\sigma_s \otimes \lambda_s)W^* = 1 \otimes \lambda_s$ for $s \in G$, and that $W(a \otimes 1)W^* = \pi(a)$. Therefore $A \rtimes G$ may be (and is often) alternatively defined as the von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $(\pi(A) \cup 1 \otimes \lambda(G))$.

Exercise 1.10. Let $G \curvearrowright (X, \mu)$ be a p.m.p. action of an ICC group G and set $A = L^\infty(X, \mu)$. Show that the commutant of $\{u_g : g \in G\}$ in $A \rtimes G$

[16] $\mathcal{H}^{\oplus\infty}$ denotes the countably infinite Hilbert space direct sum of copies of $\mathcal{H}$ and, for $k \in \mathbb{N}^*$, the Hilbert space direct sum of k copies of $\mathcal{H}$ is denoted by $\mathcal{H}^{\oplus k}$.

is the fixed-point algebra A^G. Conclude that $A \rtimes G$ is a II_1 factor if and only if the action is ergodic.

EXERCISE 1.11. Let $G \curvearrowright (X,\mu)$ be a p.m.p. action and $A = L^\infty(X,\mu)$. Let $x = \sum_{g\in G} x_g u_g \in A \rtimes G$ and $\xi = \sum_{g\in G} \xi_g u_g \in L^2(X,\mu) \otimes \ell^2(G)$. We set $x\xi = \sum_{g\in G} (x\xi)_g u_g \in L^2(X,\mu)\otimes\ell^2(G)$. Show that $(x\xi)_g = \sum_{h\in G} x_h \sigma_h(\xi_{h^{-1}g})$, where the convergence holds in $L^1(X,\mu)$, and that

$$\|(x\xi)_g\|_{L^1(X)} \leq \Big(\sum_{h\in G} \|x_h\|^2_{L^2(X)}\Big)^{1/2}\Big(\sum_{h\in G} \|\xi_h\|^2_{L^2(X)}\Big)^{1/2}.$$

EXERCISE 1.12. Let $G \curvearrowright (X,\mu)$ be a generalized Bernoulli action as defined in Remark 1.4.7.

(i) Show that this action is ergodic if and only if every orbit of the action $G \curvearrowright I$ is infinite.

(ii) Whenever ν has no atom, show that this generalized Bernoulli action $G \curvearrowright (X,\mu)$ is free if and only if the action $G \curvearrowright I$ is faithful, that is, for every $g \neq e$ there exists $i \in I$ with $gi \neq i$. In case ν has atoms, show that the generalized Bernoulli action is free if and only if for every $g \neq e$ the set $\{i \in I : gi \neq i\}$ is infinite.

EXERCISE 1.13. Show that the canonical action of $SL(n,\mathbb{Z})$ on $(\mathbb{T}^n, m)$ is free and ergodic (Hint: to prove ergodicity, use the Fourier transform from $L^2(\mathbb{T}^n, m)$ onto $\ell^2(\mathbb{Z}^n)$).

Observe that $SL(n,\mathbb{Z})$ can be replaced by any subgroup whose orbits on $\mathbb{Z}^n$ are infinite, except the trivial one.

EXERCISE 1.14. Let $\mathcal{R}$ be a countable equivalence relation on X.

(i) Let C be a Borel subset of $\mathcal{R}$ with

$$\sup_{x\in X} |C^x| < +\infty, \quad \sup_{x\in X} |C_x| < +\infty.$$

Show that there is a partition $C = \bigsqcup C_n$ into Borel subsets such that the second projection p_2 is injective on each C_n and $p_2(C_m) \supset p_2(C_n)$ for $m < n$. Conclude that there are only finitely many such non-empty subsets. Show that C is the disjoint union of finitely many Borel subsets such that both projections from $X \times X \to X$ are injective when restricted to them (use Theorem B.5).

(ii) Show that every $F \in \mathcal{M}_b(\mathcal{R})$ may be written as a finite sum $F(x,y) = \sum f_\varphi(x) S_\varphi(x,y)$, where $\varphi \in [[\mathcal{R}]]$ and $f_\varphi : R(\varphi) \to \mathbb{C}$ is a bounded Borel function.

EXERCISE 1.15. Let $\mathcal{R}$ be a countable equivalence relation on X.

(i) Show that there exists a partition $\mathcal{R} \setminus \Delta = \bigsqcup D_n$ into Borel subsets such that both projections p_1, p_2 restricted to each D_n are injective with $p_1(D_n) \cap p_2(D_n) = \emptyset$.

(ii) Use this partition to construct a countable group of Borel isomorphisms of X such $\mathcal{R} = \mathcal{R}_{G\curvearrowright X}$ (see [**FM77a**, Theorem 1]).

Exercise 1.16. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X, μ). We identify $L(\mathcal{R})$ to a subspace of $L^2(\mathcal{R}, \nu)$ by sending $T \in L(\mathcal{R})$ onto $F_T = T\mathbf{1}_\Delta$. Then we denote by L_{F_T} the operator T.

(i) Let $F \in L(\mathcal{R})$ and $\xi \in L^2(\mathcal{R}, \nu)$. Show that

$$(L_F\xi)(x, y) = \sum_z F(x, z)\xi(z, y) \text{ for } a.e. \ (x, y) \in \mathcal{R}.$$

(ii) Let $F_1, F_2 \in L(\mathcal{R})$. Show that $L_{F_1} \circ L_{F_2} = L_{F_1 * F_2}$ where

$$(F_1 * F_2)(x, y) = \sum_z F_1(x, z)F_2(z, y) \ a.e.$$

(iii) Let $F \in L(\mathcal{R})$. Show that $(L_F)^* = L_{F^*}$ where $F^*(x, y) = \overline{F(y, x)}$ *a.e.*

(iv) Let $F \in L(\mathcal{R})$. Show that $|F(x, y)| \leq \|L_F\|$ *a.e.*

Exercise 1.17. Let $G \curvearrowright (X, \mu)$ a p.m.p. action and let $(\mathcal{R}_{G\curvearrowright X}, \mu)$ be the corresponding p.m.p. equivalence relation. Show that a Borel isomorphism φ between two Borel subsets A, B of X belongs to $[[\mathcal{R}_{G\curvearrowright X}]]$ if and only if there exists a partition $A = \cup_{g\in G} A_g$ of A into Borel subsets such that $\varphi(x) = gx$ for *a.e.* $x \in A_g$.

Notes

The main part of this chapter is taken from the founding papers of Murray and von Neumann [**MVN36, MvN37, vN39, MvN43**], where von Neumann algebras were called *rings of operators*.

These algebras can also be abstractly defined as C^*-algebras that are duals of some Banach space[17]. Indeed, Dixmier [**Dix53**] proved that every von Neumann algebra is the dual of a Banach space and Sakai has shown [**Sak56**] that if a unital C^*-algebra A is the dual of a Banach space F, there is an injective homomorphism π from A into some $\mathcal{B}(\mathcal{H})$ such that $(\pi(A), \mathcal{H})$ is a von Neumann algebra. Moreover, this predual F is unique. It is called the predual of M (see [**Tak02**, Theorem III.3.5 and Corollary III.3.9] for instance). For the case of tracial von Neumann algebras, see Section 7.4.2.

The importance of factors as basic building blocks for general von Neumann algebras was already recognized in the seminal paper [**MVN36**] which is a sequel of von Neumann's article [**vN30**]. In [**MVN36**] the first examples of II_1 factors were exhibited as crossed products. Soon after, constructions of factors as infinite tensor products of matrix algebras were investigated by von Neumann in [**vN39**]. Later, group von Neumann algebras were defined and studied in [**MvN43**]. In this paper, among many other outstanding results, it was shown that the hyperfinite factor R is the unique hyperfinite separable II_1 factor, up to isomorphism. This will be made precise and proved in Chapter 11. In particular, Murray and von Neumann discovered that R is isomorphic to $L(S_\infty)$ but is not isomorphic to $L(\mathbb{F}_2)$.

[17]Thus defined, they are often called W^*-algebras.

Automorphisms of crossed products associated with free ergodic p.m.p. actions were first studied in the pioneering work of Singer [**Sin55**]. This was followed by Dye's deep analysis of the notion of orbit equivalence of group actions, in connection with the associated crossed products [**Dye59, Dye63**]. The von Neumann algebras of countable measured equivalence relations are studied in detail in [**FM77a, FM77b**]. Previously, Krieger had led the way by showing how the freeness of a group action $G \curvearrowright X$ could be relaxed in order to get a factor [**Kri70**].

CHAPTER 2

Fundamentals on von Neumann algebras

This chapter contains the most essential notions to start the study of von Neumann algebras.

We first introduce two key results: the von Neumann bicommutant theorem and the Kaplansky density theorem.

Next, we point out that an immediate consequence of the spectral theory is the abundance of projections in von Neumann algebras. We state some useful facts to know about the geometry of projections.

We observe that the definition of von Neumann algebras as concretely represented on Hilbert spaces, although easily accessible, has some drawbacks. For instance, the w.o. and s.o. topologies are not intrinsic, and so the notion of continuity for these topologies is not intrinsic either. To get around this difficulty, we introduce the notion of normal positive linear map, whose continuity is defined by using the order, and therefore is preserved under isomorphism.

However, the situation is not so bad. On its unit ball, the w.o. and s.o. topologies do not depend on the concrete representation of the von Neumann algebra. A normal positive linear map is characterized by the fact that its restriction to the unit ball is continuous with respect to either of these topologies. In the last section we show that a tracial von Neumann algebra has a natural representation, called the standard representation. We will highlight later, in Chapter 8, its central role in the classification of the representations of the algebra.

2.1. Von Neumann's bicommutant theorem

We begin by showing that, although different for infinite dimensional Hilbert spaces (see Exercise 1.1), the s.o. and w.o. topologies introduced in the first chapter have the same continuous linear functionals. Recall that for ξ, η in a Hilbert space $\mathcal{H}$ we denote by $\omega_{\xi,\eta}$ the linear functional $x \mapsto \langle \xi, x\eta \rangle$ on $\mathcal{B}(\mathcal{H})$. We set $\omega_\xi = \omega_{\xi,\xi}$.

PROPOSITION 2.1.1. *Let ω be a linear functional on $\mathcal{B}(\mathcal{H})$. The following conditions are equivalent:*

(i) *there exist $\xi_1, \dots, \xi_n, \eta_1, \dots \eta_n \in \mathcal{H}$ such that $\omega(x) = \sum_{i=1}^n \omega_{\eta_i,\xi_i}(x)$ for all $x \in \mathcal{B}(\mathcal{H})$;*

(ii) *ω is w.o. continuous;*

(iii) *ω is s.o. continuous.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is obvious. It remains to show that (iii) $\Rightarrow$ (i). Let ω be a s.o. continuous linear functional. There exist vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$ such that, for all $x \in \mathcal{B}(\mathcal{H})$,

$$|\omega(x)| \leq \Big(\sum_{i=1}^{n} \|x\xi_i\|^2\Big)^{1/2}.$$

Let $\mathcal{H}^{\oplus n} = \overbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}^{n\,\text{times}}$ be the Hilbert direct sum of n copies of $\mathcal{H}$. We set $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{H}^{\oplus n}$ and for $x \in \mathcal{B}(\mathcal{H})$,

$$\theta(x)\xi = (x\xi_1, \dots, x\xi_n).$$

The linear functional $\psi : \theta(x)\xi \mapsto \omega(x)$ is continuous on the vector subspace $\theta(\mathcal{B}(\mathcal{H}))\xi$ of $\mathcal{H}^{\oplus n}$. Therefore it extends to a linear continuous functional on the norm closure $\mathcal{K}$ of $\theta(\mathcal{B}(\mathcal{H}))\xi$. It follows that there exists $\eta = (\eta_1, \dots, \eta_n) \in \mathcal{K}$ such that, for $x \in \mathcal{B}(\mathcal{H})$,

$$\omega(x) = \psi(\theta(x)\xi) = \langle \eta, \theta(x)\xi \rangle_{\mathcal{H}^{\oplus n}} = \sum_{i=1}^{n} \langle \eta_i, x\xi_i \rangle. \qquad \square$$

Corollary 2.1.2. *The above proposition remains true when $\mathcal{B}(\mathcal{H})$ is replaced by any von Neumann subalgebra M.*

Proof. Immediate, since by the Hahn-Banach theorem, continuous w.o. (resp. s.o.) linear functionals on M extend to linear functionals on $\mathcal{B}(\mathcal{H})$ with the same continuity property. $\square$

In the sequel, the restrictions of the functionals $\omega_{\xi,\eta}$ and $\omega_\xi = \omega_{\xi,\xi}$ to any von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ will be denoted by the same symbols.

Let us observe that every w.o. continuous linear functional is a linear combination of at most four positive ones, as easily seen by polarization.

Recall that two locally convex topologies for which the continuous linear functionals are the same have the same closed convex subsets. Therefore, the s.o. and w.o. closures of any convex subset of $\mathcal{B}(\mathcal{H})$ coincide.

The following fundamental theorem shows that a von Neumann algebra may also be defined by purely algebraic conditions.

Theorem 2.1.3 (**von Neumann's bicommutant theorem**). *Let M be a unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. The following conditions are equivalent:*

(i) $M = M''$;
(ii) *M is weakly closed;*
(iii) *M is strongly closed.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is obvious. Let us show that (iii) $\Rightarrow$ (i). Since the inclusion $M \subset M''$ is trivial, we only have to prove that every $x \in M''$ belongs to the s.o. closure of M (which is M, by assumption (iii)). More precisely, given $\varepsilon > 0$ and $\xi_1, \dots, \xi_n \in \mathcal{H}$, we have to show the existence of $y \in M$ such that

$$\text{for} \quad 1 \le i \le n, \quad \|x\xi_i - y\xi_i\| \le \varepsilon.$$

We consider first the case $n = 1$. Given $\xi \in \mathcal{H}$, we denote by $[M\xi]$ the orthogonal projection from $\mathcal{H}$ onto the norm closure $\overline{M\xi}$ of $M\xi$. Since this vector space is invariant under M, the projection $[M\xi]$ is in the commutant M'. Hence

$$x\xi = x[M\xi]\xi = [M\xi]x\xi,$$

and so we have $x\xi \in \overline{M\xi}$. Therefore, given $\varepsilon > 0$, there exists $y \in M$ such that $\|x\xi - y\xi\| \le \varepsilon$.

We now reduce the general case to the case $n = 1$ thanks to the following very useful and basic matrix trick. We identify the algebra $\mathcal{B}(\mathcal{H}^{\oplus n})$ with the algebra $M_n(\mathcal{B}(\mathcal{H}))$ of n by n matrices with entries in $\mathcal{B}(\mathcal{H})$. We denote by $\theta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}^{\oplus n})$ the diagonal map

$$y \mapsto \begin{pmatrix} y & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & y \end{pmatrix}.$$

We set $N = \theta(M)$. Of course, N is s.o. closed. A straightforward computation shows that the commutant N' of N is the algebra of $n \times n$ matrices with entries in M'. It follows that for every $x \in M''$, we have $\theta(x) \in N''$. We apply the first part of the proof to $\theta(x)$ and N. Given $\varepsilon > 0$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{H}^{\oplus n}$, we get an element $\theta(y) \in N$ such that $\|\theta(x)\xi - \theta(y)\xi\| \le \varepsilon$, that is, $\|x\xi_i - y\xi_i\| \le \varepsilon$ for $i = 1, \dots, n$. □

2.2. Bounded Borel functional calculus

In this section, we deduce some immediate applications of the bicommutant theorem to the Borel functional calculus.

Let $x \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator, and $\mathrm{Sp}(x) \subset [-\|x\|, \|x\|]$ its spectrum. The continuous functional calculus defines an isometric isomorphism $f \mapsto f(x)$ from the C^*-algebra $C(\mathrm{Sp}(x))$ of complex-valued continuous functions on $\mathrm{Sp}(x)$ onto the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by x and 1 (see Appendix A.1). In particular, $f(x)$ is the limit in norm of the sequence $(p_n(x))$, where (p_n) is any sequence of polynomials converging to f uniformly on $\mathrm{Sp}(x)$.

Let us recall briefly how this functional calculus extends to the $*$-algebra $B_b(\mathrm{Sp}(x))$ of bounded Borel functions on $\mathrm{Sp}(x)$. First, given $\xi, \eta \in \mathcal{H}$, using the Riesz-Markov theorem, we get a bounded, countably additive, complex-valued measure $\mu_{\xi,\eta}$ on $\mathrm{Sp}(x)$ defined by

$$\int f \,\mathrm{d}\mu_{\xi,\eta} = \langle \xi, f(x)\eta \rangle$$

for every continuous function f on $\mathrm{Sp}(x)$. We say that $\mu_{\xi,\eta}$ is the *spectral measure of x associated with ξ, η*. We set $\mu_\xi = \mu_{\xi,\xi}$. The simple observation that, for a bounded Borel complex-valued function f on $\mathrm{Sp}(x)$, the map $(\xi,\eta)\mapsto\int f\,\mathrm{d}\mu_{\xi,\eta}$ is a bounded sesquilinear functional implies, by the Riesz lemma, the existence of a unique operator, denoted $f(x)$, such that

$$\int f\,\mathrm{d}\mu_{\xi,\eta} = \langle \xi, f(x)\eta\rangle$$

for every $\xi,\eta\in\mathcal{H}$. In particular, for every Borel subset Ω of $\mathrm{Sp}(x)$, if we denote by $\mathbf{1}_\Omega$ the characteristic function of Ω, then the operator $E(\Omega)=\mathbf{1}_\Omega(x)$ is a projection, called the *spectral projection of x associated with Ω*. The map $\Omega\mapsto E(\Omega)$ defined on the Borel subsets of $\mathrm{Sp}(x)$ is a *projection-valued measure* called the *spectral (projection-valued) measure of x*. The usual notation

$$f(x) = \int_{\mathrm{Sp}(x)} f(t)\,\mathrm{d}E_t$$

is convenient. It is interpreted as

$$\langle \xi, f(x)\eta\rangle = \int_{\mathrm{Sp}(x)} f(t)\,\mathrm{d}\langle \xi, E_t\eta\rangle$$

for every $\xi,\eta\in\mathcal{H}$, the integral being the Stieltjes integral with respect to the function $t\mapsto\langle\xi,E_t\eta\rangle$, where E_t is the spectral projection of x corresponding to $]-\infty,t]$.[1] Let us just remind the reader that the *Borel functional calculus* $f\mapsto f(x)$ is a $*$-homomorphism from $B_b(\mathrm{Sp}(x))$ into $\mathcal{B}(\mathcal{H})$ with $\|f(x)\|\le\|f\|_\infty$. The operator $f(x)$ is self-adjoint whenever f is real-valued; it is positive whenever $f\ge 0$. Moreover, if $y\in\mathcal{B}(\mathcal{H})$ commutes with x, then it commutes with $f(x)$ for every $f\in B_b(\mathrm{Sp}(x))$. Therefore, the bicommutant theorem implies the following result.

PROPOSITION 2.2.1. *Let x be a self-adjoint element of a von Neumann algebra $(M,\mathcal{H})$. Then, for every bounded Borel function f on $\mathrm{Sp}(x)$, we have $f(x)\in M$. In particular, the spectral measure of x takes its values in M.*

The continuous and Borel functional calculi have several easy and important consequences. Let us introduce first some notation[2]. Given a von Neumann algebra M,

- $M_{s.a}$ is the subspace of its self-adjoint elements,
- M_+ is the cone of its positive elements,
- $\mathcal{U}(M)$ is the group of its unitary elements u, that is such that $u^*u=1_M=uu^*$,
- $\mathcal{P}(M)$ is the set of its *projections*, that is of the self-adjoint idempotents.

We have recalled in Appendix A.2 that every element $x\in M$ may be expressed as a linear combination of four positive elements. Moreover, it follows from the continuous functional calculus that every $x\in M$ is the

[1] For details on these facts we refer to [**RS80**, Chapter VII] or [**Arv02**, Chapter II].

[2] We invite the reader to make explicit the sets introduced below when $M=L^\infty(X,\mu)$.

linear combination of at most four unitary operators in M. Indeed, it suffices to consider the case of a self-adjoint element x with $\|x\| \leq 1$. Then $1 - x^2$ is a positive operator and an immediate computation shows that

$$u = x + i(1 - x^2)^{1/2}$$

is a unitary operator in M. Moreover, $x = \frac{1}{2}(u + u^*)$.

Proposition 2.2.1 implies that a von Neumann algebra has plenty of projections.

Corollary 2.2.2. *Let M be a von Neumann algebra. The linear span of $\mathcal{P}(M)$ is dense in M equipped with the norm topology.*

Proof. It is enough to show that every self-adjoint element x of M can be approximated by linear combinations of elements of $\mathcal{P}(M)$. Given $\varepsilon > 0$, let $\mathrm{Sp}(x) = \cup_{i=1}^n \Omega_i$ be a finite partition of $\mathrm{Sp}(x)$ by Borel subsets, such that $|t - s| \leq \varepsilon$ for every $s, t \in \Omega_i$ and $1 \leq i \leq n$. We choose an element t_i in each Ω_i. Then we have $\|x - \sum_{i=1}^n t_i E(\Omega_i)\| \leq \varepsilon$ since $\sup_{t \in \mathrm{Sp}(x)} |t - \sum_{i=1}^n t_i \mathbf{1}_{\Omega_i}(t)| \leq \varepsilon$. □

We may even obtain a dyadic expansion of every positive element of $(M)_1$ in term of projections.

Corollary 2.2.3. *Let $x \in M$ with $0 \leq x \leq 1$. Then x can be written as the sum of a norm-convergent series*

$$x = \sum_{n=1}^{+\infty} \frac{1}{2^n} p_n,$$

where the p_n are projections in M.

Proof. Observe that if p_1 is the spectral projection $E([1/2, +\infty[)$ of x we have

$$0 \leq x - 2^{-1} p_1 \leq 1/2.$$

We perform the same construction with $2(x - 2^{-1}p_1)$ and we get a projection p_2 such that

$$0 \leq x - 2^{-1} p_1 - 2^{-2} p_2 \leq 2^{-2}.$$

By induction, we get the sequence $(p_n)_{n \geq 1}$ which satisfies, for all n,

$$0 \leq x - \sum_{k=1}^n 2^{-k} p_k \leq 2^{-n}. \qquad \square$$

The polar decomposition is another fundamental tool in operator theory. Given $x \in \mathcal{B}(\mathcal{H})$, recall that its *absolute value* is $|x| = (x^*x)^{1/2}$. There exists a unique partial isometry[3] u such that $x = u|x|$ and $\mathrm{Ker}\, u = \mathrm{Ker}\, x = \mathrm{Ker}\, |x|$. In particular, u^*u is the smallest projection $p \in \mathcal{B}(\mathcal{H})$ such that $xp = x$, that is, the projection on $(\mathrm{Ker}\, x)^\perp = \overline{\mathrm{Im}\, x^*}$.[4] We denote this projection by $s_r(x)$.

[3] *i.e.*, such that u^*u, and thus uu^*, are projections.

[4] As usual, $\mathrm{Ker}\,(x)$ and $\mathrm{Im}\,(x)$ denote the kernel and the image of x respectively.

It is called the *right support of* x. We have $\operatorname{Im} u = \overline{\operatorname{Im} x}$ and therefore uu^* is the smallest projection $p \in \mathcal{B}(\mathcal{H})$ such that $px = x$. It is denoted by $s_l(x)$ and is called the *left support of* x. Whenever x is self-adjoint, we set $s(x) = s_r(x) = s_l(x)$.

The factorization $x = u|x|$ is called the *polar decomposition* of x.

Proposition 2.2.4. *Let x be an element of a von Neumann algebra $(M, \mathcal{H})$.*

(i) *The left and right supports of x belong to M.*
(ii) *Let $x = u|x|$ be the polar decomposition of x. Then u and $|x|$ belong to M.*

Proof. (i) To prove that $s_r(x) \in M$, we check that $s_r(x)$ commutes with every unitary element v of the commutant M'. Since $vx = xv$, for every projection $p \in \mathcal{B}(\mathcal{H})$ satisfying $xp = x$ we have $xvpv^* = x$ and therefore $s_r(x) \leq vs_r(x)v^*$. Replacing v by v^*, we get $s_r(x) = vs_r(x)v^*$. The proof for $s_l(x)$ is similar. We may also remark that $s_l(x) = s_r(x^*)$.

(ii) We prove that u commutes with every unitary element v of the commutant M'. We have $x = vxv^* = (vuv^*)|x|$. Since $\operatorname{Ker} vuv^* = \operatorname{Ker} x$, we get $u = vuv^*$ by uniqueness of the polar decomposition. $\square$

2.3. The Kaplansky density theorem

The following theorem is an important technical result which allows approximations by *bounded* sequences.

Theorem 2.3.1 (**Kaplansky density theorem**). *Let A be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ and M its w.o. closure. The unit ball $(A)_1$ of A (resp. the unit ball of the self-adjoint part $A_{s.a}$ of A) is s.o. dense in the unit ball $(M)_1$ of M (resp. the unit ball of $M_{s.a}$).*

Proof. Obviously, we may assume that A is norm-closed. Using Proposition 2.1.1, we remark first that M is also the s.o. closure of the convex set A. Moreover, since the map $x \mapsto \frac{1}{2}(x + x^*)$ is w.o. continuous, $M_{s.a}$ is the w.o. closure of $A_{s.a}$, and so its s.o. closure, still by convexity.

The continuous function $f : t \in \mathbb{R} \mapsto \frac{2t}{1+t^2} \in [-1, 1]$ is a bijection onto $[-1, 1]$ when restricted to $[-1, 1]$. We set $g = (f_{|[-1,1]})^{-1}$.

We first consider the case of a self-adjoint element $x \in M$ with $\|x\| \leq 1$, and put $y = g(x) \in M_{s.a}$. Let (y_i) be a net in $A_{s.a}$ such $\lim_i y_i = y$ in the s.o. topology. Since $f(y_i)$ is in the unit ball of $A_{s.a}$, it suffices to show that $\lim_i f(y_i) = f(y) = x$ in the s.o. topology to conclude this part of the theorem. We have

$$\begin{aligned}
&f(y_i) - f(y) = 2y_i(1 + y_i^2)^{-1} - 2y(1 + y^2)^{-1} \\
&= 2(1 + y_i^2)^{-1}\Big(y_i(1 + y^2) - (1 + y_i^2)y\Big)(1 + y^2)^{-1} \\
&= 2(1 + y_i^2)^{-1}(y_i - y)(1 + y^2)^{-1} + 2(1 + y_i^2)^{-1} y_i(y - y_i)y(1 + y^2)^{-1}.
\end{aligned}$$

Since $\|(1+y_i^2)^{-1}\| \le 1$ and $\|(1+y_i^2)^{-1}y_i\| \le 1$, we get

$$\|(f(y_i)-f(y))\xi\| \le 2\|(y_i-y)(1+y^2)^{-1}\xi\| + 2\|(y-y_i)y(1+y^2)^{-1}\xi\|,$$

for every vector $\xi \in \mathcal{H}$. This shows our assertion.

The general case is reduced to the self-adjoint one by using once again a matrix trick. We consider the inclusions[5]

$$M_2(A) \subset M_2(M) \subset M_2(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^{\oplus 2})$$

and we observe that the s.o. convergence in $M_2(\mathcal{B}(\mathcal{H}))$ is the same as the s.o. entry-wise convergence. So $M_2(A)$ is s.o. dense into $M_2(M)$. Take $x \in M$ with $\|x\| \le 1$ and put

$$\widetilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}.$$

Then $\widetilde{x}$ is a self-adjoint element of $M_2(M)$ with $\|\widetilde{x}\| \le 1$. By the first part of the proof, there exists a net (y_i) in the unit ball of $M_2(A)_{s.a}$ which converges to $\tilde{x}$ in the s.o. topology. Writing

$$y_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix},$$

we have $\|b_i\| \le 1$ and $\lim_i b_i = x$ in the s.o. topology. This concludes the proof. $\square$

As a first application of this theorem we have:

Corollary 2.3.2. *Let M be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, with $\mathrm{Id}_{\mathcal{H}} \in M$. Then M is a von Neumann algebra if and only if its unit ball is compact (or equivalently closed) in the w.o. topology.*

Proof. If M is a von Neumann algebra, its unit ball is w.o. compact, being the intersection of the w.o. closed set M with the w.o. compact unit ball of $\mathcal{B}(\mathcal{H})$.

Conversely, assume that the unit ball of M is w.o. closed. Let x be an element of the w.o. closure of M. We may assume that $\|x\| \le 1$, and by the Kaplansky density theorem, there is a net (x_i) in the unit ball of M converging to x in the w.o. topology. Therefore, we have $x \in M$. $\square$

2.4. Geometry of projections in a von Neumann algebra

Let $\mathcal{H}$ be a Hilbert space. The set $\mathcal{P}(\mathcal{B}(\mathcal{H}))$ of its projections is equipped with the partial order induced by the partial order on the space $\mathcal{B}(\mathcal{H})_{s.a}$ of self-adjoint operators: for $p, q \in \mathcal{P}(\mathcal{B}(\mathcal{H}))$, we have $p \le q$ if and only if $\langle \xi, p\xi \rangle \le \langle \xi, q\xi \rangle$ (or equivalently $\|p\xi\| \le \|q\xi\|$) for every $\xi \in \mathcal{H}$. We remark that this is also equivalent to the inclusion $p(\mathcal{H}) \subset q(\mathcal{H})$. Given a set $\{p_i : i \in I\}$ of projections, there is a smallest projection p such $p \ge p_i$ for all $i \in I$. We denote it by $\bigvee_i p_i$ (or $\sup_i p_i$). It is the orthogonal projection on

[5] For every von Neumann algebra A, $M_n(A)$ denotes the von Neumann algebra of $n \times n$ matrices with entries in A.

the norm closure of the linear span of $\bigcup_{i\in I} p_i(\mathcal{H})$. There is also a greatest projection p with $p \le p_i$ for all i. We denote it by $\bigwedge_i p_i$ (or $\inf_i p_i$). It is the orthogonal projection on $\bigcap_i p_i\mathcal{H}$. Thus $\mathcal{P}(\mathcal{B}(\mathcal{H}))$ is a complete lattice. This fact is true in any von Neumann algebra M. For the proof, we need the following proposition which connects the order and the s.o. topology on the real vector space $M_{s.a}$, partially ordered by its cone M_+.

Theorem 2.4.1. *Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Let $(x_i)_{i\in I}$ be a bounded increasing net of self-adjoint elements in M, i.e., $\sup_i \|x_i\| = c < +\infty$ and $x_i \le x_j$ whenever $i \le j$. Then (x_i) converges in the s.o. topology to some $x \in M$. Moreover, x is the least upper bound of $\{x_i : i \in I\}$ in the partially ordered space $\mathcal{B}(\mathcal{H})_{s.a}$. We write $x = \sup_i x_i$.*

Proof. Using the polarization of the sesquilinear functional $(\xi, \eta) \mapsto \langle \xi, x_i\eta\rangle$, we see that the net $(\langle \xi, x_i\eta\rangle)_{i\in I}$ converges for every $\xi, \eta \in \mathcal{H}$. We set $b(\xi, \eta) = \lim_i \langle \xi, x_i\eta\rangle$. Obviously, b is a bounded sesquilinear functional on $\mathcal{H}$, and by the Riesz theorem there exists $x \in \mathcal{B}(\mathcal{H})$ such that $b(\xi, \eta) = \langle \xi, x\eta\rangle$ for every $\xi, \eta \in \mathcal{H}$. It is straightforward to check that x is self-adjoint with $\|x\| \le c$ and that $x_i \le x$ for every $i \in I$. Since $0 \le (x - x_i)^2 \le 2c(x - x_i)$ and since $\lim_i x_i = x$ in the w.o. topology, we get that $\lim_i x_i = x$ in the s.o. topology, as well.

Of course, x is in M. Now, if y is a self-adjoint element of $\mathcal{B}(\mathcal{H})$ with $y \ge x_i$ for all $i \in I$, we have $\langle \xi, y\xi\rangle \ge \langle \xi, x_i\xi\rangle$ and so $\langle \xi, y\xi\rangle \ge \langle \xi, x\xi\rangle$ for every $\xi \in \mathcal{H}$. Hence, $y \ge x$. □

Proposition 2.4.2. *If $\{p_i : i \in I\}$ is a set of projections in a von Neumann algebra M, then $\bigvee_i p_i$ and $\bigwedge_i p_i$ are in M.*

Proof. We set $p_F = \bigvee_{i\in F} p_i$ for any finite subset F of I. It is easily seen that p_F is the support of $\sum_{i\in F} p_i$, that is, the smallest projection $p \in \mathcal{B}(\mathcal{H})$ with $(\sum_{i\in F} p_i)p = \sum_{i\in F} p_i$. Therefore $p_F \in M$ by Proposition 2.2.4. Now, (p_F) where F ranges over the set of finite subsets of I is an increasing net converging, by Theorem 2.4.1, to $\bigvee_i p_i$ in the s.o. topology, and therefore $\bigvee_i p_i \in M$.

To show the second assertion, we remark that

$$\bigwedge_i p_i = 1 - \bigvee_i (1 - p_i). \qquad \square$$

When $(p_i)_{i\in I}$ is a family of mutually orthogonal projections, $\bigvee_i p_i$ is rather written $\sum_{i\in I} p_i$. It is the s.o. limit of the increasing net $(\sum_{i\in F} p_i)$ where F ranges over the finite subsets of I.

We introduce now a relation comparing the "sizes" of projections.

Definition 2.4.3. Let p and q be two projections in a von Neumann algebra M. We say that p and q are *equivalent* and we write $p \sim q$ if there exists a partial isometry $u \in M$ with $u^*u = p$ and $uu^* = q$. We write $p \precsim q$ if there exists a partial isometry $u \in M$ with $u^*u = p$ and $uu^* \le q$, *i.e.*, p is equivalent to a projection $p_1 \in M$ with $p_1 \leq q$. If $p \precsim q$ but p and q are not equivalent, we write $p \prec q$.

It is easy to see that $\sim$ is indeed an equivalence relation and that the relation $\precsim$ is transitive: $p \precsim q$ and $q \precsim r$ implies $p \precsim r$. It is also a straightforward exercise to show that if $\{p_i : i \in I\}$ and $\{q_i : i \in I\}$ are two sets of mutually orthogonal projections in M such that $p_i \sim q_i$ for all $i \in I$, then $\sum_{i\in I} p_i \sim \sum_{i\in I} q_i$. Less obvious is the following.

Theorem 2.4.4. *If $p \precsim q$ and $q \precsim p$, then $p \sim q$.*

Proof. We have $p \sim p' \leq q$ and $q \sim q' \leq p$ and therefore p' is equivalent to a projection e with $e \leq q'$; so $p \sim e \leq q' \leq p$. We claim that $p \sim q'$. Let u be a partial isometry in M such that $u^*u = p$ and $uu^* = e$. We set $p_{2n} = u^n p (u^*)^n$ and $p_{2n+1} = u^n q' (u^*)^n$ and we observe that $p_0 = p$, $p_1 = q'$, $p_2 = e$ and that $up_n u^* = p_{n+2}$ for $n \geq 0$, so that the sequence (p_n) of projections is decreasing. We set $f = \bigwedge_n p_n$. Then p is the sum

$$p = f + (p_0 - p_1) + (p_1 - p_2) + (p_2 - p_3) + (p_3 - p_4) + \cdots \tag{2.1}$$

of mutually orthogonal projections, and similarly

$$q' = f + (p_1 - p_2) + (p_2 - p_3) + (p_3 - p_4) + (p_4 - p_5) + \cdots,$$

that we write rather as

$$q' = f + (p_2 - p_3) + (p_1 - p_2) + (p_4 - p_5) + (p_3 - p_4) + \cdots, \tag{2.2}$$

since we immediately see, under this form, that the mutually orthogonal projections of the decompositions (2.1) of p and (2.2) of q' are two by two equivalent and so $p \sim q'$. $\square$

Proposition 2.4.5. *Let p, q be two projections in M. Then we have*

$$(p \vee q) - p \sim q - (p \wedge q).$$

Proof. Consider the operator $(1-p)q$. The projection on its kernel is $(1-q) + (q \wedge p)$. Therefore the left support of $q(1-p)$, which is the right support of $(1-p)q$, is $q - (q \wedge p)$. Similarly, the right support of $q(1-p)$ is $1 - \big(p + (1-p) \wedge (1-q)\big) = 1 - \big(p + (1 - (p \vee q))\big) = (p \vee q) - p$. The conclusion follows from the fact that the left and right supports of the same operator are equivalent. $\square$

We denote by $\mathcal{Z}(M)$ the center of the von Neumann algebra M.

Lemma 2.4.6. *Let $(M, \mathcal{H})$ be a von Neumann algebra.*

(i) *Let $p \in \mathcal{P}(M)$. There exists a smallest projection z in the center of M such that $zp = p$. We call it the* central support *of p and denote it $z(p)$. It is the orthogonal projection onto the closure of* $\mathrm{span}(Mp\mathcal{H})$*, the space of linear combinations of elements of $Mp\mathcal{H}$.*
(ii) *For $p \in \mathcal{P}(M)$, we have $z(p) = \bigvee_{u\in\mathcal{U}(M)} upu^*$.*
(iii) *If $p, q \in \mathcal{P}(M)$ are such that $p \sim q$, then $z(p) = z(q)$.*

Proof. (i) By definition, $z(p)$ is the infimum of the set of projections $z \in \mathcal{Z}(M)$ with $zp = p$. Since the closed linear span of $Mp\mathcal{H}$ is invariant under M and M', the orthogonal projection onto it belongs to $\mathcal{Z}(M)$ and is obviously majorized by $z(p)$, so is $z(p)$.

The other assertions are also very easy to establish and we leave their proof to the reader. □

Lemma 2.4.7. *Let M be a von Neumann algebra and let p, q be two projections in M. The following conditions are equivalent:*

(i) $z(p)z(q) \neq 0$;
(ii) $pMq \neq 0$;
(iii) *there exist non-zero projections $p_1 \le p$ and $q_1 \le q$ that are equivalent.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $pMq = 0$. Then for every $u, v \in \mathcal{U}(M)$ we have $upu^*vqv^* = 0$ and so $z(p)z(q) = 0$ by the previous lemma.

(ii) $\Rightarrow$ (iii). Let $x \in M$ such that $pxq \neq 0$. Then the right support q_1 of pxq and its left support p_1 satisfy the conditions of (iii).

(iii) $\Rightarrow$ (i). Let p_1, q_1 be as in (iii). Then we have $z(p) \ge z(p_1) = z(q_1)$ and $z(q) \ge z(q_1)$ and therefore $z(p)z(q) \neq 0$. □

The following theorem provides a useful tool which reduces the study of pairs of projections to the case where they are comparable.

Theorem 2.4.8 (**Comparison theorem**). *Let p, q be two projections in a von Neumann algebra M. Then there exists a projection z in the center of M such that $pz \precsim qz$ and $q(1-z) \precsim p(1-z)$.*

Proof. Using Zorn's lemma, we see that there exists a maximal (relative to the inclusion order) family $\mathcal{M} = \{(p_i, q_i) : i \in I\}$ where (p_i, q_i) are pairs of equivalent projections and the p_i (resp. q_i), $i \in I$, are mutually orthogonal and majorized by p (resp. by q). We have $\sum_{i\in I} p_i \sim \sum_{i\in I} q_i$. We set $p_0 = p - \sum_{i\in I} p_i$ and $q_0 = q - \sum_{i\in I} q_i$. We claim that $p_0 M q_0 = 0$ and therefore $z(p_0)z(q_0) = 0$. Otherwise, taking $x \neq 0$ in $p_0 M q_0$, we have $s_l(x) \sim s_r(x)$ with $s_l(x) \le p_0$ and $s_r(x) \le q_0$, which contradicts the maximality of $\mathcal{M}$.

We put $z = z(q_0)$. We have

$$pz = \big(\sum_{i\in I} p_i\big)z \sim \big(\sum_{i\in I} q_i\big)z \le qz$$

and

$$q(1-z) = \big(\sum_{i\in I} q_i\big)(1-z) \sim \big(\sum_{i\in I} p_i\big)(1-z) \le p(1-z),$$

that is $pz \precsim qz$ and $q(1-z) \precsim p(1-z)$. □

We deduce the following important consequence.

Corollary 2.4.9. *Let M be a factor and let p, q be two projections in M. Then, either $p \precsim q$ or $q \precsim p$.*

REMARK 2.4.10. Conversely, whenever any two projections are comparable, then M is a factor. Indeed, a non-trivial projection z in the center of M cannot be compared with $1 - z$.

COROLLARY 2.4.11. *Let M be a factor with a faithful tracial state*[6] τ *and let p, q be two projections in M. Then $p \precsim q$ if and only if $\tau(p) \leq \tau(q)$ and therefore $p \sim q$ if and only if $\tau(p) = \tau(q)$.*

DEFINITION 2.4.12. A projection p in a von Neumann algebra $(M, \mathcal{H})$ is said to be *minimal* if $p \neq 0$ and if for every projection $q \in M$ with $0 \leq q \leq p$, we have either $q = 0$ or $q = p$.

A von Neumann M is *diffuse* if it has no minimal projections.

Note that if $p \in \mathcal{P}(M)$, then $pMp = \{pxp : x \in M\}$ is a von Neumann algebra on $p\mathcal{H}$: indeed its unit ball is w.o. compact and then use Corollary 2.3.2. It is called the *reduced von Neumann algebra*[7] of M with respect to p. The projection p is minimal in M if and only if $pMp = \mathbb{C}p$.

Whenever $M = L^\infty(X, \mu)$, its projections are the characteristic functions of Borel subsets of X (up to null sets) and the minimal projections correspond to atoms. In $\mathcal{B}(\mathcal{H})$ the minimal projections are exactly the rank one projections.

The following proposition tells us that the type I factors are exactly those having minimal projections.

By definition, a (system of) *matrix units* in M is a family of partial isometries $(e_{i,j})_{i,j \in I}$ in M such that $e_{j,i} = (e_{i,j})^*$ and $e_{i,j} e_{k,l} = \delta_{j,k} e_{i,l}$ for every i, j, k, l. For instance the set of elementary matrices in $\mathcal{B}(\ell^2(I))$ is a matrix units.

PROPOSITION 2.4.13. *A factor M has a minimal projection if and only if it is isomorphic to $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$.*

PROOF. Assume that M has a minimal projection. Corollary 2.4.9 implies that the minimal projections in M are mutually equivalent and that for any non-zero projection $p \in M$ there is a minimal projection $q \leq p$. Using Zorn's lemma, we see that there exists a family $(e_i)_{i \in I}$ of minimal mutually orthogonal projections with $\sum_{i \in I} e_i = 1$. We fix $i_0 \in I$ and for $i \in I$, let $u_{0,i}$ be a partial isometry with $u_{0,i}^* u_{0,i} = e_i$ and $u_{0,i} u_{0,i}^* = e_{i_0}$. We put $e_{i,j} = u_{0,i}^* u_{0,j}$ and so $e_{i,i} = e_i$. Then $(e_{i,j})$ is a matrix units with $\sum_i e_{i,i} = 1_M$. For $x \in M$, let $x_{i,j} \in \mathbb{C}$ be such that $e_i x e_{j,i} = x_{i,j} e_i$. We have $e_i x e_j = (e_i x e_{j,i}) e_{i,j} = x_{i,j} e_{i,j}$. Then $x \mapsto [x_{i,j}]$ is an isomorphism from M onto $\mathcal{B}(\ell^2(I))$. $\square$

Since a diffuse von Neumann algebra is infinite dimensional, the next corollary follows.

[6] We will see in Corollary 6.4.2 that any tracial state on a factor is automatically faithful.

[7] One also says that pMp is a *corner of* M. Whenever p is in the center of M, one says that $pMp = pM$ is a *direct summand* of M.

COROLLARY 2.4.14. *A tracial factor is isomorphic to some matrix algebra when it is non-diffuse and is a* II_1 *factor otherwise.*

Let M be a von Neumann algebra and z be a projection in the center of M. Then Mz is a two-sided w.o. closed ideal in M. We see below that all such ideals are of this form Mz. As a consequence, a factor has only trivial two-sided w.o. closed ideals.

PROPOSITION 2.4.15 (**Two-sided ideals**). *Let I be a two-sided ideal in a von Neumann algebra M.*

(i) *I is self-adjoint.*
(iii) *Let $x \in I_+ = I \cap M_+$ and $t \in]0, +\infty[$. The spectral projection e_t of x relative to $[t, +\infty[$ belongs to I.*
(iii) *Assume in addition that I is w.o. closed. Then there exists a unique projection $z \in \mathcal{Z}(M)$ such that $I = Mz$.*

PROOF. (i) Given $x \in I$, we have $|x| \in I$. Indeed, consider the polar decomposition $x = u|x|$. Then $|x| = u^*x \in I$. It follows that $x^* = |x|u^*$ belongs to I; hence, the ideal I is self-adjoint.

(ii) Denote by f the bounded Borel function on the spectrum of x with $f(s) = 0$ for $s < t$ and $f(s) = s^{-1}$ for $s \geq t$. Since $sf(s) = \mathbf{1}_{[t,+\infty[}(s)$ for every s, the bounded Borel functional calculus results tell us that $xf(x) = e_t$ and so e_t belongs to the set $\mathcal{P}(I)$ of projections in I.

(iii) The support of $x \in I_+$, which is $\bigvee_{t>0} e_t$, belongs to I when I is w.o. closed. We set $z = \bigvee_{p \in \mathcal{P}(I)} p$. We have $z \in I$, whence $Mz \subset I$. But z majorizes the left support of every $x \in I$, and so $I \subset Mz$.

Finally, being a two-sided ideal, $I = uIu^*$, and therefore $z = uzu^*$ for every unitary operator $u \in M$, so that $z \in \mathcal{Z}(M)$. □

2.5. Continuity and order

As before, without further mention, M is a von Neumann algebra on a Hilbert space $\mathcal{H}$. Recall that a linear functional ω on M is said to be *positive* if $\omega(M_+) \subset \mathbb{R}_+$. We introduce a notion of continuity for ω which is expressed in term of the order on the space of self-adjoint elements.

DEFINITION 2.5.1. Let ω be a positive linear functional on M. We say that

(i) ω is *normal* if for every bounded increasing net (x_i) of positive elements in M, we have $\omega(\sup_i x_i) = \sup_i \omega(x_i)$;
(ii) ω is *completely additive* if for every family $\{p_i : i \in I\}$ of mutually orthogonal projections in M, we have $\omega(\sum_i p_i) = \sum_i \omega(p_i)$.

Complete additivity is reminiscent of the analogous property for integrals in measure theory.

REMARKS 2.5.2. **(a)** It is a straightforward exercise to show that whenever ω is normal, every positive linear functional φ with $\varphi \leq \omega$ is normal.

(b) Every w.o. continuous positive linear functional is normal (see Theorem 2.4.1). However, there exist normal positive linear functionals which are not w.o. continuous. For instance, assume that $\mathcal{H}$ is separable and infinite dimensional. Let (ϵ_n) be an orthonormal basis of $\mathcal{H}$ and set $\omega(x) = \sum_{n\geq 1} n^{-2}\langle \epsilon_n, x\epsilon_n\rangle$ for $x \in \mathcal{B}(\mathcal{H})$. Then ω is normal but not w.o. continuous. Otherwise, by Proposition 2.5.4 below there would exist $\eta_1, \dots, \eta_k \in \mathcal{H}$ with $\omega(x) = \sum_{i=1}^{k} \langle \eta_i, x\eta_i\rangle$ for every $x \in \mathcal{B}(\mathcal{H})$. If p denotes the orthogonal projection on the linear span of $\{\eta_i : 1 \leq i \leq k\}$ then $1 - p$ is a non-zero projection with $\omega(1 - p) = 0$. This is impossible, since ω is a faithful linear functional on $\mathcal{B}(\mathcal{H})$.

(c) Recall that every w.o. continuous linear functional is a linear combination of at most four positive w.o. continuous linear functionals (hence normal).

We will provide in Theorem 2.5.5 several characterisations of normality. Before then, we give the general form of a w.o. continuous positive linear functional. For that, we need the following elementary Radon-Nikodým type lemma.

Lemma 2.5.3. *Let ω be a positive linear functional on M and $\xi \in \mathcal{H}$ such that $\omega(x) \leq \langle \xi, x\xi\rangle$ for $x \in M_+$. There exists $x' \in M'_+$ such that $\omega(x) = \langle (x'\xi), x(x'\xi)\rangle$ for all $x \in M$.*

Proof. The Cauchy-Schwarz inequality gives, for $x, y \in M$,

$$|\omega(x^*y)|^2 \leq \omega(x^*x)\omega(y^*y) \leq \|x\xi\|^2\|y\xi\|^2.$$

Therefore, we get a well-defined bounded sesquilinear form on $M\xi$ by setting

$$(x\xi|y\xi) = \omega(x^*y).$$

Hence, there exists a positive operator z on the Hilbert space $\overline{M\xi}$ such that $\omega(x^*y) = \langle x\xi, zy\xi\rangle$. For $x, y, t \in M$, we have

$$\langle x\xi, zty\xi\rangle = \omega(x^*ty) = \omega((t^*x)^*y) = \langle t^*x\xi, zy\xi\rangle = \langle x\xi, tzy\xi\rangle,$$

so that $tz = zt$ on $\overline{M\xi}$. We denote by p the orthogonal projection onto $\overline{M\xi}$ and we let x' be the square root of the positive element zp in M'. Obviously,

$$\omega(x) = \langle \xi, xzp\xi\rangle = \left\langle x'\xi, xx'\xi\right\rangle$$

for all $x \in M$. $\square$

Proposition 2.5.4. *Let ω be a w.o. continuous positive linear functional on M. Then there exist $\zeta_1, \dots, \zeta_n \in \mathcal{H}$ such that $\omega = \sum_{i=1}^{n} \omega_{\zeta_i}$.*

Proof. By Proposition 2.1.1, ω is of the form $\sum_{i=1}^{n} \omega_{\eta_i,\xi_i}$. Thanks to the classical trick by which we replace $\mathcal{H}$ by $\mathcal{H}^{\oplus n}$ and M by $\theta(M)$ where $\theta(x)(\zeta_1, \dots, \zeta_n) = (x\zeta_1, \dots, x\zeta_n)$, it suffices to consider the case $\omega = \omega_{\eta,\xi}$. But since ω is positive, we have, for $x \in M_+$,

$$\begin{aligned} 4\langle \eta, x\xi\rangle &= \langle (\eta+\xi), x(\eta+\xi)\rangle - \langle (\eta-\xi), x(\eta-\xi)\rangle \\ &\leq \langle (\eta+\xi), x(\eta+\xi)\rangle. \end{aligned}$$

The conclusion follows from Lemma 2.5.3. $\square$

THEOREM 2.5.5. *Let ω be a positive linear functional on M. The following conditions are equivalent:*

(1) *ω is normal;*
(2) *ω is completely additive;*
(3) *ω is the limit in norm, in the dual M^* of M, of a sequence of w.o. continuous positive linear functionals;*
(4) *the restriction of ω to the unit ball of M is w.o. continuous;*
(5) *the restriction of ω to the unit ball of M is s.o. continuous.*

PROOF. We show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1). The only non-immediate implication is (2) $\Rightarrow$ (3).

Assume that ω is completely additive. Let $(p_i)_{i\in I}$ be a maximal family of mutually orthogonal projections in M such that, for every i, there exists $\xi_i \in p_i\mathcal{H}$ with $\omega(x) = \langle \xi_i, x\xi_i\rangle$ on p_iMp_i. Note that $\|\xi_i\| = \omega(p_i)^{1/2}$. We put $q = \sum p_i$. Lemma 2.5.6 below, applied to ω restricted to $(1-q)M(1-q)$, shows that $\sum p_i = 1$. Thanks to the complete additivity of ω, we have $\sum_i \omega(p_i) = \omega(1) < +\infty$, and therefore the subset I_0 of indices i for which $\omega(p_i) \neq 0$ is countable.

By the Cauchy-Schwarz inequality, we have, for $x \in M$,

$$|\omega(xp_i)| \leq \omega(1)^{1/2}\omega(p_ix^*xp_i)^{1/2} = \omega(1)^{1/2}\|x\xi_i\|. \tag{2.3}$$

It follows that $x\xi_i \mapsto \omega(xp_i)$ is a well-defined and bounded linear functional on $M\xi_i$. Hence, there exists $\eta_i \in \overline{M\xi_i}$ such that $\omega(xp_i) = \langle \eta_i, x\xi_i\rangle$ for $x \in M$. For every finite subset F of I_0, we set $q_F = \sum_{i\in F} p_i$ and denote by ω_F the positive linear functional $x \mapsto \omega(q_F x q_F)$ on M. We have, for $x \in M$,

$$\omega_F(x) = \sum_{i\in F} \omega(q_F x p_i) = \sum_{i\in F} \langle q_F\eta_i, x\xi_i\rangle.$$

Therefore, ω_F is w.o. continuous. Moreover, $\lim_F \|\omega - \omega_F\| = 0$, where the limit is taken along the net of finite subsets of I_0. Indeed,

$$\begin{aligned}
\Big|\omega(x) - \omega_F(x)\Big| &\leq \Big|\omega(x) - \omega(xq_F)\Big| + \Big|\omega(xq_F) - \omega(q_F x q_F)\Big| \\
&\leq \Big|\omega(x(1-q_F))\Big| + \Big|\omega((1-q_F)xq_F)\Big| \\
&\leq 2\omega(1)^{1/2}\|x\|\omega(1-q_F)^{1/2},
\end{aligned}$$

so that $\|\omega - \omega_F\| \leq 2\omega(1)^{1/2}\omega(1-q_F)^{1/2}$. But $\lim_F \omega(1-q_F) = 0$. $\square$

LEMMA 2.5.6. *Let φ be a completely additive positive linear functional on M. There exist a non-zero projection $p \in M$ and $\xi \in p\mathcal{H}$ such that $\varphi(x) = \langle \xi, x\xi\rangle$ for every $x \in pMp$.*

PROOF. We choose a vector $\eta \in \mathcal{H}$ such that $\varphi(1) < \langle \eta, \eta\rangle$. It suffices to prove the existence of p such that $\varphi(x) \leq \langle \eta, x\eta\rangle$ for $x \in (pMp)_+$. Then the conclusion will follow from Lemma 2.5.3. Let (p_i) be a maximal family of mutually orthogonal projections with $\varphi(p_i) \geq \langle \eta, p_i \eta \rangle$ for all i. The complete additivity of φ implies that

$$\varphi(\sum_i p_i) \geq \langle \eta, (\sum_i p_i)\eta \rangle.$$

We put $p = 1 - \sum_i p_i$. Observe that $p \neq 0$ since $\varphi(1) < \langle \eta, \eta \rangle$. By the maximality of (p_i), we have $\varphi(q) < \langle \eta, q\eta \rangle$ for every non-zero projection $q \leq p$. Using spectral theory, we approximate $x \in (pMp)_+$, in norm, by appropriate linear combinations of its spectral projections, with positive coefficients, and we get $\varphi(x) \leq \langle \eta, x\eta \rangle$, since φ is norm continuous. $\square$

REMARK 2.5.7. With more effort[8], we can get that the conditions of Theorem 2.5.5 are also equivalent to the following condition (3') which is stronger than (3):

(3') $\omega = \sum_n \omega_{\zeta_n}$ with $\sum_n \|\zeta_n\|^2 < +\infty$;

Let us give a proof that (2) implies (3') when ω is a trace. We keep the notation of the proof of (2) $\Rightarrow$ (3) in Theorem 2.5.5. Using the equality $\omega(xp_i) = \omega(p_i x p_i)$ we get $|\omega(xp_i)| \leq \omega(p_i)^{1/2} \|x\xi_i\|$ instead of the inequality (2.3) and so now we have $\|\eta_i\| \leq \omega(p_i)^{1/2}$. We have

$$\begin{aligned}\Big|\omega(x) - \sum_{i\in F} \omega(xp_i)\Big|^2 &\leq \omega\big((1 - \sum_{i\in F} p_i)x^*x(1 - \sum_{i\in F} p_i)\big)\omega(1 - \sum_{i\in F} p_i) \\ &\leq \|x\|^2 \omega(1)\, \omega(1 - \sum_{i\in F} p_i).\end{aligned}$$

Passing to the limit, we get

$$\omega(x) = \sum_{i\in I_0} \langle \eta_i, x\xi_i \rangle,$$

with $\sum_{i\in I_0} \|\eta_i\|^2 \leq \sum_{i\in I_0} \omega(p_i) = \sum_{i\in I_0} \|\xi_i\|^2 < +\infty$. To conclude that $\omega = \sum \omega_{\zeta_i}$ with $\sum \|\zeta_i\|^2 < +\infty$, we argue as in the proof of Proposition 2.5.4.

We say that a linear map Φ from a von Neumann algebra $(M, \mathcal{H})$ into a von Neumann algebra $(N, \mathcal{K})$ is *positive* if $\Phi(M_+) \subset N_+$. We say that such a positive linear map is *normal* if for every bounded increasing net (x_i) of positive elements in M, we have $\Phi(\sup_i x_i) = \sup_i \Phi(x_i)$.

PROPOSITION 2.5.8. *Let $\Phi : M \to N$ be a positive linear map. The following conditions are equivalent:*

(1) *Φ is normal;*
(2) *$\omega \circ \Phi$ is a normal positive linear functional on M for every such functional ω on N;*
(3) *the restriction of Φ to the unit ball of M is continuous with respect to the w.o. topologies.*

[8] see [**Dix81**, Theorem 1, page 57]

Whenever Φ is a homomorphism, the above conditions are also equivalent to:

(4) *the restriction of Φ to the unit ball of M is continuous with respect to the s.o. topologies.*

Proof. (1) $\Rightarrow$ (2) is obvious. Assume that (2) holds. To show that the restriction of Φ to the unit ball $(M)_1$ is continuous with respect to the w.o. topologies, we have to check that $x \mapsto \omega \circ \Phi(x)$ is w.o. continuous on $(M)_1$ for every w.o. continuous linear functional ω on N. We may assume that ω is positive. Then $\omega \circ \Phi$ is a normal positive linear functional and the assertion (3) follows from Theorem 2.5.5.

(3) $\Rightarrow$ (1). We now assume the w.o. continuity of the restriction of Φ to the unit ball of M. Let (x_i) be an increasing net of positive elements in the unit ball of M. Its supremum x is the w.o. limit of (x_i) and therefore $\Phi(x)$ is the w.o. limit of $(\Phi(x_i))$. But then, $\Phi(x) = \sup_i \Phi(x_i)$, and therefore Φ is normal.

Assume now that Φ is a homomorphism. If $\lim_i x_i = x$ strongly in $(M)_1$, then $\lim_i (x - x_i)^*(x - x_i) = 0$ in the w.o. topology and so, if (3) holds we have $\lim_i \Phi((x - x_i)^*(x - x_i)) = 0$ in the w.o. topology. Since $\Phi((x - x_i)^*(x - x_i)) = \Phi(x - x_i)^*\Phi(x - x_i)$, we see that $\lim_i \Phi(x_i) = \Phi(x)$ strongly. Therefore (3) implies (4). The proof of (4) $\Rightarrow$ (1) is similar to that of (3) $\Rightarrow$ (1). $\square$

Corollary 2.5.9. *Every isomorphism $\alpha : M \to N$ is normal and therefore its restriction to the unit ball of M is continuous with respect to the w.o. topologies, as well as with respect to the s.o. topologies.*

Proof. Obviously, α preserves the positivity in M, and since it is an isomorphism, we have $\alpha(\sup_i x_i) = \sup_i \alpha(x_i)$ for every bounded increasing net (x_i) of positive elements in M. $\square$

Remark 2.5.10. As a consequence of this corollary, whereas the w.o. topology on a von Neumann algebra depends on the Hilbert space on which it acts (see Exercise 1.3), the w.o. topology on its unit ball is intrinsic. The same observation applies to the s.o. topology.

Proposition 2.5.11. *Let ω and ψ be positive linear functionals on von Neumann algebras M and N respectively, and let $\Phi : M \to N$ be a positive linear map such that $\psi \circ \Phi \leq \omega$. We assume that ω and ψ are normal and that ψ is faithful. Then Φ is normal.*

Proof. We set $\varphi = \psi \circ \Phi$. Since $\varphi \leq \omega$, we see that φ is normal. Now, let (x_i) be a bounded increasing net of positive elements in M and put $y = \Phi(\sup_i x_i)$. We have $y \geq \sup_i \Phi(x_i)$ and

$$\begin{aligned}\sup_i \varphi(x_i) &= \varphi(\sup_i x_i) = \psi \circ \Phi(\sup_i x_i)\\ &= \psi(y) \geq \psi(\sup_i \Phi(x_i)) = \sup_i \psi(\Phi(x_i)) = \sup_i \varphi(x_i).\end{aligned}$$

Since ψ is faithful, we deduce that $y = \sup_i \Phi(x_i)$. □

PROPOSITION 2.5.12. *Let $\pi : M \to \mathcal{B}(\mathcal{K})$ be a normal unital homomorphism. Then $\pi(M)$ is a von Neumann algebra on $\mathcal{K}$.*

PROOF. Let us show that $\pi(M)$ is w.o. closed. We first claim that the kernel of π is a w.o. closed two-sided ideal of M. Indeed, let x be in the w.o. closure of $\operatorname{Ker}\pi$ with $\|x\| \leq 1$. By the Kaplansky density theorem, there exists a net (x_i) in the unit ball of $\operatorname{Ker}\pi$ that converges to x in the w.o. topology. It follows that $\pi(x) = 0$. Proposition 2.4.15 shows that $\operatorname{Ker}\pi$ is of the form Mz where z is a projection in $\mathcal{Z}(M)$. Now the restriction of π to $M(1-z)$ is an injective homomorphism, and so is an isometry. Since the unit ball of $M(1-z)$ is w.o.compact, its image under π, namely the unit ball of $\pi(M)$ is also w.o. compact and $\pi(M)$ is w.o. closed by Corollary 2.3.2. □

REMARK 2.5.13. Let the abelian von Neumann algebra $L^\infty(X,\mu)$ act on $L^2(X,\mu)$. Let $\omega = \sum_n \omega_{\zeta_n}$, with $\sum_n \|\zeta_n\|_2^2 < +\infty$, be a positive normal linear functional on $L^\infty(X,\mu)$ (see Remark 2.5.7). Setting $\xi = \sum_n |\zeta_n|^2 \in L^1(X,\mu)_+$, we see that

$$\omega(f) = \int_X f\xi \,\mathrm{d}\mu = \left\langle \xi^{1/2}, f\xi^{1/2} \right\rangle$$

for every $f \in L^\infty(X,\mu)$. It follows that the positive normal linear functionals on $L^\infty(X,\mu)$ are w.o. continuous and that they are exactly the positive $\sigma(L^\infty(X,\mu), L^1(X,\mu))$-continuous linear functionals. We deduce from this observation that a positive linear map $\Phi : L^\infty(X,\mu) \to L^\infty(Y,\nu)$ is normal if and only if it is continuous with respect to the weak* topologies defined by the L^1-L^∞ duality.

2.6. GNS representations

Just as $L^\infty(X,\mu)$ has a natural representation on $L^2(X,\mu)$, we will see that every tracial von Neumann algebra (M,τ) has a privileged normal faithful representation, called the standard representation.

2.6.1. The GNS construction. Since a tracial von Neumann algebra is given with a specific state, it is natural to study the corresponding Gelfand-Naimark-Segal representation. We begin by recalling this construction.

Let M be a von Neumann algebra, or more generally a unital C^*-algebra, and let φ be a positive linear functional on M. We define a sesquilinear form on M by

$$\langle x, y \rangle_\varphi = \varphi(x^* y).$$

Let $N_\varphi = \{x \in M : \varphi(x^* x) = 0\}$. Using the Cauchy-Schwarz inequality, we see that N_φ is the space of all $x \in M$ such that $\langle x, y \rangle_\varphi = 0$ for every $y \in M$, and therefore it is a linear subspace of M. We define $\mathcal{H}_\varphi$ as the completion of the pre-Hilbert space M/N_φ with respect to the inner product

$$\langle \hat{x}, \hat{y} \rangle_\varphi = \varphi(x^*y),$$

where $\hat{x}$ denotes the class of x in the quotient. We set $\|\hat{x}\|_\varphi = \varphi(x^*x)^{1/2}$.

For $x, y \in M$, we put

$$\pi_\varphi(x)\hat{y} = \widehat{xy}.$$

We have

$$\begin{aligned} \|\pi_\varphi(x)\hat{y}\|_\varphi^2 = \|\widehat{xy}\|_\varphi^2 &= \varphi(y^*x^*xy) \\ &\leq \|x^*x\|\varphi(y^*y) = \|x\|^2\|\hat{y}\|_\varphi^2. \end{aligned}$$

It follows that $\pi_\varphi(x)$ extends to an element of $\mathcal{B}(\mathcal{H}_\varphi)$, still denoted $\pi_\varphi(x)$. It is easy to check that π_φ is a homomorphism from M into $\mathcal{B}(\mathcal{H}_\varphi)$. Moreover, if we put $\xi_\varphi = \hat{1}$, we have, for $x \in M$,

$$\varphi(x) = \langle \xi_\varphi, \pi_\varphi(x)\xi_\varphi \rangle_\varphi. \tag{2.4}$$

We say that $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ is the *Gelfand-Naimark-Segal* (GNS) representation associated with φ. Note that the vector ξ_φ is cyclic for $\pi_\varphi(M)$.

If we start from a faithful state φ, it follows from Equation (2.4) that π_φ is an injective homomorphism and that ξ_φ is separating for $\pi_\varphi(M)$. In this case we will identify M with $\pi_\varphi(M)$ and write $x\xi$ for $\pi_\varphi(x)\xi$. Also, we identify $x \in M$ with $x\xi_\varphi$ and view M as a dense subspace of $\mathcal{H}_\varphi$. Sometimes, we will write $\hat{x}$ instead of $x = x\xi_\varphi$ to emphasize the fact that x is seen as an element of $\mathcal{H}_\varphi$. If we start from $M = L^\infty(X, \mu)$ and $\varphi = \tau_\mu$, then π_φ is the representation by multiplication on the Hilbert space $L^2(X, \mu)$. For that reason, in general we write $L^2(M, \varphi)$ for the Hilbert space $\mathcal{H}_\varphi$ and $\|\cdot\|_2$ instead of $\|\cdot\|_\varphi$.

2.6.2. Normal GNS representations. Returning to the general case, it is of course important for us to know when $\pi_\varphi(M)$ is a von Neumann algebra on $\mathcal{H}_\varphi$.

Theorem 2.6.1. *Let φ be a state on a von Neumann algebra M and let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the GNS construction. The state φ is normal if and only if π_φ is normal. Moreover, in this case $\pi_\varphi(M)$ is a von Neumann algebra on $\mathcal{H}_\varphi$.*

Proof. Obviously, if π_φ is normal, so is φ by Equation (2.4). Conversely, assume that φ is normal. Then for $a, b \in M$, the map

$$x \mapsto \langle a\xi_\varphi, \pi_\varphi(x)b\xi_\varphi \rangle_\varphi = \varphi(a^*xb)$$

is w.o. continuous on the unit ball $(M)_1$ of M, and thanks to the density of $\pi_\varphi(M)\xi_\varphi$ in $\mathcal{H}_\varphi$, we see that $x \mapsto \langle \xi, \pi_\varphi(x)\eta \rangle_\varphi$ is w.o. continuous on $(M)_1$ for every $\xi, \eta \in \mathcal{H}_\varphi$. So π_φ is normal. Proposition 2.5.12 tells us that in this case $\pi_\varphi(M)$ is a von Neumann algebra. $\square$

Definition 2.6.2. Let (M, τ) be a tracial von Neumann algebra. Its GNS representation on $L^2(M, \tau)$ is called the *standard representation*[9].

Remark 2.6.3. For a detailed study of this representation, the reader may go directly to Chapter 7. We only note here that for $x, y \in M$ we have $\|\widehat{yx}\|_2 \leq \|x\|_\infty \|\widehat{y}\|_2$, so that M acts also to the right on $L^2(M, \tau)$ by $\widehat{y}\, x = \widehat{yx}$ (see Subsection 7.1.1).

2.6.3. An abstract characterisation. We sometimes meet the situation where M is a unital C^*-algebra equipped with a faithful tracial state τ and we want to know whether $\pi_\tau(M)$ is a von Neumann algebra on $L^2(M, \tau)$ (see for instance Section 5.4). A useful answer is provided by the study of the metric d_2 defined by the norm $\|x\|_2 = \|x\xi_\tau\|_\tau$ on the unit ball $(M)_1$ of M. Note that since $\|x\hat{y}\|_\tau \leq \|y\|_\infty \|x\|_2$, *the topology induced on $(M)_1$ by the s.o. topology of $\mathcal{B}(L^2(M, \tau))$ is the same as the topology defined by the metric d_2*. This no longer holds on M (Exercise 2.13).

Proposition 2.6.4. *Let M be a unital C^*-algebra equipped with a faithful tracial state τ. Then M (identified with $\pi_\tau(M)$) is a von Neumann algebra on $L^2(M, \tau)$ if and only if its unit ball $(M)_1$ is complete with respect to the metric d_2 induced by the norm $\|\cdot\|_2$. Moreover, τ is normal when this condition is satisfied.*

Proof. Assume first that M is s.o. closed in $\mathcal{B}(L^2(M, \tau))$. Let (x_n) be a Cauchy sequence in $((M)_1, d_2)$. Since

$$\|x_n\hat{y} - x_m\hat{y}\|_\tau \leq \|y\|_\infty \|x_n - x_m\|_2$$

whenever $y \in M$, we see that the sequence $(x_n\hat{y})$ is convergent in $L^2(M, \tau)$. Setting $x\hat{y} = \lim_n x_n\hat{y}$, we define an element $x \in \mathcal{B}(L^2(M, \tau))$ with $\|x\| \leq 1$. Obviously, (x_n) converges to x in the s.o. topology, so $x \in M$. Of course, we have $\lim_n \|x_n - x\|_2 = 0$.

Conversely, assume that $(M)_1$, equipped with the metric d_2, is complete. Let N be the closure of M in the s.o. topology. We extend τ to a normal tracial state on N by setting $\tau(x) = \langle \xi_\tau, x\xi_\tau \rangle$ for every $x \in N$. Due to the inequality $\|x\hat{y}\|_\tau \leq \|y\|_\infty \|x\xi_\tau\|_\tau$, which is still valid for $x \in N$ and $y \in M$, we see that τ is faithful on N. By the Kaplansky density theorem, $(M)_1$ is s.o. dense in the unit ball $(N)_1$ of N. Since the s.o. topology coincides with the $\|\cdot\|_2$ topology on $(N)_1$, we see that $(M)_1 = (N)_1$, whence $M = N$. □

2.6.4. Separable tracial von Neumann algebras.

Definition 2.6.5. We say that a von Neumann algebra is *countably decomposable* (or σ-finite) if every family of mutually orthogonal non-zero projections is at most countable.

Of course, every tracial von Neumann algebra is countably decomposable. We now introduce a stronger form of separability.

[9]We will see in Proposition 7.5.1, that this representation does not depend on the choice of the trace.

Definition 2.6.6. We say that a von Neumann algebra is *separable* if it has a faithful normal representation on a separable Hilbert space.

Proposition 2.6.7. *Let (M,τ) be a tracial von Neumann algebra. The following conditions are equivalent:*

(i) *M is separable;*
(ii) *The unit ball $(M)_1$ contains a s.o. dense sequence (equivalently, dense in the metric induced by $\|\cdot\|_2$);*
(iii) *$L^2(M,\tau)$ is a separable Hilbert space.*

Proof. (i) $\Rightarrow$ (ii). Assume that M acts on a separable Hilbert space $\mathcal{H}$. Then on $(M)_1$ the s.o. topology is second countable and therefore $(M)_1$ contains a s.o. dense countable subset.

(ii) $\Rightarrow$ (iii). Let $D \subset (M)_1$ be countable and dense in $(M)_1$ in the topology defined by $\|\cdot\|_2$. Since M is dense in $L^2(M,\tau)$ we see that $\mathrm{span}(\mathbb{Q}D)$ is dense in $L^2(M,\tau)$.

(iii) $\Rightarrow$ (i) is obvious. $\square$

Exercises

Exercise 2.1. Let $(M,\mathcal{H})$ be a von Neumann algebra and $(z_i)_i \in I$ be a family of mutually orthogonal projections in $\mathcal{Z}(M)$ such that $\sum_{i\in I} z_i = 1_M$. Show that $(M,\mathcal{H})$ is (isomorphic to) the direct sum $\sum_i^{\oplus}(z_iM, z_i\mathcal{H})$.

Exercise 2.2. Let M be a finite dimensional von Neumann algebra. Show that M is isomorphic to a finite direct sum of matrix algebras.

Exercise 2.3. Let M and N be two von Neumann algebras on a Hilbert space $\mathcal{H}$. Show that $(M\cap N)' = (M'\cup N')''$ and conclude that M is a factor if and only if $(M\cup M')'' = \mathcal{B}(\mathcal{H})$.

Exercise 2.4. Let e be a projection in M with central support 1 and let p be a non-zero projection in M. Show that there is a non-zero partial isometry $u \in M$ with $u^*u \leq e$ and $uu^* \leq p$.

Exercise 2.5. Let e be a projection in M and let (f_i) be a maximal family of mutually orthogonal projections in M such that $f_i \precsim e$ for every i. Show that $\sum_i f_i$ is the central support of e.

Exercise 2.6. Let M be a von Neumann algebra, $p \in M$ a minimal projection and $z(p)$ its central support. Show that $Mz(p)$ is a type I factor, *i.e.,* is isomorphic to some $\mathcal{B}(\mathcal{K})$.

Exercise 2.7. Let M be a II_1 factor, N a subfactor of type I_n, and α an automorphism of M. Show that there is a unitary element $u \in M$ such that $\alpha(x) = uxu^*$ for every $x \in N$.

Exercise 2.8. Let M be a von Neumann algebra, $x \in M_{s.a}$ and $t \in \mathbb{R}$. Denote by Λ_t the ordered set of continuous functions $f : \mathrm{Sp}(x) \to [0,1]$ such that $f(s) = 0$ for $s \geq t$. Show that $(f(x))_{f\in\Lambda_t}$ converges in the s.o. topology to the spectral projection $E(]-\infty, t[)$ of x (along the net Λ_t).

Exercise 2.9. Let M be a von Neumann algebra on $\mathcal{H}$ and let $x, y \in M_+$ with $y \leq x$. Show that there exists a unique element $a \in \mathcal{B}(\mathcal{H})$ with $y^{1/2} = a x^{1/2}$ and $s_r(a) = s(x)$. Show that $a \in M$.

Exercise 2.10. Let I be a two-sided ideal in a von Neumann algebra M.

(i) Let $x \in I_+$ and let $y \in M$ with $0 \leq y \leq x$. Show that $y \in I_+$.
(ii) Show that I is linearly generated by I_+.

Exercise 2.11. Let I be a two-sided ideal in a von Neumann algebra M.

(i) We assume that $\overline{I}^{wo} = M$. Let p be a non-zero projection in M. Show that there is a non-zero projection $q \in I$ with $q \leq p$.
(ii) Show that $\overline{I}^{wo} = M$ if and only if there is an orthogonal family (q_i) of projections in I such that $\sum_i q_i = 1$.
(iii) Assume that $\overline{I}^{wo} = M$. Show that every $x \in M_+$ is the least upper bound of an increasing net of elements of I_+.

Exercise 2.12. Let (M, τ) be a tracial von Neumann algebra, acting on $L^2(M, \tau)$.

(i) Show that $x \mapsto x^*$ is s.o. continuous on the unit ball of M.
(ii) Let A be a $*$-subalgebra of M. Show that A is dense in M in the s.o. topology if and only if for every $x \in M$ with $\|x\| \leq 1$, there is a sequence (a_n) in the unit ball of A such that $\lim_n \|x - a_n\|_2 = 0$.

Exercise 2.13. Find a sequence (f_n) in $L^\infty([0,1])$ such that $\lim_n \|f_n\|_2 = 0$ while (f_n) does not converge to 0 in the s.o. topology.

Exercise 2.14. Let $\alpha : M \to N$ be an isomorphism between two von Neumann algebras and let $\mathcal{M}$ be a $*$-subalgebra of M. Show that $\mathcal{M}$ is s.o. dense in M if and only if $\alpha(\mathcal{M})$ is s.o. dense in N (thus, s.o. density of $\mathcal{M}$ is intrinsic).

Exercise 2.15. Let (M_i, τ_i), $i = 1, 2$, be two tracial von Neumann algebras and let $\mathcal{M}_i$ be a $*$-subalgebra s.o. dense in M_i. Let $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ be a $*$-isomorphism such that $\tau_2 \circ \alpha(x) = \tau_1(x)$ for every $x \in \mathcal{M}_1$. Show that there is a unitary operator $U : L^2(M_1, \tau_1) \to L^2(M_2, \tau_2)$ such that $UxU^* = \alpha(x)$ for every $x \in \mathcal{M}_1$ and therefore that α extends to an isomorphism from M_1 onto M_2.

Exercise 2.16. Let (M, τ) be a tracial von Neumann algebra. We denote by M^{op} the *opposite von Neumann algebra*: it is M as a vector space, the involution is the same, but the multiplication in M^{op} is defined by $x \cdot y = yx$. If M is a group von Neumann algebra or the von Neumann algebra of a countable p.m.p. equivalence relation, show that M is isomorphic to M^{op}.

The first example of a II_1 factor not anti-isomorphic to itself was found in [**Con75**]

Exercise 2.17. Let M_1, M_2 be two tracial von Neumann algebras such that there exist increasing sequences $(N_1^k)_{k\geq 1}$, $(N_2^k)_{k\geq 1}$ of matrix algebras, with $N_1^k \simeq N_2^k$ for every k and $\overline{\cup_k N_i^k}^{s.o.} = M_i$, $i = 1, 2$. Show that M_1 and M_2 are isomorphic II_1 factors.

Exercise 2.18. Let G be a group.

(i) Show that $L(G)$ is the unique (up to isomorphism) tracial von Neumann algebra (M, τ) generated by unitary elements $(u_g)_{g\in G}$ such that $u_g u_h = u_{gh}$ for all $g, h \in G$ and $\tau(u_g) = 0$ for all $g \neq e$.

(ii) Show that if H is a subgroup of G then $L(H)$ is canonically isomorphic to the von Neumann subalgebra of $L(G)$ generated by $\{u_h : h \in H\}$.

(iii) Let $G \curvearrowright (X, \mu)$ be a p.m.p. action of G. Show that $L(G)$ is canonically isomorphic to the von Neumann subalgebra of $L^\infty(X, \mu) \rtimes G$ generated by its canonical unitaries $\{u_g : g \in G\}$.

Notes

The content of this chapter is the outcome of advances due to von Neumann, Murray and von Neumann, Dixmier, Dye, Kaplansky and many others from 1929 up to the early fifties. The bicommutant theorem 2.1.3 is one of the main results of the pioneering paper [**vN30**] of von Neumann on rings of operators. The Kaplansky density theorem is proved in [**Kap51**]. Most of the results about projections are included in [**MVN36**]. Theorem 2.5.5 is due to Dixmier [**Dix53**] where the reader will also find the major part of our sections 2.5 and 2.6. For these facts, we also refer to Dye's paper [**Dye52**].

CHAPTER 3

Abelian von Neumann algebras

As we will see in this chapter, abelian von Neumann algebras are well understood, and this subject is nothing but a part of classical measure theory. Of particular importance are the abelian von Neumann algebras acting on a separable Hilbert space. In this chapter *we only consider such algebras*, since the theory is simpler in this case, and covers most of the interesting applications.

A nice fact is that there exists a unique diffuse separable abelian von Neumann algebra, up to isomorphism (Theorem 3.2.4).

3.1. Maximal abelian von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$

Let M be a von Neumann algebra on $\mathcal{H}$. Recall that a vector $\xi \in \mathcal{H}$ is *cyclic* for M if $\overline{M\xi} = \mathcal{H}$. We say that ξ is *separating* for M if, for $x \in M$, we have $x\xi = 0$ if and only $x = 0$.

LEMMA 3.1.1. *A vector $\xi \in \mathcal{H}$ is cyclic for M if and only if it is separating for M'.*

PROOF. Obviously, if ξ is cyclic for M it is separating for M'. Conversely, assume that ξ is separating for M'. The orthogonal projection p on $\overline{M\xi}$ is in M'. Since $(1-p)\xi = 0$, we conclude that $p = 1$. □

Using a maximality argument, $\mathcal{H}$ can be written as a Hilbert sum $\mathcal{H} = \oplus_{i\in I}\overline{M\xi_i}$ of subspaces which are cyclic for M. Moreover, I is countable whenever $\mathcal{H}$ is assumed to be separable.

PROPOSITION 3.1.2. *Let A be an abelian von Neumann algebra on a separable Hilbert space $\mathcal{H}$. There exists a cyclic vector for A', hence a separating vector for A.*

PROOF. We write $\mathcal{H} = \oplus_{n\geq 1}\overline{A'\xi_n}$ where the vectors ξ_n have norm-one and we set $\xi = \sum_{n\geq 1}\frac{1}{2^n}\xi_n$. Let $p_n \in A \subset A'$ be the orthogonal projection onto $\overline{A'\xi_n}$. We have $A'\xi_n = 2^n A' p_n \xi \subset A'\xi$ for every n, whence $\overline{A'\xi} = \mathcal{H}$. □

PROPOSITION 3.1.3. *Every abelian von Neumann algebra A on a separable Hilbert space is generated by a self-adjoint operator.*

PROOF. Since the unit ball of A equipped with the w.o. topology is compact and metrizable, it has a countable dense subset. Therefore there is a countable family $\{a_n : n \geq 1\}$ of self-adjoint operators in A whose linear span is w.o. dense in A. For each a_n there is a countable subset P_n of its spectral projections such that a_n belongs to the norm closure of the linear span of P_n (see Corollary 2.2.3). It follows that one may find a countable set $\{e_n : n \geq 1\}$ in $\mathcal{P}(A)$ whose linear span is w.o. dense in A. Let B be the C^*-subalgebra generated by 1 and the projections e_n, $n \geq 1$. Since B is w.o. dense in A, it suffices to show that B is generated, as a C^*-algebra, by a single self-adjoint operator.

The Gelfand transform identifies B with the C^*-algebra $C(X)$ of continuous functions on the compact spectrum X of B, and each e_n with the characteristic function of a closed and open subset E_n of X. We remark that, since $\{e_n : n \geq 1\}$ generates B, for every pair of distinct points in X there exists n such that E_n contains one of these points but not the other. We set $F_{2n} = E_n$, $F_{2n+1} = X \setminus E_n$ and

$$f = \sum_{n \geq 1} \frac{1}{2^n}(\mathbf{1}_{F_n} - 1).$$

To show that f generates B, it suffices to prove that f separates the points of X and then apply the Stone-Weierstrass theorem. Let $s \neq t$ be two distinct points in X. Let n_0 be the largest integer such that for $n < n_0$ both points either are in F_n or in $X \setminus F_n$, and assume for instance that $s \in F_{n_0}$ and $t \notin F_{n_0}$. We have

$$f(s) - f(t) = \frac{1}{2^{n_0}} + \sum_{n > n_0} \frac{1}{2^n}(\mathbf{1}_{F_n}(s) - \mathbf{1}_{F_n}(t)) \neq 0. \qquad \square$$

Theorem 3.1.4. *Let A be an abelian von Neumann algebra on a separable Hilbert space $\mathcal{H}$. The following conditions are equivalent:*

(i) *$A = A'$, i.e., A is a maximal abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$;*
(ii) *A has a cyclic vector;*
(iii) *there exist a compact metric space X, a probability measure μ on X and a unitary operator $U : L^2(X, \mu) \to \mathcal{H}$ such that $A = UL^\infty(X, \mu)U^*$ (where $L^\infty(X, \mu)$ is viewed as a von Neumann subalgebra of $\mathcal{B}(L^2(X, \mu))$, as in Proposition 1.2.1).*

Proof. (i) $\Rightarrow$ (ii) is an immediate consequence of Proposition 3.1.2.

(ii) $\Rightarrow$ (iii). Let ξ be a cyclic vector for A with $\|\xi\| = 1$. Let x be a self-adjoint operator which generates A and let E be the spectral measure of x. We denote by μ_ξ $(= \mu_{\xi,\xi})$ the probability measure $\Omega \mapsto \langle \xi, E(\Omega)\xi \rangle$ on the Borel subsets of the spectrum X of x. Let $f \in B_b(X)$ be a Borel bounded function on X. We have $\|f(x)\xi\| = \|f\|_{L^2(X,\mu_\xi)}$, so that the map $f \mapsto f(x)\xi$ extends to an isometry U from $L^2(X, \mu_\xi)$ into $\mathcal{H}$. This isometry is surjective since ξ is cyclic for A. A straightforward computation shows that $f(x) = UM_fU^*$ for every $f \in L^\infty(X, \mu_\xi)$, where M_f is the multiplication operator by f. In particular, $\Phi : M_f \mapsto f(x)$ is an isometric w.o. continuous homomorphism from $L^\infty(X, \mu_\xi)$ into A. Since $L^\infty(X, \mu_\xi)$ is a maximal abelian subalgebra of $\mathcal{B}(L^2(X,\mu_\xi))$ and since Φ is a spatial homomorphism, $\Phi(L^\infty(X,\mu_\xi))$ is a maximal abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$, whence $\Phi(L^\infty(X,\mu_\xi)) = A$.

(iii) $\Rightarrow$ (i) is proved in Proposition 1.2.1. $\square$

Note that the space X may be taken as a compact subset of $\mathbb{R}$.

Remark 3.1.5. Let $x \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator. Let $\xi \in \mathcal{H}$ be a cyclic vector for x, *i.e.*, such that the set $\{x^n\xi : n \in \mathbb{N}\}$ is total. The proof of the previous theorem includes the classical *spectral theorem*: there exists a unitary operator U from $\mathcal{H}$ onto $L^2(\mathrm{Sp}(x),\mu_\xi)$ such the UxU^* is the multiplication operator by the function $t \in \mathrm{Sp}(x) \mapsto t$.

3.2. Classification up to isomorphisms

We have seen in the proof of the previous theorem that if an abelian von Neumann algebra A on a separable Hilbert space $\mathcal{H}$ has a cyclic vector ξ, then it is spatially isomorphic to $L^\infty(X,\mu_\xi)$ acting by multiplication on $L^2(X,\mu_\xi)$. If ξ is only separating, the next theorem shows that A is still isomorphic to $L^\infty(X,\mu_\xi)$, but the isomorphism needs not be spatial.

Theorem 3.2.1. *Let A be an abelian von Neumann algebra on $\mathcal{H}$. Let x be a self-adjoint operator generating A and set $X = \mathrm{Sp}(x)$. We choose a separating vector ξ and denote by $\mu = \mu_\xi$ the spectral measure on X associated with ξ. Then the Gelfand map $f \mapsto f(x)$ extends uniquely to an isomorphism from $L^\infty(X,\mu)$ onto A.*

Proof. Let $\Phi : B_b(X) \to A$ be the $*$-homomorphism defined by the bounded Borel functional calculus. For $f \in B_b(X)$, we have $\|f\|_{L^2(X,\mu)} = \|f(x)\xi\|$. It follows that $f(x) = 0$ if and only if $f = 0$, *a.e.* Therefore, Φ defines an injective homomorphism from $L^\infty(X,\mu)$ into A.[1] In particular, Φ is an isometry.

Since $\langle \xi, \Phi(f)\xi\rangle = \int_X f \,\mathrm{d}\mu$, we deduce from Proposition 2.5.11 that Φ is normal. Then, Proposition 2.5.12 tells us that $\Phi(L^\infty(X,\mu))$ is a von Neumann algebra. Now, since $\Phi(L^\infty(X,\mu))$ contains x which generates A as a von Neumann algebra, we see that $\Phi(L^\infty(X,\mu)) = A$.

The uniqueness of Φ follows from the fact that the unit ball of $C(X)$ is weak* dense (or equivalently, w.o. dense by Remark 1.2.2) in the unit ball of $L^\infty(X,\mu)$, combined with the continuity of Φ on the unit ball relative to the w.o. topologies. $\square$

Corollary 3.2.2. *Let A be an abelian von Neumann algebra on a separable Hilbert space and let τ be a normal faithful state on A. There exist a probability measure μ on a compact subset X of $\mathbb{R}$ and an isomorphism α from A onto $L^\infty(X,\mu)$ such that $\tau_\mu \circ \alpha = \tau$.*

[1] It follows that the class of μ is independent of the choice of the separating vector ξ.

Proof. By the previous theorem and Proposition 3.1.2, there exists an isomorphism $\alpha : A \to L^\infty(X,\nu)$ where ν is a probability measure on some compact subset X of $\mathbb{R}$. For every Borel subset E of X, we set $\mu(E) = \tau \circ \alpha^{-1}(\mathbf{1}_E)$. In this way, we get a probability measure on X, which is equivalent to ν since τ is faithful. It follows that $L^\infty(X,\nu) = L^\infty(X,\mu)$ and $\tau_\mu \circ \alpha = \tau$. $\square$

Remark 3.2.3. It is not difficult to see that if μ_1 and μ_2 are two probability measures on X such that there exists an isomorphism from $L^\infty(X,\mu_1)$ onto $L^\infty(X,\mu_2)$ which is the identity on the subalgebra $C(X)$, then μ_1 is equivalent to μ_2, and the isomorphism is the identity map of $L^\infty(X,\mu_1) = L^\infty(X,\mu_2)$ (see [**Dou98**, Theorem 4.55]).

We can go further in the classification of abelian von Neumann algebras. We refer to the appendix B for the results used below. Every probability measure μ on X can be uniquely written as $\mu = \mu_c + \mu_d$ where μ_c is continuous and μ_d is discrete. Therefore $L^\infty(X,\mu)$ is isomorphic to the product of $L^\infty(X \setminus T, \mu_c)$ by $L^\infty(T,\mu_d)$ where T is the support (necessarily countable) of μ_d. Of course, $L^\infty(T,\mu_d)$ is isomorphic to the algebra ℓ^∞_n of bounded sequences indexed by a set of cardinality n equal to the cardinality $|T|$ of T. So it remains to consider the case where μ is continuous.

Recall that a von Neumann algebra A is *diffuse* if for any non-zero projection $p \in A$, there is a non-zero projection $q \in A$ with $q \leq p$ and $q \neq p$. When A is isomorphic to $L^\infty(X,\mu)$, this is equivalent to the continuity of μ.

Theorem 3.2.4. *Any diffuse abelian von Neumann algebra A on a separable Hilbert space is isomorphic to $L^\infty([0,1],\lambda)$, where λ is the Lebesgue measure on $[0,1]$. Moreover, if a faithful normal state τ is given on A, we may choose the isomorphism α such that $\tau = \tau_\lambda \circ \alpha : a \mapsto \int_0^1 \alpha(a)\,\mathrm{d}\lambda$.*

Proof. We apply Corollary 3.2.2 and observe that μ is a continuous probability measure since A is diffuse. Since (X,μ) is a standard probability measure space, the conclusion follows from Theorem B.7 in the appendix. $\square$

3.3. Automorphisms of abelian von Neumann algebras

We study in this section the pointwise realization of isomorphisms between separable abelian von Neumann algebras, when viewed as algebras of bounded measurable functions. In the course of the proof of the next theorem, we will use the following observation: every isomorphism between two measure space algebras $L^\infty(X,\mu)$ and $L^\infty(Y,\nu)$ equipped with the weak* topologies (defined by the duality with the corresponding L^1-spaces) is continuous (see Remark 2.5.13). As already mentioned, these topologies are also the w.o. topologies relative to the representations of these algebras on the corresponding L^2-spaces.

Theorem 3.3.1. *Let (X, μ) be a standard probability measure space.*

(i) *Let θ be a Borel isomorphism between two co-null subsets of X, which preserves the measure class of μ. Then $f \mapsto f \circ \theta$ is an automorphism of $L^\infty(X, \mu)$.*

(ii) *Conversely, let α be an automorphism of $L^\infty(X, \mu)$. There exists a (unique, up to null sets) isomorphism θ between two co-null subsets of X which preserves the measure class of μ and is such that $\alpha(f) = f \circ \theta$ for every $f \in L^\infty(X, \mu)$.*

Proof. (i) is obvious. Let us show (ii) in the interesting case where μ is continuous. We are reduced to consider the case $(X, \mu) = ([0,1], \lambda)$. Let us denote by ι the function $t \mapsto t$ defined on $[0,1]$. We set $\theta = \alpha(\iota)$. Since α is positive, θ is a measurable function from $[0,1]$ into itself.

Let us show that $\int f \,\mathrm{d}\,\theta_*\lambda = \int \alpha(f) \,\mathrm{d}\lambda$ for every bounded Borel function f on $[0,1]$. We will use the weak* density of the unit ball of $C([0,1])$ into the unit ball of its bidual (which is the dual of the Banach space of bounded measures on $[0,1]$) and will identify the space $B_b([0,1])$ of Borel bounded functions on $[0,1]$ with a subspace of $C([0,1])^{**}$. We deduce from these observations that for every $f \in B_b([0,1])$ there exists a net $(g_i)_{i\in I}$ of continuous functions on $[0,1]$ such that $\|g_i\|_\infty \leq \|f\|_\infty$ for all i and $\lim_i \int g_i \,\mathrm{d}\nu = \int f \,\mathrm{d}\nu$ for every bounded measure ν on $[0,1]$. In particular, if we consider for ν the bounded measures that are absolutely continuous with respect to λ, we see that $\lim_i g_i = f$ in $L^\infty([0,1], \lambda)$ equipped with the weak* topology. Therefore we have $\lim_i \alpha(g_i) = \alpha(f)$ in the weak* topology.

Since $\alpha(\iota) = \iota \circ \theta$, it follows from the Stone-Weierstrass theorem that $\alpha(g) = g \circ \theta$ for every continuous function g on $[0,1]$, and thus

$$\int \alpha(g) \,\mathrm{d}\lambda = \int g \circ \theta \,\mathrm{d}\lambda = \int g \,\mathrm{d}\,\theta_*\lambda.$$

So, we get $\int \alpha(f) \,\mathrm{d}\lambda = \int f \,\mathrm{d}\theta_*\lambda$ for every $f \in B_b([0,1])$. In particular, taking f to be the characteristic function of a Borel subset E of $[0,1]$, we see that $\lambda(E) = 0$ if and only if $(\theta_*\lambda)(E) = 0$, since $\lambda(E) = 0$ if and only if $\alpha(\mathbf{1}_E) = 0$. Therefore, the measures λ and $\theta_*\lambda$ are equivalent.

Let $f \in L^\infty([0,1], \lambda)$ and let (g_i) be a bounded net of continuous functions on X such that $\lim_i g_i = f$ in the weak* topology, as above. Since $\alpha(g_i) = g_i \circ \theta$ for every i, we conclude that $\alpha(f) = f \circ \theta$.

Similarly, there is a measurable function ρ from $[0,1]$ into itself such that $\alpha^{-1}(f) = f \circ \rho$ for every $f \in L^\infty([0,1], \lambda)$. We have

$$\iota = \alpha^{-1} \circ \alpha(\iota) = \alpha^{-1}(\theta) = \rho \circ \theta$$

and similarly $\iota = \theta \circ \rho$. Therefore, θ is a Borel isomorphism between two co-null subsets of X . □

Remark 3.3.2. It follows that every isomorphism α from $L^\infty(X, \mu)$ onto $L^\infty(Y, \nu)$ is of the form $f \mapsto f \circ \theta$, where $\theta : Y \to X$ is a Borel isomorphim such that $\theta_*\nu$ is equivalent to μ.

Let τ_μ be the integral map $f \mapsto \int_X f \,\mathrm{d}\,\mu$ on $L^\infty(X,\mu)$ and denote by $\mathrm{Aut}\,(L^\infty(X,\mu),\tau_\mu)$ the group of automorphisms of $L^\infty(X,\mu)$ which preserve τ_μ. We recall that $\mathrm{Aut}\,(X,\mu)$ is the group of μ-preserving Borel automorphisms of X.

Corollary 3.3.3. *The map $\theta \mapsto \alpha_\theta$, where $\alpha_\theta(f) = f\circ\theta^{-1}$, is a group isomorphism from* $\mathrm{Aut}\,(X,\mu)$ *onto* $\mathrm{Aut}\,(L^\infty(X,\mu),\tau_\mu)$.

Proof. Immediate. □

In the same way we have:

Theorem 3.3.4. *Let (X,μ) and (Y,ν) be two standard probability measure spaces and $\alpha : L^\infty(X,\mu) \to L^\infty(Y,\nu)$ be a homomorphism such that $\int_Y \alpha(f)\,\mathrm{d}\nu = \int_X f\,\mathrm{d}\mu$ for every $f \in L^\infty(X,\mu)$. Then there is a unique (up to null sets) Borel map $\theta : Y \to X$ such that $\theta_*\nu = \mu$ and $\alpha(f) = f\circ\theta$ for every $f \in L^\infty(X,\mu)$. Moreover θ is onto, modulo a set of measure 0, and θ is an isomorphism if and only if α is a von Neumann algebra isomorphism.*

Proof. The ideas are the same as in the previous proof. The main points to mention are that, since α preserves the integrals, it is injective and, above all, it is continuous for the weak* topologies (see Proposition 2.5.11). □

Remark 3.3.5. Two separable abelian von Neumann algebras $A \simeq L^\infty(X,\mu)$ and $B \simeq L^\infty(Y,\nu)$ are thus isomorphic if and only there is a class measure preserving isomorphism between the spaces X and Y.

Now assume that A and B are represented on separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. Recall that a *spatial isomorphism* is an isomorphism $\alpha : A \to B$ of the form $a \mapsto UaU^*$ where $U : \mathcal{H} \to \mathcal{K}$ is a unitary operator. The classification of abelian von Neumann algebras, up to spatial isomorphism, involves, in addition to a measure class, a *multiplicity invariant* as we will see in Chapter 8.

Exercises

Exercise 3.1. Let $x \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator. Show that there exist a probability measure space (X,μ), a unitary operator $U : \mathcal{H} \to L^2(X,\mu)$ and a bounded real-valued function f on X such that $UxU^* = M_f$.

Exercise 3.2. Let A be a separable abelian diffuse von Neumann algebra and τ be a normal faithful state on A.

(i) Show that there is an increasing family $(p_t)_{t\in[0,1]}$ of projections in A with $\tau(p_t) = t$ for every t.
(ii) Show that there is a unitary operator u in A with $\tau(u^n) = 0$ for every $n \neq 0$ and such that $\lim_{n\to+\infty} u^n = 0$ in the w.o. topology.

Exercise 3.3. Let (M,τ) be a diffuse tracial von Neumann algebra (for instance a II_1 factor).

(i) Show that every maximal abelian subalgebra A of M is diffuse.

(ii) Assuming moreover that M is separable, show that there exists a family $(p_t)_{0\leq t\leq 1}$ of projections in M with $p_s < p_t$ for $s < t$ and $\tau(p_t) = t$ for every t.

Notes

The main results of this chapter are due to Halmos and von Neumann [**HvN42**] and are continuations of earlier works due to von Neumann [**vN32a, vN32b**].

CHAPTER 4

II_1 factors. Some basics

Among the tracial von Neumann algebras, II_1 factors are at the opposite of abelian von Neumann algebras. We show that they are simple with a unique tracial state.

In the second section, we introduce a first invariant for these factors, their fundamental group.

4.1. Uniqueness of the trace and simplicity

Given an abelian von Neumann algebra $L^\infty(X,\mu)$, the functional $\tau_\mu : f \mapsto \int_X f \,\mathrm{d}\mu$ is a faithful normal tracial state. Of course, in this situation, it is easy to construct many other such traces. We also observe that the w.o. closed ideals of $L^\infty(X,\mu)$ are in bijective correspondence with the measurable subsets of X (up to null sets).

Let us now consider the case of a tracial von Neumann factor[1]. Recall that such a factor is either isomorphic to some matrix algebra $M_n(\mathbb{C})$ or is of type II_1, depending on its dimension[2] or, equivalently, depending on whether or not it has a minimal projection (see Corollary 2.4.14). It is a classical result of linear algebra that $M_n(\mathbb{C})$ has only one tracial state and is simple, *i.e.*, has no non-trivial two-sided ideal. We now prove that these facts hold for any tracial factor.

We need a preliminary lemma.

Lemma 4.1.1. *Let M be a diffuse factor and let $p \neq 0$ be a projection in M. There exist two projections $p_1, p_2 \in M$ with $p_1 \sim p_2$ and $p_1 + p_2 = p$.*

Proof. We first claim that for any non-zero projection e in M there exist two non-zero equivalent orthogonal projections e_1, e_2 with $e_1 + e_2 \leq e$. Indeed, since e is not minimal, there exists $f \in \mathcal{P}(M)$ with $f \leq e$, $f \neq 0$ and $f \neq e$. We have $z(f)z(e-f) = 1$ because M is a factor and our claim follows from Lemma 2.4.7.

Now, we consider the set $\mathcal{F}$ of families $(p_i, q_i)_{i\in I}$ of pairs of equivalent projections, majorized by p and such that $\{p_i, q_i : i \in I\}$ are mutually orthogonal projections. Let $(p_i, q_i)_{i\in I}$ be a maximal family in $\mathcal{F}$ and put $p_1 = \sum_{i\in I} p_i$, $p_2 = \sum_{i\in I} q_i$. Then p_1 and p_2 are equivalent. Moreover, using the maximality of the family, and applying the first part of the proof to $p - (p_1 + p_2)$ if this projection is non-zero, we see that $p_1 + p_2 = p$. $\square$

[1]Later, such factors will be called finite factors (see Chapter 6).

[2]Recall that a II_1 factor is an infinite dimensional, tracial factor.

COROLLARY 4.1.2. *Let M be a diffuse factor. For every integer $n \geq 1$ there are mutually orthogonal and equivalent projections $p_1, \dots, p_{2^n}$ such that $\sum_{i=1}^{2^n} p_i = 1$. In particular, if M carries a tracial state τ, we have $\tau(p_i) = 2^{-n}$.*

PROOF. Obvious. □

PROPOSITION 4.1.3. *A von Neumann factor M has at most one tracial state*[3].

PROOF. It is enough to consider the case where M is diffuse. Let $q \in \mathcal{P}(M)$, $q \neq 1$, and consider $p_1, \dots, p_{2^n}$ as in the previous corollary. Thanks to the comparison result 2.4.9, we see that there is a unique integer k such that

$$\sum_{i \leq k} p_i \precsim q \prec \sum_{i \leq k+1} p_i.$$

It follows that for every tracial state τ on M we have

$$\frac{k}{2^n} \leq \tau(q) < \frac{k+1}{2^n}.$$

Therefore, the real number $\tau(q)$ does not depend on the choice of τ. This prove the uniqueness of τ, because the linear span of $\mathcal{P}(M)$ is dense in M with respect to the norm topology (see Corollary 2.2.2). □

PROPOSITION 4.1.4. *Let (M, τ) be a tracial von Neumann algebra. Then M is a factor if and only if τ is the unique normal faithful tracial state on M.*

PROOF. The uniqueness when M is a factor is proved in the previous proposition. Now, assume that M is not a factor and let z be a non-trivial central projection. Let α be any number in $]0,1[$ with $\alpha \neq \tau(z)$ Then $\widetilde{\tau}$ defined on M by

$$\widetilde{\tau}(x) = \frac{\alpha}{\tau(z)}\tau(xz) + \frac{1-\alpha}{1-\tau(z)}\tau(x(1-z))$$

is a normal faithful tracial state with $\widetilde{\tau} \neq \tau$. □

PROPOSITION 4.1.5. *A tracial factor (M, τ) contains no non-trivial two-sided ideal.*

PROOF. Let $I \neq 0$ be a two-sided ideal and let x be a non-zero positive element in I. We take $t > 0$ small enough so that the spectral projection e of x relative to $[t, +\infty[$ is non-zero. We have $e \in I$ (see Proposition 2.4.15). Since the normal tracial state τ on M is faithful we have $\tau(e) \neq 0$, and thus any maximal family of mutually orthogonal projections in M, all equivalent to e, is finite. Therefore, we can find mutually orthogonal projections $p_1 = e, p_2, \dots, p_k$ with $1 = \sum_{i=1}^k p_i$, $p_i \sim e$ for $i < k$ and $p_k \precsim e$. There exist partial isometries $u_1, u_2, \dots, u_k$ in M with $u_i = u_i e$ and $u_i u_i^* = p_i$ for $i = 1, \dots, k$. It follows that $p_i \in I$ for all i, whence $1 \in I$ and $I = M$. □

[3] As already said, it is also automatically normal and faithful, but this is much more difficult to show (see Theorem 6.3.5).

We have seen in Corollary 2.4.11 that any two projections of a tracial factor (M, τ) are equivalent if and only if they have the same trace. If M is isomorphic to $M_n(\mathbb{C})$, then $\tau(\mathcal{P}(M)) = \{0, 1, \dots, n\}$. For II_1 factors, we have:

Proposition 4.1.6. *Let M be a II_1 factor and τ its tracial normal state. Then $p \mapsto \tau(p)$ induces a bijection from the set of equivalence classes of projections in M onto $[0,1]$.*

Proof. We only need to show that for that for every $t \in]0,1[$ there is a projection $p \in \mathcal{P}(M)$ with $\tau(p) = t$. Let $t = \sum_k 2^{-n_k}$ be the dyadic expansion of t. Using the comparison theorem of projections in a factor and the fact that M has projections of trace 2^{-n} for every n since it is diffuse, we construct by induction a sequence of mutually orthogonal projections $p_1, p_2, \dots, p_k, \dots$ such that $\tau(p_k) = 2^{-n_k}$ for every k. We set $p = \sum_k p_k$. Since τ is normal, we get $\tau(p) = t$. $\square$

The number $\tau(p)$ is viewed as the "dimension of p". It is a very important feature of II_1 factors that their projections have a continuum of dimensions.

4.2. The fundamental group of a II_1 factor

Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Given a projection $p \in M$, we have already introduced the reduced von Neumann algebra $pMp = \{pxp : x \in M\}$. If q is a projection in M' then Mq is a von Neumann algebra on $q\mathcal{H}$, since $x \mapsto xq$ is a normal representation of M on $q\mathcal{H}$ (see Proposition 2.5.12). It is called the *induced von Neumann algebra* of M with respect to q. When M is a factor, then pMp and Mq are factors, as a consequence of the following facts.

Proposition 4.2.1. *Let M and p as above. Let e be a projection in $\mathcal{Z}(pMp)$. Then $e = z(e)p$ where $z(e)$ is the central support of e in M. It follows that $\mathcal{Z}(pMp) = \mathcal{Z}(M)p$.*

Proof. We have $(p-e)Me = (p-e)pMpe = 0$ and therefore

$$(p-e)ueu^* = 0$$

for every $u \in \mathcal{U}(M)$. It follows, by Lemma 2.4.6, that $(p-e)z(e) = 0$, hence $e = z(e)p$.

The inclusion $\mathcal{Z}(pMp) \subset \mathcal{Z}(M)p$ is then a consequence of Corollary 2.2.3. The opposite inclusion $\mathcal{Z}(M)p \subset \mathcal{Z}(pMp)$ is obvious. $\square$

Proposition 4.2.2. *Let M be a von Neumann on a Hilbert space $\mathcal{H}$.*

(i) *Let p be a projection in M. Then the commutant $(pMp)'$ of pMp in $\mathcal{B}(p\mathcal{H})$ is pM' .*
(ii) *Let q be a projection in M'. Then the commutant $(Mq)'$ of Mq in $\mathcal{B}(q\mathcal{H})$ is $qM'q$.*

PROOF. (i) We have obviously $pMp \subset (M'p)'$. Let $x \in (M'p)' \subset \mathcal{B}(p\mathcal{H})$, and set $\tilde{x} = xp = pxp \in \mathcal{B}(\mathcal{H})$. For $y \in M'$, we have

$$y\tilde{x} = ypxp = (xp)(yp) = xpy = \tilde{x}y$$

and so $\tilde{x} \in M$. It follows that $x = p\tilde{x}p \in pMp$ and therefore we have $(M'p)' \subset pMp$. The bicommutant theorem 2.1.3 gives $(pMp)' = M'p$.

(ii) is a consequence of (i), after replacing M by M' and p by q. $\square$

For every integer $n \geq 1$, we may as well enlarge the Hilbert space $\mathcal{H}$ and introduce the algebra $M_n(M)$ of $n \times n$ matrices with entries in M, acting on $\mathcal{H}^{\oplus n}$. A routine proof shows that $M_n(M)$ is a von Neumann algebra, whose commutant is the algebra of diagonal matrices with equal diagonal entries in M'. Writing $\mathcal{H}^{\oplus n}$ as the Hilbert tensor product $\mathbb{C}^n \otimes \mathcal{H}$, the algebra $M_n(M)$ appears as the algebraic tensor product $M_n(\mathbb{C}) \otimes M$. Embedding $M_n(M)$ into $M_{n+1}(M)$ (as $(n+1) \times (n+1)$ matrices with coefficients 0 placed in the last line and the last column), we introduce the $*$-algebra

$$\mathcal{M}(M) = \cup_{n \geq 1} M_n(M).$$

We may view the elements of $\mathcal{M}(M)$ as matrices $[m_{i,j}]_{i,j \geq 1}$ such that there exists n with $m_{i,j} = 0$ whenever $i > n$ or $j > n$. This algebra acts on $\mathcal{H}^{\oplus \infty} = \ell^2(\mathbb{N}^*) \otimes \mathcal{H}$ in an obvious way. It is not w.o. closed. Its closure is the von Neumann tensor product $\mathcal{B}(\ell^2(\mathbb{N}^*)) \overline{\otimes} M$ to be defined in the next chapter.

From now on in this section, M will be a II_1 factor and, as usual, τ is its trace. We observe first that each $M_n(M)$ is a II_1 factor. We denote by $\mathrm{Tr}_n \otimes \tau$ its (non-normalized) trace defined by $(\mathrm{Tr}_n \otimes \tau)([x_{i,j}]) = \sum_i \tau(x_{i,i})$ and by $\mathrm{Tr} \otimes \tau$ the trace on $\mathcal{M}(M)$ whose restriction to $M_n(M)$ is $\mathrm{Tr}_n \otimes \tau$ for every n. Since any two projections $p, q \in \mathcal{M}(M)$ belong to some $M_n(M)$, we see that there exists $u \in \mathcal{M}(M)$ such that $u^*u = p$ and $uu^* = q$ if and only if $(\mathrm{Tr} \otimes \tau)(p) = (\mathrm{Tr} \otimes \tau)(q)$. It follows that the spatial isomorphism class of $p\mathcal{M}(M)p = pM_n(M)p$ only depends on the real number $t = (\mathrm{Tr} \otimes \tau)(p)$. We set $M^t = pM_n(M)p$, which is, as such, well defined up to isomorphism. Usually, $M_n(\mathbb{C}) \otimes M$ is called an amplification of M, and so, more generally, we will say that any M^t is an *amplification* of M. Moreover, since $\{(\mathrm{Tr} \otimes \tau)(p) : p \text{ projection} \in M_n(M)\} = [0, n]$ for every n, we see that M^t is defined for every $t > 0$.

Given two von Neumann algebras M and N, recall that we write $M \simeq N$ whenever they are isomorphic.

LEMMA 4.2.3. *Let M be a II_1 factor and s, t be two real numbers > 0. Then $(M^s)^t \simeq M^{st}$.*

PROOF. We take $M^s = pM_m(M)p$ with

$$p \in \mathcal{P}(M_m(M)), \quad (\mathrm{Tr}_m \otimes \tau)(p) = s,$$

and $(M^s)^t = q(M_n(M^s))q$ with

$$q \in \mathcal{P}(M_n(M^s)), \quad (\mathrm{Tr}_n \otimes \tau_s)(q) = t,$$

where $\tau_s = (1/s)(\mathrm{Tr}_m \otimes \tau)_{|M^s}$. We view q as a projection in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \otimes M$ smaller than $1_{M_n(\mathbb{C})} \otimes p$. Then $(M^s)^t = qM_{nm}(M)q$ with $(\mathrm{Tr}_{nm} \otimes \tau)(q) = st$. $\square$

Definition 4.2.4. Let M be a II_1 factor. We denote by $\mathfrak{F}(M)$ the subset of $\mathbb{R}_+^*$ formed of the positive real numbers t such that $M^t \simeq M$. The previous lemma shows that $\mathfrak{F}(M)$ is a subgroup of $\mathbb{R}_+^*$. It is called the *fundamental group* of M.

It is immediate that $\mathfrak{F}(M)$ is the set of $\tau(p)/\tau(q)$, where p and q run over the non-zero projections in M such that pMp and qMq are isomorphic. The computation of this invariant (up to isomorphism) is one of the major problems in the theory of II_1 factors.

The next proposition shows that $M \simeq M_n(\mathbb{C}) \otimes M^{1/n}$.

Proposition 4.2.5. *Let M be a II_1 factor and let $p \in \mathcal{P}(M)$ with $\tau(p) = 1/n$. Then M is isomorphic to $M_n(pMp) \simeq M_n(\mathbb{C}) \otimes (pMp)$.*

Proof. Using the comparison theorem of projections, we find mutually orthogonal and equivalent projections $p_1, p_2, \dots, p_n$ with $p_1 = p$ and $\sum_{i=1}^n p_i = 1$. Let u_i, $i = 1, \dots, n$, be partial isometries such that $u_i^* u_i = p_1$ and $u_i u_i^* = p_i$. Then

$$x \mapsto [u_i^* x u_j]_{1 \leq i,j \leq n}$$

is an isomorphism from M onto $M_n(pMp)$. Note that $(u_i u_j^*)_{1 \leq i,j \leq n}$ is a set of matrix units in M. $\square$

Proposition 4.2.6. *The hyperfinite II_1 factor R can be embedded as a von Neumann subfactor of any II_1 factor M.*

Proof. Using the previous proposition we construct an increasing sequence (Q_n) of subalgebras of M such that Q_n is isomorphic to $M_{2^n}(\mathbb{C})$ for every n. Then the s.o. closure of $\cup_{n \geq 1} Q_n$ is isomorphic to R (Exercise 2.17). $\square$

Notes

The fundamental group is one of the three invariants introduced by Murray and von Neumann in [**MvN43**] in order to distinguish between II_1 factors. They proved that the fundamental group of the hyperfinite factor R is $\mathbb{R}_+^*$ (see Remark 11.2.3) but the existence of II_1 factors with fundamental group distinct from $\mathbb{R}_+^*$ was only established in 1980 by Connes [**Con80a**] (see Section 14.3). It is only in 2001 (results published in [**Pop06a**]) that the first explicit computations were achieved, providing examples with fundamental groups reduced to $\{1\}$ (see Chapter 18). Notice that such examples M are not isomorphic to $M_n(M)$ for any integer $n \geq 2$.

CHAPTER 5

More examples

We have now the sufficient background to introduce new constructions of tracial von Neumann algebras, and in particular II_1 factors: tensor products, general crossed products, free products and ultraproducts.

We will need later to have some basic knowledge of the structure of tracial von Neumann algebras beyond the now familiar case of abelian ones and factors. In the last section of this chapter we provide elementary informations on this subject and examples.

5.1. Tensor products

Given two abelian von Neumann algebras $L^\infty(X_i, \mu_i)$, $i = 1, 2$, the classical notion of product in measure theory gives rise to the abelian von Neumann algebra $L^\infty(X_1 \times X_2, \mu_1 \times \mu_2)$. This construction is extended to the general setting of von Neumann algebras in the following way.

5.1.1. Tensor product of two von Neumann algebras. Let $(M_1, \mathcal{H}_1)$ and $(M_2, \mathcal{H}_2)$ be two von Neumann algebras. The algebraic tensor product $M_1 \odot M_2$ of M_1 and M_2 acts on the Hilbert tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ as follows:

$$\forall x_i \in M_i, \forall \xi_i \in \mathcal{H}_i, i = 1, 2, \quad (x_1 \otimes x_2)(\xi_1 \otimes \xi_2) = (x_1\xi_1) \otimes (x_2\xi_2).$$

The s.o. closure of $M_1 \odot M_2$ is denoted $M_1 \overline{\otimes} M_2$ and $(M_1 \overline{\otimes} M_2, \mathcal{H}_1 \otimes \mathcal{H}_2)$ is called the *von Neumann tensor product* of $(M_1, \mathcal{H}_1)$ by $(M_2, \mathcal{H}_2)$.

One may wonder how the von Neumann tensor product $M_1 \overline{\otimes} M_2$ depends on the given spatial representations. In fact, it is intrinsic, up to isomorphism (see Exercise 8.14 for the case of II_1 factors).

Examples 5.1.1. **(a)** Starting from $(M_i, \mathcal{H}_i) = (L^\infty(X_i, \mu_i), L^2(X_i, \mu_i))$, $i = 1, 2$, one gets

$$(M_1 \overline{\otimes} M_2, \mathcal{H}_1 \otimes \mathcal{H}_2) = (L^\infty(X_1 \times X_2, \mu_1 \times \mu_2), L^2(X_1 \times X_2, \mu_1 \times \mu_2)).$$

(b) We take $M_1 = \mathcal{B}(\ell^2(\mathbb{N}))$ and $M_2 = M$ acting on $\mathcal{H}$. Then the von Neumann tensor product $\mathcal{B}(\ell^2(\mathbb{N})) \overline{\otimes} M$ acts on $\ell^2(\mathbb{N}) \otimes \mathcal{H} = \mathcal{H}^{\oplus\infty}$. We denote by $u_i : \mathcal{H} \to \mathcal{H}^{\oplus\infty}$ the isometry sending $\xi \in \mathcal{H}$ onto the sequence (ξ_n) with $\xi_n = 0$ for all n but $n = i$ where $\xi_i = \xi$. Any bounded operator T on $\mathcal{H}^{\oplus\infty}$ may be written as the infinite matrix $[T_{i,j}]_{i,j\in\mathbb{N}}$ with $T_{i,j} = u_i^* T u_j \in \mathcal{B}(\mathcal{H})$. The set N of all bounded operators with entries in M is w.o. closed because $T \mapsto T_{i,j}$ is continuous with respect to the w.o. topology. A decomposable operator $[t_{i,j}] \otimes x \in \mathcal{B}(\ell^2(\mathbb{N})) \odot M$ is identified to the matrix $[t_{i,j}x]$ and so belongs to N. Clearly, the $*$-algebra $\mathcal{M}(M)$ consisting in the matrices $[T_{i,j}]$, with entries in M and such that there exists some integer n with $T_{i,j} = 0$ whenever $i > n$ or $j > n$, is contained in $\mathcal{B}(\ell^2(\mathbb{N})) \odot M$ and is s.o. dense in N. From these observations we deduce that $N = \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$.

Obviously, $\mathbb{N}$ can be replaced by any set I and so $\ell^2(\mathbb{N})$ can be replaced by any Hilbert space $\mathcal{K}$. Note in particular that $\mathcal{B}(\mathcal{K})\overline{\otimes}\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{K} \otimes \mathcal{H})$.

The simplest case is $I = \{1, \dots, n\}$. Then, for any von Neumann algebra M, the von Neumann tensor product $M_n(\mathbb{C})\overline{\otimes}M$ coincides with the algebraic tensor product (denoted $M_n(\mathbb{C}) \otimes M$ rather than $M_n(\mathbb{C}) \odot M$) and with the von Neumann algebra $M_n(M)$ of $n \times n$ matrices with entries in M.

Let $(M_1, \mathcal{H}_1)$ be a von Neumann algebra and $\mathcal{H}_2$ a Hilbert space. We leave as an exercise to check that $(M_1\overline{\otimes}\mathcal{B}(\mathcal{H}_2))' = M_1' \otimes \mathrm{Id}_{\mathcal{H}_2}$ and that $(M_1 \otimes \mathrm{Id}_{\mathcal{H}_2})' = M_1'\overline{\otimes}\mathcal{B}(\mathcal{H}_2)$.[1]

Clearly, for $M_1\overline{\otimes}M_2$ to be a factor, each component needs to be a factor. Conversely:

Proposition 5.1.2. *$(M_1\overline{\otimes}M_2, \mathcal{H}_1 \otimes \mathcal{H}_2)$ is a factor when $(M_1, \mathcal{H}_1)$ and $(M_2, \mathcal{H}_2)$ are factors.*

Proof. We claim that

$$\Big((M_1\overline{\otimes}M_2)' \cap (M_1\overline{\otimes}M_2)\Big)' = \mathcal{B}(\mathcal{H}_1)\overline{\otimes}\mathcal{B}(\mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

The left handside contains $M_1\overline{\otimes}\mathrm{Id}_{\mathcal{H}_2}$ and $M_1'\overline{\otimes}\mathrm{Id}_{\mathcal{H}_2}$ and so it contains

$$(M_1 \cup M_1')''\overline{\otimes}\mathrm{Id}_{\mathcal{H}_2} = \mathcal{B}(\mathcal{H}_1)\overline{\otimes}\mathrm{Id}_{\mathcal{H}_2}.$$

Similarly we see that it contains $\mathrm{Id}_{\mathcal{H}_1}\overline{\otimes}\mathcal{B}(\mathcal{H}_2)$.

Since $\mathcal{B}(\mathcal{H}_1)\overline{\otimes}\mathrm{Id}_{\mathcal{H}_2} \cup \mathrm{Id}_{\mathcal{H}_1}\overline{\otimes}\mathcal{B}(\mathcal{H}_2)$ generates $\mathcal{B}(\mathcal{H}_1)\overline{\otimes}\mathcal{B}(\mathcal{H}_2)$, our claim is proved. □

When (M_1, τ_1) and (M_2, τ_2) are two tracial von Neumann algebras, we implicitly consider them as represented on $L^2(M_1, \tau_1)$ and $L^2(M_2, \tau_2)$ respectively, in order to define their von Neumann tensor product.

Proposition 5.1.3. *With the above assumptions, $M_1\overline{\otimes}M_2$ is in a natural way a tracial algebra: it carries a unique tracial normal state τ such that $\tau(x_1 \otimes x_2) = \tau_1(x_1)\tau_2(x_2)$ for $x_1 \in M_1$, $x_2 \in M_2$, and this trace is faithful. Moreover the Hilbert spaces $L^2(M_1, \tau_1) \otimes L^2(M_2, \tau_2)$ and $L^2(M_1\overline{\otimes}M_2, \tau)$ are canonically isomorphic. We write $\tau = \tau_1 \otimes \tau_2$.*

When M_1, M_2 are tracial factors and at least one of them is of type II_1, *then $M_1\overline{\otimes}M_2$ is a* II_1 *factor.*

[1] More generally, given $(M_1, \mathcal{H}_1)$ and $(M_2, \mathcal{H}_2)$, it is true that $(M_1\overline{\otimes}M_2)' = M_1'\overline{\otimes}M_2'$. In this generality it is a deep result that was obtained in the 1960s, using Tomita's theory of modular Hilbert algebras (see [**Tak70**] for details and history and [**Tak02**, Theorem 5.9] for a simplified proof).

PROOF. The vector $1_{M_1} \otimes 1_{M_2} \in L^2(M_1, \tau_1) \otimes L^2(M_2, \tau_2)$ is cyclic for $M_1 \overline{\otimes} M_2$. This vector is also cyclic for the right action of $M_1 \odot M_2$ and therefore separating for $M_1 \overline{\otimes} M_2$. It defines the faithful tracial normal state of the above statement. The rest of the proposition is also obvious. $\square$

PROPOSITION 5.1.4. *The von Neumann tensor product (M, τ) of (M_1, τ_1) by (M_2, τ_2) is characterized, up to isomorphism, in the following way: it is the unique tracial von Neumann algebra (M, τ) containing $M_1 \odot M_2$ as a s.o. dense subalgebra, and such that $\tau(x_1 \otimes x_2) = \tau_1(x_1)\tau_2(x_2)$ for every $x_1 \in M_1$, $x_2 \in M_2$.*

PROOF. Let $(\widetilde{M}, \tilde{\tau})$ be another tracial von Neumann algebra with the same properties. Then there is a unitary operator

$$U : L^2(\widetilde{M}, \tilde{\tau}) \to L^2(M_1, \tau_1) \otimes L^2(M_2, \tau_2)$$

that induces a spatial isomorphism from $(\widetilde{M}, \tilde{\tau})$ onto $(M_1 \overline{\otimes} M_2, \tau_1 \otimes \tau_2)$ (see Exercise 2.15). $\square$

5.1.2. Infinite tensor products. The construction of Section 1.6 relative to infinite tensor products of matrix algebras is easily extended to the case of tracial von Neumann algebras. So let $(M_i, \tau_i)_{i \in \mathbb{N}}$ be a sequence of such algebras. We set $(N_k, \varphi_k) = (\overline{\otimes}_{i=0}^k M_i, \otimes_{i=1}^k \tau_i)$ and we embed N_k into N_{k+1} in the obvious way. Then $\mathcal{M} = \cup_{k \in \mathbb{N}} N_k$ is equipped with the unique trace τ such that $\tau(x) = \varphi_k(x)$ for any k such that $x \in N_k$. As in Section 1.6 we introduce the completion $\mathcal{H}$ of $\mathcal{M}$ with respect to the inner product $\langle x, y \rangle = \tau(x^* y)$ and we denote by π the corresponding representation of $\mathcal{M}$ on $\mathcal{H}$. It is obviously injective and the s.o. closure of $\pi(\mathcal{M})$ is written $\overline{\otimes}_{i \in \mathbb{N}} M_i$. Again, exactly as in Section 1.6, we see that τ extends in a unique way to a normal faithful tracial state on $\overline{\otimes}_{i \in \mathbb{N}} M_i$. The tracial von Neumann algebra $(\overline{\otimes}_{i \in \mathbb{N}} M_i, \tau)$ is called the *infinite tensor product* of $(M_i, \tau_i)_{i \in \mathbb{N}}$.

It remains to check that $\overline{\otimes}_{i \in \mathbb{N}} M_i$ is a factor whenever each component is so. To that purpose, we claim that τ is the unique normal faithful tracial state on $\overline{\otimes}_{i \in \mathbb{N}} M_i$ (see Proposition 4.1.4). This is an immediate consequence of the fact that each N_k is a factor and so its tracial state is unique (Proposition 4.1.3).

Given a finite subset F of $\mathbb{N}$, the von Neumann tensor product $\overline{\otimes}_{i \in F} M_i$, when viewed as a von Neumann subalgebra of $\overline{\otimes}_{i \in \mathbb{N}} M_i$, will be denoted $(\overline{\otimes}_{i \in F} M_i) \otimes \mathrm{Id}^{\otimes \mathbb{N} \setminus F}$.

5.2. Crossed products

In the first chapter, we introduced the group measure space von Neumann algebra associated with a probability measure preserving action $G \curvearrowright (X, \mu)$, or equivalently, a trace preserving action of the group G on $(L^\infty(X, \mu), \tau_\mu)$. This construction extends easily to the case of a trace preserving action of G on any tracial von Neumann algebra.

Let (B, τ) be a tracial von Neumann algebra and let $\mathrm{Aut}\,(B, \tau)$ be the group of automorphisms of B which preserve τ. We observe that every $\alpha \in \mathrm{Aut}\,(B, \tau)$ extends to a unitary operator of $L^2(B, \tau)$, still denoted α, such that

$$\forall x \in B, \quad \alpha(\hat{x}) = \widehat{\alpha(x)}.$$

A trace preserving action $G \curvearrowright (B, \tau)$ is a group homomorphism σ from G into $\mathrm{Aut}\,(B, \tau)$. The *crossed product* $B \rtimes G$ associated with this action is defined exactly as in Section 1.4. We introduce the algebra $B[G]$ of finitely supported formal sums

$$\sum_{g \in G} b_g u_g$$

with $b_g \in B$, the product and involution being defined by

$$(b_1 u_g)(b_2 u_h) = b_1 \sigma_g(b_2) u_{gh}, \quad (b u_g)^* = \sigma_{g^{-1}}(b^*) u_{g^{-1}}.$$

Of course, B will be identified with the subalgebra $B u_e$ of $B[G]$. We represent $B[G]$ in the Hilbert space

$$\mathcal{H} = L^2(B, \tau) \otimes \ell^2(G) = \ell^2(G, L^2(B, \tau))$$

by the formula

$$(b u_g)(\xi \otimes \delta_h) = (b \sigma_g(\xi)) \otimes \delta_{gh}.$$

Again, we find it convenient to write ξu_h instead of $\xi \otimes \delta_h \in \mathcal{H}$, so that the previous formula becomes

$$(b u_g)(\xi u_h) = b \sigma_g(\xi) u_{gh}.$$

The s.o. closure of $B[G]$ in $\mathcal{B}(\mathcal{H})$ is $B \rtimes G$, by definition. We may similarly let $B[G]$ act to the right on $\mathcal{H}$ by

$$(\xi u_h)(b u_g) = \xi \sigma_h(b) u_{hg}.$$

Let us state briefly the main properties of this construction, which are proved exactly as in the commutative case. The vector $u_e = \hat{1} \otimes \delta_e$ is cyclic and separating for $B \rtimes G$. Therefore, the map $x \mapsto x u_e$ identifies $B \rtimes G$ with a subspace of $L^2(B, \tau) \otimes \ell^2(G)$. So we write x under the form $\sum_{g \in G} x_g u_g \in L^2(B, \tau) \otimes \ell^2(G)$.[2]

The trace τ on B extends to a trace on $B \rtimes G$, which we will still denote by τ, by the formula

$$\tau(x) = \langle u_e, x u_e \rangle_{\mathcal{H}} = \tau(x_e) \quad \text{for} \quad x = \sum_{g \in G} x_g u_g \in B \rtimes G.$$

This trace is normal and faithful. Note that

$$\tau(x^* x) = \sum_{g \in G} \tau(x_g^* x_g) = \sum_{g \in G} \|x_g\|^2_{L^2(B,\tau)} = \|x\|^2_{\ell^2(G, L^2(B,\tau))}.$$

[2] The notation ξu_g for $\xi \otimes \delta_g$ is compatible with the inclusion of $B[G]$ into $L^2(B, \tau) \otimes \ell^2(G)$.

Again the x_g are called the *Fourier coefficients* of x and the u_g are the *canonical unitaries* of the crossed product. The convergence of the expansion $x = \sum_{g\in G} x_g u_g$ holds in $L^2(B,\tau)\otimes \ell^2(G)$ with its Hilbert norm.

We end this section with the definitions of the non-commutative analogues of freeness and ergodicity.

DEFINITION 5.2.1. Let B be a von Neumann algebra and α an automorphism of B. We say that α is *free* or *properly outer* if there is no element $y \in B$, other than 0, such that $y\alpha(x) = xy$ for every $x \in B$.

The reader will easily check that whenever B is a factor, α is properly outer if and only if it is outer (*i.e.*, not inner, that is, not of the form $x \mapsto uxu^*$ for some unitary operator $u \in B$).

DEFINITION 5.2.2. Let σ be a homomorphism from a group G into the group $\mathrm{Aut}\,(B)$ of automorphisms of a von Neumann algebra B. We say that the action σ is

(a) *ergodic* if $\mathbb{C}1_B = \{x \in B : \sigma_g(x) = x, \forall g \in G\}$.
(b) *free* or *properly outer* if for every $g \neq e$, the automorphism σ_g is properly outer.

Here is the non-commutative version of Proposition 1.4.5, whose proof is similar.

PROPOSITION 5.2.3. *Let (B,τ) be a tracial von Neumann algebra, and let $\sigma : G \curvearrowright (B,\tau)$ be a trace preserving action.*

(i) *$B' \cap (B \rtimes G) = \mathcal{Z}(B)$ if and only if the action is properly outer.*
(ii) *Assume that the action is properly outer. Then $B \rtimes G$ is a factor (and thus a II_1 factor) if and only if the action on the center of B is ergodic.*

EXAMPLE 5.2.4. Let G be a countable group and let (N,τ) be a tracial von Neumann algebra. Let $(B = \overline{\otimes}_{g\in G} N_g, \tau^{\otimes G})$ be the tensor product of copies of N indexed by G. The *Bernoulli action* on B is well defined by

$$(\sigma_g(x))_h = x_{gh}$$

for every $x = (\otimes_{g\in F} x_g) \otimes \mathrm{Id}^{\otimes G\setminus F}$, F finite subset of G.

This action is ergodic. Even more, it is *mixing*: for $x, y \in B$, we have $\lim_{g\to\infty} \tau(x\sigma_g(y)) = \tau(x)\tau(y)$. This is easily seen by approximating x, y by elements in some $(\overline{\otimes}_{g\in F} N_g) \otimes \mathrm{Id}^{\otimes G\setminus F}$ where F is a finite subset of G.

Moreover, for every $g \neq e$, the automorphism σ_g is properly outer. Indeed, let $b \in B$ with $\|b\|_2 = 1$ and $b\sigma_g(y) = yb$ for every $y \in B$. We fix a non-trivial projection p in N. Given $\varepsilon > 0$, there exists a finite subset F of G and $b' \in (\otimes_{k\in F} N_k) \otimes \mathrm{Id}^{\otimes G\setminus F}$ with $\|b - b'\|_2 < \varepsilon$. Then we have

$$\left\| b'\sigma_g(y) - yb' \right\|_2 \leq 2\varepsilon$$

for every y in the unit ball $(B)_1$ of B. Let $h \notin F \cup gF$, and let y be the element of B whose only non-trivial component is p in the position h. Then we have

$$4\varepsilon^2 \geq \left\|b'\sigma_g(y) - yb'\right\|_2^2 = \left\|b'(\sigma_g(y) - y)\right\|_2^2 = 2\left\|b'\right\|_2^2(\tau(p) - \tau(p)^2),$$

with $\|b'\|_2 \geq 1 - \varepsilon$. It follows that $\tau(p) - \tau(p)^2 \leq 2\varepsilon^2(1-\varepsilon)^{-2}$ for every $\varepsilon \in]0,1[$, hence $\tau(p) \in \{0,1\}$, a contradiction.

5.3. Free products

5.3.1. Free subalgebras. Given two groups G_1, G_2, the von Neumann algebra associated with their product $G_1 \times G_2$ is the tensor product:

$$L(G_1 \times G_2) = L(G_1)\overline{\otimes} L(G_2).$$

There is another familiar and useful construction in group theory, namely the *free product* $G = G_1 * G_2$. Recall that G is generated by G_1 and G_2 and is such that, given any group H and any homomorphisms $f_i : G_i \to H$, $i = 1, 2$, there is a (unique) homomorphism $f : G \to H$ with $f_{|G_i} = f_i$. Every element s in $G \setminus \{e\}$ is an irreducible word $s = s_1 \cdots s_n$, that is, $s_i \in G_{k_i} \setminus \{e\}$ with $k_i \neq k_{i+1}$ for $i = 1, \dots n-1$. The product is defined by concatenation and reduction. We are interested in the construction of $L(G_1 * G_2)$ from $L(G_1)$ and $L(G_2)$. Let τ be the canonical tracial state on $L(G)$. Let $x_1, \dots, x_n$ in $L(G)$ be such that $x_i \in L(G_{k_i})$, with $k_i \neq k_{i+1}$ for $i = 1, \dots, n-1$, and $\tau(x_i) = 0$ for all i. A straightforward computation shows that $\tau(x_1 x_2 \cdots x_n) = 0$. This means that $L(G_1)$ and $L(G_2)$ sit as freely independent subalgebras of $L(G)$ in the following sense (compare with the tensor product $L(G_1)\overline{\otimes} L(G_2)$).

Definition 5.3.1. Let M_1, M_2 be two von Neumann subalgebras of a von Neumann algebra M equipped with a faithful normal state φ. We say that M_1, M_2 are *free with respect to* φ if $\varphi(x_1 x_2 \cdots x_n) = 0$ whenever $x_i \in M_{k_i}$ with $k_1 \neq k_2 \neq \cdots \neq k_n$ and $\varphi(x_i) = 0$ for all i. We say that two elements a_1, a_2 of M are free with respect to φ if the von Neumann algebras they generate are free.

Proposition 5.3.2. *Let M_1, M_2 be two von Neumann subalgebras of M that are free with respect to a faithful normal state φ. We assume that M is generated (as a von Neumann algebra) by $M_1 \cup M_2$.*

(i) *φ is completely determined by its restrictions to M_1 and M_2.*
(ii) *If each restriction is a trace, then φ is a trace.*

Proof. Given $x_i \in M_{k_i}$ with $k_1 \neq k_2 \neq \cdots \neq k_n$, we claim that $\varphi(x_1 x_2 \cdots x_n)$ is uniquely determined. This will conclude the proof of (i) since the linear span $\mathcal{M}$ of such products is a w.o. dense $*$-subalgebra of M and φ is normal. We proceed inductively on n. We write $x_i = \varphi(x_i)1 + \overset{o}{x}_i$. Note that $\varphi(\overset{o}{x}_i) = 0$. Replacing each x_i by its expression, expanding, and using the fact that $\varphi(\mathring{x}_1\mathring{x}_2 \cdots \mathring{x}_n) = 0$, we see that we are reduced to computations involving at most $n-1$ products.

Assume now that the restrictions of φ to M_1 and M_2 are tracial. It suffices to show that the restriction of φ to $\mathcal{M}$ is a trace, and even, by linearity, that $\varphi(xy) = \varphi(yx)$ for $x = x_1x_2\cdots x_m$ and $y = y_1y_2\cdots y_n$ where $x_i \in M_{k_i}$, $y_j \in M_{l_j}$ with $k_1 \neq k_2 \neq \cdots \neq k_m$, $l_1 \neq l_2 \neq \cdots \neq l_n$, $n \leq m$, and $\varphi(x_i) = 0 = \varphi(y_j)$ for all i, j.

First, we obviously have that $\varphi(xy) = 0$ whenever $k_m \neq l_1$. Assuming $k_m = l_1$ we get

$$\begin{aligned}\varphi(xy) &= \varphi(x_1\cdots x_{m-1}(x_my_1 - \varphi(x_my_1)1)y_2\cdots y_n) + \\ &\qquad\qquad\qquad\qquad \varphi(x_my_1)\varphi(x_1\cdots x_{m-1}y_2\cdots y_n) \\ &= \varphi(x_my_1)\varphi(x_1\cdots x_{m-1}y_2\cdots y_n).\end{aligned}$$

Iterating this computation we see that $\varphi(xy) = 0$, except possibly when $n = m$ and $k_i = l_{m-i+1}$ for every i, and then we have

$$\varphi(xy) = \varphi(x_1y_m)\varphi(x_2y_{m-1})\cdots\varphi(x_my_1).$$

Similarly, we see that $\varphi(yx) = 0$ except possibly in the same conditions as above and then we have

$$\varphi(yx) = \varphi(y_1x_m)\varphi(y_2x_{m-1})\cdots\varphi(y_mx_1).$$

The conclusion follows from the tracial property of the restrictions of φ to M_1 and M_2. $\square$

Proposition 5.3.3. *Let (M_1,τ_1), (M_2,τ_2) be two tracial von Neumann algebras. There exists (up to isomorphism) at most one triple $((M,\tau),\phi_1,\phi_2)$, where τ is a normal faithful tracial state and $\phi_i : M_i \to M$, $i = 1,2$, are homomorphisms, satisfying the following properties:*

(i) *$\tau_i = \tau\circ\phi_i$ for $i = 1,2$;*
(ii) *$\phi_1(M_1), \phi_2(M_2)$ sit in M as free von Neumann subalgebras with respect to τ and M is generated by $\phi_1(M_1)\cup\phi_2(M_2)$.*

Proof. Note first that every homomorphism ϕ_i satisfying Condition (i) is normal by Proposition 2.5.11 and so $\phi_i(M_i)$ is a von Neumann subalgebra of M by Proposition 2.5.12.

Let (M,τ_M) and (N,τ_N) be two solutions. We denote by $\phi_i : M_i \to M$ and $\psi_i : M_i \to N$ the trace preserving inclusions ($i = 1,2$). Let $\mathcal{M}$ and $\mathcal{N}$ be the $*$-algebras generated by $\phi_1(M_1)\cup\phi_2(M_2)$ and $\psi_1(M_1)\cup\psi_2(M_2)$ respectively.

There is a well defined $*$-homomorphism α from $\mathcal{M}$ onto $\mathcal{N}$ such that $\alpha(\phi_i(x)) = \psi_i(x)$ for $x \in M_i$, $i = 1,2$. Indeed, for $y = \phi_{k_1}(x_1)\cdots\phi_{k_n}(x_n)$ with $x_i \in M_{k_i}$, we set $\alpha(y) = \psi_{k_1}(x_1)\cdots\psi_{k_n}(x_n)$, and whenever y is a linear combination of such terms we extend α by linearity. Of course, such an expression of y is not unique. If $y = Y_1$ and $y = Y_2$ are two such expressions, to see that $\alpha(Y_1)$ and $\alpha(Y_2)$ defined in this way coincide, we first observe that, by Proposition 5.3.2 (i), we have

$$\tau_N(\psi_{k_1}(x_1)\cdots\psi_{k_n}(x_n)) = \tau_M(\phi_{k_1}(x_1)\cdots\phi_{k_n}(x_n))$$

whenever $x_i \in M_{k_i}$, $i = 1, \dots, n$. It follows that

$$0 = \tau_M((Y_1 - Y_2)^*(Y_1 - Y_2)) = \tau_N\big((\alpha(Y_1) - \alpha(Y_2))^*(\alpha(Y_1) - \alpha(Y_2))\big).$$

Since the trace τ_N is faithful, we conclude that $\alpha(Y_1) = \alpha(Y_2)$.

Finally, since $\tau_N \circ \alpha = \tau_M$ on $\mathcal{M}$, it follows that α extends (uniquely) to a trace preserving isomorphism from M onto N (Exercise 2.15). □

Definition 5.3.4. Let (M_1, τ_1), (M_2, τ_2) be two tracial von Neumann algebras and let $((M, \tau), \phi_1, \phi_2)$ be such that the conditions (i) and (ii) of the previous proposition are satisfied. Then we say that (M, τ) is the *free product* of (M_1, τ_1) and (M_2, τ_2) and we write $(M, \tau) = (M_1, \tau_1) * (M_2, \tau_2)$ or simply $M = M_1 * M_2$. Usually, we identify M_1 and M_2 with their ranges in M.

For instance, $(L(G_1), \tau_1)$ and $(L(G_2), \tau_2)$ (with their canonical tracial states) satisfy the conditions of Proposition 5.3.3 with respect to $L(G_1 * G_2)$ equipped with its trace τ and so $(L(G_1 * G_2), \tau)$ is isomorphic to $(L(G_1), \tau_1) * (L(G_2), \tau_2)$.

We now prove the existence of $(M_1, \tau_1) * (M_2, \tau_2)$ for any pair of tracial von Neumann algebras.

5.3.2. Construction of $M_1 * M_2$. For $i = 1, 2$ we set $\mathcal{H}_i = L^2(M_i, \tau_i)$ and $\xi_i = \widehat{1}_{M_i}$. The first step is to represent M_1 and M_2 on the Hilbert space free product of $(\mathcal{H}_1, \xi_1)$ by $(\mathcal{H}_2, \xi_2)$.

We denote by $\overset{o}{\mathcal{H}}_i$ the orthogonal complement of $\mathbb{C}\xi_i$ in $\mathcal{H}_i$. The *Hilbert space free product* $(\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$ is $(\mathcal{H}, \xi)$ given by the direct hilbertian sum

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \Big(\bigoplus_{i_1 \neq i_2 \neq \cdots \neq i_n} \overset{o}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{i_n} \Big),$$

where ξ is a unit vector. We set

$$\mathcal{H}_l(i) = \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \Big(\bigoplus_{\substack{i_1 \neq i_2 \neq \cdots \neq i_n \\ i_1 \neq i}} \overset{o}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{i_n} \Big),$$

and we define a unitary operator $V_i : \mathcal{H}_i \otimes \mathcal{H}_l(i) \to \mathcal{H}$ as follows:

$$\begin{cases} \xi_i \otimes \xi \mapsto \xi \\ \xi_i \otimes \eta \mapsto \eta, \ \forall \eta \in \overset{o}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{i_n}, i_1 \neq i \\ \eta \otimes \xi \mapsto \eta, \ \forall \eta \in \overset{o}{\mathcal{H}}_i \\ \eta \otimes \eta' \mapsto \eta \otimes \eta', \ \forall \eta \in \overset{o}{\mathcal{H}}_i, \eta' \in \overset{o}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{i_n}, i_1 \neq i. \end{cases}$$

Similarly, we set

$$\mathcal{H}_r(i) = \mathbb{C}\xi \oplus \bigoplus_{n\geq 1}\Big(\bigoplus_{\substack{i_1\neq i_2\neq\cdots\neq i_n\\ i_n\neq i}} \overset{o}{\mathcal{H}}_{i_1}\otimes\cdots\otimes \overset{o}{\mathcal{H}}_{i_n}\Big),$$

and we define the corresponding unitary operator $W_i : \mathcal{H}_r(i)\otimes\mathcal{H}_i \to \mathcal{H}$.

We faithfully represent M_i on $\mathcal{H}$ by

$$\forall x\in M_i,\quad \lambda_i(x) = V_i(x\otimes \mathrm{Id}_{\mathcal{H}_l(i)})V_i^*,$$

and similarly, we represent faithfully the commutant M_i' of M_i (in $\mathcal{B}(\mathcal{H}_i)$) by

$$\rho_i(x) = W_i(\mathrm{Id}_{\mathcal{H}_r(i)}\otimes x)W_i^*.$$

A straightforward computation shows that $\lambda_i(x)\rho_j(y) = \rho_j(y)\lambda_i(x)$ for every $x\in M_i, y\in M_j'$, $i,j\in\{1,2\}$. So, if we set $M = \big(\lambda_1(M_1)\cup\lambda_2(M_2)\big)''$, and $N = \big(\rho_1(M_1')\cup\rho_2(M_2')\big)''$, we see that these von Neumann algebras commute. We will see in Subsection 7.1.3 (d) that $N = M'$.

We set $\overset{o}{M}_i = \ker\tau_i$. Note that $\overset{o}{\mathcal{H}}_i$ is the norm closure of $\overset{o}{M}_i\xi_i$. Moreover, we have $\lambda_i(x)\xi = x\xi_i$ whenever $x\in\overset{o}{M}_i$, and an easy induction argument shows that

$$(\lambda_{k_1}(x_1)\cdots\lambda_{k_n}(x_n))\xi = x_1\xi_{k_1}\otimes\cdots\otimes x_n\xi_{k_n}\in\overset{o}{\mathcal{H}}_{k_1}\otimes\cdots\otimes\overset{o}{\mathcal{H}}_{k_n}, \tag{5.1}$$

for $x_i\in\overset{o}{M}_{k_i}$ with $k_1\neq k_2\neq\cdots\neq k_n$. It follows that ξ is cyclic for M. Similarly, it is cyclic for N, and finally we get that ξ is cyclic and separating for both algebras.

In particular, the vector state ω_ξ is faithful on M. We claim that $((M,\omega_\xi),\lambda_1,\lambda_2)$ satisfies the conditions stated in Proposition 5.3.3. Since $V_i^*\xi = \xi_i\otimes\xi$, a straightforward computation shows that $\omega_\xi\circ\lambda_i = \tau_i$. Moreover, we deduce immediately from Equation (5.1) that the von Neumann algebras $\lambda_1(M_1)$ and $\lambda_2(M_2)$ are free with respect to ω_ξ. Since ω_ξ is a tracial state (by Proposition 5.3.2), we denote it by τ. Hence (M,τ) is the free product of $\lambda_1(M_1)$ and $\lambda_2(M_2)$ we were looking for.

Remark 5.3.5. Since ξ is separating for M, the map $x\in M\mapsto x\xi$ is injective and so we may identify M with a subspace of $\mathcal{H}$. In particular, we identify $\lambda_{k_1}(x_1)\cdots\lambda_{k_n}(x_n)$ with $x_1\otimes\cdots\otimes x_n\in\overset{o}{M}_{k_1}\otimes\cdots\otimes\overset{o}{M}_{k_n}$, thanks to (5.1). We set

$$\mathcal{M} = \mathbb{C}1\oplus\bigoplus_{n\geq 1}\Big(\bigoplus_{i_1\neq i_2\neq\cdots\neq i_n}\overset{o}{M}_{i_1}\otimes\cdots\otimes\overset{o}{M}_{i_n}\Big). \tag{5.2}$$

Then $\mathcal{M}$ is a $*$-subalgebra of M, w.o. dense, called the *algebraic free product* of M_1 and M_2. We observe that the components of the decomposition of $\mathcal{M}$ in (5.2) are mutually orthogonal with respect to the inner product defined by τ.

We end this section by giving a sufficient condition for the free product of two tracial von Neumann algebras to be a factor. We make use of the notion of conditional expectation defined in Chapter 9 and the reading of the proof of the next lemma may be postponed.

LEMMA 5.3.6. *Let* (M_1, τ_1), (M_2, τ_2) *be two tracial von Neumann algebras and set* $(M, \tau) = (M_1, \tau_1) * (M_2, \tau_2)$. *Let* Q *be a diffuse von Neumann subalgebra of* M_1. *Then* $Q' \cap M \subset M_1$. *In particular,* M *is a* II_1 *factor whenever* M_1 *is so.*

PROOF. We denote by E_{M_1} the trace preserving conditional expectation from M onto M_1 (see Theorem 9.1.2). Let (u_n) be a sequence of unitary operators in Q such that $\lim_n u_n = 0$ in the w.o. topology (see Exercise 3.2). We claim that for $x, y \in M$ such that $E_{M_1}(x) = E_{M_1}(y) = 0$, we have

$$\lim_n \|E_{M_1}(x u_n y)\|_2 = 0. \tag{5.3}$$

A crucial observation is that $E_{M_1}(z) = 0$ whenever $z \notin \overset{o}{M}_1$ is an alternated product[3] of elements in $\overset{o}{M}_i$. This follows from the fact that $\tau_1(m_1 E_{M_1}(z)) = \tau(m_1 z) = 0$ for every $m_1 \in M_1$, where the latter equality results from straightforward computations.

Let us prove (5.3). Using the Kaplansky density theorem (and the observation preceding Proposition 2.6.4), we see that x is the limit in $\|\cdot\|_2$-norm of elements of $\mathcal{M}$. Moreover, since $E_{M_1}(x) = 0$ and E_{M_1} is $\|\cdot\|_2$-continuous we may assume that these elements have no component on $\mathbb{C}1 \oplus \overset{o}{M}_1 = M_1$. The same argument applies to y and finally it suffices to consider the case where x, y are in some $\overset{o}{M}_{i_1} \cdots \overset{o}{M}_{i_n}$, $i_1 \neq i_2 \neq \cdots \neq i_n$, $n \geq 2$, or in $\overset{o}{M}_2$. So we write $x = x_1 a b$ and $y = d c y_1$, where $b, d \in \{1\} \cup \overset{o}{M}_1$, $a, c \in \overset{o}{M}_2$, and x_1 (resp. y_1), if $\neq 1$, is an alternated product of elements in $\overset{o}{M}_i$, ending (resp. beginning) with some element in $\overset{o}{M}_1$. Then, we have

$$\begin{aligned} x u_n y &= (x_1 a)(b u_n d)(c y_1) \\ &= (x_1 a)\big(b u_n d - \tau_1(b u_n d)1\big)(c y_1) + \tau_1(b u_n d)(x_1 a c y_1). \end{aligned}$$

We set $v = b u_n d - \tau_1(b u_n d)1$ and note that $E_{M_1}(x_1 a v c y_1) = 0$ by our previous observation. It follows that $E_{M_1}(x u_n y) = \tau_1(b u_n d) E_{M_1}(x_1 a c y_1)$. But $\lim_n \tau_1(b u_n d) = 0$, and our claim (5.3) is proved.

Now let $x \in Q' \cap M$. Subtracting $E_{M_1}(x)$, we may assume that $E_{M_1}(x) = 0$. Then, we have

$$E_{M_1}(x u_n x^*) = u_n E_{M_1}(x x^*)$$

and $\|u_n E_{M_1}(x x^*)\|_2 = \|E_{M_1}(x x^*)\|_2$. Together with (5.3), this implies that $E_{M_1}(x x^*) = 0$ and thus $x = 0$ since E_{M_1} is faithful. $\square$

[3] An alternated product is of the form $x_1 \ldots x_n$ with $x_k \in \overset{o}{M}_{i_k}$, $i_1 \neq i_2 \neq \cdots \neq i_n$.

REMARK 5.3.7. The same proof applies to the case where $1_Q \neq 1_M$: if $Q \subset M_1$ is diffuse, then $Q' \cap 1_Q M 1_Q \subset M_1$.

COROLLARY 5.3.8. *Let (M_1, τ_1), (M_2, τ_2) be two tracial von Neumann algebras. We assume that M_1 is diffuse and that M_2 is non-trivial. Then $(M, \tau) = (M_1, \tau_1) * (M_2, \tau_2)$ is a II_1 factor.*

PROOF. We keep the notation of Section 5.3.2. By the previous lemma, we have $\mathcal{Z}(M) \subset \mathcal{Z}(M_1)$. Let $z \in \mathcal{Z}(M)$ with $\tau_1(z) = 0$ and let y be a non-zero element in M_2 with $\tau_2(y) = 0$. We have on the one hand $\|zy\xi\|_2 = \|z\xi_1\|_2 \|y\xi_2\|_2$ by (5.1) and, on the other hand,

$$\|zy\xi\|_2^2 = \tau(y^* z^* z y) = \tau(z^* y^* z y) = 0.$$

It follows that $z\xi_1 = 0$ and thus $z = 0$. Therefore we get $\mathcal{Z}(M) = \mathbb{C}1$. $\square$

5.4. Ultraproducts

As we will see later, the technique of ultraproducts is a very useful tool when studying the behaviour of families of sequences. We fix a *free ultrafilter* ω. Recall that ω is an element of $\beta\mathbb{N} \setminus \mathbb{N}$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$, *i.e.*, the spectrum of the C^*-algebra $\ell^\infty(\mathbb{N})$. For any bounded sequence (c_n) of complex numbers, $\lim_\omega c_n$ is defined as the value at ω of this sequence, viewed as a continuous function on $\beta\mathbb{N}$.

Let (M_n, τ_n) be a sequence of tracial von Neumann algebras. The product algebra $\prod_{n \geq 1} M_n$ is the C^*-algebra of bounded sequences $x = (x_n)_n$ with $x_n \in M_n$ for every n, endowed with the norm $\|x\| = \sup_n \|x_n\|$. The (tracial) *ultraproduct* $\prod_\omega M_n$ is the quotient of $\prod_{n\geq 1} M_n$ by the ideal I_ω of all sequences $(x_n)_n$ such that $\lim_\omega \tau_n(x_n^* x_n) = 0$. It is easily seen that I_ω is a normed closed two-sided ideal, so that $\prod_\omega M_n$ is a C^*-algebra. If x_ω denotes the class of $x \in \prod_{n\geq 1} M_n$, then $\tau_\omega(x_\omega) = \lim_\omega \tau_n(x_n)$ defines without ambiguity a faithful tracial state on $\prod_\omega M_n$. We set $\|y\|_{2,\omega} = \tau_\omega(y^*y)^{1/2}$ whenever $y \in \prod_\omega M_n$.

When the (M_n, τ_n) are the same tracial von Neumann algebra (M, τ), we set $M^\omega = \prod_\omega M$, and we say that (M^ω, τ_ω) is the (tracial) *ultrapower of (M, τ) along ω*.

PROPOSITION 5.4.1. *$(\prod_\omega M_n, \tau_\omega)$ is a tracial von Neumann algebra. Moreover, if the M_n are tracial factors such that* $\lim_n \dim M_n = +\infty$, *then $\prod_\omega M_n$ is a II_1 factor*[4].

PROOF. For simplicity of notation, we deal with the case M^ω, the proof in the general case being the same. We use the characterisation given in Proposition 2.6.4, and show that the unit ball of M^ω is complete for the

[4]This factor is not separable in general. It is the only example where we really need to work with non separable II_1 factors in this monograph.

metric induced by $\|\cdot\|_{2,\omega}$. Let $(x(p))_p$ be a sequence in M^ω such that, for every p

$$\|x(p)\|_\infty < 1, \quad \|x(p+1) - x(p)\|_{2,\omega} < 2^{-(p+1)}.$$

We choose inductively a representing sequence $(x_n(p))_n$ for $x(p)$ such that

$$\sup_n \|x_n(p)\| \leq 1, \quad \sup_n \|x_n(p+1) - x_n(p)\|_2 \leq 2^{-(p+1)}.$$

Then, for each $n \in \mathbb{N}$, the sequence $(x_n(p))_p$ is a Cauchy sequence in the unit ball $(M)_1$ of M equipped with the $\|\cdot\|_2$ metric, and therefore converges to some $x_n \in (M)_1$. Now, we have $\|x_n - x_n(p)\|_2 \leq 2^{-p}$, whence, if x denotes the class of the sequence $(x_n)_n$,

$$\|x - x(p)\|_{2,\omega} = \lim_\omega \|x_n - x_n(p)\|_2 \leq 2^{-p}.$$

Assume now that M is a factor. We claim that any two projections $p, q \in M^\omega$ are comparable and so M^ω is a factor (see Remark 2.4.10). Indeed, using the lemma to follow, we choose representatives (p_n) and (q_n) of p, q respectively, consisting in sequences of projections such that $\tau(p_n) = \tau_\omega(p)$ and $\tau(q_n) = \tau_\omega(q)$ for every n. Assume that $\tau_\omega(p) \leq \tau_\omega(q)$. Since M is a factor, there exists a partial isometry u_n in M with $u_n^* u_n = p_n$ and $u_n u_n^* \leq q_n$. Let u_ω be the class of the sequence $(u_n)_n$. Then we have $u_\omega^* u_\omega = p$ and $u_\omega u_\omega^* \leq q$. $\square$

LEMMA 5.4.2. *Let (M,τ) be a tracial von Neumann algebra, ω a free ultrafilter and p a projection in M^ω.*

(i) *There exists a representative (p_n) of p such that p_n is a projection for every n.*
(ii) *If in addition M is a factor, we may choose the p_n such that $\tau(p_n) = \tau_\omega(p)$ for every n.*

PROOF. (i) Let (x_n) be a representative of p such that $0 \leq x_n \leq 1$ for every n. We have $\lim_\omega \left\|x_n - x_n^2\right\|_2 = 0$. We may assume that $\left\|x_n - x_n^2\right\|_2 = \delta_n < 1/4$ for every n. Let p_n be the spectral projection of x_n relative to the interval $[1 - \delta_n^{1/2}, 1)$. Then $\lim_\omega \|x_n - p_n\|_2 = 0$ by Lemma 5.4.3 below.

(ii) We only consider the case where M is a II_1 factor, the case of matrix algebras being trivial. We set $\tau_\omega(p) = \lambda$. Let q be a projection in M with $\tau(q) = \lambda$. We have either $q \precsim p_n$ or $p_n \precsim q$. We choose a projection $q_n \in M$ with $\tau(q_n) = \lambda$ and either $q_n \leq p_n$ or $p_n \leq q_n$. Then we have

$$\|q_n - p_n\|_2^2 = |\tau(q_n - p_n)| = |\lambda - \tau(p_n)|$$

and so $\lim_\omega \|q_n - p_n\|_2 = 0$. $\square$

LEMMA 5.4.3. *Let $0 \leq x \leq 1$ be an element of a tracial von Neumann algebra such that $\left\|x - x^2\right\|_2 = \delta < 1/4$. Let p be the spectral projection of x relative to $[1 - \sqrt{\delta}, 1]$. Then we have $\|x - p\|_2 \leq (3\delta)^{1/2}$.*

PROOF. Let μ be the spectral probability measure of x associated with the vector $\widehat{1} \in L^2(M,\tau)$. We have

$$\int_0^1 (t-t^2)^2 \,\mathrm{d}\mu(t) = \int_0^1 (t-t^2)^2 \,\mathrm{d}\langle \widehat{1}, E_t \widehat{1}\rangle = \left\|x - x^2\right\|_2^2 = \delta^2.$$

Put $\delta_1 = \delta^{1/2}$. We have

$$\delta^2 \geq \int_{\delta_1}^{1-\delta_1} (t-t^2)^2 \,\mathrm{d}\mu(t) \geq (\delta_1 - \delta_1^2)^2 \mu([\delta_1, 1-\delta_1]),$$

hence $\mu([\delta_1, 1-\delta_1]) \leq \delta(1-\delta_1)^{-2}$. It follows that

$$\begin{aligned} \|x-p\|_2^2 &= \int_0^{\delta_1} t^2 \,\mathrm{d}\mu(t) + \int_{\delta_1}^{1-\delta_1} t^2 \,\mathrm{d}\mu(t) + \int_{1-\delta_1}^1 (1-t)^2 \,\mathrm{d}\mu(t) \\ &\leq 2\delta + (1-\delta_1)^2 \mu([\delta_1, 1-\delta_1]) \leq 3\delta. \end{aligned}$$

□

REMARK 5.4.4. Let M be a separable II_1 factor. One may wonder whether the ultraproduct M^ω depends on the free ultrafilter ω. Ge and Hadwin proved that, assuming the Continuum Hypothesis, all these ultraproducts are isomorphic [**GH01**]. It has been proved more recently by Farah, Hart and Sherman that, conversely, if all these ultraproducts are isomorphic then the Continuum Hypothesis holds [**FHS13**].

5.5. Beyond factors and abelian von Neumann algebras

A tracial factor is either isomorphic to some matrix algebra or is of type II_1 depending on the existence or not of a minimal projection (see Corollary 2.4.14). In the non-factor case, the distinction is made via the existence of non-zero abelian projections, which generalize the notion of minimal projection.

DEFINITION 5.5.1. Let M be a von Neumann algebra. A projection $p \in M$ is called *abelian* if $p \neq 0$ and the reduced von Neumann algebra pMp is abelian.

A useful feature of abelian projections is the following one.

PROPOSITION 5.5.2. *Let p, q be two projections in a von Neumann algebra M. We assume that p is abelian and that $p \leq z(q)$ where $z(q)$ is the central support of q. Then we have $p \precsim q$. In particular, two abelian projections with the same central support are equivalent.*

PROOF. Since there exists a central projection z such that $zp \precsim zq$ and $(1-z)q \precsim (1-z)p$, it suffices to show that $(1-z)q \sim (1-z)p$. So, we may assume that $q \sim q_1 \leq p \leq z(q)$.

Since p is abelian, we have $q_1 = pe$ where $e \in \mathcal{Z}(M)$ (see Proposition 4.2.1). Then we have

$$e \geq z(q_1) = z(q) \geq z(p)$$

and so $p = pe = q_1$. □

Definition 5.5.3. We say that a von Neumann algebra is of *type* I if it has an abelian projection whose central support is 1. A von Neumann algebra is said to be of *type* II_1 if it is finite (*i.e.*, its unit is not equivalent to a strictly smaller projection, see the next chapter) and does not have any abelian projection.

Remark 5.5.4. For factors, abelian projections are the same as minimal projections. The above definitions are compatible with the definitions of type I and type II_1 given in the case of factors (see Theorem 6.3.5 for II_1 factors).

There is almost no loss of generality to deal with tracial von Neumann algebras instead of finite ones. Indeed, finite von Neumann algebras are direct sums of tracial von Neumann algebras and, in the countably decomposable case, they are exactly the tracial von Neumann algebras (Exercise 6.2).

Examples 5.5.5. **(a)** For instance, any tensor product $L^\infty(X,\mu)\overline{\otimes}\mathcal{B}(\mathcal{H})$ is easily seen to be of type I. Such an algebra M is said to be *d-homogeneous* where d is the dimension of $\mathcal{H}$. The cardinal d only depends on M (see Exercise 5.5 for countable cardinals).

More generally, every product of the form $\prod_{i\in I} A_i\overline{\otimes}\mathcal{B}(\mathcal{H}_i)$ where the A_i's are abelian is still of type I.

(b) Let (N,τ) be a II_1 factor and $L^\infty(X,\mu)$ an abelian von Neumann algebra. Then $M = L^\infty(X,\mu)\overline{\otimes}N$ is a type II_1 von Neumann algebra. Indeed, assume that M has an abelian projection p. Let (e_n) be a decreasing sequence of projections in N such $\lim_n \tau(e_n) = 0$. Since the central support of $1\otimes e_n$ is 1_M (Exercise 5.4 (ii)), we deduce from the previous proposition that $p \precsim 1\otimes e_n$ and so

$$(\tau_\mu\otimes\tau)(p) \leq (\tau_\mu\otimes\tau)(1\otimes e_n) = \tau(e_n)$$

for every n, in contradiction with the fact that $p\neq 0$.

More generally, every product $\prod_{i\in I} A_i\overline{\otimes}N_i$ where the A_i's are abelian and the N_i's are II_1 factors is a type II_1 von Neumann algebra.

Remark 5.5.6. Every type I von Neumann algebra can be written as $\prod_{i\in I} A_i\overline{\otimes}\mathcal{B}(\mathcal{H}_i)$ where the A_i's are abelian(see [**Tak02**, Theorem V.1.27]). Finite type I von Neumann algebras are exactly those with $\dim\mathcal{H}_i < +\infty$ for all i.

On the other hand, not every type II_1 von Neumann algebra is of the form $\prod_{i\in I} A_i\overline{\otimes}N_i$ where the A_i's are abelian and the N_i's are II_1 factors, but separable II_1 von Neumann algebras are direct integrals of II_1 factors (see [**Dix81**, Chapitre II, §3 and §5]).

Theorem 5.5.7. *Every von Neumann algebra M has a unique decomposition as a direct sum $M_1\oplus M_2$ where M_1 is a type* I *von Neumann algebra and M_2 is without abelian projection (with possibly one of the two components degenerated to $\{0\}$).*

PROOF. Assume that M has at least an abelian projection (otherwise there is nothing to prove). Let $(p_i)_{i\in I}$ be a maximal family of abelian projections p_i in M whose central supports $z(p_i)$ are mutually orthogonal and set $p = \sum_{i\in I} p_i$, $z = \sum_{i\in I} z(p_i)$. Then p is an abelian projection whose central support is z. Moreover, thanks to the maximality of $(p_i)_{i\in I}$, we see that $M(1-z)$ has no abelian projection. So $M = (Mz) \oplus (M(1-z))$ is a decomposition of M as a direct sum of a type I von Neumann algebra by a von Neumann algebra without any abelian projection.

Let $M = (Mz_1) \oplus (M(1-z_1))$ be another such decomposition. We have $z_1 \leq z$. Indeed, let p_1 be an abelian projection having z_1 as central support. If $z_1(1-z) \neq 0$, then $p_1 z_1(1-z)$ is an abelian projection in $M(1-z)$, but this cannot occur. Similarly, we see that $z \leq z_1$. □

We have seen in Corollary 4.1.2 that in any diffuse factor, for all $n \geq 1$ every projection is the sum of 2^n equivalent projections. This result extends to any von Neumann algebra without abelian projections.

PROPOSITION 5.5.8. *Let M be a von Neumann algebra without abelian projection. Then any projection in M is the sum of two equivalent orthogonal projections and therefore is the sum of 2^n equivalent orthogonal projections for all $n \geq 1$.*

PROOF. It suffices to show that for any non-zero projection e in M there exist two non-zero equivalent orthogonal projection e_1, e_2 with $e_1 + e_2 \leq e$. Then the end of the proof will be exactly the same as that of Lemma 4.1.1.

The crucial observation is that eMe is not abelian. So, there is a projection f in eMe but not in $\mathcal{Z}(eMe)$. Therefore we have $fM(e-f) \neq 0$. Then, it follows from Lemma 2.4.7 that there exist non-zero projections $e_1 \leq f$ and $e_2 \leq e - f$ that are equivalent. □

Exercises

EXERCISE 5.1. We keep the notation of Example 5.1.1 (b).

(i) Prove the assertions stated in this example.
(ii) Let $T = [T_{i,j}]$ be a matrix with coefficients in M. For every n we denote by $T(n)$ the matrix with $T(n)_{i,j} = T_{i,j}$ if $i, j \leq n$ and $T(n)_{i,j} = 0$ otherwise. Show that $T \in \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes} M$ if and only if the sequence $(\|T(n)\|)_n$ is bounded.
(iii) Extend these results, when $\mathbb{N}$ is replaced by any set I.

EXERCISE 5.2. Let M be a von Neumann algebra and let $(p_i)_{i\in I}$ be a family of mutually equivalent projections such that $\sum_{i\in I} p_i = 1$. Show that M is isomorphic to $\mathcal{B}(\ell^2(I))\overline{\otimes}(p_{i_0} M p_{i_0})$ with $i_0 \in I$.

EXERCISE 5.3. Let M_1 and M_2 be two factors. Show that

$$(1_{M_1}\overline{\otimes} M_2)' \cap (M_1 \overline{\otimes} M_2) = M_1 \overline{\otimes} 1_{M_2}.$$

Exercise 5.4. Let (X,μ) be a probability measure space and $(N,\mathcal{H})$ a factor.

(i) Show that the commutant of $L^\infty(X,\mu)\overline{\otimes}1_{\mathcal{B}(\mathcal{H})}$ in $\mathcal{B}(L^2(X,\mu)\otimes\mathcal{H})$ is $L^\infty(X,\mu)\overline{\otimes}\mathcal{B}(\mathcal{H})$.

(ii) Show that the center of $L^\infty(X,\mu)\overline{\otimes}N$ is $L^\infty(X,\mu)\overline{\otimes}1_N$.

More generally, one shows that the center of a tensor product of two von Neumann algebras is the tensor product of their centers (see [**Tak02**, Corollary IV.5.11]).

Exercise 5.5. Let M be a von Neumann algebra and let $(e_i)_{i\in I}$, $(f_j)_{j\in J}$ be two countable families of abelian projections having 1 as central support and such that $\sum_{i\in I} e_i = 1 = \sum_{j\in J} f_j$. Show that all these projections are equivalent and that $\mathrm{card}\, I = \mathrm{card}\, J$.

Exercise 5.6. We set $\ell^2_\infty = \ell^2(\mathbb{N}^*)$ and ℓ^2_i denotes the canonical Hilbert space of finite dimension i. Let I, J be two subsets of $\mathbb{N}^*\cup\{\infty\}$ and $(A_i)_{i\in I}$, $(B_j)_{j\in J}$ be two families of abelian von Neumann algebras. Let

$$\alpha : \sum\nolimits^{\oplus}_{i\in I} A_i\overline{\otimes}\mathcal{B}(\ell^2_i) \to \sum\nolimits^{\oplus}_{j\in J} B_j\overline{\otimes}\mathcal{B}(\ell^2_j)$$

be an isomorphism. Show that $I = J$ and that $\alpha\big(A_i\overline{\otimes}\mathcal{B}(\ell^2_i)\big) = B_i\overline{\otimes}\mathcal{B}(\ell^2_i)$ for every $i\in I$.

Exercise 5.7. When M is a factor, show that $\alpha\in \mathrm{Aut}\,(M)$ is inner if and only if their exists a non-zero element $y\in M$ such that $y\alpha(x) = xy$ for every $x\in M$.

Exercise 5.8. Let (B,τ) be a tracial von Neumann algebra and $\sigma : G\curvearrowright B$ a trace preserving action of a group G. Show that $B\rtimes G$ is spatially isomorphic to the von Neumann algebra of operators on $\mathcal{H} = L^2(B,\tau)\otimes \ell^2(G) = \ell^2(G, L^2(B,\tau))$ generated by $\{\pi(B)\}\cup\{1\otimes\lambda_g : g\in G\}$ where $\pi(b)$ is defined by $(\pi(b)f)(g) = \sigma_{g^{-1}}(b)f(g)$ for $b\in B$ and $f\in\mathcal{H}$ (see Exercise 1.9 for the case where B is abelian).

Exercise 5.9. Let (B,τ) be a tracial von Neumann algebra and $\sigma : G\curvearrowright B$ a trace preserving action of a group G. Show that the crossed product $B\rtimes G$ is the unique (up to isomorphism) tracial von Neumann algebra (M,τ) generated by a trace preserving copy of B and unitary elements $(u_g)_{g\in G}$ satisfying the following properties:

$$u_g b u_g^* = \sigma_g(b) \text{ for all } g\in G, b\in B, \quad u_g u_h = u_{gh} \text{ for all } g,h\in G,$$
$$\tau(bu_g) = 0 \text{ for all } b\in B, g\neq e.$$

Exercise 5.10. Let G be an ICC group and let $\sigma : G\curvearrowright (B,\tau)$ be a trace preserving action. We identify $L(G)$ in the obvious way with a von Neumann subalgebra of $B\rtimes G$ (*i.e.*, the von Neumann subalgebra generated by the $u_g, g\in G$).

(i) Show that $L(G)'\cap(B\rtimes G) = B^G$ (the algebra of G-fixed elements in B).

(ii) Show that $B \rtimes G$ is a factor (and so a II_1 factor) if and only if the G-action on the center of B is ergodic.

Exercise 5.11. Let G be a finite group and $\sigma : G \curvearrowright B$ a properly outer trace preserving action on a tracial von Neumann algebra (B,τ). For $g \in G$, we denote by v_g the unitary operator on $L^2(B,\tau)$ defined by $v_g\widehat{x} = \widehat{\sigma_g(x)}$ for every $x \in B$. Let M be the $*$-subalgebra of $\mathcal{B}(L^2(B,\tau))$ generated by $B \cup \{v_g : g \in G\}$. Let $\phi : B \rtimes G \to M$ by defined by $\phi(\sum_{g\in G} b_g u_g) = \sum_{g\in G} b_g v_g$.

(i) Show that $\phi : B \rtimes G \to \mathcal{B}(L^2(B,\tau))$ is a normal homomorphism.
(ii) Show that ϕ is injective (use the fact that the center of $B \rtimes G$ is contained in $\mathcal{Z}(B)$).
(iii) Conclude that M is a von Neumann algebra isomorphic to $B \rtimes G$.

Exercise 5.12. Let M be a II_1 factor and ω a free ultrafilter on $\mathbb{N}$.

(i) Let $f^1, \dots, f^k$ be mutually orthogonal projections in M^ω. Show that we can find, for $i = 1, \dots, k$, a representative $(f_n^i)_{n\in\mathbb{N}}$ in $\ell^\infty(\mathbb{N}, M)$ of f^i, such that for every n, the f_n^i, $i = 1, \dots, k$, are mutually orthogonal projections in M. Show that these projections can be chosen mutually equivalent whenever one starts with mutually equivalent projections in M^ω.
(ii) Let u be a partial isometry in M^ω and set $f^1 = u^*u$, $f^2 = uu^*$. Choose representatives $(f_n^i)_{n\in\mathbb{N}}$ of f^i, $i = 1, 2$, such that for every n the projections f_n^1 and f_n^2 are equivalent. Show that u can be lifted into a sequence $(u_n)_n$ satisfying $u_n^*u_n = f_n^1$ and $u_n u_n^* = f_n^2$ for every n.
(iii) Show that every matrix units in M^ω can be lifted to a sequence of matrix units in M.

Exercise 5.13. Let M be a von Neumann algebra. Show the equivalence of the following two conditions:

(i) M is of type I;
(ii) every non-zero projection of M majorizes an abelian projection.

Notes

The tensor product of two von Neumann algebras was introduced in the first joint paper of Murray and von Neumann [**MVN36**]. Infinite tensor products of von Neumann algebras were defined by von Neumann [**vN39**] very soon after. The notion of crossed product for a group action on an abelian von Neumann algebra goes back to the pioneering work of Murray and von Neumann [**MVN36**]. The setting of group actions on tracial von Neumann algebras was originated by the Japanese school of operator algebras in the late fifties. ([**Tur58, Suz59, NT58**] to cite a few)

The notion of free product of two von Neumann algebras appears for the first time in [**Chi73**], but was developed and used in its full strength by

Voiculescu, his students and others. It gave rise in the eighties to the very active and powerful theory of free probability (see [**Voi85**] for the beginning).

Ultraproducts constructions appeared in model theory in the fifties. However, they are already implicit in the operator algebra setting in Wright's paper [**Wri54**] and later in Sakai's notes [**Sak62**] although these authors do not use the ultrapower terminology. Ultraproducts are a crucial ingredient in McDuff's characterisation of those II_1 factors M that are isomorphic to $M \overline{\otimes} R$ (the so-called *McDuff factors* [**McD70**]), in Connes' characterisation of full factors [**Con74**] (see Chapter 15) and in his celebrated work on the classification of injective factors [**Con76**]. Ultraproduct techniques are nowadays a classical useful tool when one wants to replace approximate properties by some precise version.

CHAPTER 6

Finite factors

It is now the time to clarify the definition of II_1 factors. We have introduced them in term of the existence of an appropriate trace. The main purpose of this chapter is to give an equivalent definition, which relies on the behaviour of the projections: an infinite dimensional factor is of type II_1 if and only if its unit is not equivalent to a strictly smaller projection. The notion of dimension function will be the key of the proof of the equivalence of the two definitions. We will also see that whenever a factor has a tracial state, this trace is automatically normal and faithful. It is also unique, as already shown in Chapter 4.

We end this chapter by a general averaging result for factors which, when applied to a finite factor, gives a nice description of the trace.

6.1. Definitions and basic observations

Definition 6.1.1. A projection p in a von Neumann algebra M is *finite* if p is not equivalent to a projection q strictly smaller than p. In other terms, p is finite if for every partial isometry $u \in M$ with $u^*u = p$ and $uu^* \leq p$, then $uu^* = p$.

If p is not finite, we say that p is *infinite*.

Every projection $q \in M$ smaller than a finite projection $p \in M$ is also finite. Indeed, if there exists a partial isometry $u \in M$ with $u^*u = q$ and $uu^* < q$ then $v = u + (p - q)$ will be a partial isometry with $v^*v = p$ and $vv^* < p$, a contradiction. In particular, when the unit element 1 of M is a finite projection, every projection in M is finite.

Definition 6.1.2. We say that a von Neumann algebra M (in particular a factor) is *finite* if 1 is a finite projection. Otherwise, we say that M is *infinite*.

Obviously, abelian von Neumann algebras are finite. Every von Neumann algebra which has a faithful tracial state τ is finite. Indeed, let u be a partial isometry in M such that $u^*u = 1$. Then $\tau(1 - uu^*) = \tau(1 - u^*u) = 0$, so that $1 = uu^*$ since τ is faithful.

Whenever $p \sim q$, it is not true in general that $1 - p \sim 1 - q$. This can be observed for instance in the von Neumann algebra $\mathcal{B}(\ell^2(\mathbb{N}))$.

On the other hand, when 1 is finite we have:

Lemma 6.1.3. *Let M be a finite von Neumann algebra.*

(i) *Let $p, q \in M$ be two projections such that $p \sim q$. Then $1-p \sim 1-q$. In particular, there exists a unitary $u \in M$ with $upu^* = q$.*

(ii) *Let $v \in M$ be a partial isometry. There exists a unitary operator u with $v = u(v^*v)$.*

PROOF. (i) The comparison theorem 2.4.8 tells us that there exists a projection $z \in \mathcal{Z}(M)$ with $(1-p)z \precsim (1-q)z$ and $(1-q)(1-z) \precsim (1-p)(1-z)$. By considering separately the situations in Mz and $M(1-z)$, we may assume for instance that $1-p \precsim 1-q$. If $1-p \sim r \leq 1-q$, we have

$$1 \geq q + r \sim p + (1-p) = 1$$

so that $q + r = 1$, whence $1 - p \sim 1 - q$.

Let $v \in M$ be a partial isometry with $v^*v = p$, $vv^* = q$ and let $w \in M$ be such that $w^*w = 1-p$, $ww^* = 1-q$. Then $u = v + w$ is a unitary that has the required property.

(ii) is immediate from (i). □

In contrast with the case of finite projections, any infinite projection in a factor can be cut up in two pieces equivalent to itself.

PROPOSITION 6.1.4. *Every infinite projection p in a factor M can be written as $p = p_1 + p_2$ where p_1, p_2 are projections in M such that $p_1 \sim p \sim p_2$.*

PROOF. Replacing M by the factor pMp we may assume that $p = 1$. Let $e_1 \in \mathcal{P}(M)$ be such that $e_1 \sim 1$ with $e_1 \neq 1$ and let $u \in M$ be such that $u^*u = 1$ and $uu^* = e_1$. We put $e_0 = 1$, $e_n = u^n(u^n)^*$ for $n > 0$. Then $(e_n)_{n\geq 0}$ is a strictly decreasing sequence of projections which are all equivalent to 1. Morever the projections $f_n = e_n - e_{n+1}$, $n \geq 0$, are equivalent, since, if we set $v_n = u(e_n - e_{n+1})$, we have $v_n^*v_n = e_n - e_{n+1}$ and $v_nv_n^* = e_{n+1} - e_{n+2}$.

Let $(q_i)_{i\in I}$ be a maximal family of mutually orthogonal and equivalent projections, which contains the sequence $\{f_n : n \in \mathbb{N}\}$. Since M is a factor, the maximality of the family implies that $q = 1 - \sum_{i\in I} q_i \precsim q_i$. We consider a partition $I_1 \cup I_2$ of I into two subsets of the same cardinal as I and we put $p_1 = \sum_{i\in I_1} q_i$, $p_2 = \sum_{i\in I_2} q_i + q$. We have $p_1 + p_2 = 1$ and due to the fact that the cardinal of I is infinite, we immediately see that $p_1 \sim p_2 \sim 1$. □

We remark that, by using a partition of I into a countable family of subsets of the same cardinality, we even get $p = \sum_{n\geq 1} p_n$ where the p_n are all equivalent to p.

It follows from this proposition that an infinite factor has no tracial state. Indeed, if 1 is infinite, we may write $1 = p_1 + p_2$ where p_1, p_2 are two projections equivalent to 1. Assuming that M has a tracial state τ, we obtain the contradiction

$$1 = \tau(p_1) + \tau(p_2) = 2.$$

6.2. Construction of the dimension function

The aim of the next two sections is to prove that a finite factor carries a normal tracial state. Since a finite factor that has a minimal projection is isomorphic to some matrix algebra $M_n(\mathbb{C})$ (see Proposition 2.4.13) we only have to deal with finite diffuse factors. The first step is to construct a function that, like tracial states, measures the dimension of projections. We begin by the introduction of a kind of dyadic expansion for projections.

PROPOSITION 6.2.1. *Let M be a diffuse factor. There exists a sequence $(p_n)_{n\geq 1}$ of mutually orthogonal projections in M such that*

$$p_{n+1} \sim 1 - \sum_{i=1}^{n+1} p_i \quad for \quad n \geq 0. \tag{6.1}$$

PROOF. By Proposition 4.1.1 we find two equivalent projections $p_1, q_1 \in M$ with $p_1 + q_1 = 1$. By the same argument, we find two equivalent projections p_2, q_2 with $p_2 + q_2 = 1 - p_1$ and therefore $1 - p_1 - p_2 \sim p_2$. Repeating this process, we get a sequence $(p_n)_{n\geq 1}$ of projections with the required properties. □

REMARK 6.2.2. We observe that 1 is the orthogonal sum of two projections equivalent to p_1 and that every p_n is the orthogonal sum of two projections equivalent to p_{n+1}.

We now turn to the case of finite factors.

PROPOSITION 6.2.3. *Let M be a diffuse finite factor and let (p_n) be as in Proposition 6.2.1.*

(i) *Let $p \in \mathcal{P}(M)$ be such that $p \precsim p_n$ for every $n \geq 1$. Then $p = 0$.*
(ii) *We have $\sum_{k=1}^{+\infty} p_k = 1$, and therefore $\sum_{k=n+1}^{+\infty} p_k \sim p_n$ for $n \geq 1$.*
(iii) *If p is a non-zero projection in M, there exists an integer n such that $p_n \precsim p$.*

PROOF. (i) Suppose $p \neq 0$. We have $p \sim q_n \leq p_n$ for all n. Setting

$$q = \sum_{n\,\mathrm{odd}} q_n \quad \text{and} \quad q' = \sum_{n\geq 1} q_n,$$

we have $q \sim q'$ and $q < q'$, in contradiction with the fact that q' is finite.

(ii) If we put $p = 1 - \sum_{k=1}^{+\infty} p_k$, we get from (6.1) that $p \precsim p_n$ for every n and so $p = 0$.

(iii) follows immediately from (i). □

DEFINITION 6.2.4. A projection $p \in M$ which is equivalent to one of the above constructed projections $p_n, n \geq 1$, is called a *fundamental projection*. We denote by $\mathcal{FP}(M)$ the set of fundamental projections in M.

The (equivalence classes of) fundamental projections play, for projections, the role of dyadic rationals for numbers in $[0, 1]$.

PROPOSITION 6.2.5. *Let M and (p_n) be as above and let $p \in M$ be a non-zero projection. There exists a unique increasing sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of integers and a sequence $(p'_{n_k})_{k\geq 1}$ of mutually orthogonal projections in M with the following properties:*

(i) $p'_{n_k} \sim p_{n_k}$ *for every* $k \geq 1$*;*
(ii) $p = \sum_{i=1}^{+\infty} p'_{n_i}$.

PROOF. We first observe that condition (ii) is equivalent to

$$p - \sum_{i=1}^{k} p'_{n_i} \precsim p_{n_k} \quad \text{for} \quad k \geq 1. \qquad \text{(ii')}$$

Indeed, (ii') implies that

$$p - \sum_{i=1}^{\infty} p'_{n_i} \leq p - \sum_{i=1}^{k} p'_{n_i} \precsim p_{n_k} \precsim p_k$$

for all k, and so $p = \sum_{i=1}^{\infty} p'_{n_i}$ by Proposition 6.2.3 (i). Conversely, if (ii) holds, then we have, for all k,

$$p - \sum_{i=1}^{k} p'_{n_i} \sim \sum_{i=n_k+1}^{\infty} p_{n_i} \precsim p_{n_k}.$$

We may assume that $p \neq 1$ since for $1 = \sum_{k=1}^{+\infty} p_k$ the only possible choice is $n_k = k$ for all k. By Proposition 6.2.3 we see that there exists n such that $p_n \precsim p$, and we define n_1 to be the smallest integer with this property. Let $p'_{n_1} \sim p_{n_1}$ be such that $p'_{n_1} \leq p$. We have $p - p'_{n_1} \prec p_{n_1}$, otherwise we would have $p_{n_1} \precsim p - p'_{n_1}$, and n_1 would not be the smallest integer with $p_n \precsim p$. In case $p - p'_{n_1} = 0$, we have, by Proposition 6.2.3,

$$p = \sum_{k=n_1+1}^{\infty} p'_k\,,$$

where the projections are mutually orthogonal, with $p'_k \sim p_k$ for every $k \geq n_1 + 1$. We easily see that it is the only possible infinite expansion, up to equivalence of projections[1].

If $p - p'_{n_1} \neq 0$, we repeat the process and we choose n_2 to be the smallest integer with $p_{n_2} \precsim p - p'_{n_1}$. By induction, we get a strictly increasing sequence $(n_k)_{k\geq 1}$ of integers and a sequence (p'_{n_k}) of mutually orthogonal projections with the required properties (i) and (ii'), where we make the convention that if we get at some stage the equality $p = \sum_{i=1}^{k} p'_{n_i}$, then we choose the expansion of the form

$$p = \sum_{i=1}^{k-1} p'_{n_i} + \sum_{i=n_k+1}^{\infty} p'_i,$$

[1] This choice of an infinite decomposition is similar to the convention of choosing the infinite expansion of a dyadic rational number instead of the finite one.

with $p_i' \sim p_i$ for every $i \geq n_k + 1$.

The uniqueness of the sequence $(n_k)_{k\geq 1}$ is also easily checked by induction. $\square$

By a slight abuse of language, we will say that $p = \sum_{i=1}^{\infty} p'_{n_i}$ is *the dyadic expansion* of p.

We now define the notion of dimension function and prove its existence and uniqueness.

DEFINITION 6.2.6. Let M be a diffuse finite factor. A *dimension function* on M is a map $\Delta : \mathcal{P}(M) \to [0,1]$ such that

(i) $\Delta(1) = 1$;
(ii) $p \sim q \Rightarrow \Delta(p) = \Delta(q)$;
(iii) $\Delta(p+q) = \Delta(p) + \Delta(q)$ for every pair (p,q) of orthogonal projections.

LEMMA 6.2.7. *Let Δ be dimension function on a diffuse finite factor M. Then we have*

(i) *$\Delta(p) = 0$ if and only if $p = 0$;*
(ii) *$p \precsim q$ if and only if $\Delta(p) \leq \Delta(q)$;*
(iii) *Δ is completely additive, i.e., for any family $(q_i)_{i\in I}$ of mutually orthogonal projections in M, we have $\Delta(\sum_{i\in I} q_i) = \sum_{i\in I} \Delta(q_i)$.*

PROOF. (i) If $p \neq 0$, by Proposition 6.2.3 (iii) there exists a fundamental projection p_n with $p_n \precsim p$. We have $\Delta(p_n) = 2^{-n}$ (see Remark 6.2.2), whence $\Delta(p) \neq 0$.

(ii) is immediate. It remains to show the complete additivity. Let $(q_i)_{i\in I}$ be a family of mutually orthogonal projections in M and set $q = \sum_{i\in I} q_i$. For every finite subset F of I we have $\sum_{i\in F} \Delta(q_i) \leq \Delta(q)$ and hence $\sum_{i\in I} \Delta(q_i) \leq \Delta(q)$. Since the sum $\sum_{i\in I} \Delta(q_i)$ is finite, the set of indices i with $q_i \neq 0$ is countable and we may assume that $I = \mathbb{N}^*$.

Assume that $\sum_{n\geq 1} \Delta(q_n) < \Delta(q)$ and choose an integer k with

$$2^{-k} + \sum_{n\geq 1} \Delta(q_n) \leq \Delta(q).$$

Let $r \in \mathcal{P}(M)$ with $\Delta(r) = 2^{-k}$. We construct, by induction, a sequence $(r_n)_{n\geq 0}$ of mutually orthogonal projections with $r_0 \sim r$, $r_n \sim q_n$ for $n \geq 1$ and $r_n \leq q$ for $n \geq 0$. First, since $\Delta(r) \leq \Delta(q)$, there is $r_0 \sim r$ with $r_0 \leq q$. Suppose now that we have constructed $r_0, r_1, \dots, r_{n-1}$. We have

$$\Delta(q - \sum_{i=0}^{n-1} r_i) = \Delta(q) - \sum_{i=0}^{n-1} \Delta(r_i) \geq \sum_{i=n}^{\infty} \Delta(q_i) \geq \Delta(q_n).$$

Thus we have

$$q_n \precsim q - \sum_{i=0}^{n-1} r_i$$

and therefore there exists $r_n \sim q_n$ with $r_n \leq q - \sum_{i=0}^{n-1} r_i$.

Finally, we obtain

$$q = \sum_{i=1}^{\infty} q_i \sim \sum_{i=1}^{\infty} r_i < \sum_{i=0}^{\infty} r_i \leq q$$

which is impossible since q is a finite projection. □

THEOREM 6.2.8. *Let M be a diffuse finite factor. There exists a unique dimension function $\Delta : \mathcal{P}(M) \to [0,1]$. It is defined by $\Delta(0) = 0$ and for $p \neq 0$ by the expression*

$$\Delta(p) = \sum_{i=1}^{\infty} \frac{1}{2^{n_i}}, \tag{6.2}$$

where $p = \sum_{i=1}^{\infty} p'_{n_i}$ is the dyadic expansion of p.

PROOF. We first prove the uniqueness of Δ. We keep the notation of Proposition 6.2.1. Using Remark 6.2.2, we see that we must have $\Delta(p_n) = 2^{-n}$ for every n. Therefore, using the complete additivity of Δ, we obtain that $\Delta(p)$ must be given by the expression (6.2) if $p \neq 0$..

So, we define Δ by this expression. Obviously, we have $\Delta(p) = 0$ if and only if $p = 0$.

We check first that whenever $p \precsim q$, then $\Delta(p) \leq \Delta(q)$. Let $p = \sum_{i=1}^{\infty} p'_{n_i}$, $q = \sum_{i=1}^{\infty} q'_{m_i}$ be the dyadic expansions of p and q. Assume that $\Delta(p) > \Delta(q)$. Then we denote by i_0 the smallest integer i with $n_i \neq m_i$. We have $n_{i_0} < m_{i_0}$. By Proposition 6.2.3 (ii), we get

$$p'_{n_{i_0}} \sim p_{n_{i_0}} \succsim \sum_{i=i_0}^{\infty} p_{m_i} \sim \sum_{i=i_0}^{\infty} q'_{m_i}$$

and we deduce the contradiction $p \succ q$.

Let us show now that Δ is a dimension function. Condition (ii) of Definition 6.2.6 is immediate. We claim that $\Delta(p+q) = \Delta(p) + \Delta(q)$ when $pq = 0$. We first consider the case where p is a fundamental projection, say $p \sim p_n$. Let $q = \sum_{i=1}^{\infty} q'_{m_i}$ be the dyadic expansion of q. Then either $n \notin \{m_i : i \geq 1\}$ and then $p + \sum_{i=1}^{\infty} q'_{m_i}$ is the dyadic expansion of $p+q$ and we get immediately the additivity, or there is m_{i_0} with $n = m_{i_0}$. In this case, $p + q'_{m_{i_0}} \sim p_{n-1}$ and we iterate the argument with $(p + q'_{m_{i_0}}) + \sum_{i \neq i_0} q'_{m_i}$. In a finite number of steps we get the dyadic expansion of $p+q$, from which we again deduce the additivity.

We now study the general case where p has the dyadic expansion $\sum_{i=1}^{\infty} p'_{n_i}$. For every k, we write $p = \sum_{i=1}^{k} p'_{n_i} + r_k$ and $q = \sum_{i=1}^{k} q'_{n_i} + r'_k$ and we notice that $r_k \precsim p'_{n_k} \precsim p_k$ and $r'_k \precsim q'_{n_k} \precsim p_k$, so that $r_k + r'_k \precsim p_{k-1}$. From the above observations, we see that

$$\begin{aligned}\Delta(p+q) &= \Delta\Big(\big(\sum_{i=1}^{k} p'_{n_i} + \sum_{i=1}^{k} q'_{n_i}\big) + (r_k + r'_k)\Big)\\ &= \sum_{i=1}^{k} \Delta(p'_{n_i}) + \sum_{i=1}^{k} \Delta(q'_{n_i}) + \Delta(r_k + r'_k)\end{aligned}$$

with $0 \le \Delta(r_k + r'_k) \le \frac{1}{2^{k-1}}$.

It follows that

$$\Delta(p+q) = \sum_{i=1}^{\infty} \Delta(p'_{n_i}) + \sum_{i=1}^{\infty} \Delta(q'_{n_i}) = \Delta(p) + \Delta(q).$$

It is now easy to prove (2). Assume that $\Delta(p) \le \Delta(q)$ and that $q \sim q' \le p$. Then $\Delta(p) = \Delta(q) + \Delta(p - q')$, so that $p = q'$ and $p \sim q$. □

6.3. Construction of a tracial state

We keep the assumptions and notations of Theorem 6.2.8. We show that the dimension function extends in a unique way to a normal faithful tracial state on M.

Lemma 6.3.1. *Let φ and ψ be two non-zero completely additive maps from $\mathcal{P}(M)$ into $\mathbb{R}_+$. We assume that φ is faithful, i.e., $\varphi(e) \neq 0$ whenever $e \neq 0$. Given $\varepsilon > 0$, there is a non-zero fundamental projection $p \in \mathcal{FP}(M)$ and a constant $\theta > 0$ such that for every projection $q \le p$, we have*

$$\theta\varphi(q) \le \psi(q) \le \theta(1+\varepsilon)\varphi(q). \tag{6.3}$$

Proof. We may assume that $\varphi(1) = \psi(1) \neq 0$. We first show that there exists a fundamental projection e such that $\varphi(e_1) \le \psi(e_1)$ for every fundamental projection $e_1 \le e$.

Suppose, on the contrary, that for every $e \in \mathcal{FP}(M)$ there exists $e_1 \in \mathcal{FP}(M)$ with $e_1 \le e$ and $\varphi(e_1) > \psi(e_1)$. Take a maximal family $(e_i)_{i\in I}$ of mutually orthogonal fundamental projections such that $\varphi(e_i) > \psi(e_i)$. Using Proposition 6.2.3 (iii) we see that $\sum_{i\in I} e_i = 1$, whence

$$\varphi(1) = \sum_{i\in I} \varphi(e_i) > \sum_{i\in I} \psi(e_i) = \psi(1),$$

thanks to the complete additivity of φ and ψ.

Therefore there exists e with the required property. We set

$$\theta = \sup\{\eta : \eta\varphi(e_1) \le \psi(e_1), \forall e_1 \le e, e_1 \in \mathcal{FP}(M)\}.$$

We have $\theta \in [1, +\infty[$ and $\theta\varphi(e_1) \le \psi(e_1)$ for $e_1 \in \mathcal{FP}(M)$ and $e_1 \le e$.

Let us assume now that for every projection $p \in \mathcal{FP}(M)$ with $p \le e$ there exists a fundamental projection $e_1 \le p$ with

$$\theta(1+\varepsilon)\varphi(e_1) \le \psi(e_1).$$

Using a maximality argument as above, but in the von Neumann algebra pMp, this would imply that $\theta(1+\varepsilon)\varphi(p) \le \psi(p)$, in contradiction with the definition of θ.

Hence there exists a fundamental projection $p \le e$ such that $\theta(1+\varepsilon)\varphi(e_1) \ge \psi(e_1)$ for every fundamental projection $e_1 \le p$, and the inequality $\theta\varphi(e_1) \le \psi(e_1)$ is of course satisfied. Thanks to the dyadic expansion of any projection $q \le p$ and the complete additivity of φ and ψ, we get (6.3). $\square$

LEMMA 6.3.2. *Let ψ be a positive linear functional on M and $\varepsilon > 0$ such that*

$$\forall q \in \mathcal{P}(M), \quad \Delta(q) \le \psi(q) \le (1+\varepsilon)\Delta(q).$$

Then for every $x \in M_+$ and every unitary operator u in M, we have

$$\psi(uxu^*) \le (1+\varepsilon)\psi(x). \tag{6.4}$$

PROOF. For $q \in \mathcal{P}(M)$, we have

$$\psi(uqu^*) \le (1+\varepsilon)\Delta(uqu^*) = (1+\varepsilon)\Delta(q) \le (1+\varepsilon)\psi(q).$$

By Corollary 2.2.3, every $x \in M_+$ is the sum $\sum_n 2^{-n} q_n$ of a series which converges in norm, with $q_n \in \mathcal{P}(M)$. The inequality (6.4) follows immediately. $\square$

A positive linear functional ψ is called an *ε-trace* if it satisfies the inequality (6.4) for every $x \in M_+$ and $u \in \mathcal{U}(M)$. Note that we have then

$$\forall y \in M, \quad \psi(yy^*) \le (1+\varepsilon)\psi(y^*y), \tag{6.5}$$

because the polar decomposition of y and Lemma 6.1.3 imply that y may be written as $y = u|y|$ with $u \in \mathcal{U}(M)$. Then

$$\psi(yy^*) = \psi(u|y|^2u^*) \le (1+\varepsilon)\psi(|y|^2) = (1+\varepsilon)\psi(y^*y).$$

Conversely, the property (6.5) easily implies that ψ is an ε-trace.

LEMMA 6.3.3. *Let M be a diffuse finite factor and let Δ be its dimension function. Then for every $\varepsilon > 0$ there is a normal ε-trace ψ_ε such that*

$$\frac{1}{1+\varepsilon}\Delta(q) \le \psi_\varepsilon(q) \le (1+\varepsilon)^2\Delta(q) \tag{6.6}$$

for all $q \in \mathcal{P}(M)$.

PROOF. We apply Lemma 6.3.1 with $\varphi = \Delta$ and ψ a non-zero normal linear functional ω. Replacing ω by $\theta^{-1}\omega$ we obtain the existence of a non-zero fundamental projection p such that for any $q \in \mathcal{P}(M)$ with $q \le p$, we have

$$\Delta(q) \le \omega(q) \le (1+\varepsilon)\Delta(q). \tag{6.7}$$

Applying Lemma 6.3.2 to the diffuse factor pMp instead of M, we see that ω restricted to pMp is a normal ε-trace. Now, since p is a fundamental projection, there exists an integer n and fundamental projections $q_1, \dots, q_{2^n}$ such that $\sum_{i=1}^{2^n} q_i = 1$, $q_1 = p$ and $q_i \sim p$ for every i. Let w_i be a partial isometry with $w_i^*w_i = p$ and $w_iw_i^* = q_i$. We set $\psi_\varepsilon(x) = \sum_{i=1}^{2^n} \omega(w_i^*xw_i)$. Then we have, for $y \in M$,

$$\begin{aligned}
\psi_\varepsilon(yy^*) &= \sum_{i=1}^{2^n} \omega\big(w_i^*y(\sum_{j=1}^{2^n} w_jw_j^*)y^*w_i\big)\\
&= \sum_{i,j=1}^{2^n} \omega\big((w_i^*yw_j)(w_i^*yw_j)^*\big)\\
&\le (1+\varepsilon)\sum_{i,j=1}^{2^n} \omega\big((w_j^*y^*w_i)(w_i^*yw_j)\big)\\
&= (1+\varepsilon)\psi_\varepsilon(y^*y),
\end{aligned}$$

and therefore ψ_ε is an ε-trace. Moreover $\psi_\varepsilon \neq 0$ since $\psi_\varepsilon(p) = \omega(p) \ge \Delta(p)$.

It remains to show that the inequalities (6.6) are satisfied. Let $q \in \mathcal{P}(M)$ and let $q = \sum_{i\ge 1} p'_{n_i}$ be its dyadic expansion. By comparing the fundamental projections p'_{n_i} with the fundamental projections q_j we first see that there is a unitary operator $u \in M$ such that $up'_{n_i}u^*$ commutes with q_j for every i, j. We set $q' = uqu^*$. This projection commutes with each q_j. Then $w_j^*q'w_j$ is a projection in pMp and furthermore, the projections $w_j^*q'w_j$ and $q'q_j$ are equivalent via the partial isometry $q'w_j$. The inequality (6.7) gives

$$\begin{aligned}
\Delta(q'q_j) &= \Delta(w_j^*q'w_j) \le \omega(w_j^*q'w_j)\\
&\le (1+\varepsilon)\Delta(w_j^*q'w_j) = (1+\varepsilon)\Delta(q'q_j),
\end{aligned}$$

and after addition,

$$\Delta(q') \le \psi_\varepsilon(q') \le (1+\varepsilon)\Delta(q').$$

Since ψ_ε is an ε-trace, we get

$$\psi_\varepsilon(q) \le (1+\varepsilon)\psi_\varepsilon(q') \le (1+\varepsilon)^2\Delta(q') = (1+\varepsilon)^2\Delta(q),$$

and

$$\psi_\varepsilon(q) \ge \frac{1}{1+\varepsilon}\psi_\varepsilon(q') \ge \frac{1}{1+\varepsilon}\Delta(q') = \frac{1}{1+\varepsilon}\Delta(q). \qquad \square$$

Theorem 6.3.4. *Let M be a diffuse finite factor. Its dimension function extends in a unique way to a normal faithful tracial state on M.*

Proof. Let (ε_n) be a decreasing sequence of positive real numbers with $\lim_n \varepsilon_n = 0$. By Lemma 6.3.3, there is a sequence of normal ε_n-traces ψ_n such that

$$\frac{1}{1+\varepsilon_n}\Delta(q) \le \psi_n(q) \le (1+\varepsilon_n)^2\Delta(q)$$

for every $q \in \mathcal{P}(M)$. In particular, we have $\lim_n \psi_n(q) = \Delta(q)$. In fact, this sequence $(\psi_n)_n$ converges uniformly on the unit ball of M. Indeed, writing any x of the unit ball of M as $x = (x_1 - x_2) + i(x_3 - x_4)$ where the $0 \le x_i \le 1$, it suffices to consider the case $0 \le x \le 1$. By Corollary 2.2.3, we can write x as a sum $\sum_{n=1}^{\infty} \frac{1}{2^n} q_n$ with $q_n \in \mathcal{P}(M)$ for all n. Then we have, for $n \ge m$,

$$\begin{aligned}|\psi_n(x) - \psi_m(x)| &\le \sum_{k=1}^{\infty} \frac{1}{2^k} |\psi_n(q_k) - \psi_m(q_k)| \\ &\le \sum_{k=1}^{\infty} \frac{1}{2^k} \Delta(q_k) \Big((1+\varepsilon_m)^2 - \frac{1}{1+\varepsilon_m} \Big) \\ &\le \Big((1+\varepsilon_m)^2 - \frac{1}{1+\varepsilon_m} \Big).\end{aligned}$$

It follows that (ψ_n) is a Cauchy sequence and therefore converges in norm to a linear functional ψ on M.

We easily check that ψ is a normal tracial state. Let us show that ψ is faithful. Let $x \in M_+$ with $\psi(x) = 0$. For every real number $t > 0$, we denote by e_t the spectral projection relative to the interval $[t, +\infty[$. Since $te_t \le x$, we get $\Delta(e_t) = \psi(e_t) = 0$ and so $e_t = 0$. Hence, we have $x = 0$.

The uniqueness of the extension of the dimension function follows from the expansion $x = \sum_{n=1}^{\infty} \frac{1}{2^n} q_n$ of every $0 \le x \le 1$, obtained in Corollary 2.2.3. □

THEOREM 6.3.5. *Let M be a factor. The following conditions are equivalent:*

(i) *M has a normal tracial state;*
(ii) *M has a (norm continuous) tracial state;*
(iii) *1 is a finite projection (i.e., M is finite).*

Moreover, the tracial state, when it exists is unique and faithful.

PROOF. It suffices to consider the case where M is diffuse.

(i) $\Rightarrow$ (ii) is obvious and (ii) $\Rightarrow$ (iii) is an immediate consequence of Proposition 6.1.4. That (iii) $\Rightarrow$ (i) follows from the previous theorem.

The uniqueness of the tracial state has been proved in Proposition 4.1.3. The faithfulness of the tracial state τ follows from Theorem 6.3.4, but can also be shown directly. Indeed, if τ is a tracial state, the set $\{x \in M : \tau(x^*x) = 0\}$ is a two-sided ideal, which is reduced to zero (see Proposition 4.1.5). □

Thus, for an infinite dimensional factor, to say that it is of type II_1 or finite is the same.

6.4. Dixmier averaging theorem

Let $M = M_n(\mathbb{C})$ be a matrix algebra. We observe that its unique tracial state τ can be obtained by averaging over the compact group $\mathcal{U}_n(\mathbb{C})$ of unitary $n \times n$ matrices, with respect to its Haar probability measure, that is,

$$\tau(x)1 = \int_{\mathcal{U}_n(\mathbb{C})} uxu^* \,\mathrm{d}u. \tag{6.8}$$

Consider now a II_1 factor M. We will extend formula (6.8) to this setting, in an appropriate way (see Corollary 6.4.2).

For $x \in M$, we denote by C_x the $\|\cdot\|_2$-closed convex hull of

$$\{uxu^* : u \in \mathcal{U}(M)\}$$

in $L^2(M,\tau)$. We may assume that $\|x\|_\infty \le 1$. Then by Proposition 2.6.4, C_x is contained in the unit ball of M. Let $y \in M$ be the unique element of C_x with smallest $\|\cdot\|_2$-norm. This element commutes with the unitary group of M, and so is scalar, say $y = \alpha 1$. Since the tracial state τ is constant on C_x, we see that $\alpha = \tau(x)$. Therefore, we have $C_x \cap \mathbb{C}1 = \{\tau(x)1\}$.

In fact, we have a stronger useful result, where the $\|\cdot\|_2$-closure of the convex hull of $\{uxu^* : u \in \mathcal{U}(M)\}$ is replaced by its $\|\cdot\|_\infty$-closure, a smaller set that we denote by K_x.

Theorem 6.4.1 (**Dixmier averaging theorem**). *Let M be a factor and let $x \in M$.*

(i) *Given $\varepsilon > 0$, there are unitaries $u_1, \dots, u_n \in \mathcal{U}(M)$ and $\alpha \in \mathbb{C}$ such that*
$$\Big\|\frac{1}{n}\sum_{i=1}^n u_i x u_i^* - \alpha 1\Big\|_\infty \le \varepsilon.$$

(ii) *The set $K_x \cap \mathbb{C}1$ is not empty.*

Proof. (i) We first consider the case $x = x^*$ and we may of course assume $x \notin \mathbb{C}1$. We denote by $\mathrm{Sp}(x)$ its spectrum and set

$$c = \min \mathrm{Sp}(x), \quad C = \max \mathrm{Sp}(x), \quad t = (c+C)/2.$$

We introduce the spectral projection $p = E(]-\infty, t])$ of x. We remark that since $c \neq C$, we have $0 < p < 1$. By the the comparison theorem 2.4.9, we have either $p \precsim 1-p$ or $1-p \precsim p$. Let us assume for instance that $p \precsim 1-p$, the other case being treated similarly.

Let v be a partial isometry in M such that $p = v^*v \sim vv^* \le 1-p$. We set $p' = vv^*$ and

$$w = v + v^* + (1 - p - p').$$

Then w is a unitary operator. The main point in the proof is the evaluation of the diameter $\mathrm{diam}\,\mathrm{Sp}(T_w(x))$ of the spectrum of the self-adjoint operator $T_w(x) = \frac{1}{2}(x + wxw^*)$. We claim that

$$\mathrm{diam}\,\mathrm{Sp}(T_w(x)) \le \frac{3}{4}\mathrm{diam}\,\mathrm{Sp}(x) = \frac{3}{4}(C-c). \tag{6.9}$$

Since $T_w(x) \le C1$, it suffices to show that $x + wxw^* \ge (t+c)1$. Using the functional calculus, we see that

$$cp \le xp \le tp \quad \text{and} \quad t(1-p) \le x(1-p) \le C(1-p).$$

We will also use the facts that $w = w^*$ and $wpw^* = p'$, so that

$$w(1-p-p')w^* = 1-p-p'.$$

It follows that

$$\begin{aligned} x + wxw^* &\geq \big(t(1-p) + cp\big) + \big(wcpw^* + tw(1-p-p'+p')w^*\big) \\ &= t(1-p) + cp + cp' + t(1-p-p') + tp \\ &= (t+c)p + (t+c)p' + 2t(1-p-p') \\ &\geq (t+c)1, \end{aligned}$$

which proves our claim.

We now put $w_1 = w$ and choose n such that $(\frac{3}{4})^n \operatorname{diam} \mathrm{Sp}(x) \leq \varepsilon$. By applying the preceding process n times, we get unitaries $w_1, \dots, w_n$ such that

$$\operatorname{diam} \mathrm{Sp}(T_{w_n} \dots T_{w_1}(x)) \leq \Big(\frac{3}{4}\Big)^n \operatorname{diam} \mathrm{Sp}(x) \leq \varepsilon.$$

We put

$$y = T_{w_n} \dots T_{w_1}(x) = \frac{1}{2^n} \sum_{i=1}^{2^n} u_i x u_i^*$$

and $\alpha = \frac{1}{2}(\min \mathrm{Sp}(y) + \max \mathrm{Sp}(y)) \in \mathbb{R}$. Then we have

$$\|y - \alpha 1\|_\infty \leq \operatorname{diam} \mathrm{Sp}(y) \leq \varepsilon.$$

Let now x be an arbitrary element of M. Applying the first part of the proof to $\Re(x) = \frac{1}{2}(x + x^*)$ and $\varepsilon/2$, we get unitaries $w_1, \dots, w_k$ in M and $\alpha \in \mathbb{R}$ with

$$\Big\| \frac{1}{k} \sum_{i=1}^{k} w_i \Re(x) w_i^* - \alpha 1 \Big\|_\infty \leq \varepsilon/2.$$

We set $y = \frac{1}{k} \sum_{i=1}^{k} w_i \Im(x) w_i^*$, where $\Im(x) = \frac{1}{2i}(x - x^*)$, and apply the first part of the proof to the self-adjoint element y and $\varepsilon/2$, to get unitaries $w'_1, \dots, w'_l$ and $\alpha' \in \mathbb{R}$. Since $x = \Re(x) + i\Im(x)$, we finally obtain

$$\Big\| \frac{1}{kl} \sum_{j=1}^{l} \sum_{i=1}^{k} w'_j w_i x w_i^* (w'_j)^* - (\alpha + i\alpha')1 \Big\|_\infty \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \qquad \square$$

(ii) For every $n \geq 1$, there exist $y_n \in K_x$ and $\alpha_n \in \mathbb{C}$ such that $\|y_n - \alpha_n 1\|_\infty \leq 1/n$. The sequence (α_n) is clearly bounded, so we may assume that it converges to some $\alpha \in \mathbb{C}$. Then, (y_n) is a Cauchy sequence, and so it converges to an element $y \in K_x$, and of course $y = \alpha 1$. Therefore, we have $K_x \cap \mathbb{C}1 \neq \emptyset$.

Corollary 6.4.2. *Let τ be a tracial state on a von Neumann factor M. Then, for every $x \in M$, we have*

$$K_x \cap \mathbb{C}1 = \{\tau(x)1\}.$$

Proof. We remark that since τ is tracial and norm continuous, it takes the constant value $\tau(x)$ on K_x. It follows that $K_x \cap \mathbb{C}1 = \{\tau(x)1\}$. $\square$

Note that this gives another proof of the uniqueness of a tracial state on a factor.

REMARK 6.4.3. The previous results of this section can be extended without assuming that M is a factor. In particular, we still have $K_x \cap \mathcal{Z}(M) \neq \emptyset$ (see [**Dix81**, Chapter III, §5]).

When M is finite, instead of the existence of a tracial state one shows the existence of a center-valued conditional expectation $E : M \to \mathcal{Z}(M)$ (see Definition 9.1.5), and one has $K_x \cap \mathcal{Z}(M) = \{E(x)\}$.

Theorem 6.3.5 is replaced by the following one:

THEOREM 6.4.4. *Let M be a von Neumann algebra. The following conditions are equivalent:*

(i) *M has sufficiently many normal traces, i.e., for every non-zero $x \in M_+$ there is a normal trace τ on M with $\tau(x) \neq 0$;*
(ii) *M has a center-valued conditional expectation;*
(iii) *M is finite.*

A proof of this result is given in [**Dix81**, Chapter III, §8]. For an elegant proof of the fact (iii) implies (i), using the Ryll-Nardzewski fixed point theorem, see [**Yea71**].

Exercises

EXERCISE 6.1. Let τ be a normal trace on a von Neumann algebra M. Show that

$$\{x \in M : \tau(x^*x) = 0\}$$

is a w.o. closed ideal of M, hence of the form Mz where z is a projection in the center of M. Check that z is the largest projection p in M such that $\tau(p) = 0$. The projection $1 - z$ is called the *support* of τ.

EXERCISE 6.2. Let M be a von Neumann algebra.

(i) If M is a direct sum of tracial von Neumann algebras, show that M is finite.
(ii) If M is finite, show that there is a family $(p_i)_{i\in I}$ of mutually orthogonal projections in $\mathcal{Z}(M)$ such that $\sum_{i\in I} p_i = 1$ and each p_iM is tracial, so that M is a direct sum of tracial von Neumann algebras (Hint: consider a maximal family of normal traces with mutually orthogonal supports, and use Theorem 6.4.4).
(iii) If M is finite, show that M is countably decomposable if and only if it is tracial.

EXERCISE 6.3. Let M be a finite factor and F a finite subset of M. Given $\varepsilon > 0$, show that there exist unitaries $u_1, \dots, u_n$ in M such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} u_i x u_i^* - \tau(x)1 \right\| \leq \varepsilon$$

for $x \in F$.

Notes

The main arguments for the proof of Theorem 6.3.5 come from [**MVN36, MvN37**]. Later, this theorem was extended as Theorem 6.4.4, by Dixmier [**Dix49**], to the case of any von Neumann algebra. The theorem 6.4.1, and more generally the result mentioned in Remark 6.4.3, were also established in [**Dix49**].

CHAPTER 7

The standard representation

In this chapter, we show that a tracial von Neumann algebra (M,τ) behaves in many respects as any commutative one $(L^\infty(X,\mu),\tau_\mu)$. The set $\widetilde{M}$ of closed densely defined operators affiliated with M on $L^2(M,\tau)$ forms a $*$-algebra analogous to the $*$-algebra of complex-valued measurable functions on X. The Hilbert space $L^2(M,\tau)$ is embedded into $\widetilde{M}$ as the space of square integrable operators. We also introduce the Banach space $L^1(M,\tau)$ of integrable operators, whose dual is M. Classical results such as the Hölder inequalities or the Radon-Nikodým theorem are extended to this setting and we prove the Powers-Størmer inequality, which is specific to the non-commutative case.

Finally, we show that the group $\mathrm{Aut}\,(M)$ of automorphisms of M has a canonical implementation by unitaries in $\mathcal{B}(L^2(M,\tau))$, a generalisation of the Koopman representation in the commutative case.

7.1. Definition and basic properties

One of the main features of the representation of (M,τ) in $L^2(M,\tau)$ that we study below is that it makes M anti-isomorphic to its commutant. It plays a crucial role in the study of all normal representations of M, as we will see in the next chapter.

7.1.1. The standard representation. The GNS representation

$$(\pi_\tau, L^2(M,\tau), \xi_\tau)$$

of the *tracial von Neumann algebra* (M,τ) has been introduced in Section 2.6. It is called the *standard representation of* (M,τ). We also say that M is in *standard form on* $L^2(M,\tau)$. We recall from Section 2.6 that π_τ is a normal faithful representation. In particular, Theorem 2.6.1 implies that $\pi_\tau(M)$ is a von Neumann algebra on $L^2(M,\tau)$. We will identify $x\in M$ and $\pi_\tau(x)$ and write $x\xi$ for $\pi_\tau(x)\xi$. Also, we identify x with $x\xi_\tau$ and view M as a dense subspace of $L^2(M,\tau)$. Finally, we use the notation $\hat{x}$ when we want to stress the point that x is considered as an element of $L^2(M,\tau)$. Its norm $\tau(x^*x)^{1/2}$ will be written $\|x\|_2$, $\|\hat{x}\|_2$, $\|\hat{x}\|_\tau$ or even $\|\hat{x}\|_{2,\tau}$ depending on the context.

Since for $x,y\in M$, we have $\pi_\tau(x)\hat{y} = \widehat{xy}$, it is natural to view $\pi_\tau(x)$ as the operator of multiplication to the left by x and to denote it by L_x.

Similarly, the map $\hat{y} \mapsto \widehat{yx}$ is continuous:

$$\|\widehat{yx}\|_2^2 = \tau(x^*y^*yx) = \tau(yxx^*y^*) \le \|x\|^2\|\hat{y}\|_2^2.$$

We denote by R_x the extension of this operator to $L^2(M,\tau)$. Then $x \mapsto R_x$ is an injective homomorphism from the opposite algebra M^{op} into $\mathcal{B}(L^2(M,\tau))$. We usually write ξx instead of $R_x\xi$. The ranges of L and R are respectively denoted by $L(M)$ and $R(M)$. Clearly, these two algebras commute. Note that $L(M) = \pi_\tau(M) = M$.

The operator $J : \hat{x} \mapsto \widehat{x^*}$ is an antilinear isometry from $\widehat{M}$ onto itself, which extends to an antilinear surjective isometry of $L^2(M,\tau)$ still denoted by J (or J_M in case of ambiguity). We say that J is the *canonical conjugation operator* on $L^2(M,\tau)$. A straightforward computation shows that $JL_xJ = R_{x^*}$ for every $x \in M$, whence $JL(M)J = R(M)$. We will prove that $L(M) = R(M)'$ and give simultaneously another description of these algebras $L(M)$ and $R(M)$.

Let $\xi \in L^2(M,\tau)$. We define the following two operators from $\widehat{M}$ into $L^2(M,\tau)$:

$$\begin{aligned} L_\xi^0(\hat{y}) &= R_y(\xi) = \xi y,\\ R_\xi^0(\hat{y}) &= L_y(\xi) = y\xi. \end{aligned}$$

These operators are not bounded in general, but they are closable. Let us show for instance this property for L_ξ^0. Let (x_n) be a sequence in M such that $\lim_n \widehat{x_n} = 0$ and $\lim_n L_\xi^0(\widehat{x_n}) = \eta$. Then, for $y \in M$, we have, on one hand,

$$\langle \eta, \hat{y}\rangle = \lim_n \left\langle L_\xi^0(\widehat{x_n}), \hat{y}\right\rangle$$

and, on the other hand,

$$\begin{aligned} \left|\left\langle L_\xi^0(\widehat{x_n}), \hat{y}\right\rangle\right| &= |\langle R_{x_n}\xi, \hat{y}\rangle| = \left|\left\langle \xi, R_{x_n}^*\hat{y}\right\rangle\right|\\ &= \left|\left\langle \xi, \widehat{yx_n^*}\right\rangle\right| \le \|\xi\|_2\|yx_n^*\|_2\\ &\le \|\xi\|_2\|y\|_\infty\|x_n^*\|_2 = \|\xi\|_2\|y\|_\infty\|\widehat{x_n}\|_2. \end{aligned}$$

It follows that $\lim_n \left\langle L_\xi^0(\widehat{x_n}), \hat{y}\right\rangle = 0$, whence $\langle \eta, \hat{y}\rangle = 0$ for every $y \in M$ and so $\eta = 0$.[1]

We will denote by L_ξ and R_ξ the closures of L_ξ^0 and R_ξ^0 respectively. Whenever L_ξ is a bounded operator, we say that L_ξ (or ξ) is a *left convolver* or that the vector ξ is *left bounded*. The set of left convolvers is denoted by $LC(M)$. Similarly, we define the set $RC(M)$ of *right convolvers*. We have the following generalisation of Theorem 1.3.6. It tells us in particular that $JMJ = M'$, hence M and its commutant in $\mathcal{B}(L^2(M,\tau))$ "have the same size".

[1] One may also observe that $\widehat{M}$ is contained in the domain of $(L_\xi^0)^*$.

THEOREM 7.1.1. *Let (M,τ) be a tracial von Neumann algebra. Then*

$$L(M) = LC(M) = R(M)'$$
$$JL(M)J = R(M) = RC(M) = L(M)'.$$

In particular, for $\xi \in L^2(M,\tau)$, the closed densely defined operator L_ξ (resp. R_ξ) is bounded if and only if $\xi \in \widehat{M}$.

PROOF. We have obviously $L(M) \subset LC(M)$ and $R(M) \subset RC(M)$. Let us show that $L_\xi \circ R_\eta = R_\eta \circ L_\xi$ for $\xi \in LC(M)$ and $\eta \in RC(M)$. Let (x_n) and (y_n) be sequences in M such that $\lim_n \widehat{x_n} = \xi$ and $\lim_n \widehat{y_n} = \eta$. Then for $a, b \in M$, we have

$$\left\langle \hat{b}, x_n \hat{a} y_p \right\rangle = \left\langle R_b \widehat{x_n^*}, L_a \widehat{y_p} \right\rangle,$$

so that $\lim_{n,p\to\infty} \left\langle \hat{b}, x_n \hat{a} y_p \right\rangle = \langle R_b(J\xi), L(a)\eta \rangle$. But

$$\begin{aligned}\lim_n \left\langle \hat{b}, x_n \hat{a} y_p \right\rangle &= \lim_n \left\langle \hat{b}, R_{a y_p} \widehat{x_n} \right\rangle \\ &= \left\langle \hat{b}, R_{a y_p} \xi \right\rangle = \left\langle \hat{b}, L_\xi (\widehat{a y_p}) \right\rangle = \left\langle L_\xi^* \hat{b}, L_a \widehat{y_p} \right\rangle\end{aligned}$$

and therefore we have

$$\lim_p \lim_n \left\langle \hat{b}, x_n \hat{a} y_p \right\rangle = \left\langle L_\xi^* \hat{b}, R_\eta \hat{a} \right\rangle.$$

Similarly, we get

$$\lim_n \lim_p \left\langle \hat{b}, x_n \hat{a} y_p \right\rangle = \left\langle R_\eta^* \hat{b}, L_\xi \hat{a} \right\rangle.$$

It follows that

$$\left\langle \hat{b}, L_\xi R_\eta \hat{a} \right\rangle = \left\langle \hat{b}, R_\eta L_\xi \hat{a} \right\rangle$$

and we conclude that L_ξ and R_η commute. Hence we have

$$\begin{aligned} & L(M) \subset LC(M) \subset RC(M)' \subset R(M)' \\ \text{and} \quad & R(M) \subset RC(M) \subset LC(M)' \subset L(M)'. \end{aligned}$$

Let us show that $R(M)' \subset LC(M)$. We take $T \in R(M)'$ and put $\xi = T\hat{1}$. Then for $x \in M$, we have

$$T\hat{x} = TR_x \hat{1} = R_x T \hat{1} = R_x \xi = \xi x = L_\xi^0 \hat{x}.$$

Hence $T = L_\xi^0$ on $\widehat{M}$ and therefore $T = L_\xi$ since T is bounded and L_ξ is the closure of its restriction to $\widehat{M}$. Thus we have shown that $LC(M) = R(M)' = RC(M)'$ and similarly we have $RC(M) = L(M)' = LC(M)'$. We conclude the proof, using the bicommutant theorem, as we did in the proof of Theorem 1.3.6. $\square$

7.1.2. The standard bimodule. We now introduce the notion of bimodule over a pair of von Neumann algebras. As we will see in the sequel, this is nowadays an essential tool in the study of these algebras.

Definition 7.1.2. Let M and N be two von Neumann algebras.

(i) A *left M-module* is a Hilbert space $\mathcal{H}$, equipped with a normal unital homomorphism $\pi_l : M \to \mathcal{B}(\mathcal{H})$.
(ii) A *right N-module* is a Hilbert space $\mathcal{H}$, equipped with a normal unital anti-homomorphism $\pi_r : N \to \mathcal{B}(\mathcal{H})$ (*i.e.*, a normal unital representation of the opposite algebra N^{op}).
(iii) A *M-N-bimodule* is a Hilbert space $\mathcal{H}$ which is both a left M-module and a right N-module, such that the representations π_l and π_r commute.

We will sometimes write ${}_M\mathcal{H}$, $\mathcal{H}_N$ and ${}_M\mathcal{H}_N$ to insist on the side of the actions. Usually, for $\xi \in \mathcal{H}$, $x \in M$ and $y \in N$, we will just write $x\xi y$ instead of $\pi_l(x)\pi_r(y)\xi$.

The Hilbert space $L^2(M,\tau)$ is the most basic example of M-M-bimodule. It is called the *trivial* (or *identity*) or *standard M-M-bimodule*. Its structure of M-M-bimodule is given by:

$$\forall x, y \in M, \forall \xi \in L^2(M,\tau), \quad x\xi y = L_x R_y \xi = xJy^*J\xi.$$

7.1.3. Examples of standard representations. Let (M,τ) be a tracial von Neumann algebra and let π be a normal representation on a Hilbert space $\mathcal{H}$, and suppose that there exists a norm-one cyclic vector ξ_0 in $\mathcal{H}$ such that $\omega_{\xi_0} \circ \pi = \tau$. Then π is naturally equivalent to the standard representation. More precisely, let U be the operator from $\pi(M)\xi_0$ into $L^2(M,\tau)$ sending $\pi(x)\xi_0$ onto $\hat{x}$. Then U extends to a unitary operator, still denoted by U, from $\mathcal{H}$ onto $L^2(M,\tau)$ such that $U\pi(x)U^* = \pi_\tau(x)$ for every $x \in M$. Viewed as acting on $\mathcal{H}$, the canonical conjugation operator is defined by $J\pi(x)\xi_0 = \pi(x^*)\xi_0$.

In particular, we remark below that the main examples of von Neumann algebras given in Chapter 1 were indeed in standard form. We keep the notation of this chapter.

(a) First, let us consider the case of the group von Neumann algebra $L(G)$ acting by convolution on $\ell^2(G)$ (see Section 1.3). The natural tracial state τ on $L(G)$ is defined by the cyclic and separating vector $\delta_e \in \ell^2(G)$. Therefore, $L(G)$ is in standard form on $\ell^2(G)$. In this example, J is defined by $J\xi(t) = \overline{\xi(t^{-1})}$ and, for every $t \in G$, we have $J\lambda(t)J = \rho(t)$. It follows that $JL(G)J = R(G)$ and we retrieve the fact that $R(G) = L(G)'$.

(b) Second, let M be the crossed product $L^\infty(X,\mu) \rtimes G = L(A,G)$ relative to a probability measure preserving action $G \curvearrowright (X,\mu)$, where we put $A = L^\infty(X,\mu)$. We use the following convenient notation introduced in Section 1.4: for $f \in L^2(X,\mu)$,

$$fu_g = f \otimes \delta_g \in L^2(X,\mu) \otimes \ell^2(G).$$

The vector $u_e = \hat{1} \otimes \delta_e$ is cyclic for M and defines the canonical trace on M which is therefore in standard form on $L^2(X, \mu) \otimes \ell^2(G)$. The conjugation operator J is defined by

$$J f u_g = \sigma_{g^{-1}}(\overline{f}) u_{g^{-1}}.$$

and it is also straightforward to check that

$$JL\big(\sum_{g \in G} a_g u_g\big)J = R\big(\sum_{g \in G} u_g^* a_g^*\big) = R\big(\sum_{g \in G} \sigma_g(a_{g^{-1}}^*) u_g\big).$$

This shows that $L(A, G)' = R(A, G)$.

When $L^2(X, \mu) \otimes \ell^2(G)$ is identified with $L^2(X \times G, \mu \otimes \lambda)$ (where λ is the counting measure on G), we have

$$J\xi(x, t) = \overline{\xi(t^{-1}x, t^{-1})}.$$

For group actions which are free, this is the formula given in the next paragraph, after identification of $X \times G$ with the graph of the orbit equivalence relation.

(c) Let us consider now the case of a countable probability measure preserving equivalence relation $\mathcal{R}$ on (X, μ). With the notation of Section 1.5.2, the representation of $L(\mathcal{R})$ on $L^2(\mathcal{R}, \nu)$ introduced there is standard since $\mathbf{1}_\Delta$ is a cyclic vector which defines the canonical trace on $L(\mathcal{R})$. For $\xi \in L^2(\mathcal{R}, \nu)$ we have $J\xi(x, y) = \overline{\xi(y, x)}$ and, given $F \in \mathcal{M}_b(\mathcal{R})$, one sees that $JL_F J = R_{F^*}$. Therefore we obtain the equality $L(\mathcal{R})' = R(\mathcal{R})$.

(d) For our last example, we keep the notation of Section 5.3.2. Let (M_1, τ_1), (M_2, τ_2) be two tracial von Neumann algebras. The representation of $(M, \tau) = (M_1, \tau_1) * (M_2, \tau_2)$ on the Hilbert space $\mathcal{H}$ constructed in this section is standard since there is a vector $\xi \in \mathcal{H}$ which induces the trace τ and is cyclic.

The canonical conjugation operator J is defined by $J\xi = \xi$ and

$$J(x_1 \xi_{k_1} \otimes \cdots \otimes x_n \xi_{k_n}) = x_n^* \xi_{k_n} \otimes \cdots \otimes x_1^* \xi_{k_1},$$

for $x_i \in \overset{o}{M}_{k_i}$ with $k_1 \neq k_2 \neq \cdots \neq k_n$. For $x \in M_i$, we have $J\lambda_i(x)J = \rho_i(J_i x J_i)$, where J_i is the canonical conjugation operator on $L^2(M_i, \tau_i)$. It follows that $M' = JMJ$ is the von Neumann algebra N defined in Section 5.3.2.

7.2. The algebra of affiliated operators

Let (M, τ) be a tracial von Neumann algebra on a Hilbert space $\mathcal{H}$. We show in this section that the closed densely defined operators on $\mathcal{H}$ affiliated with M behave nicely.

7.2.1. Closed densely defined operators. We recall here a few important facts concerning unbounded operators and the spectral theory of (unbounded) self-adjoint operators[2].

Let x be a self-adjoint operator on a Hilbert space $\mathcal{H}$, that is, a densely defined operator (possibly unbounded) such that $x = x^*$. Its spectrum $\mathrm{Sp}(x)$ is a closed subset of $\mathbb{R}$. The bounded Borel functional calculus defines an algebraic $*$-homomorphism $f \mapsto f(x)$ from the algebra $B_b(\mathrm{Sp}(x))$ of bounded Borel complex-valued functions on $\mathrm{Sp}(x)$ into $\mathcal{B}(\mathcal{H})$.

This functional calculus enables the construction of the *spectral measure* $E : \Omega \to E(\Omega) = \mathbf{1}_\Omega(x)$ of x, defined on the Borel subsets of $\mathrm{Sp}(x)$. As in Section 2.2, setting $E_t = E(]-\infty, t])$, we use the notation

$$f(x) = \int_{\mathrm{Sp}(x)} f(t)\,\mathrm{d}E_t.$$

The functional calculus may be extended to the algebra $B(\mathrm{Sp}(x))$ of all Borel complex-valued functions on $\mathrm{Sp}(x)$, as follows. Let $f \in B(\mathrm{Sp}(x))$. Then $f(x)$ is the operator with domain

$$\mathrm{Dom}\,(f(x)) = \left\{\eta \in \mathcal{H} : \int_{\mathrm{Sp}(x)} |f(t)|^2\,\mathrm{d}\langle \eta, E_t\eta\rangle < +\infty\right\}, \tag{7.1}$$

and defined, for $\xi \in \mathcal{H}$ and $\eta \in \mathrm{Dom}\,(f(x))$ by

$$\langle \xi, f(x)\eta\rangle = \int_{\mathrm{Sp}(x)} f(t)\ \mathrm{d}\langle \xi, E_t\eta\rangle.$$

We get a closed densely defined operator, which is self-adjoint whenever f is real-valued. Again, we write $f(x) = \int_{\mathrm{Sp}(x)} f(t)\ \mathrm{d}E_t$. In particular, we have $x = \int_{\mathrm{Sp}(x)} t\ \mathrm{d}E_t$. It is useful to have in mind the following formula:

$$\forall \eta \in \mathrm{Dom}\,(f(x)), \quad \|f(x)\eta\|^2 = \int_{\mathrm{Sp}(x)} |f(t)|^2\,\mathrm{d}\langle \eta, E_t\eta\rangle. \tag{7.2}$$

We say that $y \in \mathcal{B}(\mathcal{H})$ commutes with an unbounded operator z if $yz \subset zy$, that is, $\mathrm{Dom}\,(yz) \subset \mathrm{Dom}\,(zy)$ and $zy = yz$ on $\mathrm{Dom}\,(yz)$. Equivalently, we have $y\big(\mathrm{Dom}\,(z)\big) \subset \mathrm{Dom}\,(z)$ and $zy = yz$ on $\mathrm{Dom}\,(z)$. An operator $y \in \mathcal{B}(\mathcal{H})$ commutes with a self-adjoint operator x if and only if it commutes with all its spectral projections $E(\Omega)$, and if so, it commutes with $f(x)$ for every $f \in B(\mathrm{Sp}(x))$.

As in the bounded case, the polar decomposition is a useful tool.

PROPOSITION 7.2.1. *Let x be a closed densely defined operator on $\mathcal{H}$. Then*

(i) *x^*x is a positive self-adjoint operator;*
(ii) *there exists a unique partial isometry u such that $x = u|x|$ and $\mathrm{Ker}\,x = \mathrm{Ker}\,u$ where, by definition, $|x| = (x^*x)^{1/2}$.*

[2]For details we refer to [**RS80**, Chapter VIII].

The expression $x = u|x|$ is called the *polar decomposition* of x. The delicate part of the proof is to show that x^*x is a self-adjoint operator[3]. Morever x^*x is positive, that is $\langle \xi, x^*x\xi \rangle \geq 0$ for $\xi \in \mathrm{Dom}\,(x^*x)$. Then $|x|$ is defined via the functional calculus, and the rest of the proof is easy[4]. We recall that u^*u is the projection $s_r(x)$ on $(\mathrm{Ker}\, x)^\perp$ and that uu^* is the projection $s_l(x)$ on the norm closure of $\mathrm{Im}\, x$. These projections $s_r(x)$ and $s_l(x)$ are called respectively the *right* and *left support* of x.

Note also that an operator $y \in \mathcal{B}(\mathcal{H})$ commutes with x if and only if it commutes with u and $|x|$.

7.2.2. Operators affiliated with a tracial von Neumann algebra.

Definition 7.2.2. Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$. We say that an (unbounded) operator x is *affiliated with* M, and we write $x \,\tilde{\epsilon}\, M$, if for every unitary operator $u \in \mathcal{U}(M')$, we have $ux = xu$.

This means that the operators ux and xu have the same domains and coincide on this common domain. In particular, we have

$$u(\mathrm{Dom}\,(x)) = \mathrm{Dom}\,(x)$$

for every $u \in \mathcal{U}(M')$. Since every $y \in M'$ is a linear combination of four unitary operators in M', we see that $x \,\tilde{\epsilon}\, M$ if and only if x commutes with every $y \in M'$.

We denote by $\widetilde{M}$ the set of all *closed densely defined operators affiliated with* M. Let us record the following consequence of the bicommutant theorem and of the results recalled in the previous section.

Proposition 7.2.3. *Let M be a von Neumann algebra on $\mathcal{H}$. Let x be a closed densely defined operator on $\mathcal{H}$ and let $x = u|x|$ be its polar decomposition. Then $x \in \widetilde{M}$ if and only if u and the spectral projections of $|x|$ are in M.*

In particular, when $x \in \widetilde{M}$, its left and right supports $s_l(x)$ and $s_r(x)$ belong to M.

We now consider the case where M is equipped with a faithful normal tracial state τ. We will see that, under this assumption, $\widetilde{M}$ behaves nicely.

Proposition 7.2.4. *Let (M, τ) be a tracial von Neumann algebra on $\mathcal{H}$ and let $x, y \in \widetilde{M}$ be such that $x \subset y$. Then $x = y$.*

Proof. Recall that $x \subset y$ means that $\mathrm{Dom}\,(x) \subset \mathrm{Dom}\,(y)$ with $x = y|_{\mathrm{Dom}\,(x)}$. Let $G(x) = \{(\xi, x\xi) : \xi \in \mathrm{Dom}\,(x)\}$ be the graph of x. Note that x is a closed operator precisely when $G(x)$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H} = \mathcal{H}^{\oplus 2}$. Similarly, we introduce the graph $G(y)$ of y. Let $[G(x)]$ and $[G(y)]$ be the orthogonal projections of $\mathcal{H}^{\oplus 2}$ onto $G(x)$ and $G(y)$ respectively.

[3]See for instance [**Yos95**, Theorem 2, p. 200].

[4]For another proof see [**RS80**, Theorem VIII.32].

The algebra $M_2(M)$ of two by two matrices with entries in M is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H}^{\oplus 2})$. Its commutant is

$$M_2(M)' = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in M' \right\}.$$

We claim that the projections $[G(x)]$ and $[G(y)]$ are in $M_2(M)$. Indeed, since $x \,\tilde{\epsilon}\, M$, for every $u' \in \mathcal{U}(M')$ we have $u'x = xu'$, from which we get

$$\begin{pmatrix} u' & 0 \\ 0 & u' \end{pmatrix} [G(x)] = [G(x)] \begin{pmatrix} u' & 0 \\ 0 & u' \end{pmatrix}.$$

It follows that $[G(x)] \in M_2(M)'' = M_2(M)$ and similarly for $[G(y)]$.

Set $p_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(M)$. Then p_1 is the left support of $p_1[G(x)]$ and $[G(x)]$ is its right support. The same observation holds for $[G(y)]$. It follows that, in $M_2(M)$,

$$[G(x)] \sim p_1 \sim [G(y)].$$

Now, since $x \subset y$, we have $G(x) \subset G(y)$ and therefore $[G(x)] \leq [G(y)]$. Since $M_2(M)$ has a faithful tracial state, we conclude that $[G(x)] = [G(y)]$, that is, $G(x) = G(y)$, whence $x = y$. $\square$

Let x, y be two closed densely defined operators on $\mathcal{H}$. Then

$$\mathrm{Dom}\,(x+y) = \mathrm{Dom}\,(x) \cap \mathrm{Dom}\,(y).$$

In general this space is not dense in $\mathcal{H}$ and can even be reduced to 0 (see Exercise 7.3). When $x, y \in \widetilde{M}$, we will see that $x+y$ is a densely defined closable operator and we will be able to define an addition in $\widetilde{M}$. Similarly, we will define a product. These facts rely on the following lemmas.

Lemma 7.2.5. *Let $x \in \widetilde{M}$. Then, for every $\varepsilon > 0$, there exists a projection $p \in M$ such that $p\mathcal{H} \subset \mathrm{Dom}\,(x)$ and $\tau(1-p) \leq \varepsilon$.*

Proof. Let $x = u|x|$ be the polar decomposition of x and denote by p_n the spectral projection of $|x|$ relative to $[0,n]$. Then, we have $p_n\mathcal{H} \subset \mathrm{Dom}\,(x)$ and $\lim_n \tau(p_n) = 1$. We choose n large enough so that $1 - \tau(p_n) \leq \varepsilon$. $\square$

Lemma 7.2.6. *Let V be a vector subspace of $\mathcal{H}$ such that for every $\varepsilon > 0$, there exists a projection $p \in M$ with $p\mathcal{H} \subset V$ and $\tau(1-p) \leq \varepsilon$. Then V is dense in $\mathcal{H}$*

Proof. It suffices to construct an increasing sequence (q_n) of projections with $\bigvee_n q_n = 1$ and $q_n\mathcal{H} \subset V$. For every integer $k \geq 1$, we choose a projection $p_k \in M$ such that $p_k\mathcal{H} \subset V$ and $\tau(1-p_k) \leq 2^{-k}$. We put

$$q_n = \bigwedge_{k>n} p_k.$$

Then we have $q_n\mathcal{H} \subset V$ and

$$\tau(1-q_n) = \tau\Big(\bigvee_{k>n}(1-p_k)\Big) \leq \sum_{k>n} \tau(1-p_k) \leq \sum_{k>n} 2^{-k} = 2^{-n}$$

by Lemma 7.2.7 below. Since τ is normal, we get $\tau(1-\bigvee_n q_n)=0$, whence $\bigvee_n q_n = 1$. $\square$

Lemma 7.2.7. *Let $(p_i)_{i\in I}$ be a family of projections in (M,τ). Then*

$$\tau(\bigvee_{i\in I} p_i) \leq \sum_{i\in I} \tau(p_i). \tag{7.3}$$

Proof. Given two projections $p,q\in M$, by Proposition 2.4.5 we have

$$\tau(p\vee q - p) = \tau(q - p\wedge q),$$

and therefore $\tau(p\vee q)\leq \tau(p)+\tau(q)$. By induction, we get the inequality (7.3) when I is a finite set, and the general case uses the normality of τ and the fact that

$$\bigvee_{i\in I} p_i = \bigvee_F (\bigvee_{i\in F} p_i)$$

where F ranges over the finite subsets of I. $\square$

Theorem 7.2.8. *Let (M,τ) be a tracial von Neumann algebra on a Hilbert space $\mathcal{H}$.*

(i) *Let $x\in\widetilde{M}$. Then $x^*\in\widetilde{M}$.*
(ii) *Let $x,y\in\widetilde{M}$. Then $x+y$ and xy are closable and densely defined, and their closures belong to $\widetilde{M}$.*
(iii) *$\widetilde{M}$, equipped with the three above operations, is a $*$-algebra.*

Proof. (i) is obvious. Let us show that if $x,y\in\widetilde{M}$, then $x+y$ is densely defined. To that purpose, we show that $\mathrm{Dom}\,(x+y)$ satisfies the condition stated in Lemma 7.2.6. Given $\varepsilon>0$, let $p,q\in\mathcal{P}(M)$ be such that $p\mathcal{H}\subset\mathrm{Dom}\,(x)$, $q\mathcal{H}\subset\mathrm{Dom}\,(y)$, and $\tau(1-p)\leq\varepsilon/2$, $\tau(1-q)\leq\varepsilon/2$. Then we have

$$(p\wedge q)\mathcal{H} = p\mathcal{H}\cap q\mathcal{H}\subset \mathrm{Dom}\,(x)\cap\mathrm{Dom}\,(y) = \mathrm{Dom}\,(x+y),$$

and

$$\begin{aligned}\tau(1-p\wedge q) &= \tau((1-p)\vee(1-q))\\ &\leq \tau(1-p)+\tau(1-q)\leq\varepsilon.\end{aligned}$$

Hence $x+y$ is densely defined and of course affiliated with M. Since x^* and y^* are also affiliated with M, we get that x^*+y^* is densely defined. Since $x+y\subset(x^*+y^*)^*$, we see that $x+y$ is closable. We denote by $x\dot{+}y$ its closure. It is a routine verification to check that the closure of an operator afiliated with M retains the same property. Therefore, $x\dot{+}y\in\widetilde{M}$.

To prove that xy is closable, we consider the projections p,q as above. The operator yq is closed and everywhere defined, hence bounded. Let r denote the projection on the kernel of $(1-p)yq$. Then $r\mathcal{H}\subset\mathrm{Dom}\,(xyq)$ and thus $(q\wedge r)\mathcal{H}\subset\mathrm{Dom}\,(xy)$. Note that $1-r\precsim 1-p$. It follows that

$$\tau(1-(q\wedge r)) = \tau((1-q)\vee(1-r))\leq\tau(1-q)+\tau(1-p)\leq\varepsilon.$$

Thus, xy is densely defined, and we conclude as for $x+y$ that xy is closable. Its closure $x^{\cdot}y$ belongs to $\widetilde{M}$.

It remains to show that these operations give to $\widetilde{M}$ the structure of a $*$-algebra. Let us explain for instance how to prove the distributivity property $x^{\cdot}(y\dotplus z) = (x^{\cdot}y)\dotplus(x^{\cdot}z)$. From the inclusion $xy + xz \subset x(y+z)$, we deduce that $(x^{\cdot}y)\dotplus(x^{\cdot}z) \subset x^{\cdot}(y\dotplus z)$. Then we use Proposition 7.2.4 to deduce the equality. $\square$

For simplicity of notation, in the sequel, we will often write $x+y$ instead of $x\dotplus y$, and similarly for the product.

Example 7.2.9. Let $M = L^\infty(X,\mu)$ where (X,μ) is a probability measure space and take for τ the integral with respect to μ. We consider the standard representation of M on $L^2(M,\mu)$. Let $f : X \to \mathbb{C}$ be a measurable function. Denote by M_f the multiplication operator by f, with $\mathrm{Dom}\,(M_f) = \big\{\xi \in L^2(M,\mu) : f\xi \in L^2(M,\mu)\big\}$. Then M_f is closed, densely defined, and affiliated with M. Conversely, every closed densely defined operator affiliated with M is of this form[5]. Therefore, $\widetilde{M}$ can be identified with the $*$-algebra of complex-valued measurable functions on X (modulo null sets).

In particular, the spaces $L^p(X,\mu)$, $p \in [1,+\infty]$, are canonically embedded in $\widetilde{M}$. This property still holds for any tracial von Neumann algebra. We will study this fact for L^2 in the next section.

7.3. Square integrable operators

In this section, (M,τ) is a tracial von Neumann algebra represented in standard form on $L^2(M,\tau)$.

7.3.1. Square integrable operators.

Definition 7.3.1. A closed densely defined operator x on $L^2(M,\tau)$ is said to be *square integrable* if it is affiliated with M and is such that $\hat{1} \in \mathrm{Dom}\,(x)$.

Given $\xi \in L^2(M,\tau)$, we have introduced in Section 7.1.1 the closed densely defined operator L_ξ and proved (Theorem 7.1.1) that this operator is bounded if and only if $\xi \in \widehat{M} \subset L^2(M,\tau)$. In the general case, the operator L_ξ has the following characterisation.

Theorem 7.3.2. *For every $\xi \in L^2(M,\tau)$, the operator L_ξ is square integrable. Moreover, the map $\xi \mapsto L_\xi$ is a linear bijection from $L^2(M,\tau)$ onto the space of square integrable operators.*

Proof. Let $\xi \in L^2(M,\tau)$. Every unitary operator in $M' = R(M)$ is of the form R_u, where $u \in \mathcal{U}(M)$. Then, for $x \in M$, we have

$$R_u L_\xi (R_u)^* \hat{x} = R_u L_\xi \widehat{xu^*} = R_u R_{xu^*}\xi = R_{xu^*u}\xi = R_x\xi = L_\xi\hat{x}.$$

[5]See for instance [**KR97**, Theorem 5.6.4].

Since $\widehat{M}$ is an essential domain of both operators $R_u L_\xi (R_u)^*$ and L_ξ, they coincide. Therefore, L_ξ is affiliated with M. Moreover, $\hat{1}$ is in the domain of L_ξ, and so L_ξ is square integrable.

Let us show that the map $\xi \mapsto L_\xi$ is linear. For $\xi, \eta \in L^2(M,\tau)$, we have

$$L^0_{\xi+\eta} = L^0_\xi + L^0_\eta \subset L_\xi \dot{+} L_\eta,$$

and therefore the closure $L_{\xi+\eta}$ of $L^0_{\xi+\eta}$ is such that $L_{\xi+\eta} \subset L_\xi \dot{+} L_\eta$. Then, Proposition 7.2.4 implies the equality.

The map $\xi \mapsto L_\xi$ is obviously injective since $\xi = L_\xi \hat{1}$. It remains to show the surjectivity. Let T be a square integrable operator, and set $\xi = T\hat{1}$. Then for $x \in M$, we have

$$L_\xi \hat{x} = R_x T \hat{1} = T R_x \hat{1} = T\hat{x}.$$

We deduce that $L_\xi \subset T$ and again $L_\xi = T$, thanks to Proposition 7.2.4. $\square$

We will freely consider the elements of $L^2(M,\tau)$ as operators. Under this identification, for $x \in M$ and $\xi \in L^2(M,\tau)$, we may view $x\xi$ and ξx as the product of two operators in $\widetilde{M}$. The adjoint corresponds to the conjugation operator J introduced in Section 7.1.1: for $\xi \in L^2(M,\tau)$, we have $(L_\xi)^* = L_{J\xi}$. Indeed, let $x, y \in M$. Then,

$$\begin{aligned}\langle \hat{x}, L_\xi \hat{y}\rangle &= \langle \hat{x}, R_y \xi\rangle = \left\langle \widehat{xy^*}, \xi \right\rangle \\ &= \left\langle J\xi, \widehat{yx^*} \right\rangle = \langle R_x J\xi, \hat{y}\rangle = \langle L_{J\xi}\hat{x}, \hat{y}\rangle.\end{aligned}$$

We deduce that $L_{J\xi} \subset (L_\xi)^*$, whence $L_{J\xi} = (L_\xi)^*$ by Proposition 7.2.4.

It is therefore natural to write $J\xi = \xi^*$ and to say that ξ is *self-adjoint* if $\xi = J\xi$, or equivalently if $L_\xi = (L_\xi)^*$.

PROPOSITION 7.3.3. *Let $\xi \in L^2(M,\tau)$. The following conditions are equivalent:*

(i) *ξ is self-adjoint;*
(ii) *$\langle \xi, \hat{x}\rangle \in \mathbb{R}$ for every $x \in M_{s.a}$;*
(iii) *there exists a sequence (x_n) in $M_{s.a}$ such that $\lim_n \|\widehat{x_n} - \xi\|_2 = 0$.*

PROOF. For $\xi \in L^2(M,\tau)$ and $x \in M_{s.a}$, we have

$$\langle \xi, \hat{x}\rangle = \overline{\langle J\xi, \hat{x}\rangle}$$

from which we immediately deduce the equivalence between (i) and (ii).

Let us now show that (i) $\Rightarrow$ (iii). Let $\xi = J\xi \in L^2(M,\tau)$. There exists a sequence (x_n) in M such that $\lim_n \|\widehat{x_n} - \xi\|_2 = 0$. We put $y_n = (x_n + x_n^*)/2 \in M_{s.a}$. Since $\left\|\widehat{x_n^*} - J\xi\right\|_2 = \|\widehat{x_n} - \xi\|_2$, we see that $\lim_n \|\widehat{y_n} - \xi\|_2 = 0$. The converse is also straightforward. $\square$

Hence, the real subspace $L^2(M,\tau)_{s.a}$ of self-adjoint elements is the norm closure of $\widehat{M_{s.a}}$.

We say that an element $\xi \in L^2(M,\tau)$ is *positive* if the corresponding operator L_ξ is self-adjoint and positive, *i.e.*, $\langle \eta, L_\xi \eta\rangle \geq 0$ for all $\eta \in \mathrm{Dom}\,(L_\xi)$. We will denote by $L^2(M,\tau)_+$ the subset of such ξ.

PROPOSITION 7.3.4. *Let $\xi \in L^2(M,\tau)$. The following conditions are equivalent:*

(i) *ξ is positive;*
(ii) *$\langle \hat{x}, \xi\rangle \geq 0$ for every $x \in M_+$;*
(iii) *there exists a sequence (x_n) in M_+ such that* $\lim_n \|\widehat{x_n} - \xi\|_2 = 0$.

PROOF. We remark that ξ is positive if and only if $\langle \hat{x}, L_\xi \hat{x}\rangle \geq 0$ for every $x \in M$. We write $x \in M_+$ as $x = yy^*$, and get

$$\langle \hat{x}, \xi\rangle = \langle \hat{y}, L_\xi \hat{y}\rangle,$$

from which we deduce the equivalence between (i) and (ii).

Let us prove that (iii) $\Rightarrow$ (i). Suppose that there exists a sequence (x_n) in M_+ such that $\lim_n \|\widehat{x_n} - \xi\|_2 = 0$. Then we have $\xi = J\xi$ by the previous proposition. Morever, if we write x_n as $x_n = y_n^* y_n$ we have, for every $x \in M$,

$$\langle \hat{x}, x_n \hat{x}\rangle = \langle \widehat{y_n x}, \widehat{y_n x}\rangle \geq 0,$$

and

$$\langle \hat{x}, L_\xi \hat{x}\rangle = \langle \hat{x}, R_x \xi\rangle = \lim_n \langle \hat{x}, R_x \widehat{x_n}\rangle = \lim_n \langle \hat{x}, x_n \hat{x}\rangle \geq 0.$$

It remains to show that (i) $\Rightarrow$ (iii). Assume that ξ is positive and for $n \in \mathbb{N}$, denote by e_n the spectral projection of ξ relative to $[0,n]$. Then $e_n \xi \in M_+$ and we have $\lim_n \|e_n \xi - \xi\|_2 = 0$. $\square$

We remark that the polar decomposition $\xi = u|\xi|$ allows us to write any element in $L^2(M,\tau)$ as the product of a partial isometry in M and an element of $L^2(M,\tau)_+$. Let us observe also that if $\xi \in L^2(M,\tau)$ is self-adjoint, then $\xi_+ = \frac{1}{2}(|\xi| + \xi)$ and $\xi_- = \frac{1}{2}(|\xi| - \xi)$ are in $L^2(M,\tau)_+$. Thus, every element of $L^2(M)$ is a linear combination of four elements in $L^2(M)_+$.

REMARK 7.3.5. Let $x \in \widetilde{M}$ and let E be the spectral measure of $|x|$. Then x is square integrable if and only if $\hat{1} \in \mathrm{Dom}\,(|x|)$, that is, if and only if

$$\int_{\mathbb{R}} |t|^2 \,\mathrm{d}\tau(E_t) = \int_{\mathbb{R}} |t|^2 \,\mathrm{d}\langle \hat{1}, E_t \hat{1}\rangle < +\infty,$$

and then, after having identified x with $x\hat{1}$, we have

$$\|x\|_2^2 = \int_{\mathbb{R}} |t|^2 \,\mathrm{d}\tau(E_t) \tag{7.4}$$

(see (7.1) and (7.2)).

7.3.2. The Powers-Størmer inequality. For $a \in M$, we will denote by τ_a the linear functional $x \mapsto \tau(ax)$ defined on M. Note that $\tau_a = \omega_{\hat{1},\hat{a}}$. If $a \in M_+$, then τ_a is a positive linear functional and we have $\tau_a \le \|a\|\tau$, since

$$\tau_a(x) = \tau(x^{1/2}ax^{1/2}) \le \|a\|\tau(x)$$

for every $x \in M_+$. We have the (easy) Radon-Nikodým type converse.

Proposition 7.3.6 (**Little Radon-Nikodým theorem**). *Let φ be a positive linear functional on M and assume the existence of $\lambda \in \mathbb{R}_+$ such that $\varphi \le \lambda\tau$. Then there exists a unique $a \in M$ with $\varphi = \tau_a$, and we have $0 \le a \le \lambda 1$.*

Proof. We may assume that $\lambda = 1$. We define a linear functional ψ on $\widehat{M}$ by $\psi(\hat{x}) = \varphi(x)$. By the Cauchy-Schwarz inequality, we have

$$|\psi(\hat{x})|^2 \le \varphi(1)\varphi(x^*x) \le \tau(x^*x) = \|\hat{x}\|_2^2,$$

and so ψ extends to a continuous linear functional on $L^2(M,\tau)$, still denoted by ψ. Therefore, there exists ξ in $L^2(M,\tau)$ such that $\psi(\eta) = \langle \xi, \eta\rangle$ for every $\eta \in L^2(M,\tau)$, and in particular $\varphi(x) = \langle \xi, \hat{x}\rangle$. Using Proposition 7.3.4, we see that ξ is positive. Similarly, we have

$$\left\langle \hat{1} - \xi, \hat{x}\right\rangle = \tau(x) - \varphi(x) \ge 0$$

for all $x \ge 0$, and so $\hat{1} - \xi \ge 0$. Since $0 \le \xi \le \hat{1}$, we get that $\xi = \hat{a}$ with $a \in M$ and $0 \le a \le 1$, whence $\varphi = \tau_a$.

Assume that $\varphi = \tau_b$ for another $b \in M$. We get $\tau((a-b)(a-b)^*) = 0$, and therefore $a = b$. $\square$

The element a is called the *Radon-Nikodým derivative* of φ with respect to τ.

For $\xi \in L^2(M,\tau)$, recall that ω_ξ is the positive linear functional $x \mapsto \langle \xi, x\xi\rangle$ on M. The following very useful result is a substitute for the obvious fact in the commutative case, saying that whenever ξ, η are two positive functions, then $|\xi - \eta|^2 \le \left|\xi^2 - \eta^2\right|$.

Theorem 7.3.7 (**Powers-Størmer inequality**). *We have*

$$\|\xi - \eta\|_2^2 \le \|\omega_\xi - \omega_\eta\| \le \|\xi - \eta\|_2\|\xi + \eta\|_2, \tag{7.5}$$

for every $\xi, \eta \in L^2(M,\tau)_+$.

Proof. The right hand side inequality follows immediately from the identity

$$\omega_\xi - \omega_\eta = (1/2)\big(\omega_{\xi-\eta,\xi+\eta} + \omega_{\xi+\eta,\xi-\eta}\big).$$

Let us prove the left hand side inequality. We begin by the study of the case where $\xi = \hat{a}$ and $\eta = \hat{b}$ with $a, b \in M_+$. Then $\omega_\xi = \tau_{a^2}$, $\omega_\eta = \tau_{b^2}$ and we have to prove that

$$\|\hat{a} - \hat{b}\|_2^2 \le \|\tau_{a^2} - \tau_{b^2}\|.$$

Let p, q be the spectral projections of $a - b$ corresponding respectively to the intervals $[0, +\infty[$ and $]-\infty, 0[$ so that $a - b = (p - q)|a - b|$.

Since $\|p-q\| \le 1$, we get the inequality

$$\big|\tau\big((a^2-b^2)(p-q)\big)\big| = |(\tau_{a^2}-\tau_{b^2})(p-q)| \le \|\tau_{a^2}-\tau_{b^2}\|.$$

The goal of the rest of the proof is to establish the inequality

$$\|\hat{a}-\hat{b}\|_2^2 \le \big|\tau\big((a^2-b^2)(p-q)\big)\big|.$$

We first claim that

$$\tau\big((a^2-b^2)p\big) \ge \tau\big((a-b)^2 p\big). \tag{7.6}$$

Indeed we have

$$\begin{aligned}\tau\big((a^2-b^2)p\big)-\tau\big((a-b)^2p\big) &= \tau\big(b(a-b)p\big)+\tau\big((a-b)bp\big)\\ &= 2\tau\big(b^{1/2}(a-b)pb^{1/2}\big) \ge 0,\end{aligned}$$

since $(a-b)p \ge 0$.

Similarly, we get

$$\tau\big((b^2-a^2)q\big) \ge \tau\big((b-a)^2 q\big). \tag{7.7}$$

Adding up (7.6) and (7.7), we obtain

$$\begin{aligned}\tau\big((a^2-b^2)(p-q)\big) &= \tau\big((a^2-b^2)p\big)+\tau\big((b^2-a^2)q\big)\\ &\ge \tau\big((p+q)(a-b)^2\big) = \tau\big((a-b)^2\big) = \|\hat{a}-\hat{b}\|_2^2.\end{aligned}$$

We now consider the general case. We chose sequences (a_n) and (b_n) in M_+ such that $\lim_n \|\widehat{a_n}-\xi\|_2 = 0$ and $\lim_n \|\widehat{b_n}-\eta\|_2 = 0$. Passing to the limit in the inequality

$$\|\widehat{a_n}-\widehat{b_n}\|_2^2 \le \|\omega_{\widehat{a_n}}-\omega_{\widehat{b_n}}\|$$

gives the first inequality of (7.5). □

The following theorem says that any normal positive linear functional on M is canonically written as a vector state.

Theorem 7.3.8. *The map $\xi \mapsto \omega_\xi$ is a homeomorphism from $L^2(M,\tau)_+$ onto the cone of all normal positive linear functionals on M.*

Proof. The injectivity is a consequence of the left inequality in (7.5). Let us prove the surjectivity. Let φ be positive normal linear functional on M. We first claim that for every $\varepsilon > 0$, there is $a \in M_+$ such that $\|\varphi - \omega_{\hat{a}}\| \le \varepsilon$. Indeed, by Theorem 2.5.5 (3) there exist $\xi_1, \dots, \xi_m$ in $L^2(M,\tau)$ such that $\|\varphi - \sum_{k=1}^m \omega_{\xi_k}\| \le \varepsilon/2$ and so we see that we may find $a_1, \dots, a_m \in M$ with

$$\|\varphi - \sum_{k=1}^m \omega_{\widehat{a_k}}\| \le \varepsilon.$$

But $\sum_{k=1}^m \omega_{\widehat{a_k}} = \tau_b$ with $b = \sum_{k=1}^m a_k a_k^* \ge 0$. To conclude our claim it suffices to put $a = b^{1/2}$.

Now, we take a sequence (b_n) in M_+ such that $\lim_n \omega_{\widehat{b_n}} = \varphi$. Thanks to the Powers-Størmer inequality, we see that $(\widehat{b_n})$ is a Cauchy sequence which, therefore, converges to an element $\xi \in L^2(M,\tau)_+$, and then $\varphi = \omega_\xi$.

The fact that $\xi \mapsto \omega_\xi$ is a homeomorphism follows from Theorem 7.3.7. $\square$

As a consequence, a positive normal linear functional is w.o. continuous on M in standard representation, (and not only when restricted to its unit ball). For the abelian case, see Remark 2.5.13.

Proposition 7.3.9. *Let (M,τ) be a tracial von Neumann algebra and let Z be its center. The restriction of the trace to Z is still denoted by τ. We identify $L^2(Z,\tau)$ to a subspace of $L^2(M,\tau)$. The map $\xi \in L^2(Z,\tau)_+ \mapsto \omega_\xi$ is a bijection onto the cone of normal traces on M. In particular, if τ_1 and τ_2 are two normal traces on M with the same restriction to Z, then $\tau_1 = \tau_2$.*

Proof. For $\xi \in L^2(M,\tau)_+$, the functional ω_ξ is a trace if and only if the positive (possibly unbounded) operator ξ commutes with M. One only needs to observe that, for $x \in M$ and $u \in \mathcal{U}(M)$,

$$\omega_\xi(uxu^*) = \omega_{u^*\xi u}(x)$$

with $u^*xu \in L^2(M,\tau)_+$. The proposition follows immediately. $\square$

7.4. Integrable operators. The predual

Still, (M,τ) is a tracial von Neumann algebra represented in standard form on $L^2(M,\tau)$.

Definition 7.4.1. Let $x \in \widetilde{M}$ and let E be the spectral measure of $|x|$. We say that x is *integrable* if

$$\int_{\mathbb{R}} |t| \, \mathrm{d}\tau(E_t) = \int_{\mathbb{R}} |t| \, \mathrm{d}\langle \hat{1}, E_t \hat{1} \rangle < +\infty,$$

where $E_t = E(]-\infty, t])$ as always.

We denote by $L^1(M,\tau)$ the set of integrable operators. More generally, for $p \geq 1$, we may define $L^p(M,\tau)$ as the set of $x \in \widetilde{M}$ with $\int_{\mathbb{R}} |t|^p \, \mathrm{d}\tau(E_t) < +\infty$. Of course, we set $L^\infty(M,\tau) = M$. These L^p-spaces behave as in the commutative case. In the previous section, we have studied the case $p = 2$. The additional case $p = 1$ will be enough for our needs. Obviously, we have

$$M \subset L^2(M,\tau) \subset L^1(M,\tau).$$

We will see that $L^1(M,\tau)$ is a Banach space whose dual is M.

7.4.1. Integration on $\widetilde{M}_+$. First, we extend $\tau_{|M_+}$ to a map τ from the cone $\widetilde{M}_+$ of positive elements of $\widetilde{M}$ into $[0,+\infty]$, by the formula

$$\tau(x) = \int_{\mathbb{R}_+} t \,\mathrm{d}\tau(E_t) = \int_{\mathbb{R}_+} t \,\mathrm{d}\langle \hat{1}, E_t \hat{1}\rangle,$$

where E is the spectral measure of x.

Lemma 7.4.2. *For $x, y \in \widetilde{M}_+$ and $\lambda \geq 0$, we have*

$$\tau(x+y) = \tau(x) + \tau(y), \quad \tau(\lambda x) = \lambda\tau(x).$$

Moreover, for every $x \in \widetilde{M}$, we have

$$\tau(x^*x) = \tau(xx^*). \tag{7.8}$$

Proof. We denote by e_n the spectral projection of $x+y$ corresponding to the interval $[0,n]$. Then we have

$$\tau(x+y) = \lim_n \tau((x+y)e_n) = \lim_n \tau(e_n x e_n + e_n y e_n).$$

But the operators $e_n x e_n$ and $e_n y e_n$ are bounded and both sequences $(\tau(e_n x e_n))_n$, $(\tau(e_n y e_n))_n$ are increasing. Thus

$$\begin{aligned}\tau(x+y) &= \lim_n(\tau(e_n x e_n) + \tau(e_n y e_n))\\ &= \lim_n \tau(e_n x e_n) + \lim_n \tau(e_n y e_n)\\ &= \lim_n \left\|x^{1/2}\widehat{e_n}\right\|_2^2 + \lim_n \left\|x^{1/2}\widehat{e_n}\right\|_2^2.\end{aligned}$$

When $\tau(x) < +\infty$ then $x^{1/2} \in L^2(M,\tau)$ and we get

$$\lim_n \left\|x^{1/2}\widehat{e_n}\right\|_2^2 = \left\|x^{1/2}\right\|_2^2 = \tau(x).$$

Therefore we see that $\tau(x+y) = \tau(x) + \tau(y)$ when $\tau(x) < +\infty$ and $\tau(y) < +\infty$. Whenever $\tau(x) = +\infty$ we claim that $\lim_n \tau(e_n x e_n) = +\infty$. Otherwise, $(x^{1/2}\widehat{e_n})$ is a Cauchy sequence in $L^2(M,\tau)$, thus converging to some ξ. Since $x^{1/2}$ is a closed operator, we deduce that $\widehat{1}$ is in its domain with $x^{1/2}\widehat{1} = \xi$, a contradiction. Then, since

$$\tau(e_n(x+y)e_n) \geq \tau(e_n x e_n),$$

we get that $\tau(x+y) = +\infty = \tau(x) + \tau(y)$.

The proof of $\tau(\lambda x) = \lambda\tau(x)$ is immediate. Finally, given $x = u|x| \in \widetilde{M}$ where $u \in \mathcal{U}(M)$ (by Lemma 6.1.3), to see that $\tau(x^*x) = \tau(xx^*)$, or equivalently that $\tau(|x|^2) = \tau(u|x|^2u^*)$, it suffices to observe that $E_t(u|x|^2u^*) = uE_t(|x|^2)u^*$, where $E_t(k)$ is here the spectral projection of $k \in \widetilde{M}_+$ relative to $]-\infty,t]$. It follows that $\tau(E_t(u|x|^2u^*)) = \tau(uE_t(|x|^2)u^*) = \tau(E_t(|x|^2))$. $\square$

We set

$$\mathfrak{n} = \left\{ x \in \widetilde{M} : \tau(x^*x) < +\infty \right\},$$
$$\mathfrak{m} = \left\{ \sum_{i=1}^{n} x_i y_i : x_i, y_i \in \mathfrak{n} \right\}.$$

LEMMA 7.4.3. *Let $\tau : \widetilde{M}_+ \to [0, +\infty]$ as above. Then*

(a) *$\mathfrak{n}$ and $\mathfrak{m}$ are linear self-adjoint subspaces of $\widetilde{M}$ which are stable under left and right multiplications by elements of M;*
(b) *$\mathfrak{m} \cap \widetilde{M}_+ = \left\{ x \in \widetilde{M}_+ : \tau(x) < +\infty \right\}$ and $\mathfrak{m}$ is linearly generated by $\mathfrak{m} \cap \widetilde{M}_+$;*
(c) *the restriction of τ to $\mathfrak{m} \cap \widetilde{M}_+$ extends in a unique way to a linear functional on $\mathfrak{m}$ (still denoted τ) and we have $\tau(x^*) = \overline{\tau(x)}$ for every $x \in \mathfrak{m}$;*
(d) *$\tau(xy) = \tau(yx)$ if either $x, y \in \mathfrak{n}$ or $x \in M$ and $y \in \mathfrak{m}$.*

PROOF. (a) Let $x, y \in \mathfrak{n}$. We have

$$(x+y)^*(x+y) + (x-y)^*(x-y) = 2(x^*x + y^*y),$$

whence $\tau((x+y)^*(x+y)) < +\infty$. Thus, $\mathfrak{n}$ is a linear subspace of $\widetilde{M}$, of course self-adjoint. Obviously, $ux \in \mathfrak{n}$ for every $u \in \mathcal{U}(M)$, and so, $\mathfrak{n}$ is a M-bimodule. The corresponding assertion for $\mathfrak{m}$ is immediate

(b) Let $z = \sum_{j=1}^n x_j^* y_j$ with $x_j, y_j \in \mathfrak{n}$. Since

$$4z = \sum_{j=1}^{n} \sum_{k=0}^{3} i^{-k} (x_j + i^k y_j)^* (x_j + i^k y_j)$$

we see that $\mathfrak{m}$ is linearly spanned by $\mathfrak{m} \cap \widetilde{M}_+$. Whenever z is self-adjoint, we get

$$4z = \sum_{j=1}^{n} (x_j + y_j)^*(x_j + y_j) - \sum_{j=1}^{n} (x_j - y_j)^*(x_j - y_j).$$

So z is the difference of two elements of $\mathfrak{m} \cap \widetilde{M}_+$. Moreover we have

$$z \leq \sum_{j=1}^{n} (x_j + y_j)^*(x_j + y_j),$$

and it follows that $\mathfrak{m} \cap \widetilde{M}_+ \subset \left\{ x \in \widetilde{M}_+ : \tau(x) < +\infty \right\}$. The opposite inclusion is obvious.

(c) Every element $x \in \mathfrak{m}$ is written as $x_1 - x_2 + i(x_3 - x_4)$, where the x_i are in $\mathfrak{m} \cap \widetilde{M}_+$. Then we set $\tau(x) = \tau(x_1) - \tau(x_2) + i(\tau(x_3 - \tau(x_4)))$. Since τ is additive on $\widetilde{M}_+$, we see that this definition is not ambiguous. Moreover, this extension τ is linear and self-adjoint.

(d) The equality $\tau(xy) = \tau(yx)$ for $x, y \in \mathfrak{n}$ is deduced from (7.8), by polarization. Finally, for $x \in M$ and $y_1, y_2 \in \mathfrak{n}$, we get

$$\begin{aligned}\tau(x(y_1 y_2)) &= \tau((xy_1)y_2) = \tau(y_2(xy_1) \\ &= \tau((y_2 x)y_1) = \tau(y_1(y_2 x)) = \tau((y_1 y_2)x).\end{aligned}$$

The second assertion of (d) follows by linearity. $\square$

Remark 7.4.4. We could have observed from the beginning that $\mathfrak{n} = L^2(M, \tau)$, showing in this way that $\mathfrak{n}$ is a M-bimodule linearly generated by its positive cone. Indeed, this observation follows immediately from Remark 7.3.5 and the definition of $\tau(x^* x) = \tau(|x|^2)$, since $E_t(|x|) = E_{t^2}(|x|^2)$, where, for $s \geq 0$ and $y \in \widetilde{M}_+$, we denote by $E_s(y)$ the spectral projection of y relative to $]-\infty, s]$.

So, when $\xi \in L^2(M, \tau)$, we have $\|\xi\|_2 = \||\xi|\|_2 = \tau(\xi^* \xi)^{1/2}$, and by polarization we get

$$\forall \xi, \eta \in L^2(M, \tau), \quad \tau(\xi^* \eta) = \langle \xi, \eta \rangle_{L^2(M)}. \tag{7.9}$$

In particular, $\omega_{\xi,\eta}$ is the linear functional $x \in M \mapsto \tau(\xi^* x \eta)$.

On the other hand, we see that

$$\mathfrak{m} = \left\{ x \in \widetilde{M} : \tau(|x|) < +\infty \right\} = L^1(M, \tau),$$

after writing x as $x = (u|x|^{1/2})(|x|^{1/2})$. Moreover, $L^1(M, \tau)$ is the set of products of two elements of $L^2(M, \tau)$.

7.4.2. The predual of M. Given $a \in L^1(M, \tau)$, we set $\|a\|_1 = \tau(|a|)$. Moreover, we denote by τ_a the linear functional $x \mapsto \tau(ax)$ defined on M. This is compatible with the definition of τ_a previously introduced when $a \in M$. We also observe that $\omega_{\xi,\eta} = \tau_{\eta \xi^*}$ for every $\xi, \eta \in L^2(M, \tau)$. In particular, $\tau_a = \omega_{\xi,\eta}$ with $\xi = |a|^{1/2}$ and $\eta = u|a|^{1/2}$ is w.o. continuous.

Theorem 7.4.5. *Let (M, τ) be a tracial von Neumann algebra acting on $L^2(M, \tau)$.*

(i) *The map $a \mapsto \tau_a$ is linear, injective, from $L^1(M, \tau)$ onto the space M_* of w.o. continuous linear functionals on M in standard form, and we have*

$$\|\tau_a\| = \tau(|a|) = \|a\|_1. \tag{7.10}$$

Moreover, the linear form τ_a is positive if and only if the operator a is in $L^1(M, \tau)_+$.

(ii) *$(L^1(M, \tau), \|\cdot\|_1)$ is a Banach space whose dual is M when $x \in M$ is viewed as the functional $a \mapsto \tau(ax)$.*

(iii) *The topology $\sigma(M, M_*)$ is the w.o. topology associated with the standard representation.*

PROOF. (i) Let us prove Equality (7.10), which will imply the injectivity of the map $a \mapsto \tau_a$. Let $a = u|a|$ be the polar decomposition of a. For $x \in M$, the Cauchy-Schwarz inequality gives

$$\begin{aligned} |\tau_a(x)| &= \left|\tau(|a|^{1/2}(xu|a|^{1/2}))\right| = \left\langle |a|^{1/2}, xu|a|^{1/2} \right\rangle \\ &\le \left\| |a|^{1/2} \right\|_2 \left\| xu|a|^{1/2} \right\|_2 \le \tau(|a|)^{1/2} \|x\|_\infty \tau(|a|)^{1/2} \\ &\le \tau(|a|)\|x\|_\infty, \end{aligned}$$

whence $\|\tau_a\| \le \tau(|a|)$. Taking $x = u^*$ we get $\tau_a(u^*) = \tau(|a|)$, and so the equality in (7.10).

The map is surjective since every positive element in M_* is of the form $\omega_\xi = \tau_{\xi^2}$ with $\xi \in L^2(M,\tau)_+$ (see Theorem 7.3.8) and since M_* is linearly generated by its positive elements (by polarization).

The last assertion of (i) is immediate.

(ii) We will identify M_* and $L^1(M,\tau)$. Using the polar decomposition $a = u|a|$, we observe that every $\varphi = \tau_a \in M_*$ may be written $\varphi = \omega_{\xi,\eta}$ with $\xi = |a|^{1/2}, \eta = u|a|^{1/2}$ and so $\|\xi\|_2 = \|\eta\|_2 = \|\varphi\|^{1/2}$.

We claim that M_* is closed in M^*. Let ω be in the norm closure of M_*. We have $\omega = \sum_{k=1}^\infty \varphi_k$ where φ_k is w.o. continuous and $\|\varphi_k\| \le 2^{-k}$ for $k \ge 2$. So, by the above observation, we get $\omega = \sum_k \omega_{\xi_k,\eta_k}$ with $\sum_k \|\xi_k\|_2^2 < +\infty$ and $\sum_k \|\eta_k\|_2^2 < +\infty$. By polarization, we see that ω is a linear combination of positive normal linear functionals, and so $\omega \in M_*$ (by Theorem 7.3.8). As a consequence, M_* is complete and so is $L^1(M,\tau)$.

Finally, we prove that M is the dual of $L^1(M,\tau)$, or of M_*. For $x \in M$ let $\widetilde{x}$ be the linear functional $\omega \mapsto \omega(x)$ defined on M_*. It is easily checked that $\|x\| = \|\widetilde{x}\|$ for $x \in M$. Now, let $v \in (M_*)^*$. The map $(\xi,\eta) \mapsto \langle v, \omega_{\xi,\eta}\rangle$ is sesquilinear and continuous and therefore there exists an operator $x \in \mathcal{B}(L^2(M,\tau))$ such that

$$\langle v, \omega_{\xi,\eta}\rangle = \langle \xi, x\eta\rangle$$

for every $\xi,\eta \in L^2(M,\tau)$. Given $y \in M'$, the functionals induced on M by $\omega_{\xi,y\eta}$ and $\omega_{y^*\xi,\eta}$ are the same. It follows that x commutes with y, whence $x \in M$, and finally $\widetilde{x} = v$.

(iii) is obvious. $\square$

REMARK 7.4.6. Note that M is naturally embedded in $L^1(M,\tau)$. Since M is the dual of $L^1(M,\tau)$, to show the density of this embedding, it suffices to check that if $x \in M$ satisfies $\tau_a(x) = 0$ for all $a \in M$, then $x = 0$. This is obvious, because τ is faithful: taking $a = x^*$, we get $\tau(x^*x) = 0$ and so $x = 0$. So $L^1(M,\tau)$ may be defined abstractly as the completion of M for the norm $\|\cdot\|_1$.

REMARK 7.4.7. Observe that M_* is the subspace of M^* linearly generated by the positive normal linear functionals, which coincide in the standard representation with the w.o. continuous positive ones. Since normality only depends on the ordered cone M_+, we see that M_* does not depend on the choice of τ. It is called *the predual of* M.

More generally, for any von Neumann algebra M we may introduce the subspace M_* of M^* linearly generated by the positive normal linear functionals. One of the basic results in the subject states that M is canonically identified to the dual of M_*, that M_* is a closed subspace of M^* and that M_* is the unique predual of M, up to isomorphism [**Dix53, Sak56**]. In addition to the example of tracial von Neumann algebras just studied, we mention the well-known fact that the von Neumann algebra $\mathcal{B}(\mathcal{H})$ is the dual of the Banach space $\mathcal{S}^1(\mathcal{H})$ of all trace-class operators, *i.e.,* operators T on $\mathcal{H}$ such that $\mathrm{Tr}(|T|) < +\infty$, where Tr is the usual trace on $\mathcal{B}(\mathcal{H})_+$ (see [**Ped89**, Section 3.4]).

Remark 7.4.8. It is a classical fact that the unit ball of M_* is weak*-dense in the unit ball of its bidual M^*. Moreover, every (norm continuous) state ψ on M is the weak* limit of a net of normal states. Indeed, if ψ in not in the closure of the convex set C formed by the normal states, the Hahn-Banach separation theorem implies the existence of a self-adjoint element $x \in M$ and a real number α such that $\psi(x) > \alpha$ and $\varphi(x) \leq \alpha$ for every $\varphi \in C$. But then $x \leq \alpha 1$, so that $\psi(x) \leq \alpha$, a contradiction.

The general non-commutative version of the *Radon-Nikodým theorem* is contained in the statement of Theorem 7.4.5. Let us spell out this important result.

Theorem 7.4.9 (**Radon-Nikodým theorem**). *Let (M, τ) be a tracial von Neumann algebra. For every $\varphi \in M_*$, there is a unique $a \in L^1(M, \tau)$ such that $\varphi = \tau_a$. The operator a is called the Radon-Nikodým derivative of φ with respect to τ.*

For further use, we also record in another form the *Hölder inequalities* (Exercise 7.6):

$$\forall a \in L^1(M,\tau), \forall x \in M, \quad |\tau(ax)| \leq \|ax\|_1 \leq \|a\|_1 \|x\|_\infty, \tag{7.11}$$

$$\forall \xi, \eta \in L^2(M,\tau), \quad |\tau(\xi\eta)| \leq \|\xi\eta\|_1 \leq \|\xi\|_2 \|\eta\|_2, \tag{7.12}$$

and the *Powers-Størmer inequality*:

$$\forall \xi, \eta \in L^2(M,\tau)_+, \quad \|\xi - \eta\|_2^2 \leq \left\|\xi^2 - \eta^2\right\|_1 \leq \|\xi - \eta\|_2 \|\xi + \eta\|_2 \tag{7.13}$$

(recall that for $\xi \in L^2(M)_+$, we have $\omega_\xi = \tau_{\xi^2}$).

Finally, note that given $\xi, \eta \in L^2(M, \tau)$, the classical inequality

$$\||\xi| - |\eta|\|_2 \leq \|\xi - \eta\|_2$$

is no longer true in the non abelian case, but is replaced by the following one:

Lemma 7.4.10. *For $\xi, \eta \in L^2(M, \tau)$ we have*

$$\||\xi| - |\eta|\|_2^2 \leq 2 \max(\|\xi\|_2, \|\eta\|_2) \|\xi - \eta\|_2.$$

Proof. As a consequence of the Powers-Størmer inequality we get

$$\begin{aligned}\||\xi|-|\eta|\|_2^2 &\leq \left\||\xi|^2-|\eta|^2\right\|_1\\ &= \|\omega_{\xi^*}-\omega_{\eta^*}\|\\ &\leq \|\xi+\eta\|_2\|\xi-\eta\|_2,\end{aligned}$$

the last by (7.5), whence the wanted inequality. □

7.5. Unitary implementation of the automorphism group

7.5.1. Uniqueness of the standard form. The following theorem shows that the standard form of a tracial von Neumann algebra is unique, up to a canonical isomorphism. In particular, this is why we will often write $L^2(M)$ instead of $L^2(M,\tau)$.

Proposition 7.5.1. *Let τ_1 and τ_2 be two normal faithful tracial states on a von Neumann algebra M. There exists one, and only one, unitary operator U from $L^2(M,\tau_1)$ onto $L^2(M,\tau_2)$ with the following properties:*

(i) *U is M-M linear (with respect to the structures of M-M-bimodules) and intertwines the canonical conjugation operators J_1 and J_2 relative to τ_1 and τ_2 respectively;*
(ii) $U\big(L^2(M,\tau_1)\big)_+ = L^2(M,\tau_2)_+$.

Proof. The Radon-Nikodým theorem implies the existence of a positive element h in $L^1(M,\tau_2)$ such that $\tau_1=\tau_2(h\cdot)$ and since τ_1 is a trace, we see that h is affiliated with the center $\mathcal{Z}(M)$ acting on $L^2(M,\tau_2)$.

Let $U:\widehat{M}\to L^2(M,\tau_2)$ be defined by $U(\widehat{m})=h^{1/2}m$. We have $\|\widehat{m}\|_{2,\tau_1}=\left\|h^{1/2}m\right\|_{2,\tau_2}$, and so U extends to an isometry from $L^2(M,\tau_1)$ into $L^2(M,\tau_2)$. The space $\overline{h^{1/2}M}$ is stable under the right action of M and therefore is of the form $pL^2(M,\tau_2)$ for some projection $p\in M$. Since $(1-p)h=0$, we get $\tau_1(1-p)=0$, whence $p=1$ and U is an isometry from $L^2(M,\tau_1)$ onto $L^2(M,\tau_2)$.

Obviously, U is M-M linear. Moreover, for $m\in M$, we have

$$U\circ J_1(\widehat{m})=h^{1/2}m^*=m^*h^{1/2}=J_2\circ U(\widehat{m}),$$

whence $U\circ J_1=J_2\circ U$.

We claim that $U\big(L^2(M,\tau_1)_+\big)=L^2(M,\tau_2)_+$. Indeed, using Proposition 7.3.4 we get $U(\widehat{M_+})\subset L^2(M,\tau_2)_+$ and then $U\big(L^2(M,\tau_1)_+\subset L^2(M,\tau_2)_+$. This proposition also gives $U^*\big(L^2(M,\tau_2)_+\big)\subset L^2(M,\tau_1)_+$ because $\langle U^*\xi,\widehat{m}\rangle=\langle\xi,U\widehat{m}\rangle\geq 0$ for $\xi\in L^2(M,\tau_2)_+$ and $m\in M_+$.

Finally, let V be a unitary operator with the same properties. Then $V^*U=W$ is a unitary operator in the center of M. The equality $J_1\circ W=W\circ J_1$ gives $W=W^*$. In addition, W is a positive unitary operator since $W\hat{1}\in L^2(M,\tau_1)_+$. It follows that $W=1$. □

7.5.2. Unitary implementation of $\mathrm{Aut}(M)$**.** We recall that $\mathrm{Aut}(M)$ is the *group of automorphisms* of M.

PROPOSITION 7.5.2. *Let (M,τ) be a tracial von Neumann algebra. There exists a unique group homomorphism $\alpha \mapsto u_\alpha$ from* $\mathrm{Aut}(M)$ *into the unitary group of $\mathcal{B}\big(L^2(M,\tau)\big)$ such that, for every $\alpha \in \mathrm{Aut}(M)$,*

(i) $\alpha(x) = u_\alpha x u_\alpha^*$ *for every $x \in M$;*
(ii) $u_\alpha J = J u_\alpha$, *and* $u_\alpha\big(L^2(M,\tau)_+\big) = L^2(M,\tau)_+$.

The map $\alpha \mapsto u_\alpha$ is called the *unitary implementation of* $\mathrm{Aut}(M)$.

PROOF. Let u, v be two unitary operators that satisfy conditions (i) and (ii) above. Then $v^*u \in M'$, and since $Jv^*uJ = v^*u$, we also have $v^*u \in M$. Moreover, $v^*u\big(L^2(M,\tau)_+\big) = L^2(M,\tau)_+$ and thus $u = v$ by Proposition 7.5.1.

Now, given $\alpha \in \mathrm{Aut}(M)$, let $v : L^2(M,\tau\circ\alpha) \to L^2(M,\tau)$ be the unitary operator such that $v(\widehat{m}) = \widehat{\alpha(m)}$ and let $w : L^2(M,\tau) \to L^2(M,\tau\circ\alpha)$ be defined in Proposition 7.5.1 with $\tau_1 = \tau$ and $\tau_2 = \tau\circ\alpha$. It is a routine verification to check that vw fulfills the above conditions (i) and (ii). $\square$

REMARK 7.5.3. More generally, let (M_1,τ_1) and (M_2,τ_2) be two tracial von Neumann algebras and let α be an isomorphism from M_1 onto M_2. There exists a unique unitary $U : L^2(M_1,\tau_1) \to L^2(M_2,\tau_2)$ such that $\alpha(x) = UxU^*$ for $x \in M_1$, $UJ_1 = J_2U$ and $U\big(L^2(M_1,\tau_1)_+\big) = L^2(M_2,\tau_2)_+$.

REMARK 7.5.4. The unitary implementation is an isomorphism from $\mathrm{Aut}(M)$ onto the subgroup of unitary operators u on $L^2(M,\tau)$ such that $uMu^* = M$, $uJ = Ju$, and $u\big(L^2(M,\tau)_+\big) = L^2(M,\tau)_+$. The subgroup $\mathrm{Aut}(M,\tau)$ of *trace preserving automorphisms* is sent on those unitaries which in addition satisfy $u\hat{1} = \hat{1}$. We observe that for $\alpha \in \mathrm{Aut}(M,\tau)$, the unitary operator u_α is defined, for $m \in M$, by $u_\alpha(\widehat{m}) = \widehat{\alpha(m)}$. We also note that the subgroups of $\mathcal{U}(\mathcal{B}(L^2(M,\tau)))$ corresponding to $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M,\tau)$ are closed with respect to the s.o. topology (or the w.o. topology).

REMARK 7.5.5. Let $M = L^\infty(X,\mu)$ equipped with its canonical tracial state τ_μ. Given any Borel automorphism θ of X such that $\theta_*\mu$ is equivalent to μ, the map $\alpha : f \in L^\infty(X,\mu) \mapsto f\circ\theta$ is an automorphism of M and every automorphism of M is of this form (see Theorem 3.3.4). If r denotes the Radon-Nikodým derivative $\mathrm{d}\theta_*\mu/\,\mathrm{d}\mu$, we immediately see that $u_\alpha\xi = \sqrt{r}\,\xi\circ\theta$ for $\xi \in L^2(X,\mu)$. This unitary implementation of $\mathrm{Aut}(M)$ is sometimes called its *Koopman representation*.

7.5.3. $\mathrm{Aut}(M,\tau)$ **is a Polish group when** M **is separable.** We equip $\mathrm{Aut}(M,\tau)$ with the topology for which a net $(\alpha_i) \in \mathrm{Aut}(M,\tau)$ converges to α if for every $x \in M$ we have

$$\lim_i \|\alpha_i(x) - \alpha(x)\|_2 = 0.$$

Then $\mathrm{Aut}(M,\tau)$ is a topological group.

As seen above, the unitary implementation of $\alpha \mapsto u_\alpha$ of $\mathrm{Aut}\,(M,\tau)$ is an an isomorphism onto a closed subgroup of the unitary group of $\mathcal{B}(L^2(M))$, equipped with the s.o. topology. Moreover, for $\alpha, \beta \in \mathrm{Aut}\,(M,\tau)$ and $x \in M$ we have

$$\|\alpha(x) - \beta(x)\|_2 = \|u_\alpha \hat{x} - u_\beta \hat{x}\|_2,$$

and therefore $\alpha \mapsto u_\alpha$ is a homeomorphism.

Recall that a *Polish group* is a topological group whose topology is Polish, that is, metrizable, complete and separable. In particular, the group $\mathcal{U}(\mathcal{H})$ of unitary operators on a separable Hilbert space $\mathcal{H}$, equipped with the s.o. topology is a Polish group. Indeed, let $\{\xi_n\}$ be a countable dense subset of the unit ball of $\mathcal{H}$. Then

$$d(u,v) = \sum_n \frac{1}{2^n}\big(\|u\xi_n - v\xi_n\| + \|u^*\xi_n - v^*\xi_n\|\big)$$

is a metric compatible with the s.o. topology on $\mathcal{U}(\mathcal{H})$, and $\mathcal{U}(\mathcal{H})$ is complete and separable with respect to this metric. As a consequence, $\mathrm{Aut}\,(M,\tau)$ is a Polish group when M is separable.

Exercises

Exercise 7.1. Let $G \curvearrowright (B,\tau)$ be a trace preserving action of a countable group on a tracial von Neumann algebra. Show that $M = B \rtimes G$ is on standard form on $L^2(B,\tau) \otimes \ell^2(G)$. Spell out the conjugation operator J and the right action of M.

Exercise 7.2. Show that a vector $\xi \in L^2(M,\tau)$ is separating for M if and only if it is cyclic.

Exercise 7.3. Let $\mathcal{H} = L^2([-1/2,1/2],\lambda)$ (where λ is the Lebesgue measure) equipped with the orthonormal basis $(e_n)_{n\in\mathbb{Z}}$, with $e_n(t) = \exp(2\pi i n t)$. Let χ be the function on $[-1/2,1/2]$ such that $\chi(t) = -1$ if $t \in [-1/2,0]$ and $\chi(t) = 1$ otherwise. We denote by u the multiplication operator on $\mathcal{H}$ by χ. Finally, let x be the self-adjoint operator such that $xe_n = \exp(n^2)e_n$ for $n \in \mathbb{Z}$.

(i) Show that $\mathrm{Im}\,(x^{-1}) \cap u\big(\mathrm{Im}\,(x^{-1})\big) = 0$ (Hint: if

$$\sum_n \exp(-n^2)\alpha_n e_n = \chi \sum_n \exp(-n^2)\beta_n e_n,$$

consider the entire functions $f(z) = \sum_n \exp(-n^2)\alpha_n e_n(z)$ and $g(z) = \sum_n \exp(-n^2)\beta_n e_n(z)$ and compare their restrictions to $[-1/2,1/2]$).

(ii) Show that the intersection of the domains of the self-adjoint operators x and uxu is reduced to 0.

Exercise 7.4. Show that M is separable if and only if $L^1(M,\tau)$ is a separable Banach space.

EXERCISE 7.5. Let (M, τ) be a tracial von Neumann algebra. Show that every normal trace τ_1 on M is of the form $\tau(h\cdot)$ where $h \in L^1(M, \tau)_+$ is affiliated with $\mathcal{Z}(M)$.

EXERCISE 7.6. Let (M, τ) be a tracial von Neumann algebra, $a \in L^1(M, \tau)$, $x \in M$, $\xi, \eta \in L^2(M, \tau)$.

(i) Show that $\|ax\|_1 \leq \|a\|_1 \|x\|_\infty$, $\|xa\|_1 \leq \|x\|_\infty \|a\|_1$.
(ii) Show that $\|\xi\eta\|_1 \leq \|\xi\|_2 \|\eta\|_2$.

EXERCISE 7.7. Let (M, τ) be a tracial von Neumann algebra. Show that that the topology on $\mathrm{Aut}\,(M, \tau)$ defined in Section 7.5.3 is also defined by the family of semi-norms $\alpha \mapsto \|\varphi \circ \alpha\|$ where φ ranges over M_*.

EXERCISE 7.8. Let (M_1, τ_1) and (M_2, τ_2) be two tracial von Neumann algebras and set $M = M_1 \overline{\otimes} M_1$, $\tau = \tau_1 \otimes \tau_2$.

(i) Show that the Hilbert spaces $L^2(M, \tau)$ and $L^2(M_1, \tau_1) \otimes L^2(M_2, \tau_2)$ are canonically isomorphic.
(ii) Given two other tracial von Neumann algebras (N_1, τ_1), (N_2, τ_2) and isomorphisms $\alpha_i : M_i \to N_i$, $i = 1, 2$, show that there is a unique isomorphism $\alpha : M \to N$ such that $\alpha(x_1 \otimes x_2) = \alpha_1(x_1) \otimes \alpha_2(x_2)$ for $x_1 \in M_1$ and $x_2 \in M_2$.

This isomorphism α is called the *tensor product of the isomorphisms α_1 and α_2* and is denoted by $\alpha_1 \otimes \alpha_2$. Such tensor products can be defined for any pair of von Neumann algebras (see [**Tak02**, Corollary 5.3]).

EXERCISE 7.9. Let M_1, M_2 be two II_1 factors and $\alpha_i \in \mathrm{Aut}\,(M_i)$, $i = 1, 2$. Show that $\alpha_1 \otimes \alpha_2$ is inner if and only if both automorphisms α_i are inner.

Notes

The subject of this chapter dates back to the seminal paper [**MVN36**] of Murray and von Neumann, which contains many of the results presented here, and in particular the fact that the set of all closed densely defined operators affiliated with a II_1 factor is a $*$-algebra. The theory of non-commutative integration was developed by many authors, among them Dixmier [**Dix53**] and Segal [**Seg53**] for finite or semi-finite von Neumann algebras. The Radon-Nikodým theorem 7.4.9 is due to Dye [**Dye52**]. Nowadays, the subject goes far beyond (see [**Tak03**, Chapter IX] for instance).

The notion of standard form has been extended to the case of any von Neumann algebra in [**Haa75**]. The very useful Powers-Størmer inequality was proved in [**PS70**] for Hilbert-Schmidt operators and in [**Ara74, Haa75**] in the general case.

CHAPTER 8

Modules over finite von Neumann algebras

We now study the right (or equivalently the left) modules $\mathcal{H}$ over a tracial von Neumann algebra (M,τ).[1] They have a very simple structure: they are M-submodules of multiples of the right M-module $L^2(M)$. It follows that, up to isomorphism, there is a natural bijective correspondence between them and the equivalence classes of projections in $\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$. This latter algebra is not finite in general, but belongs to the class of semi-finite von Neumann algebras, that we study succintly.

The set $\mathcal{B}(\mathcal{H}_M)$ of operators which commute with the right M-action on $\mathcal{H}$ is a semi-finite von Neumann algebra, equipped with a canonical semi-finite trace $\widehat{\tau}$, depending on τ. In the particular case $M=\mathbb{C}$, then $\mathcal{B}(\mathcal{H}_M) = \mathcal{B}(\mathcal{H})$ and $\widehat{\tau}$ is the usual trace Tr.

In the general case, $\widehat{\tau}$ may be defined with the help of appropriate orthonormal bases, made of M-bounded vectors, generalising the usual orthonormal bases of a Hilbert space. The dimension of $\mathcal{H}$ as a M-module is by definition $\widehat{\tau}(1)$ which, unfortunately, is not intrinsic, except when M is a factor, where τ is unique. In this case, a M-module is determined, up to isomorphism, by its dimension, which can be any element in $[0,+\infty]$ and there is in particular a well understood notion of finite M-module. The general case will be studied in the next chapter.

8.1. Modules over abelian von Neumann algebras.

Let M be a von Neumann algebra. Recall that a left *M-module* (resp. a right *M-module*) $(\pi,\mathcal{H})$ is a Hilbert space $\mathcal{H}$ equipped with a normal unital homomorphism (resp. anti-homomorphism) π of M. When π is faithful, we say that $(\pi,\mathcal{H})$ is a *faithful M-module.*

Definition. We say that two left M-modules $(\pi_i,\mathcal{H}_i)$, $i=1,2$, are *isomorphic* (or *equivalent*) if there exists a unitary operator $U:\mathcal{H}_1\to\mathcal{H}_2$ such that $U\pi_1(x)=\pi_2(x)U$ for every $x\in M$.

Our purpose is to describe the structure of these modules (assumed to be separable), up to isomorphism, for separable tracial von Neumann algebras.

[1]Except in Section 8.3, for simplicity of presentation, we implicitely limit ourselves to the separable case: von Neumann algebras as well as modules will be separable. The reader will easily see where these assumptions are not necessary and then make the straightforward appropriate modifications. We will mention explicitly where separability is indispensable.

We first consider the classical case of abelian von Neumann algebras, which amounts to the multiplicity theory for self-adjoint operators.

Let M be a separable abelian von Neumann algebra. We have seen in Theorem 3.2.1 that M is of the form $L^\infty(X,\mu)$ where (X,μ) is a standard probability measure space. For every Borel subset Y of X, the Hilbert space $L^2(Y,\mu)$ is obviously a M-module, when equipped with the representation by multiplication of functions. We may add such representations. A more general way to construct M-modules is as follows. Let $n : X \to \mathbb{N}^* \cup \{\infty\}$ be a measurable function and set $X_k = \{t \in X : n(t) = k\}$. If ℓ^2_k denotes the canonical Hilbert space of dimension k, then the direct Hilbert sum $\mathcal{H}(n) = \sum_{k\geq 1}^{\oplus} \big(\ell^2_k \otimes L^2(X_k,\mu)\big)$ has an obvious structure of M-module. We say that n is the *multiplicity function* of the module $\mathcal{H}(n)$. In fact, this is the most general construction of M-modules.

Theorem 8.1.1. *Let $(\pi,\mathcal{H})$ be a M-module where $M = L^\infty(X,\mu)$. There exists a unique (up to null sets) measurable function $n : X \to \mathbb{N}\cup\{\infty\}$ such that $\mathcal{H}$ is isomorphic to $\mathcal{H}(n)$.*

Proof. Consider first the case where $(\pi,\mathcal{H})$ is a cyclic M-module, *i.e.*, there exists $\xi \in \mathcal{H}$ with $\overline{\pi(M)\xi} = \mathcal{H}$. Let E be the Borel subset of X such that $\ker\pi = \mathbf{1}_E L^\infty(X,\mu)$ and set $Y = X \setminus E$. The restriction π_Y of π to $L^\infty(Y,\mu)$ is faithful. We choose a cyclic vector ξ such that $\|\xi\|^2 = \mu(Y)$ and so we have $\omega_\xi \circ \pi_Y(1) = \mu(Y)$. The $L^\infty(Y,\mu)$-modules $L^2(L^\infty(Y,\mu),\omega_\xi\circ\pi_Y)$ and $\mathcal{H}$ are isomorphic and thus we deduce from Proposition 7.5.1 that the M-modules $L^2(Y,\mu)$ and $\mathcal{H}$ are isomorphic.

In the general case, the M-module $\mathcal{H}$ is a Hilbert sum of cyclic modules and is therefore isomorphic, as a M-module, to some Hilbert sum $\sum_{k\geq 1}^{\oplus} L^2(Y_k,\mu)$, where the Y_k are Borel subsets of X, not necessarily disjoints. We may assume that π is faithful. We can build a partition (X_k) of X, to the price of introducing multiplicity, in order to show that $\mathcal{H}$ is of the form $\sum_{k\geq 1}^{\oplus} \big(\ell^2_k \otimes L^2(X_k,\mu)\big)$. We set

$$X_1 = \bigcup_k \big(Y_k \setminus \cup_{j\neq k} Y_j\big), \quad X_2 = \bigcup_{\{k\neq l\}} \big((Y_k \cap Y_l) \setminus \cup_{j\neq k, j\neq l} Y_j\big), \dots$$

We leave the details as an easy exercise.

Let us show that $\mathcal{H}(n)$ is completely determined by its multiplicity function. Let $n, n' : X \to \mathbb{N}^* \cup \{\infty\}$ be two multiplicity functions and $U : (\pi, \mathcal{H}(n)) \to (\pi', \mathcal{H}(n'))$ be an isomorphism between the two corresponding $L^\infty(X,\mu)$-modules. We write $\mathcal{H}(n)$ as $\sum_{k\geq 1}^{\oplus} \big(\ell^2_k \otimes L^2(X_k,\mu)\big)$ and $\mathcal{H}(n')$ as $\sum_{k\geq 1}^{\oplus} \big(\ell^2_k \otimes L^2(X'_k,\mu)\big)$. Then $\operatorname{Ad} U$ induces an isomorphism between the commutants of the two $L^\infty(X,\mu)$ actions, *i.e.*, from $\sum_{k\geq 1}^{\oplus} \mathcal{B}(\ell^2_k)\overline{\otimes} L^\infty(X_k,\mu)$ onto $\sum_{k\geq 1}^{\oplus} \mathcal{B}(\ell^2_k)\overline{\otimes} L^\infty(X'_k,\mu)$. But then, for every k we have

$$\operatorname{Ad} U\big(\mathcal{B}(\ell^2_k)\overline{\otimes} L^\infty(X_k,\mu)\big) = \mathcal{B}(\ell^2_k)\overline{\otimes} L^\infty(X'_k,\mu)$$

(see Exercice 5.6). Moreover, we have $U\pi(f) = \pi'(f)U$ for $f \in L^\infty(X,\mu)$. Taking $f = \mathbf{1}_{X_k}$, it follows that for $\xi \in \ell^2_k \otimes L^2(X_k,\mu)$,

$$\pi'(\mathbf{1}_{X_{k'}})U\xi = U\xi = U\pi(\mathbf{1}_{X_k})\xi = \pi'(\mathbf{1}_{X_k})U\xi = \pi'(\mathbf{1}_{X_k \cap X_{k'}})U\xi,$$

and therefore $X_{k'} \subset X_k$. Similarly, we get the opposite inclusion. $\square$

REMARK 8.1.2. Let x be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and let M be the abelian von Neumann algebra generated by x. By Theorem 3.2.1, we have $M = L^\infty(X,\mu)$ for some probability measure on the spectrum X of x. Then the structure theorem 8.1.1 for the M-module $\mathcal{H}$ gives the classical spectral multiplicity structure theorem for the self-adjoint operator x (see for instance [**RS80**, Theorem VII.6] for a precise statement). Conversely, since every abelian von Neumann algebra (on a separable Hilbert space) is generated by a self-adjoint operator (see Proposition 3.1.3) the classification of self-adjoint operators, up to unitary equivalence, provides Theorem 8.1.1. In this case, $n(t)$ expresses the "multiplicity" of t in the spectrum of x. As a particular case, if x has a finite spectrum the complete invariant is, of course, the multiplicity function $k \in X \mapsto n(k)$ of the eigenvalues of x.

8.2. Modules over tracial von Neumann algebras

Let (M,τ) be a separable tracial von Neumann algebra. We have seen in the previous chapter that $L^2(M)$ is a left M-module (and a right M-module as well). We will use here the notation $L_x\xi$ for $x\xi$ and $R_x\xi$ for ξx and denote by $L(M)$ and $R(M)$ the ranges of L and R respectively. Recall that $R(M)$ is the commutant of $L(M)$. We keep for the moment these notations $L(M)$ and $R(M)$ in order to avoid confusion with M and M' respectively when M is concretely represented on some Hilbert space $\mathcal{H}$, other then $L^2(M)$. The direct sum of countably many copies of $L^2(M)$ is still a left M-module, in an obvious way. It is denoted by $\ell^2(\mathbb{N})\otimes L^2(M)$. Given a projection p in the commutant $\mathcal{B}((\ell^2(\mathbb{N}))\overline{\otimes} R(M)$ of $\mathrm{Id}_{\ell^2(\mathbb{N})} \otimes L(M)$ in $\mathcal{B}(\ell^2(\mathbb{N}) \otimes L^2(M))$, the restriction to $p(\ell^2(\mathbb{N})\otimes L^2(M))$ of the left action of M defines a structure of left M-module on this Hilbert space. We will see now that this is the most general type of separable left M-module. As in the proof of Theorem 8.1.1, we first consider the case of a cyclic module.

LEMMA 8.2.1. *Let $\pi : M \to \mathcal{B}(\mathcal{H})$ be a normal unital representation. with a cyclic vector ξ. There exists an isometry $U : \mathcal{H} \to L^2(M)$ such that $U\pi(x) = L_xU$ for every $x \in M$. If we set $U\xi = \eta$, the range $\overline{M\eta}$ of U is of the form $pL^2(M)$ with $p \in R(M) = L(M)'$.*

PROOF. We define a normal positive functional on M by setting $\varphi(x) = \langle \xi, \pi(x)\xi\rangle$. By Theorem 7.3.8, it is is of the form $\varphi = \omega_\eta$ for some $\eta \in L^2(M)_+$. Then $U : \pi(x)\xi \to L_x\eta$ extends to an isometry with the required properties. $\square$

Proposition 8.2.2. *Let $\pi : M \to \mathcal{B}(\mathcal{H})$ be a normal unital representation. There exists an isometry $U : \mathcal{H} \to \ell^2(\mathbb{N}) \otimes L^2(M)$ such that $U\pi(x) = \big(\mathrm{Id}_{\ell^2(\mathbb{N})} \otimes L_x\big)U$ for every $x \in M$.*[2] *Moreover, we may choose U such that the projection on the range of U is of the diagonal form $\oplus_k p_k$ with $p_k \in R(M)$.*

Proof. We write $(\pi, \mathcal{H})$ as a the direct sum of cyclic sub-representations $(\pi_k, \mathcal{H}_k, \xi_k)$, $k \in \mathbb{N}$. For each k, we define $U_k : \mathcal{H}_k \to p_k L^2(M)$ as in the previous lemma. Then, the partial isometry

$$U : \mathcal{H} = \oplus_k \mathcal{H}_k \to \oplus_k \big(p_k L^2(M)\big) \subset \ell^2(\mathbb{N}) \otimes L^2(M)$$

defined in the obvious way intertwines π and the diagonal left representation of M on $\ell^2(\mathbb{N}) \otimes L^2(M)$. □

Proposition 8.2.3. *Let $\pi : M \to \mathcal{B}(\mathcal{H})$ be a normal unital representation. There is a projection p in $\mathcal{B}((\ell^2(\mathbb{N}))\overline{\otimes} R(M)$ such that $\mathcal{H}$ is isomorphic, as a left M-module, to $p(\ell^2(\mathbb{N}) \otimes L^2(M))$. This correspondence defines a bijection between the set of left M-modules, up to equivalence, and the set of projections of the commutant of $\mathrm{Id}_{\ell^2(\mathbb{N})} \otimes L(M)$, up to equivalence classes of projections in this commutant.*

Proof. With the notation of Proposition 8.2.2, it suffices to set $p = UU^*$. The second part of the statement is immediate. □

To go further, we will need tools to detect when two projections of $\mathcal{B}((\ell^2(\mathbb{N}))\overline{\otimes} R(M)$ are equivalent. This algebra belongs to the class of *semi-finite von Neumann algebras* that we briefly introduce now.

8.3. Semi-finite von Neumann algebras

8.3.1. Semi-finite tracial weights. Recall first that the cone $\mathcal{B}(\mathcal{H})_+$ of all positive operators on $\mathcal{H}$ comes equipped with a trace Tr (or $\mathrm{Tr}_{\mathcal{H}}$ in case of ambiguity) defined as follows. Let (ϵ_j) be any orthonormal basis of $\mathcal{H}$. For every $x \in \mathcal{B}(\mathcal{H})_+$, we put $\mathrm{Tr}(x) = \sum_j \langle \epsilon_j, x\epsilon_j \rangle \in [0, +\infty]$. This element is independent of the choice of the orthonormal basis and is called the *trace* of x. It is a faithful normal semi-finite trace in the following sense.

Definition 8.3.1. Let M be a von Neumann algebra. A map $\mathrm{Tr} : M_+ \to [0, +\infty]$ is called a *trace*[3] if it satisfies the following properties:

(a) $\mathrm{Tr}(x + y) = \mathrm{Tr}(x) + \mathrm{Tr}(y)$ for all $x, y \in M_+$;
(b) $\mathrm{Tr}(\lambda x) = \lambda \mathrm{Tr}(x)$ for all $x \in M_+$ and $\lambda \in \mathbb{R}_+$ (agreeing that $0 \cdot +\infty = 0$);
(c) $\mathrm{Tr}(x^* x) = \mathrm{Tr}(xx^*)$ for all $x \in M$.

It is called *semi-finite* if, in addition,

[2]Without the separability assumption, $\mathbb{N}$ has to be replaced by some (not necessarily countable) set I.

[3]We sometimes say *tracial weight* to insist on the fact that it is not necessarily finite.

(d) for every non-zero $x \in M_+$ there exists some non-zero $y \in M_+$ with $y \leq x$ and $\mathrm{Tr}(y) < +\infty$.

If

(e) $\mathrm{Tr}(\sup_i x_i) = \sup_i \mathrm{Tr}(x_i)$ for every bounded increasing net (x_i) in M_+, we say that Tr is *normal*.

It is called *faithful* if, for $x \in M_+$,

(f) $\mathrm{Tr}(x) = 0$ if and only if $x = 0$.

Whenever $\mathrm{Tr}(1) < +\infty$ then, since M is linearly generated by M_+, Tr extends uniquely to a linear functional on M, which is a trace in the usual sense.

Definition 8.3.2. A von Neumann algebra M is said to be *semi-finite* if there exists on M_+ a faithful, normal, semi-finite trace Tr.

The class of semi-finite von Neumann algebras encompasses finite von Neumann algebras.

Factors with a minimal projection are isomorphic to some $\mathcal{B}(\mathcal{H})$ (see 2.4.13), hence semi-finite. They form the class of type I factors. Diffuse semi-finite factors split into two classes: we find those such that $\mathrm{Tr}(1) = +\infty$, called *type* II_∞ *factors*, and those such that $\mathrm{Tr}(1) < +\infty$, our now familiar II_1 factors[4].

Let us give a basic way to construct semi-finite non-finite von Neumann algebras. Let (N, τ) be a tracial von Neumann algebra on $\mathcal{H}$ and I an infinite set. Consider the von Neumann tensor product $M = \mathcal{B}(\ell^2(I)) \overline{\otimes} N$ on $\ell^2(I) \otimes \mathcal{H}$. As usually, we write its elements as matrices $[m_{i,j}]$ with lines and columns indexed by I and entries in N. The diagonal entries of the elements of M_+ are in N_+. Let Tr be the usual normal faithful semi-finite trace on $\mathcal{B}(\ell^2(I))_+$. For $m = [m_{i,j}] \in M_+$, we set

$$\tau_\infty(m) = (\mathrm{Tr} \otimes \tau)(m) = \sum_{i \in I} \tau(m_{i,i}). \tag{8.1}$$

Then τ_∞ is a normal faithful semi-finite trace on M_+ with $\tau_\infty(1) = +\infty$. The two first conditions of Definition 8.3.1 are obvious. Using matrix multiplication, we also easily check that $\tau_\infty(mm^*) = \tau_\infty(m^*m)$ and that $\tau_\infty(m^*m) = 0$ if and only if $m = 0$. Furthermore, τ_∞ is semi-finite. Indeed, denote by p_i the projection on the subspace $\mathbb{C}\delta_i \otimes \mathcal{H}$. Given a non-zero $m \in M_+$, there is $i \in I$ such that $p_i m p_i \neq 0$. Then, we have

$$\tau_\infty(m^{1/2} p_i m^{1/2}) = \tau_\infty(p_i m p_i) < +\infty,$$

with $m^{1/2} p_i m^{1/2} \leq m$ and $m^{1/2} p_i m^{1/2} \neq 0$.

[4]This definition is not ambiguous since the faithful, normal, semi-finite trace Tr on a semi-finite factor is unique, up to multiplication by a positive real number (see Propositions 4.1.3 and 8.3.6).

Finally, let (m_κ) be an increasing net of elements in M_+ with $\bigvee_\alpha m_\kappa = m$. We easily get that $\tau_\infty(m) = \sup_\kappa \tau_\infty(m_\kappa)$ by observing that for every $i \in I$, the net of diagonal entries $(p_i m_\kappa p_i)$ is increasing with $p_i m p_i$ as supremum.

If in addition N is a factor, then M is a II_∞ factor. Exercise 8.1 shows that every II_∞ factor is of this form.

Proposition 8.3.3. *Let M be a tracial von Neumann algebra on a Hilbert space $\mathcal{H}$. Then M' is a semi-finite von Neumann algebra.*

Proof. We may assume that $\mathcal{H} = p(\ell^2(I) \otimes L^2(M))$ where p is a projection in $\mathcal{B}(\ell^2(I)) \overline{\otimes} R(M)$, so that $M' = p\big(\mathcal{B}(\ell^2(I)) \overline{\otimes} R(M)\big)p$. Then we observe that the reduction of a semi-finite von Neumann algebra remains semi-finite. $\square$

We end this section with some basic facts on II_∞ factors.

Lemma 8.3.4. *Let M be a factor with a normal faithful semi-finite trace Tr. Let $p, q \in \mathcal{P}(M)$, such that $Tr(p) = +\infty$ and $0 < Tr(q) < +\infty$. There exists a family $(q_i)_{i \in I}$ of mutually orthogonal projections in M with $q_i \sim q$ for every i, $\sum_{i\in I} q_i = p$. The set I is infinite and whenever M is separable it is countable.*

Proof. Obviously, we have $q \precsim p$. Using a maximality argument, we see that there exists $(q_i')_{i\in I}$, where the projections q_i' are mutually orthogonal and equivalent to q, with $q_i' \leq p$ for every i and $p - \sum_{i\in I} q_i' \precsim q$. Since the set I is infinite, the projections p and $\sum_{i \in I} q_i'$ are equivalent. Indeed, if we set $p_0 = p - \sum_{i\in I} q_i'$ and fix $i_0 \in I$, using the existence of a bijection from I onto $I \setminus \{i_0\}$, we get

$$p = \sum_{i\in I} q_i' + p_0 \simeq \sum_{i \in I\setminus\{i_0\}} q_i' + p_0 \precsim \sum_{i\in I} q_i'.$$

Let u be a partial isometry in M such that $u^*u = p$ and $uu^* = \sum_{i\in I} q_i'$. To conclude, we set $q_i = u^* q_i' u$ for $i \in I$. $\square$

We may write $I = I_1 \cup I_2$, where I_1 and I_2 are disjoint and have the same cardinal as I. Then p is the sum of the two mutually orthogonal projections $p_j = \sum_{i \in I_j} q_i$, $j = 1, 2$, which are equivalent to p. In particular, p is *infinite*. We easily deduce the following corollary.

Corollary 8.3.5. *Let M be a factor with a normal faithful semi-finite trace Tr. A projection $p \in M$ is infinite if and only if $Tr(p) = +\infty$.*

Proposition 8.3.6. *Let M be a II_∞ factor and Tr a normal faithful semi-finite trace on M.*

(i) *We have $\{Tr(p) : p \in \mathcal{P}(M)\} = [0, +\infty]$.*
(ii) *Let Tr_1 be another normal semi-finite faithful trace on M_+. There exists a unique $\lambda > 0$ such that $Tr_1 = \lambda Tr$.*

PROOF. (i) Let q be a projection such that $0 < \mathrm{Tr}(q) < +\infty$. Then qMq is a factor (see Proposition 4.2.1) which is diffuse and finite and so

$$\{\mathrm{Tr}(p) : p \in \mathcal{P}(qMq)\} = [0, c]$$

where $c = \mathrm{Tr}(q)$, by Proposition 4.1.6. Since 1 is the sum of infinitely many projections equivalent to q, we easily deduce the statement of (i).

(ii) Let p be a projection such that $\mathrm{Tr}(p) = 1$ and set $\lambda = \mathrm{Tr}_1(p)$. Then, by uniqueness of the tracial state on pMp, we have $\mathrm{Tr}_1(x) = \lambda\mathrm{Tr}(x)$ for every $x \in (pMp)_+$. Let q be the sum of finitely many projections equivalent to p. For $x \in M_+$ we have

$$\mathrm{Tr}_1(x^{1/2}qx^{1/2}) = \mathrm{Tr}_1(qxq) = \lambda\mathrm{Tr}(qxq) = \lambda\mathrm{Tr}(x^{1/2}qx^{1/2}),$$

since qMq is isomorphic to some $pMp \otimes M_n(\mathbb{C})$. Using the normality of the traces, and Lemma 8.3.4, we get the conclusion. $\square$

DEFINITION 8.3.7. Let M be a type II_∞ factor and Tr a normal faithful semi-finite trace on M. Given $\theta \in \mathrm{Aut}\,(M)$, the number $\lambda > 0$ such that $\mathrm{Tr} \circ \theta = \lambda\mathrm{Tr}$ (independent of the choice of Tr) is called the *module* of θ and denoted by $\mathrm{mod}(\theta)$.

PROPOSITION 8.3.8. *Let M be a separable type II_∞ factor and let Tr be a normal faithful semi-finite trace on M. Let $p, q \in \mathcal{P}(M)$. Then $p \precsim q$ if and only if $\mathrm{Tr}(p) \leq \mathrm{Tr}(q)$.*

PROOF. Clearly, if $p \precsim q$ then $\mathrm{Tr}(p) \leq \mathrm{Tr}(q)$. Conversely, assume that $\mathrm{Tr}(p) \leq \mathrm{Tr}(q)$. The only non trivial case to consider is when both p and q have an infinite trace. But then, given any non-zero projection $e \in M$ with $\mathrm{Tr}(e) < +\infty$, we see from Lemma 8.3.4 that there exist two sequences $(p_k)_{k\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$ of projections equivalent to e with $p = \sum_{k\in\mathbb{N}} p_k$ and $q = \sum_{k\in\mathbb{N}} q_k$, whence the equivalence of p and q. The fact that we get here sequences follows from the separability, which is a crucial assumption. $\square$

This proposition solves the comparison problem of projections in a separable semi-finite factor. In the non-factorial case, we need more sophisticated tools (see Proposition 9.2.4).

REMARK 8.3.9. So far, we have introduced the following types of factors: I, II_1 and II_∞. There are factors which do not belong to these classes, those which do not carry any normal non-zero semi-finite trace. They are called type III factors. They will not be considered in this monograph.

8.4. The canonical trace on the commutant of a tracial von Neumann algebra representation

In the rest of this chapter, (M, τ) is a tracial von Neumann algebra. Until now, we have only considered left M-modules. We may study, equivalently, right M-modules, which are nothing else than left M^{op}-modules. In the following we will more often consider right M-modules since we are rather interested in the commutant (that we let act to the left) of the right structures.

The commutant of $\mathrm{Id}_{\ell^2(\mathbb{N})} \otimes R(M)$ in $\mathcal{B}(\ell^2(\mathbb{N}) \otimes L^2(M))$ is $\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$, the von Neumann algebra of operators which, viewed as infinite matrices, have their entries in M (identified with $L(M)$). The analogue of Proposition 8.2.3 provides a bijective correspondence between the set of equivalence classes of right M-modules and the set of equivalence classes of projections in $\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$.

Given two right M-modules $\mathcal{H}$ and $\mathcal{K}$, we denote by $\mathcal{B}(\mathcal{H}_M, \mathcal{K}_M)$ the space of right M-linear bounded maps from $\mathcal{H}$ into $\mathcal{K}$. We set $\mathcal{B}(\mathcal{H}_M) = \mathcal{B}(\mathcal{H}_M, \mathcal{H}_M)$. This semi-finite von Neumann algebra (Proposition 8.3.3) is a generalisation of $\mathcal{B}(\mathcal{H})$ which corresponds to $M = \mathbb{C}$. It carries a specific tracial weight $\widehat{\tau}$ (equal to Tr when $M = \mathbb{C}$), depending on τ, that we define now.

8.4.1. First characterisation of $\widehat{\tau}$. Let $\mathcal{H}$ be a right M-module. Observe that, given S, T in $\mathcal{B}(L^2(M)_M, \mathcal{H}_M)$, we have $S^*T \in M$ and $TS^* \in \mathcal{B}(\mathcal{H}_M)$.

Lemma 8.4.1. *The linear span $\mathcal{F}(\mathcal{H}_M)$ of*

$$\left\{TS^* : T, S \in \mathcal{B}(L^2(M)_M, \mathcal{H}_M)\right\}$$

is an ideal of $\mathcal{B}(\mathcal{H}_M)$, dense in the w.o. topology.

Proof. The elements of $\mathcal{F}(\mathcal{H}_M)$ are analogous, for M-modules, to finite rank operators for Hilbert spaces. The only non trivial fact is the density of $\mathcal{F}(\mathcal{H}_M)$. Let z be the projection of the center of $\mathcal{B}(\mathcal{H}_M)$ such that $\overline{\mathcal{F}(\mathcal{H}_M)}^{w.o.} = \mathcal{B}(\mathcal{H}_M)z$. Then $(1-z)\mathcal{H}$ is a right M-module. Assume that there exists $\xi \neq 0$ in $(1-z)\mathcal{H}$. By Lemma 8.2.1, the M-module $\overline{\xi M}$ is isomorphic to $pL^2(M)$ for some projection $p \in M$. After identification of these two modules, we see that the map $\widehat{m} \to \widehat{pm}$ extends to a non-zero element $T \in \mathcal{B}(L^2(M)_M, \mathcal{H}_M)$ with $zT = 0$. It follows that TT^* is a non-zero element of $\mathcal{F}(\mathcal{H}_M)$ with $zTT^* = 0$, a contradiction.

∎

Proposition 8.4.2. *Let $\mathcal{H}$ be a right M-module. The commutant $\mathcal{B}(\mathcal{H}_M)$ is a semi-finite von Neumann algebra which carries a canonical normal faithful semi-finite trace $\widehat{\tau}$ characterized by the equality*

$$\widehat{\tau}(TT^*) = \tau(T^*T) \tag{8.2}$$

for every right M-linear bounded operator $T : L^2(M) \to \mathcal{H}$.

Proof. Let $U : \mathcal{H} \to \ell^2(\mathbb{N}) \otimes L^2(M)$ be a right M-linear isometry. For $x \in \mathcal{B}(\mathcal{H}_M)_+$ we set

$$\widehat{\tau}(x) = (\mathrm{Tr} \otimes \tau)(UxU^*),$$

where Tr is the usual trace on $\mathcal{B}(\ell^2(\mathbb{N}))_+$. Then, $\widehat{\tau}$ is a normal faithful semi-finite trace. Moreover, if $V : \mathcal{H} \to \ell^2(\mathbb{N}) \otimes L^2(M)$ is another M-linear isometry, we have

$$\begin{aligned}(\mathrm{Tr}\otimes\tau)(UxU^*) &= (\mathrm{Tr}\otimes\tau)((UV^*)VxV^*(VU^*))\\ &= (\mathrm{Tr}\otimes\tau)(VxV^*(VU^*)(UV^*)) = (\mathrm{Tr}\otimes\tau)(VxV^*).\end{aligned}$$

Hence, $\widehat{\tau}$ is independent of the choice of U.

Let us prove Equation (8.2). We may assume that

$$\mathcal{H} = p(\ell^2(\mathbb{N})\otimes L^2(M)),$$

where p is a projection in $\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$. Let $T\in\mathcal{B}(L^2(M)_M,\mathcal{H}_M)$ and write $T\hat{1} = \sum_{k\geq 1}\delta_k\otimes\xi_k$. For $m\in M$, we have

$$\sum_{k\in\mathbb{N}}\|\xi_k m\|_2^2 = \|T\widehat{m}\|^2 \leq \|T\|^2\|\widehat{m}\|_2^2.$$

Theorem 7.1.1 implies that $\xi_k = \widehat{m_k}\in\widehat{M}$. Moreover we have

$$\sum_{k\in\mathbb{N}} m_k^* m_k \leq \|T\|^2 1$$

where the convergence is with respect to the w.o. topology. Straightforward computations show that $T^*T = \sum_k m_k^* m_k$, and that TT^* is the matrix $[m_i m_j^*]_{i,j}$. It follows that $\widehat{\tau}(TT^*) = \tau(T^*T)$. By polarization, we get that $\widehat{\tau}(TS^*) = \tau(S^*T)$ for every $S,T\in\mathcal{B}(L^2(M)_M,\mathcal{H}_M)$. That $\widehat{\tau}$ is characterized by (8.2) follows from its normality, together with Lemma 8.4.1 and Exercise 2.11. □

We leave it to the reader to check that $\widehat{\tau}$ is the usual trace Tr on $\mathcal{B}(\mathcal{H})$ when $M=\mathbb{C}$. We note that Tr is defined via any orthonormal basis of $\mathcal{H}$. Likewise, there is a useful notion of orthonormal basis with respect to a M-module, which can be used to define the canonical trace on the commutant. We need first the notion of bounded vector.

8.4.2. Bounded vectors.

DEFINITION 8.4.3. Let (M,τ) be a tracial von Neumann and $\mathcal{H}$ a right M-module. A vector $\xi\in\mathcal{H}$ is said to be *left (M-)bounded*[5] if there exists $c>0$ such that $\|\xi x\|\leq c\|x\|_2$ for every $x\in M$. In other words, the map $\widehat{x}\mapsto\xi x$ extends to a bounded operator L_ξ from $L^2(M)$ into $\mathcal{H}$.

We denote by $\mathcal{H}^0$ the set of left bounded vectors. Obviously, $\xi\mapsto L_\xi$ is a bijection from $\mathcal{H}^0$ onto $\mathcal{B}(L^2(M)_M,\mathcal{H}_M)$. We have seen in Theorem 7.1.1 that $L^2(M)^0 = \widehat{M}$.

PROPOSITION 8.4.4. *Let $\mathcal{H}$ be a right M-module. Then $\mathcal{H}^0$ is a dense linear subspace of $\mathcal{H}$ which is stable under the actions of M and of its commutant $\mathcal{B}(\mathcal{H}_M)$. Moreover, for $\xi\in\mathcal{H}^0$, $x\in M$ and $y\in\mathcal{B}(\mathcal{H}_M)$ we have $L_{y\xi x} = yL_\xi x : \widehat{m}\to y(\xi x m)$.*

[5]We warn the reader that this notion, and therefore the notion of orthonormal basis defined in the next section, depends on the choice of τ.

PROOF. We only prove the density of $\mathcal{H}^0$, the rest of the statement being obvious. Let $\xi \in \mathcal{H}$. We have seen in Lemma 8.2.1 that the M-module $\overline{\xi M}$ is isomorphic to $pL^2(M)$ for some projection $p \in M$. The space $p\widehat{M}$ is made of left bounded vectors and is dense in $pL^2(M)$. $\square$

We now observe that for $\xi, \eta \in \mathcal{H}^0$, the operator $L_\xi^* L_\eta$ commutes with the right M-action on $L^2(M)$ and so belongs to M. We set $L_\xi^* L_\eta = \langle \xi, \eta \rangle_M$ since, as we will see now, this operation behaves like an inner-product, but with value in M.

LEMMA 8.4.5. *Given $\xi, \eta \in \mathcal{H}^0$, we have*

(i) $\langle \xi, \xi \rangle_M \geq 0$ *and* $\langle \xi, \xi \rangle_M = 0$ *if and only if* $\xi = 0$*;*
(ii) $(\langle \xi, \eta \rangle_M)^* = \langle \eta, \xi \rangle_M$*;*
(iii) $\langle \xi, \eta x \rangle_M = \langle \xi, \eta \rangle_M x$, $\langle \xi x, \eta \rangle_M = x^* \langle \xi, \eta \rangle_M$ *for every* $x \in M$*;*
(iv) $\tau(\langle \xi, y\eta x \rangle_M) = \langle \xi, y\eta x \rangle_{\mathcal{H}}$ *for every* $x \in M$ *and* $y \in \mathcal{B}(\mathcal{H}_M)$*;*
(v) $(L_\xi)^*(x\eta) = \widehat{\langle \xi, x\eta \rangle_M}$ *for every* $x \in \mathcal{B}(\mathcal{H}_M)$*.*

PROOF. Straightforward verifications. $\square$

Given a left M-module $\mathcal{K}$, we may similarly introduce the space ${}^0\mathcal{K}$ of *right bounded vectors*. It satisfies the properties stated in Lemma 8.4.5 translated to left modules. More precisely, if $\eta \in {}^0\mathcal{K}$, we denote by $R_\eta : L^2(M) \to \mathcal{K}$ the corresponding bounded right M-linear operator and, for $\xi, \eta \in {}^0\mathcal{K}$, we set $J(R_\xi^* R_\eta)J = {}_M\langle \xi, \eta \rangle$. Note that $(\xi, \eta) \to {}_M\langle \xi, \eta \rangle$ is linear with respect to the first variable and antilinear with respect to the second one.

LEMMA 8.4.6. *Given $\xi, \eta \in {}^0\mathcal{K}$, we have*

(i) ${}_M\langle \xi, \xi \rangle \geq 0$ *and* ${}_M\langle \xi, \xi \rangle = 0$ *if and only if* $\xi = 0$*;*
(ii) $({}_M\langle \xi, \eta \rangle)^* = {}_M\langle \eta, \xi \rangle$*;*
(iii) ${}_M\langle x\xi, \eta \rangle = x\, {}_M\langle \xi, \eta \rangle$, ${}_M\langle \xi, x\eta \rangle = {}_M\langle \xi, \eta \rangle x^*$ *for every* $x \in M$*;*
(iv) $\tau({}_M\langle x\xi y, \eta \rangle) = \langle \eta, x\xi y \rangle_{\mathcal{K}}$ *for every* $x \in M$ *and* $y \in \mathcal{B}({}_M\mathcal{K})$*.*

8.4.3. Orthonormal bases.

DEFINITION 8.4.7. Let $\mathcal{H}$ be a right M-module over a tracial von Neumann algebra (M, τ). An *orthonormal basis* (or *Pimsner-Popa basis*) for this M-module is a family $(\xi_i)_{i \in I}$ of non-zero left bounded vectors such that $\overline{\sum_i \xi_i M} = \mathcal{H}$ and $\langle \xi_i, \xi_j \rangle_M = \delta_{i,j} p_j \in \mathcal{P}(M)$ for all i, j. Hence, $\mathcal{H} = \oplus_i \overline{\xi_i M}$. Note that I is countable under the separability assumption on $\mathcal{H}$.

Let $\xi \in \mathcal{H}^0$ such that $\langle \xi, \xi \rangle_M = L_\xi^* L_\xi = p \in \mathcal{P}(M)$. Then $\xi = \xi p$ and $L_\xi L_\xi^*$ is the orthogonal projection on the M-submodule $\overline{\xi M}$. The verification is straightforward. The following consequence is immediate.

LEMMA 8.4.8. *Let (ξ_i) be a family of left bounded vectors. Then (ξ_i) is an orthonormal basis if and only if $\langle \xi_i, \xi_j \rangle_M = \delta_{i,j} p_j \in \mathcal{P}(M)$ for all i, j and $\sum_i L_{\xi_i} L_{\xi_i}^* = \mathrm{Id}_{\mathcal{H}}$.*

LEMMA 8.4.9 (**Polar decomposition**). *Every left bounded vector ξ in a right M-module $\mathcal{H}$ can be written in a unique way as $\xi = \xi' \langle \xi, \xi \rangle_M^{1/2}$, where ξ' is left bounded and is such that $\langle \xi', \xi' \rangle_M$ is the range projection of $\langle \xi, \xi \rangle_M^{1/2}$. Moreover $\overline{\xi' M} = \overline{\xi M}$.*

PROOF. Let $L_\xi = u\langle \xi, \xi \rangle_M^{1/2}$ be the polar decomposition of L_ξ viewed as a bounded operator from $L^2(M)$ into $\overline{\xi M}$. We set $\xi' = u(\hat{1})$. The end of the proof is immediate. □

The decomposition $\xi = \xi' \langle \xi, \xi \rangle_M^{1/2}$ is called the *polar decomposition of* ξ.

LEMMA 8.4.10. *Assume that $\mathcal{H} = \overline{\eta M}$. Then there exists a left bounded vector ξ such that $\langle \xi, \xi \rangle_M \in \mathcal{P}(M)$ and $\mathcal{H} = \overline{\xi M}$.*

PROOF. Indeed, let U be an isomorphism of M-modules from $\mathcal{H}$ onto a sub-module of $L^2(M)$, say $pL^2(M)$ with $p \in \mathcal{P}(M)$ (see Lemma 8.2.1). It suffices to set $\xi = U^{-1}(p\hat{1})$. □

PROPOSITION 8.4.11. *Every right M-module $\mathcal{H}$ has orthonormal bases.*

PROOF. Let $\{\xi_i\} \subset \mathcal{H}^0$ be a maximal family with the property that $\langle \xi_i, \xi_j \rangle_M = \delta_{i,j} p_j \in \mathcal{P}(M)$ and set $\mathcal{K} = \overline{\sum_i \xi_i M}$. If $\mathcal{K} \neq \mathcal{H}$, by the previous lemma the right M-module $\mathcal{K}^\perp$ contains a non-zero left bounded vector ξ, which can be chosen such that $\langle \xi, \xi \rangle_M \in \mathcal{P}(M)$. This contradicts the maximality of the family $\{\xi_i\}$. □

REMARK 8.4.12. An orthonormal basis (ξ_i) is indeed a basis in the following sense: every $\eta \in \mathcal{H}^0$ has a unique expression as

$$\eta = \sum_i \xi_i m_i$$

where $m_i \in p_i M$ and the series converges in norm. Indeed, we have

$$\eta = \sum_i L_{\xi_i} L_{\xi_i}^* \eta = \sum_i \xi_i \langle \xi_i, \eta \rangle_M. \tag{8.3}$$

Moreover, if $\eta = \sum_i \xi_i m_i$ then

$$\langle \xi_j, \eta \rangle_M = L_{\xi_j}^* L_\eta \widehat{1} = L_{\xi_j}^* \eta = \sum_i \langle \xi_j, \xi_i \rangle_M m_i = p_j m_j.$$

LEMMA 8.4.13. *Let $\mathcal{H}$ be a right M-module and let (ξ_i) be an orthonormal basis. Let ξ, η be two left bounded vectors. Then the series*

$$\sum \langle \xi, \xi_i \rangle_M \langle \eta, \xi_i \rangle_M$$

is convergent in M with respect to the s.o. topology and we have

$$\langle \xi, \eta \rangle_M = L_\xi^* L_\eta = \sum_i L_\xi^* L_{\xi_i} L_{\xi_i}^* L_\eta = \sum_i \langle \xi, \xi_i \rangle_M \langle \xi_i, \eta \rangle_M.$$

PROOF. Obvious since $\sum_i L_{\xi_i} L_{\xi_i}^* = \mathrm{Id}_{\mathcal{H}}$. □

REMARK 8.4.14. As already mentioned in Lemma 8.4.1, the linear span $\mathcal{F}(\mathcal{H}_M)$ of the set of operators $L_\xi L_\eta^*$, $\xi, \eta \in \mathcal{H}^0$ is a w.o. dense two-sided ideal of $\mathcal{B}(\mathcal{H}_M)$. It is the ideal of finite rank operators in case $M = \mathbb{C}1$. So, in the general case, it is useful to view the elements of $\mathcal{F}(\mathcal{H}_M)$ as "finite rank" operators.

8.4.4. Second characterisation of $\widehat{\tau}$. We may now state our second characterisation of $\widehat{\tau}$.

PROPOSITION 8.4.15. *Let $\mathcal{H}$ be a right module over a tracial von Neumann algebra (M, τ) and let $(\xi_i)_{i \in I}$ be an orthonormal basis of this module. Then, for every non-negative element $x \in \mathcal{B}(\mathcal{H}_M)$, we have*

$$\widehat{\tau}(x) = \sum_i \tau(\langle \xi_i, x\xi_i \rangle_M) = \sum_n \langle \xi_i, x\xi_i \rangle_{\mathcal{H}}. \tag{8.4}$$

PROOF. We set $p_i = \langle \xi_i, \xi_i \rangle_M$. Let U be the isometry from $\mathcal{H}$ into $\ell^2(I) \otimes L^2(M)$ such that, for $m \in M$ and all i,

$$U(\xi_i m) = \delta_i \otimes p_i \widehat{m} = \delta_i \otimes L_{\xi_i}^*(\xi_i m).$$

We know that for $x \in \mathcal{B}(\mathcal{H}_M)_+$,

$$\widehat{\tau}(x) = (\mathrm{Tr} \otimes \tau)(UxU^*) = \sum_i \tau((UxU^*)_{i,i}),$$

and we have $(UxU^*)_{i,i} = L_{\xi_i}^*(x\xi_i) = \langle \xi_i, x\xi_i \rangle_M$. $\square$

REMARK 8.4.16. From Lemma 8.4.13 and the expression (8.4) we get another proof that $\widehat{\tau}$ is a trace. Indeed, given $x \in \mathcal{B}(\mathcal{H}_M)$ we have

$$\begin{aligned}\widehat{\tau}(x^*x) &= \sum_i \tau(\langle x\xi_i, x\xi_i \rangle_M) = \sum_{i,j} \tau(\langle x\xi_i, \xi_j \rangle_M \langle \xi_j, x\xi_i \rangle_M) \\ &= \sum_{i,j} \tau(\langle x^*\xi_j, \xi_i \rangle_M \langle \xi_i, x^*\xi_j \rangle_M) = \widehat{\tau}(xx^*).\end{aligned}$$

PROPOSITION 8.4.17. *Let ξ, η be two left bounded vectors. We have*

$$\widehat{\tau}(L_\xi L_\eta^*) = \tau(\langle \eta, \xi \rangle_M) = \tau(L_\eta^* L_\xi) = \langle \eta, \xi \rangle_{\mathcal{H}}. \tag{8.5}$$

PROOF. We have

$$\begin{aligned}\widehat{\tau}(L_\xi L_\eta^*) &= \sum_i \left\langle \xi_i, (L_\xi L_\eta^*)\xi_i \right\rangle_{\mathcal{H}} = \sum_i \left\langle L_\xi^* \xi_i, L_\eta^* \xi_i \right\rangle_{L^2(M)} \\ &= \sum_i \langle \langle \xi, \xi_i \rangle_M, \langle \eta, \xi_i \rangle_M \rangle_{L^2(M)} = \tau(\langle \eta, \xi \rangle_M) = \tau(L_\eta^* L_\xi).\end{aligned}$$

$\square$

8.5. First results on finite modules

DEFINITION 8.5.1. Let (M, τ) be a tracial von Neumann algebra. We say that a right M-module $\mathcal{H}$ is *finitely generated* if there exists a finite set $\{\eta_1, \ldots, \eta_n\}$ of elements of $\mathcal{H}$ such that $\mathcal{H} = \sum_{i=1}^n \overline{\eta_i M}$.

The following orthonormalisation process will imply that $\{\eta_1, \ldots \eta_n\}$ may be chosen to be an orthonormal basis.

LEMMA 8.5.2 (**Gram-Schmidt orthonormalisation**). *Let $\eta_1, \ldots, \eta_n$ be some elements of a right M-module. There exists an orthonormal family $\xi_1, \ldots, \xi_n$ (i.e., such that $\langle \xi_i, \xi_j \rangle_M = \delta_{i,j} p_j \in \mathcal{P}(M)$) such that $\{\eta_1, \ldots, \eta_n\} \subset \oplus_{i=1}^n \overline{\xi_i M}$.*

PROOF. Using lemma 8.4.10, we may assume that the η_j are left M-bounded. Let $\eta_1 = \xi_1 m_1$ be the polar decomposition of η_1 and set $\eta_2' = \eta_2 - \xi_1 \langle \xi_1, \eta_2 \rangle_M$. We have $\langle \xi_1, \eta_2' \rangle_M = 0$. We consider the polar decomposition $\eta_2' = \xi_2 m_2$ of η_2'. Then $\eta_2 \in \xi_1 M + \xi_2 M$ and $\langle \xi_1, \xi_2 \rangle_M = 0$ since $0 = \langle \xi_1, \xi_2 \rangle_M m_2$ and thus $0 = \langle \xi_1, \xi_2 \rangle_M \langle \xi_2, \xi_2 \rangle_M = \langle \xi_1, \xi_2 \rangle_M$. Iterations of this process prove the lemma. $\square$

PROPOSITION 8.5.3. *Let $\mathcal{H}$ be a right module over a tracial von Neumann algebra M. The following conditions are equivalent:*

(i) *$\mathcal{H}$ is finitely generated;*
(ii) *there exist $n \in \mathbb{N}$ and a projection $p \in M_n(\mathbb{C}) \otimes M = M_n(M)$ such that $\mathcal{H}$ is isomorphic to the right M-module $p(\ell_n^2 \otimes L^2(M))$;*
(iii) *there exist $n \in \mathbb{N}$ and a diagonal projection $p \in M_n(\mathbb{C}) \otimes M = M_n(M)$ such that $\mathcal{H}$ is isomorphic to the right M-module $p(\ell_n^2 \otimes L^2(M))$;*
(iv) *the M-module $\mathcal{H}$ has a finite orthonormal basis;*
(v) *$\mathcal{F}(\mathcal{H}_M) = \mathcal{B}(\mathcal{H}_M)$.*

PROOF. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious and (i) $\Rightarrow$ (iv) is a consequence of the previous lemma.

(iv) $\Rightarrow$ (v). Let $(\xi_1, \ldots, \xi_n)$ be an orthonormal basis of $\mathcal{H}$. Then $\mathrm{Id}_{\mathcal{H}} = \sum_{i=1}^n L_{\xi_i} L_{\xi_i}^* \in \mathcal{F}(\mathcal{H}_M)$.

(v) $\Rightarrow$ (i). Assume that $\mathrm{Id}_{\mathcal{H}} = \sum_{i=1}^n L_{\xi_i} L_{\eta_i}^*$. Then, for $\xi \in \mathcal{H}$ we have

$$\xi = \sum_{i=1}^n L_{\xi_i}(L_{\eta_i}^*(\xi))$$

and therefore $\mathcal{H}$ is finitely generated, since $L_{\xi_i}(L_{\eta_i}^*(\xi)) \in \overline{\xi_i M}$. $\square$

DEFINITION 8.5.4. Let (M, τ) be a tracial von Neumann algebra and $\mathcal{H}$ a right M-module. The *M-dimension* of $\mathcal{H}$ is the number $\widehat{\tau}(\mathrm{Id}_{\mathcal{H}})$ or, equivalently the number $(\mathrm{Tr} \otimes \tau)(p)$, where p is any projection in $\mathcal{B}(\ell^2(\mathbb{N})) \overline{\otimes} M$ such that $\mathcal{H}$ is isomorphic to $p(\ell^2(\mathbb{N}) \otimes L^2(M))$. It is denoted by $\dim(\mathcal{H}_M)$.

One defines similarly the dimension $\dim({}_M\mathcal{K})$ of a left M-module $\mathcal{K}$. In particular, $\dim(L^2(M)_M) = 1 = \dim({}_M L^2(M))$.

The right module $\mathcal{H}$ is said to be *finite* if $\dim(\mathcal{H}_M)$ is finite.

Note that $\dim(\mathcal{H}_M)$ depends on the choice of τ and so the notation may be, unfortunately, misleading.

Given an orthonormal basis (ξ_i) of the module $\mathcal{H}$, or more generally any family (ξ_i) of left bounded vectors such that $\sum_i L_{\xi_i} L_{\xi_i}^* = \mathrm{Id}_{\mathcal{H}}$, we have

$$\dim(\mathcal{H}_M) = \sum_i \tau(\langle \xi_i, \xi_i \rangle_M) = \sum_i \|\xi_i\|_{\mathcal{H}}^2. \tag{8.6}$$

Indeed, this follows from Proposition 8.4.17, since $\widehat{\tau}(\mathrm{Id}_{\mathcal{H}}) = \sum_i \widehat{\tau}(L_{\xi_i} L_{\xi_i}^*)$.

In particular, if $\mathcal{H}$ and $\mathcal{K}$ are two right M-modules and $\mathcal{H} \oplus \mathcal{K}$ is their Hilbert direct sum, we have

$$\dim((\mathcal{H} \oplus \mathcal{K})_M) = \dim(\mathcal{H}_M) + \dim(\mathcal{K}_M).$$

Proposition 8.5.5. *Let $\mathcal{H}$ be a module on a tracial von Neumann algebra M. Consider the following conditions:*

(i) *$\mathcal{H}$ is finitely generated;*
(ii) *$\dim(\mathcal{H}_M) < +\infty$;*
(iii) *the commutant $\mathcal{B}(\mathcal{H}_M)$ of the right representation is a finite von Neumann algebra.*

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. Obvious. $\square$

In the non-factor case, the situation is quite subtle. The three above conditions are not equivalent (see Exercise 8.13). Moreover, the number $\dim(\mathcal{H}_M)$ does not determine the isomorphism class of the corresponding right module. These questions will be clarified in the next chapter. We only consider below the easy case where M is a II_1 factor.

8.6. Modules over II_1 factors

Proposition 8.6.1. *When M is a factor, the three conditions of Proposition 8.5.5 are equivalent.*

Proof. Immediate, since $\mathcal{B}(\mathcal{H}_M)$ is a factor. Indeed, whenever $\mathcal{B}(\mathcal{H}_M)$ is a finite factor, it is isomorphic to some $p\big(\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes} M\big)p$ with $(\mathrm{Tr}\otimes\tau)(p) < +\infty$; so p is equivalent to a projection in some $M_n(\mathbb{C}) \otimes M$ and Condition (ii) of Proposition 8.5.3 holds. $\square$

For the next result, the separability assumptions are essential.

Proposition 8.6.2. *Let M be a separable II_1 factor. The map $\mathcal{H}_M \mapsto \dim(\mathcal{H}_M)$ induces a bijection from the set of equivalences classes of separable right M-modules onto $[0, +\infty]$.*

Proof. We observe that $p(\ell^2(\mathbb{N})\otimes L^2(M))$ and $q(\ell^2(\mathbb{N})\otimes L^2(M))$ are isomorphic if and only if the projections p and q are equivalent in $\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$, thus if and only if $(\mathrm{Tr}\otimes\tau)(p) = (\mathrm{Tr}\otimes\tau)(q)$ (see Proposition 8.3.8).

Finally, the M-dimension can be any element of $[0,+\infty]$ since

$$\big\{(\mathrm{Tr}\otimes\tau)(p) : p\in\mathcal{P}\big(\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M\big)\big\} = [0,+\infty].$$

$\square$

Remark 8.6.3. Let M be a II_1 factor on a Hilbert space $\mathcal{H}$ such that M' is also finite. In [**MVN36**, Theorem 10], Murray and von Neumann proved the deep fact that the number $\tau_M([M'\xi])/\tau_{M'}([M\xi])$ is independent of the choice of the non-zero vector $\xi\in\mathcal{H}$ (where τ_M and $\tau_{M'}$ are the tracial states on M and M' respectively). They used this number as a tool to compare M and M'. It is called the *coupling constant* (between M and M').

We leave it as an exercise to check that this coupling constant is equal to $\dim({}_M\mathcal{H})$.

Exercises

Exercise 8.1. Let M be a II_∞ factor and Tr a normal faithful semi-finite trace on M_+. Let $p\in\mathcal{P}(M)$ be such that $\mathrm{Tr}(p) < +\infty$.

(i) Show that pMp is a II_1 factor.
(ii) Show that there exists a family $(p_i)_{i\in I}$, where I is an infinite set of indices, of mutually orthogonal projections, equivalent to p and such that $\sum_{i\in I} p_i = 1$.
(iii) Show that M is isomorphic to $\mathcal{B}(\ell^2(I))\overline{\otimes}(pMp)$.
(iv) Show that I is countable if and only if M is countably decomposable.

Exercise 8.2. Let M be a von Neumann algebra and let Tr be a trace on M_+. We set

$$\mathfrak{n} = \{x\in M : \mathrm{Tr}(x^*x) < +\infty\}$$

and

$$\mathfrak{m} = \Big\{\sum_{i=1}^n x_i y_i : x_i, y_i\in\mathfrak{n}\Big\}.$$

Prove the following assertions:

(i) $\mathfrak{n}$ and $\mathfrak{m}$ are two-sided ideals of M.
(ii) $\mathfrak{m}\cap M_+ = \{x\in M_+ : \mathrm{Tr}(x) < +\infty\}$ and $\mathfrak{m}$ is linearly generated by $\mathfrak{m}\cap M_+$.
(iii) the restriction of Tr to $\mathfrak{m}\cap M_+$ extends in a unique way to a linear functional on $\mathfrak{m}$ (still denoted by Tr).
(iv) $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ if either $x, y\in\mathfrak{n}$ or $x\in M$ and $y\in\mathfrak{m}$.

The proof is similar to that of Lemma 7.4.3. One says that $\mathfrak{m}$ is the *ideal of definition of* Tr. When Tr is the trace on $\mathcal{B}(\mathcal{H})_+$, then $\mathfrak{m}$ and $\mathfrak{n}$ are respectively the ideals of trace class operators and Hilbert-Schmidt operators.

Exercise 8.3. Let M be a von Neumann algebra and let Tr be a trace on M_+.

(i) Show that Tr is semi-finite if and only if $\mathfrak{n}$ (or $\mathfrak{m}$) is w.o. dense in M.
(ii) Assume that Tr is normal. Show that for $m \in \mathfrak{m}_+$, the positive linear functional $x \mapsto \mathrm{Tr}(xm)$ is normal on M.

Exercise 8.4. Let M be a von Neumann algebra and let Tr be a normal semi-finite trace on M_+. Show (with the help of Exercise 2.11) that there exists a family (a sequence when M is separable) (φ_i) of positive normal linear functionals on M such that $\mathrm{Tr}(x) = \sum_i \varphi_i(x)$ for $x \in M_+$. Conclude that Tr is *lower semi-continuous* in the sense that for every $c > 0$ the set $\{x \in M_+ : \mathrm{Tr}(x) \leq c\}$ is w.o. closed.

Exercise 8.5. Let M be a von Neumann algebra and let Tr be a normal faithful semi-finite trace on M_+. On the two-sided ideal

$$\mathfrak{n} = \{x \in M : \mathrm{Tr}(x^*x) < +\infty\},$$

we define the inner product $\langle x, y\rangle = \mathrm{Tr}(x^*y)$. We denote by $L^2(M, \mathrm{Tr})$ the corresponding completion of $\mathfrak{n}$. Show that $L^2(M, \mathrm{Tr})$ has a natural structure of M-M-bimodule. Observe that whenever $M = \mathcal{B}(\mathcal{H})$ with its usual trace, $L^2(M, \mathrm{Tr})$ is the space $\mathcal{S}^2(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$.

Exercise 8.6. Let $\mathcal{H}$ be a right M-module on a II_1 factor M. Assume that $\dim(\mathcal{H}_M) = c$ with $n \leq c < n+1$. Show that $\mathcal{H}$ is isomorphic, as a M-module to $L^2(M)^{\oplus n} \oplus pL^2(M)$ with $p \in \mathcal{P}(M)$ and $\tau(p) = c - n$.

Exercise 8.7. Let (M, τ) be a tracial von Neumann algebra and let $\mathcal{H} = \mathbb{C}^n \otimes L^2(M)$ be a right $M_n(M)$-module in the obvious way. Show that $\dim(\mathcal{H}_{M_n(M)}) = 1/n$.

Exercise 8.8. Let $\mathcal{H}$ be a right M-module on a II_1 factor M such that $\dim(\mathcal{H}_M) < +\infty$. Let p be a projection in the commutant $\mathcal{B}(\mathcal{H}_M)$ of the right action. Show that $\dim((p\mathcal{H})_M) = \tau_{\mathcal{B}(\mathcal{H}_M)}(p)\dim(\mathcal{H}_M)$, where $\tau_{\mathcal{B}(\mathcal{H}_M)}$ is the unique tracial state on $\mathcal{B}(\mathcal{H}_M)$.

Exercise 8.9. Let M be a II_1-factor, $\mathcal{H}$ a right M-module and let p be a projection in M. Show that $\dim(\mathcal{H}_M) = \tau(p)\dim((\mathcal{H}p)_{pMp})$.

Exercise 8.10. Let $\mathcal{H}$ be a right M-module on a II_1 factor M such that $\dim(\mathcal{H}_M) < +\infty$. Show that $\dim(\mathcal{H}_M)\dim({}_{\mathcal{B}(\mathcal{H}_M)}\mathcal{H}) = 1$.

Exercise 8.11. Let M be a II_1 factor and let $\mathcal{H}$ be a right M-module.

(i) Show that $\mathcal{H}$ has a cyclic vector if and only if $\dim(\mathcal{H}_M) \leq 1$.
(ii) Show that $\mathcal{H}$ has a separating vector if and only if $\dim(\mathcal{H}_M) \geq 1$.

Exercise 8.12. Let M be a II_1 factor on a Hilbert space $\mathcal{H}$ such that M' is also finite. Show that the coupling constant between M and M' is equal to $\dim({}_M\mathcal{H})$.

EXERCISE 8.13. Let $f : [0,1] \to \mathbb{N}$ be a Borel function such that $f \in L^1([0,1],\lambda)$ but $f \notin L^\infty([0,1],\lambda)$, where λ is the Lebesgue measure on $[0,1]$. For each integer n we chose a projection p_n of rank n in $\mathcal{B}(\ell^2(\mathbb{N}))$. Let p be the projection in $L^\infty([0,1],\lambda)\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N}))$ defined by $p(x) = p_n$ if $f(x) = n$. Show that the right module $p\big(L^2([0,1],\lambda)\otimes\ell^2(\mathbb{N})\big)$ over $L^\infty([0,1],\lambda)$ is finite but is not finitely generated.

EXERCISE 8.14. Let M_1, M_2 be two II_1 factors and $\pi_i : M_i \to \mathcal{B}(\mathcal{H}_i)$, $i = 1,2$, be normal representations. Recall that the von Neumann tensor product $(M_1\overline{\otimes}M_2, L^2(M_1,\tau_1)\otimes L^2(M_2,\tau_2))$ has been defined in Section 5.1.1. Show that there is a unique isomorphism π from $M_1\overline{\otimes}M_2$ onto $\pi_1(M_1)\overline{\otimes}\pi_2(M_2)$ (represented on $\mathcal{H}_1\otimes\mathcal{H}_2$) such that $\pi(x_1\otimes x_2) = \pi_1(x_1)\otimes\pi_2(x_2)$ for every $x_1 \in M_1$, $x_2 \in M_2$ (use the structure result about M-modules).

EXERCISE 8.15. Let M be a separable II_1 factor. Let τ_∞ be the canonical normal faithful semi-finite trace on $M_\infty = M\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N}))$. Recall from Section 4.2 that the fundamental group $\mathfrak{F}(M)$ of M is the set of $\tau_\infty(p)$ where p runs over the set of projections $p \in M_\infty$ such that M and $pM_\infty p$ are isomorphic. Show that $\mathfrak{F}(M) = \{\mathrm{mod}(\theta) : \theta \in \mathrm{Aut}\,(M_\infty)\}$.

EXERCISE 8.16. We keep the notation of the previous exercise and we set $\mathrm{Aut}\,_1(M_\infty) = \{\theta \in \mathrm{Aut}\,(M_\infty) : \mathrm{mod}(\theta) = 1\}$. Let p_0 be the rank one projection on the first vector of the canonical basis of $\ell^2(\mathbb{N})$.

(i) Show that there is a unitary $u \in \mathcal{U}(M_\infty)$ and $\alpha \in \mathrm{Aut}\,(M)$ such that $\mathrm{Ad}\,(u)\circ\theta(x\otimes p_0) = \alpha(x)\otimes p_0$ for every $x \in M$. Show that the class of α in $\mathrm{Out}\,(M)$ is well defined. We denote it by α_θ.

(ii) Show that $\theta \mapsto \alpha_\theta$ is a homomorphism from $\mathrm{Aut}\,_1(M_\infty)$ into $\mathrm{Out}\,(M)$ which defines, by passing to the quotient, a homomorphism from $\mathrm{Aut}\,_1(M_\infty)/\mathrm{Inn}\,(M_\infty)$ into $\mathrm{Out}\,(M)$.

Notes

The study of the structure of modules over factors goes back to [**MVN36, MvN43**]. In particular, the coupling constant, or in other terms the dimension of a module, had be investigated in details in [**MVN36**].

Since the eighties, this subject has been developed by many authors, mainly in view of the study of subfactors and of the ergodic theory of group actions and their associated crossed products. A major impetus is due to the influential work of V.F.R. Jones on the index of subfactors [**Jon83b**]. A large part of the above exercises is borrowed from this paper. The idea of using orthonormal bases to compute dimensions of modules comes from [**PP86**] where indices of subfactors were computed in terms of Pimsner-Popa bases (see Propositions 9.4.7 and 9.4.8 in the next chapter).

CHAPTER 9

Conditional expectations. The Jones' basic construction

In this chapter, we consider a tracial von Neumann algebra (M, τ) and a von Neumann subalgebra B. We study in details the right B-module $L^2(M)_B$. An important tool is the trace preserving conditional expectation $E_B : M \to B$ that we introduce first.

Having this notion at hand, we focus on the conditional expectation E_Z where Z is the center of M. It is tracial and intrinsic, and we call it the center-valued trace. When M is only assumed to be semi-finite, there is a more technical notion of center-valued tracial weight, which is essentially unique and plays the same role as E_Z. We use this notion to clarify the relations between the various possible definitions of a finite module over a tracial von Neumann, like B, which is not necessarily a factor.

Then we come back to the case of $L^2(M)_B$. The von Neumann algebra $\mathcal{B}(L^2(M)_B)$ of operators commuting with the right B-action is the so-called algebra of the basic construction for $B \subset M$ which plays an important role in many contexts. In this framework we translate the general results about modules obtained in the first part of this chapter and in the previous chapter.

9.1. Conditional expectations

We extend to the non-commutative setting the notion of conditional expectation which is familiar in measure theory.

Definition 9.1.1. Let M be a von Neumann algebra and B a von Neumann subalgebra. A *conditional expectation* from M to B is a linear map $E : M \to B$ which satisfies the following properties:

(i) $E(M_+) \subset B_+$;
(ii) $E(b) = b$ for $b \in B$;
(iii) $E(b_1 x b_2) = b_1 E(x) b_2$ for $b_1, b_2 \in B$ and $x \in M$.

Hence E is a positive projection from M onto B, and is left and right B-linear. Moreover, for $x \in M$ we have

$$E\big((E(x) - x)^*(E(x) - x)\big) = -E(x)^* E(x) + E(x^* x).$$

It follows that $E(x)^* E(x) \leq E(x^* x)$ and, since $x^* x \leq \|x\|^2 1$, we have $\|E(x)\|^2 \leq \|E(x^* x)\| \leq \|x\|^2$. Hence E is a norm-one projection from M onto B. All this holds as well in the setting of C^*-algebras.

Conversely, every norm-one projection is a conditional expectation (see Theorem A.4 in the appendix).

9.1.1. Existence of conditional expectations.

THEOREM 9.1.2. *Let (M,τ) be a tracial von Neumann algebra and let B be a von Neumann subalgebra. There exists a unique conditional expectation E_B from M onto B such that $\tau \circ E_B = \tau$. Moreover, E_B is normal and faithful*[1].

PROOF. We remark that $L^2(B,\tau|_B)$ is a Hilbert subspace of $L^2(M,\tau)$. For simplicity of notation, we denote these spaces by $L^2(B)$ and $L^2(M)$ respectively. We denote by e_B the orthogonal projection onto $L^2(B)$. Of course, we have $e_B(\hat{b}) = \hat{b}$ for $b \in B$. Thanks to Proposition 7.3.4 we see that $e_B(L^2(M)_+) \subset L^2(B)_+$. Now, given $x \in M$ with $0 \leq x \leq 1$, we get $0 \leq e_B(\widehat{x}) \leq \hat{1}$, whence $e_B(\widehat{x}) \in \widehat{B_+}$, and thus we deduce the inclusion $e_B(\widehat{M}) \subset \widehat{B}$. We identify M and $\widehat{M}$ and define E_B to be the restriction of e_B to M. For $b \in B$ and $x \in M$, we have $\tau(xb) = \tau(E_B(x)b)$. It follows that E_B is a conditional expectation with $\tau \circ E_B = \tau$. In particular, E_B is faithful.

We now show the uniqueness of E_B. If E is another conditional expectation with $\tau \circ E = \tau$, then for $x \in M$ and $b \in B$, we have

$$\tau\big((x - E(x))b\big) = \tau\big(E((x - E(x))b\big) = 0,$$

i.e., $\hat{x} - \widehat{E(x)}$ and $\widehat{B}$ are orthogonal. Hence, E has to be the orthogonal projection from $\widehat{M} \subset L^2(M)$ onto $\widehat{B} \subset L^2(B)$.

The normality of E_B follows from Corollary 2.5.11. □

REMARK 9.1.3. Note that $E_B(x)$ is the unique element y in B such that $\tau(xb) = \tau(yb)$ for every $b \in B$. Another way to introduce E_B is to use the Radon-Nikodým theorem 7.3.6 (see Exercise 9.1). We also remark that E_B is the restriction to M of the orthogonal projection e_B from $L^2(M)$ onto $L^2(B)$, when M and B are identified to subspaces of $L^2(M)$ and $L^2(B)$ respectively.

9.1.2. Examples. Of course, E_B depends on the choice of the trace τ. Usually, this choice is implicit and we do not mention it.

(1) Take $M = M_n(\mathbb{C})$ and let B be the subalgebra of diagonal matrices. Then E_B is the application sending a matrix x to its diagonal part.

More generally, let (M,τ) be a tracial von Neumann algebra, and let $B = \sum_{i=1}^n \mathbb{C}e_i$ be generated by non-zero projections $e_1, \dots, e_n$ with $\sum_{i=1}^n e_i = 1$. Then, for $x \in M$,

$$E_B(x) = \sum_{i=1}^n \frac{\tau(e_i x e_i)}{\tau(e_i)} e_i.$$

[1] In case of ambiguity, we will write E_B^M for E_B.

(2) We keep the same notations. Then

$$E_{B'\cap M}(x) = \sum_{i=1}^n e_i x e_i.$$

(3) Let $(e_{i,j})_{1\leq i,j\leq n}$ be a matrix units of a von Neumann subalgebra B of (M,τ) (so that B is isomorphic to $M_n(\mathbb{C})$). Then

$$E_B(x) = \sum_{1\leq i,j\leq n} n\tau(xe_{j,i})e_{i,j}.$$

(4) Let $G \curvearrowright (B,\tau)$ be a trace preserving action and set $M = B \rtimes G$. Let τ still denote the natural tracial state on M defined by

$$\tau\big(\sum_{g\in G} x_g u_g\big) = \tau(x_e)$$

(see Section 5.2). Then $E_B\big(\sum_{g\in G} x_g u_g\big) = x_e$. It follows that the Fourier coefficient x_g of $x = \sum_{g\in G} x_g u_g$ is given by $x_g = E_B(xu_g^*)$.

(5) Let $\mathcal{R}$ be a p.m.p. countable equivalence relation on (X,μ). We keep the notation of Section 1.5.2. We have seen that the von Neumann algebra $L(\mathcal{R})$ may be identified with a subset of $L^2(\mathcal{R},\nu)$. Its natural trace τ is defined by

$$\tau(F) = \int_X F(x,x)\,\mathrm{d}\mu(x).$$

Recall that $B = L^\infty(X,\mu)$ embeds into $L(\mathcal{R})$, as its diagonally supported elements. For $F \in L(\mathcal{R})$, we readily check that $E_B(F)$ is the restriction of F to the diagonal subset of $\mathcal{R}$.

9.1.3. Extensions of conditional expectations to L^1-spaces. Let (M,τ) be a tracial von Neumann algebra and let B be a von Neumann subalgebra. For $b \in B$ and $m \in M$, we have

$$|\tau(bE_B(m))| = |\tau(bm)| \leq \|b\|_\infty \|m\|_1,$$

whence $\|E_B(m)\|_1 \leq \|m\|_1$. It follows that E_B extends to a norm-one projection from $L^1(M)$ onto $L^1(B)$, still denoted E_B. Observe that $\tau(bE_B(\xi)) = \tau(b\xi)$ for every $\xi \in L^1(M)$ and $b \in B$.[2]

By definition, $E_B : M \to B$ extends to the orthogonal projection $e_B : L^2(M) \to L^2(B)$, that we also denote by E_B, for consistency reasons.

Lemma 9.1.4. *Let (M,τ) and B as above.*

(i) *The restriction of $E_B : L^1(M) \to L^1(B)$ to $L^2(M)$ is the projection $E_B = e_B : L^2(M) \to L^2(B)$;*

[2]In other terms, when $L^1(M)$ and $L^1(B)$ are identified to M_* and B_* respectively (see Theorem 7.4.5), then E_B is the map sending a functional in M_* to its restriction to B.

(ii) $E_B(L^p(M)_+) = L^p(B)_+$ *and so* $E_B(\xi^*) = E_B(\xi)^*$ *for* $\xi \in L^p(M)$, $p = 1, 2$;
(iii) $E_B(b\xi) = bE_B(\xi)$, $E_B(\xi b) = E_B(\xi)b$ *for* $b \in B$ *and* $\xi \in L^p(M)$, $p = 1, 2$;
(iv) $E_B(\eta\xi) = \eta E_B(\xi)$, $E_B(\xi\eta) = E_B(\xi)\eta$ *for* $\eta \in L^2(B)$ *and* $\xi \in L^2(M)$;
(v) *Whenever* D *is a von Neumann subalgebra of* B, *we have* $E_D = E_D \circ E_B$.

PROOF. We leave the straightforward proofs to the reader. $\square$

9.1.4. Center-valued traces.

DEFINITION 9.1.5. A *center-valued trace* on a von Neumann algebra M is a conditional expectation E from M onto its center $\mathcal{Z}(M)$ such that $E(xy) = E(yx)$ for every $x, y \in M$.

PROPOSITION 9.1.6. *Let* (M, τ) *be a tracial von Neumann algebra and* $Z = \mathcal{Z}(M)$ *its center. Then* E_Z *is a center-valued trace. It is normal and faithful, and it is the only normal center-valued trace on* M.

PROOF. Given $z \in Z$, we have

$$\tau(zE_Z(xy)) = \tau(zxy) = \tau(xzy) = \tau(zyx) = \tau(zE_Z(yx)),$$

whence $E_Z(xy) = E_Z(yx)$.

Let E be a normal center-valued trace on M. Then $\tau \circ E$ is a normal trace on M which has the same restriction to Z as τ. It follows from Proposition 7.3.9 that $\tau = \tau \circ E$ and therefore $E = E_Z$. $\square$

REMARK 9.1.7. More generally, any finite von Neumann algebra carries a unique faithful center-valued trace. Moreover this center-valued trace is normal (see [**Tak02**, Chapter V, Theorem 2.6]).

The following result generalizes the corollary 2.4.11.

PROPOSITION 9.1.8. *Let* E_Z *be the normal center-valued trace on* (M, τ). *Given two projections* p, q *in* M, *we have* $p \precsim q$ *if and only if* $E_Z(p) \leq E_Z(q)$.

PROOF. Assume that $E_Z(p) \leq E_Z(q)$ and that there exists a projection z in Z such that $qz \prec pz$. Since E_Z is faithful, we have

$$E_Z(q)z = E_Z(qz) < E_Z(pz) = E_Z(p)z,$$

in contradiction with the fact that $E_Z(p) \leq E_Z(q)$. It follows that $pz \precsim qz$ for every projection $z \in Z$, and the comparison theorem for projections implies that $p \precsim q$. $\square$

9.2. Center-valued tracial weights

A von Neumann algebra which carries a faithful normal center-valued trace is finite since it has obviously sufficiently many normal traces in the sense of Theorem 6.4.4. For semi-finite von Neumann algebras we may use, instead, center-valued tracial weights, which generalize both center-valued traces and tracial weights.

In this section and the following one, we only consider separable von Neumann algebras acting on separable Hilbert spaces.

Let M be a (separable) von Neumann algebra. We identify its center Z with $L^\infty(X,\mu)$ where (X,μ) is a standard probability measure space. We denote by $\widehat{Z}_+$ the cone of measurable functions from X into $[0,+\infty]$, where two functions which coincide almost everywhere are identified. This set has an obvious order, which extends the natural order on $L^\infty(X,\mu)_+ \subset \widehat{Z}_+$. In $\widehat{Z}_+$ every increasing net has a least upper bound.

Definition 9.2.1. A *center-valued tracial weight* on M_+ is a map $\mathcal{T}r_Z : M_+ \to \widehat{Z}_+$ such that

(a) $\mathcal{T}r_Z(x+y) = \mathcal{T}r_Z(x) + \mathcal{T}r_Z(y)$ for $x, y \in M_+$;
(b) $\mathcal{T}r_Z(zx) = z\mathcal{T}r_Z(x)$ for $z \in Z_+$ and $x \in M_+$;
(c) $\mathcal{T}r_Z(x^*x) = \mathcal{T}r_Z(xx^*)$ for $x \in M$.

It is called *semi-finite* if, in addition,

(d) for every non-zero $x \in M_+$, there exists some non-zero $y \in M_+$ with $y \leq x$ and $\mathcal{T}r_Z(y) \in Z_+$.

If

(e) $\mathcal{T}r_Z(\sup_i x_i) = \sup_i \mathcal{T}r_Z(x_i)$ for every bounded increasing net (x_i) in M_+, we say that $\mathcal{T}r_Z$ is *normal*.

The notion of faithful center-valued tracial weight is defined in the obvious way. Whenever $\mathcal{T}r_Z(1) \in Z$ (or equivalently $\mathcal{T}r_Z(x) \in Z_+$ for every $x \in M_+$), one says that $\mathcal{T}r_Z$ is finite. In particular, if $\mathcal{T}r_Z(1) = 1$, then $\mathcal{T}r_Z$ extends uniquely to a center-valued trace on M.

It is easily seen that a von Neumann algebra M which admits a normal, faithful, semi-finite center-valued tracial weight is semi-finite. Conversely, we have:

Theorem 9.2.2. *Let M be a semi-finite von Neumann algebra and $Z = L^\infty(X,\mu)$ be its center*[3].

(i) *There exists a normal faithful semi-finite center-valued tracial weight on M_+.*
(ii) *Let $\mathcal{T}r_{Z,1}$ and $\mathcal{T}r_{Z,2}$ be two such center-valued tracial weights. There exists a unique (up to null sets) Borel function $f : X \to (0,+\infty)$ such that $\mathcal{T}r_{Z,1} = f\,\mathcal{T}r_{Z,2}$.*

[3]The assertions to follow, which only involve properties up to null sets, do not depend on this choice.

Proof. (i) Let Tr be a normal faithful semi-finite tracial weight on M_+. For $y \in M_+$, we denote by Tr_y the map $z \to \mathrm{Tr}(yz)$ from Z_+ into $[0, +\infty]$. We write Tr as a sum $\sum_n \varphi_n$ of finite normal functionals (see Exercise 8.4). The classical Radon-Nikodým theorem applied to $z \mapsto \varphi_n(zy)$ gives $f_n(y) \in L^1(X,\mu)_+$ such that $\varphi_n(zy) = \int_X f_n(y) z \,\mathrm{d}\mu$ for every $z \in L^\infty(X,\mu)$. We set $\Phi(y) = \sum_n f_n(y)$. Then we have:

$$\forall z \in Z_+, \mathrm{Tr}(zy) = \int_X \Phi(y) z \,\mathrm{d}\mu.$$

It is a routine exercise to check that Φ is a normal faithful semi-finite center-valued tracial weight on M_+.

(ii) Set $\mathrm{Tr}_i(x) = \int_X \mathcal{T}r_{Z,i}(x)\,\mathrm{d}\mu$ for $x \in M_+$, $i = 1, 2$. Then f is the Radon-Nikodým derivative of Tr_1 with respect to Tr_2 (see Exercise 9.6 for the only case we will need, where one of the center-valued weights is finite)[4]. □

Example 9.2.3. Let (M, τ) be a tracial von Neumann algebra with center Z and let E_Z be its center-valued trace. We identify the center $\mathrm{Id}_{\ell^2(\mathbb{N})} \otimes Z$ of $\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes} M$ with Z. For $x \in \big(\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes} M\big)_+$, we set

$$(\mathrm{Tr} \otimes E_Z)([x_{i,j}]) = \sum_{i \in \mathbb{N}} E_Z(x_{i,i}) \in \widehat{Z}_+.$$

It is easily checked that $\mathrm{Tr} \otimes E_Z$ is a normal faithful semi-finite center-valued tracial weight on $\big(\mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes} M\big)_+$.

Given a faithful normal semi-finite tracial weight Tr on a separable semi-finite von Neumann algebra M, we may have $\mathrm{Tr}(p) = \mathrm{Tr}(q)$ despite the fact that the projections p and q are not equivalent. In contrast, center-valued tracial weights prove to be a useful tool in the classification of projections. The next result generalizes Proposition 8.3.8.

Proposition 9.2.4. *Let $\mathcal{T}r_Z$ be a normal faithful semi-finite center-valued tracial weight on a separable semi-finite von Neumann algebra M and let $p, q \in \mathcal{P}(M)$. Then $p \precsim q$ if and only if $\mathcal{T}r_Z(p) \leq \mathcal{T}r_Z(q)$.*

Proof. When one of the two functions $\mathcal{T}r_Z(p)$ or $\mathcal{T}r_Z(q)$ is finite almost everywhere, the proof is similar to that of Proposition 9.1.8. We admit the general case, that we do not really need in this monograph[5]. □

Proposition 9.2.5. *We keep the notation of the previous proposition. A projection $p \in M$ is finite if and only if $\mathcal{T}r_Z(p) < +\infty$ almost everywhere.*

Proof. Assume that $\mathcal{T}r_Z(p) < +\infty$ almost everywhere and that $p \sim p_1 \leq p$. Then $\mathcal{T}r_Z(p - p_1) = 0$ and so $p = p_1$. Conversely, assume that p is finite. We identify the center pZ of pMp with qZ, where q is the central support of p. The restriction of $\mathcal{T}r_Z$ to $(pMp)_+$ has its range into $q\widehat{Z}_+$ and

[4] We refer to [**Dix81**, Chapter III, §4]) for a complete proof.

[5] For the general case we refer the interested reader to [**Dix81**, Chapter III, §4, Exercise 4] combined with [**Dix81**, Chapter III, §1, Exercise 15].

is a center-valued normal faithful semi-finite tracial weight. Replacing M by pMp, we may assume that $p = 1$. Let E_Z be the center-valued trace on the finite von Neumann algebra M. By Theorem 9.2.2 (ii) we have $\mathcal{T}r_Z = fE_Z$ where f is finite almost everywhere, hence the conclusion since $f = \mathcal{T}r_Z(1)$. $\square$

COROLLARY 9.2.6. *Let p, q be two finite projections in a semi-finite von Neumann algebra. Then $p \vee q$ is a finite projection.*

9.3. Back to the study of finite modules

We have seen that the modules $\mathcal{H}$ over II_1 factors are classified by their dimension $\widehat{\tau}(1)$ where $1 = \mathrm{Id}_{\mathcal{H}}$ and $\widehat{\tau}$ is the canonical tracial weight on the commutant of the representation. Here $\widehat{\tau}$ is intrinsic.

In the general case of a tracial von Neumann algebra (M, τ), we must replace the tracial weight $\widehat{\tau}$, which depends on the choice of τ, by a center-valued tracial weight in order to get a complete invariant for M-modules. Let $Z = L^\infty(X, \mu)$ be the center of M, where μ comes from the restriction of τ to Z, and let E_Z be the trace-preserving conditional expectation from M onto Z. Given a right M-module $\mathcal{H}$, let $U : \mathcal{H} \to \ell^2(\mathbb{N}) \otimes L^2(M)$ be a M-linear isometry. Then we define a normal, faithful, semi-finite center-valued tracial weight $\widehat{E}_Z$ on $\mathcal{B}(\mathcal{H}_M)_+$ by the formula

$$\widehat{E}_Z(x) = (\mathrm{Tr} \otimes E_Z)(UxU^*) \in \widehat{Z}_+.$$

We easily see, as in the proof of Proposition 8.4.2, that $\widehat{E}_Z$ does not depend on the choice of U. Furthermore, $\widehat{E}_Z$ does not depend on τ since E_Z is intrinsic.

The same proof as that of Proposition 8.4.2 gives

$$\widehat{E}_Z(TT^*) = E_Z(T^*T) \tag{9.1}$$

for every bounded, right M-linear operator $T : L^2(M) \to \mathcal{H}$.

Note that, for $x \in \mathcal{B}(\mathcal{H}_M)_+$,

$$\widehat{\tau}(x) = \int_X \widehat{E}_Z(x)\, \mathrm{d}\mu. \tag{9.2}$$

The function $\widehat{E}_Z(1) = (\mathrm{Tr} \otimes E_Z)(UU^*) = (\mathrm{Tr} \otimes E_Z)(p)$, where p is any projection in $\mathcal{B}(\ell^2(\mathbb{N}) \otimes L^2(M))$ such that $\mathcal{H}$ is isomorphic to $p(\ell^2(\mathbb{N}) \otimes L^2(M))$, should be considered as the "*dimension*" of the module $\mathcal{H}$. It is independent of τ and is a complete invariant (under our separability assumptions): two projections p and q in $\mathcal{B}(\ell^2(\mathbb{N}) \otimes L^2(M))$ are equivalent if and only if $(\mathrm{Tr} \otimes E_Z)(p) = (\mathrm{Tr} \otimes E_Z)(q)$, by Proposition 9.2.4.

REMARK 9.3.1. When M has no abelian projection then, for every $z \in \widehat{Z}_+$, there is a projection $p \in \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes} M$ such that $\widehat{E}_Z(p) = z$ (see Exercise 9.5). In this case, right M-modules (up to isomorphism) are thus in bijective correspondence with $\widehat{Z}_+$. When M is a II_1 factor, this result applies with $\widehat{Z}_+ = [0, +\infty]$.

On the other hand, when $M = L^\infty(X,\mu)$, the element $\widehat{E}_Z(1)$ of $\widehat{Z}_+$ corresponding to the M-module $\mathcal{H} = \oplus_k\big(\ell^2_k \otimes L^2(X_k,\mu)\big)$, where (X_k) is a partition of X, is the multiplicity function $n : X \to [0,+\infty]$ such that $n(t) = k$ for $t \in X_k$ (which is thus a complete invariant, as already observed in Theorem 8.1.1). In this case, the M-modules are in bijective correspondence with the elements of $\widehat{Z}_+$ taking their values in $\mathbb{N} \cup \{\infty\}$ (see Theorem 8.1.1).

We now clarify the statement of Proposition 8.5.5, in term of the behaviour of the "dimension" $\widehat{E}_Z(1)$.

Proposition 9.3.2. *Let $\mathcal{H}$ be a right M-module. We set $Z = L^\infty(X,\mu)$, where μ is the probability measure defined by τ. Let $d = \widehat{E}_Z(1) \in \widehat{Z}_+$.*

(i) *$\mathcal{H}$ is a finitely generated M-module if and only if $d \in L^\infty(X,\mu)$.*
(ii) *$\mathcal{H}$ is a finite right M-module if and only if $\int_X d(t)\,\mathrm{d}\mu(t) < +\infty$ (i.e., $d \in L^1(X,\mu)$).*
(iii) *The commutant $\mathcal{B}(\mathcal{H}_M)$ of the right representation is finite if and only if $d < +\infty$ a.e.*

Proof. (i) Obviously, d is bounded whenever $\mathcal{H}$ is finitely generated. Conversely, assume that $d \leq n$. For simplicity we consider the case $n = 1$. By Proposition 8.2.2, we may take $\mathcal{H} = \oplus_k p_k L^2(M)$ with $p_k \in \mathcal{P}(M)$ for all k and we have $\sum_k E_Z(p_k) = d \leq 1$. Using Proposition 9.1.8, we see that we may choose the projections p_k to be mutually orthogonal in M. Then the right module $\mathcal{H}$ is isomorphic to $qL^2(M)$, where $q = \sum_k p_k$, and is therefore generated by $\widehat{q}$.

(ii) is obvious since $(\mathrm{Tr} \otimes \tau)(p) = \int_X d(t)\,\mathrm{d}\mu(t)$ and (iii) is a consequence of Proposition 9.2.5. $\square$

Observe that only the property stated in (ii) depends on the choice of τ.

A finitely generated M-module is finite. The converse is not too far from being true.

Corollary 9.3.3. *Let $\mathcal{H}$ be a finite right M-module. There is an increasing sequence (z_n) of projections in Z such that $\lim_n z_n = 1$ in the s.o. topology and such that $\mathcal{H}z_n$ is a finitely generated right M-module for every n.*

Proof. Take d as in the previous proposition and let z_n by the characteristic function of $\{t \in X : d(t) \leq n\}$. Since $z_n d$ is bounded, the M-module $\mathcal{H}z_n$ is a finitely generated. Moreover, $\lim_n z_n = 1$ in the s.o. topology because d is μ-integrable (in fact, $d < +\infty$ a.e. would be enough). $\square$

9.4. Jones' basic construction

In this section, we are given a tracial von Neumann algebra (M,τ) in standard form, *i.e.*, $M \subset \mathcal{B}(L^2(M))$, and a von Neumann subalgebra B. Then $L^2(M)$ has an obvious structure of right B-module, which will be our object of study.

9.4.1. Definition and first properties.

Definition 9.4.1. Let (M,τ) be a tracial von Neumann algebra and let B be a von Neumann subalgebra. The von Neumann algebra $\langle M, e_B\rangle$ generated by M and the projection e_B in $\mathcal{B}(L^2(M))$ is called the *extension of M by B*, or the von Neumann algebra of the *(Jones') basic construction for $B \subset M$*.

We give below a list of some fundamental properties of this basic construction. Recall that J is the canonical conjugation operator on $L^2(M)$. Assertion (4) states that $\langle M, e_B\rangle$ is the commutant $\mathcal{B}(L^2(M)_B)$ of the right B-action.

Proposition 9.4.2. *Let $B \subset M$ be as above. Then*

(1) $e_B x e_B = E_B(x) e_B$ *for every* $x \in M$*;*
(2) $J e_B = e_B J$*;*
(3) $B = M \cap \{e_B\}'$*;*
(4) $\langle M, e_B\rangle = JB'J = (JBJ)'$*;*
(5) *the central support of* e_B *in* $\langle M, e_B\rangle$ *is* 1*;*
(6) $\langle M, e_B\rangle = \overline{span\{x e_B y : x, y \in M\}}^{w.o}$*;*
(7) $B \ni b \mapsto b e_B$ *is an isomorphism from* B *onto* $e_B\langle M, e_B\rangle e_B$*.*

Proof. The proof of statements (1) to (4) is straightforward and left to the reader. The central support of e_B in $\langle M, e_B\rangle$ is the orthogonal projection from $L^2(M)$ onto $\overline{\langle M, e_B\rangle\, e_B L^2(M)}^{\|\cdot\|_2}$ which is obviously $L^2(M)$. So, assertion (5) is immediate.

Using (1), it is easily seen that $\mathrm{span}\{x e_B y : x, y \in M\}$ is a $*$-subalgebra of $\langle M, e_B\rangle$ and a two-sided ideal of the $*$-algebra generated by $M \cup \{e_B\}$. Thus, $I = \overline{\mathrm{span}\{x e_B y : x, y \in M\}}^{w.o}$ is a w.o. closed two-sided ideal of $\langle M, e_B\rangle$. Since I contains e_B whose central support is 1, we get (6) (see Proposition 2.4.15).

Finally, to prove (7), we observe that $B \ni b \mapsto b e_B$ is a normal homomorphism from B into $e_B\langle M, e_B\rangle e_B$ by (3). It is injective since $b e_B = 0$ implies $0 = b e_B \widehat{1} = \widehat{b}$. Moreover, for $x, y \in M$, we have $e_B(x e_B y) e_B = E_B(x) E_B(y) e_B$, so the surjectivity is a consequence of (6). $\square$

Since $\langle M, e_B\rangle$ is the commutant of the right action of B, it is a semi-finite von Neumann algebra equipped with its canonical normal semi-finite faithful center-valued tracial weight $\widehat{E}_Z$ and its tracial weight $\widehat{\tau}$, where Z is here the center of B and where the restriction to B of the trace on M is still denoted by τ. Let us make explicit these objects. Given $x \in M$, denote by $L_x : L^2(B) \to L^2(M)$ the right B-linear bounded operator such that $L_x(\hat{b}) = \widehat{xb}$ for $b \in B$. We have $(L_x)^*(\widehat{m}) = e_B(x^*\widehat{m})$ for $m \in M$ and thus

$$\forall x, y \in M, \quad L_x L_y^* = x e_B y^*, \quad L_y^* L_x = E_B^M(y^* x).$$

It follows from the equality (8.2) that

$$\widehat{\tau}(xe_By) = \tau(xy) \tag{9.3}$$

whenever $x, y \in M$. By item (6) of the previous proposition, this characterizes $\widehat{\tau}$.

Similarly, using the equality (9.1), where now, as already said, Z is the center of B (assumed to be separable), we get

$$\widehat{E}_Z(xe_By) = E_Z^B \circ E_B^M(xy) = E_Z^M(xy), \tag{9.4}$$

since $E_Z^B \circ E_B^M = E_Z^M$.

We now translate the results of Sections 8.5 and 8.6 in the setting of the right B-module $L^2(M)_B$. The elements of $\widehat{M}$ are left B-bounded but the space $(L^2(M)_B)^0$ of left B-bounded vectors can be strictly larger. Let $\xi \in (L^2(M)_B)^0$ and denote by $L_\xi : L^2(B) \to L^2(M)$ the corresponding operator. Then we have, for $\eta \in L^2(M)$,

$$L_\xi^*(\eta) = E_B(\xi^*\eta). \tag{9.5}$$

Indeed, for $b \in B$, we have

$$\begin{aligned}\left\langle L_\xi^*(\eta), \widehat{b}\right\rangle_{L^2(B)} &= \langle \eta, \xi b\rangle_{L^2(M)} = \langle \eta, Jb^*J\xi\rangle_{L^2(M)} = \langle b^*J\xi, J\eta\rangle_{L^2(M)}\\ &= \langle \xi^*, b\eta^*\rangle_{L^2(M)} = \tau(\xi b\eta^*) = \tau(\eta^*\xi b) = \tau(E_B(\eta^*\xi)b).\end{aligned}$$

It follows that for $\xi, \eta \in (L^2(M)_B)^0$,

$$\langle \xi, \eta\rangle_B = L_\xi^* L_\eta = E_B(\xi^*\eta) \in B, \tag{9.6}$$

$$L_\eta L_\xi^* = \eta \circ e_B \circ \xi^*, \tag{9.7}$$

where $\eta \circ e_B \circ \xi^* \in \langle M, e_B\rangle$ is the bounded operator on $L^2(M)$ such that $\eta \circ e_B \circ \xi^*(\widehat{m}) = \eta E_B(\xi^* m)$ for every $m \in M$.

As said in Remark 8.4.14, the operators $L_\eta L_\xi^*$ may be viewed as "finite rank" operators, and therefore the elements of the norm closure $\mathcal{I}_0(\langle M, e_B\rangle)$ of the vector space they generate are "compact" operators. We will come back to this subject in Section 16.3.

Proposition 9.4.3. *The space $\mathcal{I}_0(\langle M, e_B\rangle)$ defined above is the norm-closed ideal of $\langle M, e_B\rangle$ generated by e_B.*

Proof. Observe first that $\mathcal{I}_0(\langle M, e_B\rangle)$ is a norm-closed ideal which contains $e_B = L_1 L_1^*$. On the other hand, $L_\xi L_\eta^* = (L_\xi e_B) e_B (L_\eta e_B)^*$ sits in the ideal generated by e_B, since $L_\xi e_B$ and $L_\eta e_B$ belong to $\langle M, e_B\rangle$. This concludes the proof. $\square$

Proposition 9.4.4. *Let (M, τ) and B as above and let (ξ_i) be an orthonormal basis of $L^2(M)_B$.*

(i) *We have* $\dim(L^2(M)_B) = \widehat{\tau}(1) = \sum_i \|\xi_i\|_{L^2(M)}^2 = \sum_i \|\xi_i \xi_i^*\|_{L^1(M)}$.

(ii) *We have* $\dim(L^2(M)_B) < +\infty$ *if and only if the series* $\sum_i \xi_i\xi_i^*$ *is convergent in* $L^1(M)$. *Then* $\sum_i \xi_i\xi_i^*$ *is the Radon-Nikodým derivative of* $\widehat{\tau}_{|M}$ *with respect to* τ *and so is affiliated to the center of* M.

PROOF. (i) is Formula (8.6).

(ii) Assume that the series $\sum_i \xi_i\xi_i^*$ is convergent in $L^1(M)$. By Proposition 8.4.15 we have, for $m \in M_+$,

$$\widehat{\tau}(m) = \sum_i \langle \xi_i, m\xi_i\rangle_{L^2(M)} = \sum_i \tau(m\xi_i\xi_i^*) = \tau(m(\sum_i \xi_i\xi_i^*)).$$

It follows that $\widehat{\tau}$ is finite and that $\sum_i \xi_i\xi_i^*$ is the Radon-Nikodým derivative of $\widehat{\tau}_{|M}$ with respect to τ. This operator is affiliated to the center of M since $\widehat{\tau}$ is tracial. $\square$

9.4.2. Case where M is a II_1 factor.

PROPOSITION 9.4.5. *Let* M *be a* II_1 *factor and* $B \subset M$ *a von Neumann subalgebra such that* $\dim(L^2(M)_B) < +\infty$. *For every* $x \in \langle M, e_B\rangle$ *there exists a unique* $m \in M$ *such that* $xe_B = me_B$.

PROOF. Our assumption is that $\widehat{\tau}$ is a normal faithful finite trace on $\langle M, e_B\rangle$. We set $d = \widehat{\tau}(1)$. Then $\widehat{\tau}_{|M} = d\tau$. Let E_M be the unique conditional expectation from $\langle M, e_B\rangle$ onto M such that $\widehat{\tau} \circ E_M = \widehat{\tau}$. For $m \in M$, we have

$$d\tau(mE_M(e_B)) = \widehat{\tau}(me_B) = \tau(m),$$

whence $dE_M(e_B) = 1$.

If $xe_B = me_B$ with $m \in M$, we get $m = dE_M(xe_B)$, hence the uniqueness of m. Let us prove its existence. We first consider the case $x = m_1e_Bm_2$, with $m_1, m_2 \in M$. Then, we have

$$m_1e_Bm_2e_B = m_1E_B(m_2)e_B$$

which proves our assertion in this case. The conclusion for any x follows from (6) in Proposition 9.4.2 and the continuity property of E_M with respect to the w.o. topology. $\square$

COROLLARY 9.4.6. *Let* M *be a* II_1 *factor and* $B \subset M$ *a von Neumann subalgebra such that* $\dim(L^2(M)_B) < +\infty$. *The set of left* B*-bounded vectors in* $L^2(M)_B$ *coincides with* $\widehat{M}$.

PROOF. Let ξ be a left B-bounded vector. Then $L_\xi e_B$ belongs to $\langle M, e_B\rangle$, and by the previous proposition, there exists $m \in M$ such that $L_\xi e_B = me_B$. It follows that $\xi = L_\xi e_B(\hat{1}) = me_B(\hat{1}) = \widehat{m}$. $\square$

This fact non longer holds in general (see Exercise 9.11).

As a consequence of the corollary, we see in the next proposition, that whenever M is a factor with $\dim(L^2(M)_B) < +\infty$, the elements ξ_i of Proposition 9.4.4 are in $\widehat{M}$.

Proposition 9.4.7. *Let M be a II_1 factor and $B \subset M$ a von Neumann subalgebra. Then $d = \dim(L^2(M)_B) < +\infty$ if and only if there exists a family (m_i) in M such that*

(i) $E_B(m_i^* m_j) = \delta_{i,j} p_j \in \mathcal{P}(B)$ *for all i, j;*
(ii) $\sum_i m_i e_B m_i^* = 1$ *(convergence in the w.o. topology);*
(iii) $\sum_i m_i m_i^*$ *converges in $L^1(M)$.*

Whenever these conditions hold, we have $\sum_i m_i m_i^* = d1$. In particular, the convergence of this series also holds in the w.o. topology. Moreover, $L^2(M)_B = \oplus_i \overline{\widehat{m_i} B}$ (orthogonal Hilbert sum).

Proof. Assume the existence of a family $(m_i) \in M$ satisfying conditions (i), (ii) and (iii) of the above statement. It is an orthonormal basis. In particular, since the projections $m_i e_B m_i^*$ are mutually orthogonal with range $\overline{\widehat{m_i} B}$, we have $L^2(M)_B = \oplus_i \overline{\widehat{m_i} B}$.

By Proposition 9.4.4, we get that $\sum_i m_i m_i^*$ is a scalar operator $d\,1$ with $\dim(L^2(M_B)) = d < +\infty$. But then, $d\,1$ is the least upper bound in M of the family of finite partial sums of the series, whence the convergence in the w.o. topology.

Conversely, assume that $\dim(L^2(M_B)) = d < +\infty$. Then the "only if" part follows from Proposition 9.4.4 and Corollary 9.4.6. $\square$

Proposition 9.4.8. *Let M be a II_1 factor and $B \subset M$ a von Neumann subalgebra. Then $L^2(M)_B$ is finitely generated if and only if there exists $m_1, \dots, m_n \in M$ such that*

(i) $E_B(m_i^* m_j) = \delta_{i,j} p_j \in \mathcal{P}(B)$*for all i, j;*
(ii) $\sum_{1 \le i \le n} m_i e_B m_i^* = 1$.

Whenever these conditions hold, we have $\sum_{1 \le i \le n} m_i m_i^* = \dim(L^2(M)_B)1$ and $x = \sum_{1 \le i \le n} m_i E_B(m_i^* x)$ for every $x \in M$.

Proof. Assume that $L^2(M)_B$ is a finitely generated B-module. By Proposition 8.5.3, we know that it has a finite orthonormal basis. We conclude thanks to Corollary 9.4.6. The converse is obvious.

If these conditions hold, recall from Remark 8.4.12 that every $x \in M$ has a unique expression of the form $x = \sum_{i=1}^n m_i b_i$ with $b_i \in p_i B$, and that $b_i = \langle m_i, x \rangle_B = E_B(m_i^* x)$. $\square$

The family $(m_i)_{1 \le i \le n}$ is called a *Pimsner-Popa basis.*

Definition 9.4.9. In case B is a subfactor of a separable II_1 factor M, the *Jones' index*[6] of B in M is the number

$$[M : B] = \dim(L^2(M)_B).$$

It is finite if and only if $\langle M, e_B \rangle$ is a II_1 factor and also if and only if $L^2(M)_B$ is finitely generated (see Proposition 8.6.1). We set

$$\mathfrak{I}(M) = \{[M : B] : B \subset M, \text{ subfactor of finite index}\}.$$

[6] For an explanation of the terminology, see Exercise 9.13.

Remark 9.4.10. We observe that since $L^2(B) \subset L^2(M)$, we have $[M : B] \geq 1$ and $[M : B] = 1$ if and only if $M = B$. It is also easy to see that $\{n^2 : n \geq 1, n \in \mathbb{N}\} \subset \mathfrak{I}(M)$ (see Exercise 9.14). A remarkable result is that

$$\mathfrak{I}(M) \subset \left\{4\cos(\pi/n)^2 : n \in \mathbb{N}, n \geq 3\right\} \cup [4, +\infty[= \mathfrak{I}(R), \qquad (9.8)$$

where R is, as always, the hyperfinite II_1 factor.

9.4.3. An example. Let $\sigma : G \curvearrowright (B,\tau)$ be a trace preserving action of a countable group G and let $M = B \rtimes G$ be the corresponding crossed product. We keep the notation of Section 5.2. We observe that M is in standard form on $\mathcal{H} = L^2(B) \otimes \ell^2(G)$, which is therefore also written $L^2(M)$. This was noticed in Section 7.1.3 whenever B is commutative, and the general case is dealt with similarly.

We want to describe the extension $\langle M, e_B\rangle$. Recall that the canonical unitary $u_g \in M$ is identified with $\widehat{1} \otimes \delta_g$ and that we write ξu_g for $\xi \otimes \delta_g$. Now, we note that

$$\xi u_g = \xi \otimes \delta_g = (\xi \otimes \delta_e) u_g = u_g(\sigma_{g^{-1}}\xi \otimes \delta_e).$$

This allows us to write $L^2(M)$ as the Hilbert direct sum $\sum_{g\in G} L^2(B)u_g$, as we did until now, but also as $\sum_{g\in G} u_g L^2(B)$. The latter decomposition is more convenient to study the structure of right B-module of $L^2(M)$. Indeed, $(u_g)_{g\in G}$ is an orthonormal basis of the B-module $L^2(M)_B$. The right action of B is diagonal and so, clearly $B\overline{\otimes}\mathcal{B}(\ell^2(G))$ is the commutant $\langle M, e_B\rangle$ of B acting to the right on $L^2(M)$. Furthermore, e_B is the matrix y with entries equal to 0 except $y_{e,e} = 1_B$. It is also easy to check that the canonical trace on $\langle M, e_B\rangle$ is $\tau \otimes \mathrm{Tr}$, where Tr is the usual trace on $\mathcal{B}(\ell^2(G))_+$.

Note also that since $(u_g)_{g\in G}$ is an orthonormal basis of $L^2(M)_B$, we get that $\dim(L^2(M)_B)$ is the cardinal of G.

Exercises

Exercise 9.1. Let (M,τ) be a tracial von Neumann algebra. Given $x \in M_+$, consider the linear positive functional $\varphi : b \mapsto \tau(bx)$ defined on B. Show that $E_B(x)$ is the Radon-Nikodým derivative of φ with respect to the restriction of τ to B.

Exercise 9.2. Let (M,τ) be a tracial von Neumann algebra. Given $x \in M$, we denote by C_x the $\|\cdot\|_2$-closed convex hull of $\{uxu^* : u \in \mathcal{U}(M)\}$. Show that $C_x \cap Z = \{E_Z(x)\}$.

Exercise 9.3. Let (M,τ) be a tracial von Neumann algebra with center Z and let p be a projection in M. Show that whenever $E_Z(p)$ is a projection we have $p = pE_Z(p) = E_Z(p)$ and conclude that $E_Z(p)$ is a projection if and only if $p \in Z$.

EXERCISE 9.4. Let (M,τ) be a tracial von Neumann algebra, A a von Neumann subalgebra of M and p a projection in M. Show that the set of projections q in A such that $q \precsim p$ has a maximal element (Hint: use Proposition 9.1.8).

EXERCISE 9.5. Let (M,τ) be a separable tracial von Neumann algebra with center Z. We assume that M does not have abelian projections. Show that for every $z \in \widehat{Z}_+$ there is a projection $p \in \mathcal{B}(\ell^2(\mathbb{N}))\overline{\otimes}M$ such that $(\mathrm{Tr}\otimes E_Z)(p) = z$ (use [**Dix81**, Chapter III, §4, Exercise 1]).

EXERCISE 9.6. Let M be a separable tracial von Neumann algebra and $Z = L^\infty(X,\mu)$ its center. Let E_Z be its center-valued trace and let $\mathcal{T}r_Z$ be a normal faithful semi-finite tracial weight on M. For $x \in M_+$ we set $\mathrm{Tr}(x) = \int_X \mathcal{T}r_Z(x)\,\mathrm{d}\mu(x)$ and $\tau(x) = \int_X E_Z(x)\,\mathrm{d}\mu(x) = \tau_\mu \circ E_Z(x)$.

(i) Show that Tr is a normal faithful semi-finite trace on M_+.

(ii) Let (p_n) be an increasing sequence of projections in M with $\bigvee p_n = 1$ and $\mathrm{Tr}(p_n) < +\infty$ for every n. Let $q_n \in \mathcal{P}(Z)$ be the central support of p_n. Show the existence of $f_n \in L^1(X,\mu)_+$ with $(1-q_n)f_n = 0$ such that

$$\mathrm{Tr}(p_n x p_n) = \int_X f_n E_Z(p_n x p_n)\,\mathrm{d}\mu$$

for every $x \in M_+$.

(iii) Deduce the existence of a mesurable function $f : X \to [0,+\infty]$ such that, for $x \in M_+$,

$$\mathrm{Tr}(x) = \int_X f E_Z(x)\,\mathrm{d}\mu.$$

(iv) Show that $\mathcal{T}r_Z(x) = f E_Z(x)$ and that $0 < f < +\infty$ almost everywhere.

EXERCISE 9.7. Let $\mathcal{H}$ be a right module on a separable tracial von Neumann algebra (M,τ). We denote by $\mathcal{I}_0(\mathcal{B}(\mathcal{H}_M))$ the norm closure of the ideal $\mathcal{F}(\mathcal{H}_M)$ (defined in Lemma 8.4.1) into $\mathcal{B}(\mathcal{H}_M)$. We set $Z = \mathcal{Z}(M) = L^\infty(X,\mu)$. Let p be a projection in $\mathcal{B}(\mathcal{H}_M)$.

(i) Show that p is finite if and only if $\widehat{E}_Z(p) < +\infty$ almost everywhere.

(ii) Show that $\widehat{\tau}(p) < +\infty$ if and only if $\widehat{E}_Z(p) \in L^1(X,\mu)$.

(iii) Show that $p \in \mathcal{I}_0(\mathcal{B}(\mathcal{H}_M))$ if and only if $\widehat{E}_Z(p) \in Z$.

EXERCISE 9.8 (**Compact operators**). Let (M,Tr) be a von Neumann algebra equipped with a faithful normal semi-finite trace. Let $\mathcal{I}(M)$ (resp. $\mathcal{J}(M)$) be the norm-closed two-sided ideal of M generated by the finite projections (resp. the projections p with $\mathrm{Tr}(p) < +\infty$) of M.

(i) Let $p \in \mathcal{P}(M)$. Show that $p \in \mathcal{I}(M)$ (resp. $\mathcal{J}(M)$) if and only if p is finite (resp. $\mathrm{Tr}(p) < +\infty$).

(ii) Show that $x \in M$ belongs to $\mathcal{I}(M)$ (resp. $\mathcal{J}(M)$) if and only if the spectral projections e_s of $|x|$ relative to $[s, +\infty[$, $s > 0$, are finite (resp. such that $\mathrm{Tr}(e_s) < +\infty$).

Exercise 9.9 (**Compact operators**). We keep the notation of Exercise 9.7. Let $\mathcal{I}(\mathcal{B}(\mathcal{H}_M))$ (resp. $\mathcal{J}(\mathcal{B}(\mathcal{H}_M))$) be the norm-closed two-sided ideal of $\mathcal{B}(\mathcal{H}_M))$ generated by the finite projections (resp. the projections p with $\widehat{\tau}(p) < +\infty$) of $\mathcal{B}(\mathcal{H}_M)$.

(i) Show that $x \in \mathcal{B}(\mathcal{H}_M)$ belongs to $\mathcal{I}_0(\mathcal{B}(\mathcal{H}_M))$ if and only if the spectral projections e_s of $|x|$ relative to $[s, +\infty[$, $s > 0$, are in $\mathcal{I}_0(\mathcal{B}(\mathcal{H}_M))$.
(ii) Show that $\mathcal{I}_0(\mathcal{B}(\mathcal{H}_M)) \subset \mathcal{J}(\mathcal{B}(\mathcal{H}_M)) \subset \mathcal{I}(\mathcal{B}(\mathcal{H}_M))$.

When $M = \mathbb{C}$, these three ideals are the same, namely the usual ideal of compact operators.

When B is a von Neumann subalgebra of a tracial von Neumann algebra (M, τ) then $\mathcal{I}_0(\mathcal{B}(L^2(M)_B) = \langle M, e_B \rangle$ (Proposition 9.4.3). Therefore, we have $\langle M, e_B \rangle \subset \mathcal{J}(\mathcal{B}(L^2(M)_B)) \subset \mathcal{I}(\mathcal{B}(L^2(M)_B))$.

Exercise 9.10. Let (ξ_i) be an orthonormal basis of a right M-module $\mathcal{H}$ and let $\widehat{E}_Z$ be the canonical center-valued tracial weight on $\mathcal{B}(\mathcal{H}_M)$. Show that $\widehat{E}_Z(1) = \sum_i E_Z(L_{\xi_i}^* L_{\xi_i})$.

Exercise 9.11. Let $Y = \{n \in \mathbb{N} : n \geq 1\}$ and $X = Y \times \{0, 1\}$. We endow X with the probability measure ν such that $\nu(\{n, 1\}) = (1/n)2^{-n}$ and $\nu(\{n, 0\}) = (1 - 1/n)2^{-n}$ and let μ be the image of ν under the first projection.

(i) Show that the $L^\infty(Y, \mu)$-module $L^2(X, \nu)$ is finitely generated, and compute its dimension.
(ii) Show the existence of $L^\infty(Y, \mu)$-bounded vectors which are not in $L^\infty(X, \nu)$.

Exercise 9.12. Let $\sigma : G \curvearrowright (B, \tau)$ be a trace preserving action of a countable group G and set $M = B \rtimes G$. Let $\xi = \sum_{g \in G} u_g \eta_g$ be an element of $L^2(M) = \sum_{g \in G} u_g L^2(B)$. Show that ξ is left B-bounded if and only if $\eta_g \in B$ for every g and $\sum_{g \in G} \eta_g^* \eta_g$ converges in B with respect to the s.o. topology.

Exercise 9.13. Let H be a subgroup of a countable group G. Show that $\dim(\ell^2(G)_{L(H)}) = [G : H]$.

Exercise 9.14. Let M be a II_1 factor. Recall (Proposition 4.2.5) that for any integer $n \geq 1$ there is a II_1 factor N such that M is isomorphic to $N \overline{\otimes} M_n(\mathbb{C})$.

(i) Show that $[N \overline{\otimes} M_n(\mathbb{C}) : N \otimes 1] = n^2$.
(ii) Conclude that $\{n^2 : n \in \mathbb{N}^*\} \subset \mathfrak{I}(M)$.

EXERCISE 9.15. Let $N \subset M$ be an inclusion of II_1 factors and let $\mathcal{H}$ be a finite right M-module. Show that $\dim(\mathcal{H}_N) < +\infty$ if and only if $[M : N] < +\infty$ and that in this case

$$\dim(\mathcal{H}_N) = [M : N] \dim(\mathcal{H}_M).$$

EXERCISE 9.16. Let $N \subset M$ be an inclusion of II_1 factors with $[M : N] < +\infty$ and let p be a non-zero projection in $N' \cap M$. Show that

$$[pMp : Np] = [M : N]\tau_M(p)\tau_{N'}(p)$$

where N' is the commutant of N acting on $L^2(M)$ and τ_M, $\tau_{N'}$ are the normalized traces on M and N' respectively (Hint: use Exercises 9.15, 8.8 and 8.9).

EXERCISE 9.17. Let $N \subset M$ as in the previous exercise and let $p_1, \dots, p_n$ be pairwise orthogonal non-zero projections in $N' \cap M$ such that $\sum_{i=1}^n p_i = 1$. Show that

$$[M : N] = \sum_{i=1}^n \tau_{N'}(p_i)^{-1}[p_i M p_i : N p_i],$$

and conclude that $[M : N] \geq n^2$.

EXERCISE 9.18. Let M be a II_1 factor and r be in the fundamental group $\mathfrak{F}(M)$ of M. Let $t = t(r)$ be the unique element of $]0, 1[$ such that $t(1-t)^{-1} = r$ (so M^t and M^{1-t} are isomorphic) and let $p \in \mathcal{P}(M)$ with $\tau(p) = t$. We consider an isomorphism θ from pMp onto $(1-p)M(1-p)$ and we introduce the subfactor $N = \{x + \theta(x) : x \in pMp\}$. Show that

$$[M : N] = 1/t + 1/(1-t).$$

This defines an injective map from $\mathfrak{F}(M) \cap]0, 1]$ into $\mathfrak{I}(M)$. Therefore $\mathfrak{F}(M)$ is countable whenever $\mathfrak{I}(M)$ is countable.

EXERCISE 9.19. Let B be a subfactor of a II_1 factor M, such that $[M : B] = d < +\infty$. Let n be the integer part of d. Show that there exists an orthonormal basis $m_1, \dots, m_{n+1}$ of $L^2(M)_B$ such that $E_B(m_i^* m_i) = 1$ for $i \leq n$ and $E_B(m_{n+1}^* m_{n+1})$ is a projection with trace $d - n$.

Notes

Conditional expectations in tracial von Neumann algebras were introduced in [**Ume54**] as non-commutative extensions of the usual notion in probability theory. A related notion was studied in [**Dix53**]. Center-valued traces and tracial weights have been investigated in [**Dix49, Dix52**].

The basic construction appears in [**Ska77, Chr79**]. Given a subfactor B of a II_1 factor M, Jones [**Jon83b**] made the crucial observation that the index of B in M is the same as the index of M in $\langle M, e_B \rangle$. His deep analysis of this fact allowed him to prove his celebrated result (9.8), stated in Remark 9.4.10, on the restriction of the possible values of the index. This is quite suprising, compared with the continuum of possible values (*i.e.*, $]0, +\infty]$) of dimensions for general B-modules.

The result stating that the index of B in M is finite if and only if M is a finitely generated projective module on B is due to Pimsner and Popa [**PP86**], as well as the computation of the index from any basis of this module.

Part 2

CHAPTER 10

Amenable von Neumann algebras

The most tractable groups are certainly the so-called amenable ones. We briefly recall their definition and give, for a group G, a condition on its von Neumann algebra $L(G)$ equivalent to the amenability of G. This is the starting point for the definition, in full generality, of an amenable tracial von Neumann algebra. We give several equivalent characterisations of this notion, analogous to well-known equivalent definitions of amenability for a group: existence of a hypertrace and a Følner type condition, in particular. The main results are Theorems 10.2.9 and 10.3.1.

10.1. Amenable groups and their von Neumann algebras

10.1.1. Amenable groups. Recall that a group G is *amenable* if there exists a left invariant mean m on G, that is a state m on $\ell^\infty(G)$ such that $m({}_sf) = m(f)$ for every $s \in G$ and $f \in \ell^\infty(G)$, where $({}_sf)(t) = f(s^{-1}t)$ for all $t \in G$.

Examples 10.1.1. **(1)** Every finite group G is amenable. Indeed, the uniform probability measure m on G (*i.e.*, the Haar measure) is an invariant mean.

(2) Let G be a *locally finite group*, that is, be the union $G = \cup_n G_n$ of an increasing sequence of finite subgroups G_n. Then G is amenable. To construct a left invariant mean on G we start with the sequence (m_n) of Haar measures on the subgroups and we take an appropriate limit of the sequence. To this end, we fix a free ultrafilter ω. Recall that, for any bounded sequence (c_n) of complex numbers, $\lim_\omega c_n$ is defined as the value at $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ of this sequence, viewed as a continuous function on the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Given $f \in \ell^\infty(G)$, we set

$$m(f) = \lim_\omega m_n(f|_{G_n}).$$

It is easily checked that m is an invariant mean on G.

A basic example is the group S_∞ of all finite permutations of $\mathbb{N}$.

(3) The simplest example of non-amenable group is the free group $\mathbb{F}_2$ with two generators a and b. Indeed, for $x \in \{a, b, a^{-1}, b^{-1}\}$ let us denote by E_x the set of reduced words beginning by the letter x. We have $\mathbb{F}_2 = \{e\} \cup E_a \cup E_b \cup E_{a^{-1}} \cup E_{b^{-1}}$, together with $\mathbb{F}_2 = E_a \cup aE_{a^{-1}}$ and $\mathbb{F}_2 = E_b \cup bE_{b^{-1}}$. This makes impossible the existence of a left invariant mean.

A remarkable fact is that amenability admits many equivalent characterisations. We recall below several of them[1].

PROPOSITION 10.1.2. *Let G be a group. The following conditions are equivalent:*

(i) *G is amenable;*
(ii) *there exists a net (ξ_i) of unit vectors in $\ell^2(G)$ such that, for every $g \in G$,*
$$\lim_i \|\lambda_G(g)\xi_i - \xi_i\|_2 = 0;$$
(iii) *there exists a net of finitely supported positive definite functions on G which converges pointwise to 1;*
(iv) *there exists a net (E_i) of finite, non-empty, subsets of G such that, for every $g \in G$,*
$$\lim_i \frac{|gE_i \Delta E_i|}{|E_i|} = 0.$$

Condition (ii) means that *the left regular representation λ_G of G almost has invariant vectors* in the sense of Definition 13.3.4, or in other terms, that *the trivial representation ι_G of G is weakly contained in the left regular representation λ_G* (see Proposition 13.3.5). The notion of positive definite function on a group is recalled in Section 13.1.3. A net satisfying Condition (iv) is called a *Følner net*. This condition means that in (ii) we may take for ξ_i the normalized characteristic function $|E_i|^{1/2}\mathbf{1}_{E_i}$.

10.1.2. The von Neumann algebra of an amenable group.

PROPOSITION 10.1.3. *Let G be a group and $M = L(G)$. Then G is amenable if and only if there exists a conditional expectation E from $\mathcal{B}(L^2(M))$ onto M.*

PROOF. As always, the u_s, $s \in G$, are the canonical unitaries of $L(G)$. Assume first the existence of E. Given $f \in \ell^\infty(G)$, we denote by M_f the multiplication operator by f on $\ell^2(G)$. We set $m(f) = \tau(E(M_f))$, where τ is the canonical trace on M. Since $u_s M_f u_s^* = M_{sf}$ for every $s \in G$, we see that the state m is left invariant.

Conversely, assume that G is amenable, and let m be a left invariant mean on $\ell^\infty(G)$. Given $\xi, \eta \in \ell^2(G)$, and $T \in \mathcal{B}(L^2(M))$, we introduce the function defined by
$$f^T_{\xi,\eta}(s) = \langle \xi, \rho(s)T\rho(s^{-1})\eta \rangle$$
where ρ is the right regular representation of G. Obviously, $f^T_{\xi,\eta}$ is a bounded function on G with
$$|f^T_{\xi,\eta}(s)| \leq \|T\|\|\xi\|\|\eta\|.$$

[1]For details, see for instance [**BdlHV08**, Appendix G]. Of course, when G is countable, nets can be replaced by sequences.

We define a continuous sesquilinear functional on $\ell^2(G)$ by the formula

$$(\xi, \eta) = m(f^T_{\xi,\eta}).$$

It follows that there is a unique operator, denoted by $E(T)$, with

$$\langle \xi, E(T)\eta \rangle = m(f^T_{\xi,\eta})$$

for every $\xi, \eta \in \ell^2(G)$.

The invariance property of m implies that $\rho(s)E(T)\rho(s^{-1}) = E(T)$ for all $s \in G$. Therefore, $E(T)$ commutes with $\rho(G)$, whence $E(T) \in L(G)$. It is easily checked that E is a conditional expectation. $\square$

10.2. Amenable von Neumann algebras

The previous proposition motivates the next definition.

Definition 10.2.1. We say that a von Neumann M is *amenable*, or *injective*, if it has a concrete representation as a von Neumann subalgebra of some $\mathcal{B}(\mathcal{H})$ such that there exists a conditional expectation[2] $E : \mathcal{B}(\mathcal{H}) \to M$.

Injectivity is a more usual terminology. This is justified by the following proposition which also shows that the definition is independent of the choice of $\mathcal{H}$. For basic facts related to the notion of completely positive map used below see Section A.3 in the appendix.

Proposition 10.2.2. *Let M be a von Neumann algebra. The following conditions are equivalent:*

(i) *M is injective;*
(ii) *for every inclusion $A \subset B$ of unital C^*-algebras, every unital completely positive map $\phi : A \to M$ extends to a completely positive map from B to M;*
(iii) *for any $\mathcal{B}(\mathcal{H})$ which contains M as a von Neumann subalgebra, there is a conditional expectation from $\mathcal{B}(\mathcal{H})$ onto M.*

Proof. (i) $\Rightarrow$ (ii). Assume that M is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ and that there exists a conditional expectation $E : \mathcal{B}(\mathcal{H}) \to M$. We extend ϕ to a completely positive map $\widetilde{\phi} : B \to \mathcal{B}(\mathcal{H})$, using Arveson's extension theorem, which says that $\mathcal{B}(\mathcal{H})$ is an injective object in the category of C^*-algebras with completely positive maps as morphisms (see Theorem A.5 in the appendix). Then $E \circ \widetilde{\phi} : B \to M$ is a completely positive extension of ϕ.

(ii) $\Rightarrow$ (iii). Let $M \subset \mathcal{B}(\mathcal{H})$. We apply (ii) with $A = M$, $B = \mathcal{B}(\mathcal{H})$ and the identity map Id_M. Then there exists a completely positive map $\phi : \mathcal{B}(\mathcal{H}) \to M$ whose restriction to M is Id_M. Such a map is automatically a conditional expectation (see Theorem A.4).

(iii) $\Rightarrow$ (i) is obvious. $\square$

[2] or equivalently a norm-one projection, by Theorem A.4

We will rather use the name amenable for such von Neumann algebras to emphasize the analogy with amenability for groups. Indeed, by the previous section, a group G is amenable if and only if its von Neumann algebra $L(G)$ is amenable.

REMARKS 10.2.3. **(a)** As a consequence of Propositions 10.1.3 and 10.2.2, if G is a non-amenable ICC group, for instance $G = \mathbb{F}_n$, $n \geq 2$, the II_1 factor $L(G)$ is not isomorphic to the II_1 factor $L(S_\infty)$.

(b) Let $G \curvearrowright (X, \mu)$ be a p.m.p. action. Then the crossed product $L^\infty(X, \mu) \rtimes G$ is amenable if and only if the group G is amenable (Exercice 10.6).

10.2.1. Example: the hyperfinite II_1 factor.

THEOREM 10.2.4. *The hyperfinite factor R is amenable.*

PROOF. By definition, $R = \overline{\cup_{n\geq 1} Q_n}^{s.o}$, with $Q_n = M_{2^n}(\mathbb{C})$. Let J be the conjugation operator in $L^2(R)$. For $n \geq 1$ and $T \in \mathcal{B}(L^2(R))$ we set

$$E_n(T) = J\Big(\int_{\mathcal{U}_{2^n}} uJTJu^* \,\mathrm{d}u\Big)J,$$

where $\mathrm{d}u$ is the Haar probability measure on the (compact) group $\mathcal{U}_{2^n}$ of unitary $2^n \times 2^n$ matrices. Then $(E_n(T))_n$ is a norm bounded sequence of operators in $\mathcal{B}(L^2(R))$. Note that whenever $T \in R$, this sequence is constant, with value T.

We will construct a conditional expectation $E : \mathcal{B}(L^2(R)) \to R$ by taking the limit of the sequence of maps E_n along a free ultrafilter ω. Using the Riesz representation theorem, we check that there exists a unique bounded operator, that we denote by $E(T)$, such that

$$\langle \xi, E(T)\eta \rangle = \lim_\omega \langle \xi, E_n(T)\eta \rangle, \quad \forall \xi, \eta \in L^2(R).$$

Since $E_n(T) \in JQ'_{n_0}J$ for $n \geq n_0$, we see that

$$E(T) \in \cap_{n_0 \geq 1} JQ'_{n_0}J = R.$$

We have $E(T) = T$ if $T \in R$. It is also obvious that $E(T) \geq 0$ if $T \in \mathcal{B}(L^2(R))_+$ and that $E(xT) = xE(T)$, $E(Tx) = E(T)x$ for $x \in R$ and $T \in \mathcal{B}(L^2(R))$. $\square$

As for groups, amenability of von Neumann algebras can be defined in many equivalent ways. This is the matter of the rest of this chapter and we will come back to this subject in Section 13.4.

10.2.2. Hypertraces. Let (M, τ) be a tracial von Neumann algebra. A state ψ on $\mathcal{B}(L^2(M))$ is called a *hypertrace* (for (M, τ)) if $\psi(xT) = \psi(Tx)$ for every $x \in M$ and $T \in \mathcal{B}(L^2(M))$ and $\psi|_M = \tau$. Note that this latter condition is automatic when M is a II_1 factor. A hypertrace can be viewed as the analogue of an invariant mean on an amenable group.

PROPOSITION 10.2.5. *Let (M,τ) be a tracial von Neumann algebra. Then M is amenable if and only if it has a hypertrace.*

PROOF. If $E : \mathcal{B}(L^2(M)) \to M$ is a conditional expectation, then $\tau \circ E$ is a hypertrace.

Conversely, suppose that ψ is a hypertrace. Given $T \in \mathcal{B}(L^2(M))_+$, we define a positive linear functional φ on M by $\varphi(x) = \psi(Tx)$. For $x \in M_+$, we have

$$\begin{aligned}|\varphi(x)|^2 = \left|\psi(x^{1/2}Tx^{1/2})\right|^2 &\le \psi(x^{1/2}T^2x^{1/2})\psi(x)\\ &\le \|T\|^2\psi(x)^2.\end{aligned}$$

Since $\psi_{|_M} = \tau$, we get $\varphi(x) \le \|T\|\tau(x)$ for $x \in M_+$. In particular, φ is normal. Using the Radon-Nikodým theorem 7.3.6, we see that there is an element $E(T) \in M_+$ such that, for every $x \in M$,

$$\psi(Tx) = \tau(E(T)x).$$

Then, it is easily seen that E extends to a conditional expectation from $\mathcal{B}(L^2(M))$ onto M. □

10.2.3. Another characterisation. We will prove in Theorem 10.2.9 the analogue for von Neumann algebras of the property (ii) in Proposition 10.1.2. This will be made more specific later in Section 13.4.

In order to establish this new characterisation of amenability, we need two preliminary results.

Let $\mathcal{H}$ be a Hilbert space. Recall that the predual of $\mathcal{B}(\mathcal{H})$ is isometric to the Banach space $\mathcal{S}^1(\mathcal{H})$ of trace-class operators: each $T \in \mathcal{S}^1(\mathcal{H})$ is identified to the linear functional $\varphi_T : x \in \mathcal{B}(\mathcal{H}) \mapsto \mathrm{Tr}(Tx)$ where Tr is the usual trace on $\mathcal{B}(\mathcal{H})$. We denote by $S(\mathcal{B}(\mathcal{H}))$ the state space of $\mathcal{B}(\mathcal{H})$, *i.e.*,

$$S(\mathcal{B}(\mathcal{H})) = \{\varphi \in \mathcal{B}(\mathcal{H})^* : \varphi \ge 0, \varphi(1) = 1\}.$$

We will often write $\mathcal{B}$ instead of $\mathcal{B}(\mathcal{H})$ for simplicity.

LEMMA 10.2.6. *We denote by K_0 the set of φ_T where T runs over the convex set of positive finite rank operators on $\mathcal{H}$ with $Tr(T) = 1$. Then K_0 is contained in $S(\mathcal{B}(\mathcal{H}))$ and is dense in $S(\mathcal{B}(\mathcal{H}))$ in the weak* topology (i.e., the $\sigma(\mathcal{B}^*, \mathcal{B})$ topology).*

PROOF. The closure $\overline{K_0}$ of K_0 in the $\sigma(\mathcal{B}^*,\mathcal{B})$-topology is a $\sigma(\mathcal{B}^*,\mathcal{B})$-compact convex subset of $S(\mathcal{B})$. Assume that there is an element $\varphi \in S(\mathcal{B})$ which does not belong to $\overline{K_0}$. By the Hahn-Banach separation theorem, there is an $\alpha > 0$ and a $\sigma(\mathcal{B}^*,\mathcal{B})$-continuous linear functional on $\mathcal{B}^*$, that is an element $x \in \mathcal{B}$, with $\Re\langle x,\varphi\rangle > \alpha$ and $\Re\langle x,\psi\rangle \le \alpha$ for all $\psi \in \overline{K_0}$. Replacing x by its real part, we may assume that $x = x^*$. Hence, we have $\varphi(x) > \alpha$ and $\mathrm{Tr}(xT) \le \alpha$ for every positive finite rank operator T on $\mathcal{H}$ such that $\mathrm{Tr}(T) = 1$.

Taking T to be the rank one projection $\eta \mapsto \langle \xi, \eta \rangle \xi$ where ξ is a norm-one vector in $\mathcal{H}$, we get

$$\forall \xi \in \mathcal{H}, \|\xi\| = 1, \quad \langle \xi, x\xi \rangle \leq \alpha,$$

and therefore $x \leq \alpha \, \mathrm{Id}_{\mathcal{H}}$, so that

$$\varphi(x) \leq \alpha \, \varphi(\mathrm{Id}_{\mathcal{H}}) = \alpha,$$

which is a contradiction. □

The second result we need it the original Powers-Størmer inequality. Given a Hilbert-Schmidt operator T, we write $\|T\|_{2,\mathrm{Tr}} = \mathrm{Tr}(T^*T)^{1/2}$ its Hilbert-Schmidt norm.

Theorem 10.2.7. *Let T, S be positive finite rank operators in $\mathcal{B}(\mathcal{H})$. Then*

$$\|T - S\|_{2,Tr}^2 \leq \big\| Tr(T^2 \cdot) - Tr(S^2 \cdot)\big\| = \big\|\varphi_{T^2} - \varphi_{S^2}\big\|.$$

The proof is similar that of Theorem 7.3.7. At the same time, we also record here the following more general inequality, that we will be useful in the next chapter[3].

Theorem 10.2.8 (**Powers-Størmer inequality**). *Let (M, Tr) be a semi-finite von Neumann algebra equiped with a normal faithful semi-finite trace. Let x, y be two elements of M_+ with $Tr(x^2) < +\infty$ and $Tr(y^2) < +\infty$. Then we have*

$$\|x - y\|_{2,Tr}^2 \leq \big\| Tr(x^2 \cdot) - Tr(y^2 \cdot)\big\|.$$

The theorem below uses two main ingredients in order to show its condition (2) assuming the existence of a hypertrace: the above Powers-Størmer inequality and a convexity argument due to Day in the framework of groups.

Theorem 10.2.9. *Let (M, τ) be a tracial von Neumann algebra. The following conditions are equivalent:*

(1) *M is amenable;*
(2) *for every $\varepsilon > 0$ and every finite set F of unitaries in M there exists a positive finite rank operator T on $L^2(M)$ with $\|T\|_{2,Tr} = 1$ such that*

$$\max_{u \in F} \|uT - Tu\|_{2,Tr} \leq \varepsilon \quad \textit{and} \tag{10.1}$$

$$\sup_{x \in M, \|x\| \leq 1} \big| Tr(xT^2) - \tau(x)\big| \leq \varepsilon; \tag{10.2}$$

(3) *for every $\varepsilon > 0$ and every finite set F of unitaries in M there exists a Hilbert-Schmidt operator T on $L^2(M)$ with $\|T\|_{2,Tr} = 1$, which satisfies (10.1) and*

$$\max_{u \in F} | Tr(T^* uT) - \tau(u)| \leq \varepsilon.$$

[3]Both Theorems 7.3.7 and 10.2.8 are particular cases of [**Haa75**, Lemma 2.10].

Proof. We still set $\mathcal{B} = \mathcal{B}(L^2(M))$.

(1) $\Rightarrow$ (2). Let ψ be a hypertrace and consider $\varepsilon > 0$ and a finite set $F = \{u_1, \dots, u_n\}$ of unitaries in M. We identify the dual of $(\mathcal{B}_*)^n \times M_*$ with $\mathcal{B}^n \times M$ by

$$\langle (\varphi_1, \dots, \varphi_{n+1}), (T_1, \dots, T_{n+1}) \rangle = \sum_{i=1}^{n+1} \varphi_i(T_i).$$

We set

$$C = \big\{ \big((u_1 \varphi u_1^* - \varphi), \dots, (u_n \varphi u_n^* - \varphi), (\varphi|_M - \tau) \big) : \varphi \in K_0 \big\},$$

where $(u\varphi u^*)(x) = \varphi(u^* x u)$ and K_0 is as in Lemma 10.2.6. Using this lemma, we see that there is a net (φ_i) of elements in K_0 such that $\lim \varphi_i = \psi$ in the $\sigma(\mathcal{B}^*, \mathcal{B})$-topology. But then, for every $u \in \mathcal{U}(M)$ we have

$$\lim_i u\varphi_i u^* = u\psi u^* = \psi = \lim_i \varphi_i.$$

It follows that $(0, \dots, 0)$ is in the $\sigma((\mathcal{B}_*)^n \times M_*, \mathcal{B}^n \times M)$ closure of C. The crucial observation is that C is a convex subset of $(\mathcal{B}_*)^n \times M_*$ and so this closure is the same as the norm closure. Hence, there is a positive finite rank operator S on $L^2(M)$ with $\mathrm{Tr}(S) = 1$ and

$$\max_{u \in F} \| u\varphi_S u^* - \varphi_S \| \leq \varepsilon, \quad \| \varphi_S|_M - \tau \| \leq \varepsilon.$$

Now, we set $T = S^{1/2}$. Since $(uSu^*)^{1/2} = uTu^*$, we get from the Powers-Størmer inequality that for $u \in F$,

$$\| uTu^* - T \|_{2,\mathrm{Tr}}^2 \leq \| \varphi_{uSu^*} - \varphi_S \| = \| u\varphi_S u^* - \varphi_S \| \leq \varepsilon.$$

(2) $\Rightarrow$ (3) is obvious. Let us show that (3) $\Rightarrow$ (1). Let (T_i) be a net of Hilbert-Schmidt operators with $\|T_i\|_{2,\mathrm{Tr}} = 1$, such that for every $u \in \mathcal{U}(M)$, we have

$$\lim_i \| uT_i - T_i u \|_{2,\mathrm{Tr}} = 0, \quad \lim_i \mathrm{Tr}(T_i^* u T_i) = \tau(u).$$

For $i \in I$, we introduce the normal state $\varphi_i : x \mapsto \mathrm{Tr}(T_i^* x T_i)$ on $\mathcal{B}$. Let $\psi \in \mathcal{B}^*$ be a cluster point of the net (φ_i) in the weak* topology. Obviously, τ is the restriction of ψ to M. Moreover, for $u \in \mathcal{U}(M)$ and $x \in \mathcal{B}$, we have

$$\psi(uxu^*) - \psi(x) = \lim_i \mathrm{Tr}(T_i^* uxu^* T_i) - \lim_i \mathrm{Tr}(T_i^* x T_i) = 0$$

since, using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\mathrm{Tr}(T_i^* uxu^* T_i) - \mathrm{Tr}(T_i^* x T_i)| &= |\langle u^* T_i u, xu^* T_i u \rangle_{\mathrm{Tr}} - \langle T_i, xT_i \rangle_{\mathrm{Tr}}| \\ &\leq 2\|x\|_\infty \|u^* T_i u - T_i\|_{2,\mathrm{Tr}}. \end{aligned}$$

So, ψ is a hypertrace. □

10.3. Connes' Følner type condition

10.3.1. Følner type characterisation of amenable II_1 factors. In this section we assume that M is a II_1 factor. Note that in this situation, Condition (10.2) is unnecessary in the statement of Theorem 10.2.9, since the restriction to M of any hypertrace is the unique tracial state on M. The next theorem is an important step in the proof that an amenable II_1 factor is hyperfinite (see Chapter 11). It says that in Condition (10.1), we may take T to be the normalization of a finite rank projection. This result corresponds to the Følner characterisation of amenable groups

Given a positive operator x and $t > 0$, in the rest of this section, $E_t^c(x)$ will denote the spectral projection of x relative to the interval $(t, +\infty)$.

Theorem 10.3.1. *A II_1 factor is amenable if and only if for every $\varepsilon > 0$ and every finite set F of unitaries in M, there is a finite rank projection $P \in \mathcal{B}(L^2(M))$ such that*

$$\max_{u\in F} \|uPu^* - P\|_{2,Tr} < \varepsilon \|P\|_{2,Tr}. \tag{10.3}$$

Proof. Assume that M is amenable. Let $0 < \eta < 1$ be given. By Theorem 10.2.9, we know that there exists a positive finite rank operator T on $L^2(M)$ such that

$$\max_{u\in F} \|uTu^* - T\|_{2,\mathrm{Tr}} < \eta \|T\|_{2,\mathrm{Tr}}.$$

We have to show that we can replace T by a projection. This relies on the so-called Connes' trick, proved below in Theorem 10.3.4, which implies the existence of a $t > 0$ with

$$\max_{u\in F} \|E_t^c(uTu^*) - E_t^c(T)\|_{2,\mathrm{Tr}} < (3n\eta)^{1/2} \|E_t^c(T)\|_{2,\mathrm{Tr}},$$

where n is the number of elements in F.

Clearly, we have $E_t^c(uTu^*) = uE_t^c(T)u^*$. We take $\eta \leq \varepsilon^2/(3n)$ and set $P = E_t^c(T)$ to get (10.3). $\square$

Remark 10.3.2. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}'$ be a dense vector subspace. Given a finite rank projection $P \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists a finite rank projection Q with $Q\mathcal{H} \subset \mathcal{H}'$ and $\|P - Q\|_{2,\mathrm{Tr}} \leq \varepsilon$. Indeed, let $(\xi_1, \dots, \xi_n)$ be an orthonormal basis of $P\mathcal{H}$. We can approximate each ξ_k by a vector $\eta_k \in \mathcal{H}'$ and we can do so that the projection Q on the linear span of the $\eta_1, \dots, \eta_n$ satisfies the required inequality. We leave the details to the reader.

In particular, for $\mathcal{H} = L^2(M)$, this observation applies to the dense subspace $\widehat{M}$. It follows that in the previous theorem, we may choose P such that $PL^2(M) \subset \widehat{M}$.

We now turn to the statement of Connes' trick.

10.3.2. Connes's trick. It is a non-commutative version of a trick due to Namioka, that we first explain.

Let $\mathbf{1}_{(t,\infty)}$ denote the characteristic function of the interval $(t,+\infty)$. The main point in Namioka's trick is the following elementary observation:

$$\forall s_1, s_2 \in \mathbb{R}, \quad \int_{\mathbb{R}} |\mathbf{1}_{(t,\infty)}(s_1) - \mathbf{1}_{(t,\infty)}(s_2)| \,\mathrm{d}t = |s_1 - s_2|.$$

Now, given a σ-finite measure space (X,μ) and $f,g \in L^1(X,\mu)$, using Fubini's theorem, we get

$$\|f-g\|_1 = \int_{\mathbb{R}} \|E_t^c(f) - E_t^c(g)\|_1 \,\mathrm{d}t$$

where, in analogy with spectral theory, we write $E_t^c(f) = \mathbf{1}_{(t,\infty)} \circ f$. Applied with $g = 0$, this gives

$$\|f\|_1 = \int_{\mathbb{R}} \|E_t^c(f)\|_1 \,\mathrm{d}t.$$

In particular, if f,g are such that $\|f-g\|_1 < \varepsilon \|f\|_1$ for some $\varepsilon > 0$, we at once deduce the existence of a $t_0 \in \mathbb{R}$ with

$$\|E_{t_0}^c(f) - E_{t_0}^c(g)\|_1 < \varepsilon \|E_{t_0}^c(f)\|_1,$$

and so $E_{t_0}^c(f) \neq 0$ if $\varepsilon < 1$.

We want to obtain a non-commutative version of this Namioka's observation. The first task is to reduce computations of Hilbert-Schmidt norms of operators to computations of L^2-norms of functions.

Proposition 10.3.3. *Let $\mathcal{H}$ be Hilbert space and let x,y be two positive finite rank operators on $\mathcal{H}$. There exists a positive Radon measure ν on $\mathbb{R}_+^2$ such that for every pair f,g of Borel complex-valued functions on $\mathbb{R}_+$ with $f(0) = 0 = g(0)$ one has*

$$\|f(x) - g(y)\|_{2,Tr}^2 = \int_{\mathbb{R}_+^2} |f(\alpha) - g(\beta)|^2 \,\mathrm{d}\nu(\alpha,\beta).$$

Proof. We write

$$x = \sum_{i=1}^m \lambda_i e_i, \quad y = \sum_{j=1}^n \mu_j f_j,$$

where the λ_i's are the distinct strictly positive eigenvalues of x and the e_i's are the corresponding spectral projections (and similarly for y). We put $e_0 = 1 - \sum_{i=1}^n e_i$, $f_0 = 1 - \sum_{j=1}^m f_j$, and Tr is the usual trace on $\mathcal{B}(\mathcal{H})_+$.

We set

$$X = (\mathrm{Sp}(x) \times \mathrm{Sp}(y)) \setminus \{(0,0)\},$$

and we define a measure ν on X (and therefore on $\mathbb{R}_+^2$) by setting

$$\nu(\{(\lambda_i,\mu_j)\}) = \mathrm{Tr}(e_i f_j)$$
$$\nu(\{(\lambda_i,0)\}) = \mathrm{Tr}(e_i f_0)$$
$$\nu(\{(0,\mu_j)\}) = \mathrm{Tr}(e_0 f_j).$$

Let f, g be two real-valued Borel functions on $\mathbb{R}_+$ with $f(0) = 0 = g(0)$. Then

$$f(x) = \sum_{i=1}^{m} f(\lambda_i) e_i, \quad g(y) = \sum_{j=1}^{n} g(\mu_j) f_j$$

are still finite rank operators. We have

$$\mathrm{Tr}(f(x)g(y)) = \int_{\mathbb{R}_+^2} f(\alpha) g(\beta) \,\mathrm{d}\nu(\alpha, \beta),$$

$$\mathrm{Tr}(f^2(x)) = \int_{\mathbb{R}_+^2} f^2(\alpha) \,\mathrm{d}\nu(\alpha, \beta), \quad \mathrm{Tr}(g^2(y)) = \int_{\mathbb{R}_+^2} g^2(\beta) \,\mathrm{d}\nu(\alpha, \beta),$$

and therefore

$$\|f(x) - g(y)\|_{2,\mathrm{Tr}}^2 = \int_{\mathbb{R}_+^2} |f(\alpha) - g(\beta)|^2 \,\mathrm{d}\nu(\alpha, \beta). \tag{10.4}$$

□

We are now ready to prove the following non-commutative version of Namioka's trick, that is formulated for n elements.

Theorem 10.3.4 (**Connes' trick**). *Let $x_1, \dots, x_n$ be positive finite rank elements in $\mathcal{B}(\mathcal{H})$ and $0 < \varepsilon < 1$ such that, for $i = 1, \cdots, n$,*

$$\|x_i - x_1\|_{2,Tr} < \varepsilon \|x_1\|_{2,Tr}.$$

Then there is a $t_0 > 0$ with

$$\left\|E_{t_0}^c(x_i) - E_{t_0}^c(x_1)\right\|_{2,Tr} < (3n\varepsilon)^{1/2} \left\|E_{t_0}^c(x_1)\right\|_{2,Tr}, \quad 1 \le i \le n.$$

Proof. We apply (10.4) with $f = g = \mathbf{1}_{(t^{1/2}, +\infty)}$. Then for every pair x, y of positive finite rank operators we have

$$\|E_{t^{1/2}}^c(x) - E_{t^{1/2}}^c(y)\|_{2,\mathrm{Tr}}^2 = \int_{\mathbb{R}_+^2} |\mathbf{1}_{(t^{1/2}, +\infty)}(\alpha) - \mathbf{1}_{(t^{1/2}, +\infty)}(\beta)| \,\mathrm{d}\nu(\alpha, \beta),$$

and therefore

$$\begin{aligned}
&\int_0^\infty \|E_{t^{1/2}}^c(x) - E_{t^{1/2}}^c(y)\|_{2,\mathrm{Tr}}^2 \,\mathrm{d}t \\
&\qquad = \int_{\mathbb{R}_+^2} \big(\int_0^\infty |\mathbf{1}_{(t^{1/2}, +\infty)}(\alpha) - \mathbf{1}_{(t^{1/2}, +\infty)}(\beta)| \,\mathrm{d}t \big) \,\mathrm{d}\nu(\alpha, \beta) \\
&\qquad = \int_{\mathbb{R}_+^2} |\alpha^2 - \beta^2| \,\mathrm{d}\nu(\alpha, \beta) \\
&\qquad \le \big(\int_{\mathbb{R}_+^2} |\alpha - \beta|^2 \,\mathrm{d}\nu(\alpha, \beta) \big)^{1/2} \big(\int_{\mathbb{R}_+^2} |\alpha + \beta|^2 \,\mathrm{d}\nu(\alpha, \beta) \big)^{1/2} \\
&\qquad = \|x - y\|_{2,\mathrm{Tr}} \|x + y\|_{2,\mathrm{Tr}},
\end{aligned}$$

after having again applied (10.4).

By assumption, for $\varepsilon < 1$, we have $\|x_j + x_1\|_{2,\mathrm{Tr}} \le 3\|x_1\|_{2,\mathrm{Tr}}$. It follows that

$$\begin{aligned}\int_0^\infty \sum_{i=1}^n \|E^c_{t^{1/2}}(x_i) - E^c_{t^{1/2}}(x_1)\|^2_{2,\mathrm{Tr}}\, \mathrm{d}t &< 3n\varepsilon\|x_1\|^2_{2,\mathrm{Tr}} \\ &= 3n\varepsilon \int_0^\infty \|E^c_{t^{1/2}}(x_1)\|^2_{2,\mathrm{Tr}}\, \mathrm{d}t.\end{aligned}$$

Hence, there is a $t_0 > 0$ with

$$\sum_{i=1}^n \|E^c_{t_0}(x_i) - E^c_{t_0}(x_1)\|^2_{2,\mathrm{Tr}} < 3n\varepsilon\|E^c_{t_0}(x_1)\|^2_{2,\mathrm{Tr}}.$$

□

This theorem holds true for any semi-finite von Neumann algebra (M, Tr) instead of $(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$. For later use we record the following particular case. The details are left to the reader[4].

Proposition 10.3.5. *Let (M,τ) be a tracial von Neumann algebra. Let ξ, η be two elements of $L^2(M)_+$. There exists a positive Radon measure ν on $\mathbb{R}^2_+$ such that for every pair f, g of Borel complex-valued functions on $\mathbb{R}_+$, with $f(0) = 0 = g(0)$ and $f(\xi), g(\eta) \in L^2(M)$, the functions $(\alpha,\beta) \mapsto f(\alpha)$ and $(\alpha,\beta) \mapsto g(\beta)$ are square integrable and*

$$\|f(\xi) - g(\eta)\|_2^2 = \int_{\mathbb{R}^2_+} |f(\alpha) - g(\beta)|^2\, \mathrm{d}\nu(\alpha,\beta).$$

Theorem 10.3.6. *Let (M,τ) be a tracial von Neumann algebra. Let $\xi_1, \dots, \xi_n$ be elements of $L^2(M)_+$. Let $0 < \varepsilon < 1$ be such that*

$$\|\xi_i - \xi_1\|_2 < \varepsilon\|\xi_1\|_2\,, \quad 1 \le i \le n.$$

Then there is a $t_0 > 0$ with

$$\left\|E^c_{t_0}(\xi_i) - E^c_{t_0}(\xi_1)\right\|_2 < (3n\varepsilon)^{1/2}\left\|E^c_{t_0}(\xi_1)\right\|_2\,, \quad 1 \le i \le n.$$

Exercises

Exercise 10.1. Let M be a finite factor such that there exists an increasing sequence $(P_n)_{n\ge1}$ of matrix subalgebras P_n of M containing 1_M, of type $I_{2^{k_n}}$, with $\overline{\cup_n P_n}^{s.o} = M$. Show that M is isomorphic to the hyperfinite factor R.[5]

Exercise 10.2. Let $(M,\mathcal{H})$ be a von Neumann algebra.

(i) Let $p \in \mathcal{P}(M)$. Show that pMp is amenable whenever M is amenable.

[4] See [**Con76**, Section I.1].

[5] We will prove in the next chapter (Theorem 11.2.2) the much more general and difficult result saying that all separable AFD II_1 factors are isomorphic.

(ii) If M is amenable, show that $M\overline{\otimes}\mathcal{B}(\mathcal{K})$ is amenable for any Hilbert space $\mathcal{K}$ (Hint: consider first the case where $\mathcal{K}$ is finite dimensional and conclude by approximation).

Exercise 10.3. Let $(M, \mathcal{H})$ be a von Neumann algebra and let $(\pi, \mathcal{H})$ be a representation of an amenable group G such that $\pi(g)M\pi(g)^* = M$ for every $g \in G$. Denote by M^G the von Neumann subalgebra of fixed points of M under this action. Show that there exists a conditional expectation from M onto M^G and that M^G is amenable whenever M is so.

Exercise 10.4. Show that abelian von Neumann algebras are amenable.

Exercise 10.5. Let $(M, \mathcal{H})$ be an amenable von Neumann algebra. Show that M' is amenable (Hint: assume that M is tracial and consider first the case where $(M, \mathcal{H})$ is a standard form; then use Proposition 8.2.2 to deal with the case of a non-standard representation. If M is not tracial, the proof is the same but requires the general notion of standard form for which the interested reader may look at [**Haa75**].).

Exercise 10.6. Let $G \curvearrowright (X, \mu)$ be a p.m.p. action on a standard probability measure space. Show that the crossed product $L^\infty(X, \mu) \rtimes G$ is amenable if and only if the group G is amenable.

Exercise 10.7. Let M be a von Neumann algebra and let p be a projection having 1 as central support. Show that M is amenable if and only if pMp is amenable.

Exercise 10.8. Show that every von Neumann algebra M has a unique decomposition as a direct sum $M_1 \oplus M_2$ where M_1 is amenable and M_2 has no amenable corner.

Exercise 10.9. Let M_1 and M_2 be two von Neumann algebras such that $M_1\overline{\otimes}M_2$ is amenable. Show that M_1 and M_2 are amenable.

Exercise 10.10. A von Neumann $(M, \mathcal{H})$ is said to have *Property* (P) if for every $T \in \mathcal{B}(\mathcal{H})$ the w.o. closed convex hull of $\{uTu^* : u \in \mathcal{U}(M)\}$ has a non-empty intersection with M'.

(i) Let $(M, \mathcal{H})$ be a von Neumann algebra and suppose that there exists an increasing sequence (M_n) of finite dimensional subalgebras, with the same unit as M, such that $(\cup M_n)'' = M$. Show that $(M, \mathcal{H})$ has the property (P) of Schwartz.

(ii) Show that the hyperfinite factor R has the property (P).

Notes

The main results and techniques presented in this chapter are borrowed from Connes' seminal paper [**Con76**].

The early stage towards the notion of injective von Neumann algebra is *Property* (P) of J.T. Schwartz [**Sch63**]. In this case, Schwartz proved the existence of a conditional expectation from $\mathcal{B}(\mathcal{H})$ onto M'. He showed that a group von Neumann algebra $L(G)$ has Property (P) if and only if G is amenable.

Later, Hakeda and Tomiyama [**HT67**] introduced the *extension property* for $(M, \mathcal{H})$ by the existence of a norm-one projection from $\mathcal{B}(\mathcal{H})$ onto M. This condition, which is now known as amenability or injectivity, is *a priori* weaker than Property (P) (by Exercise 10.5). It has many advantages in comparison with Property (P): it is easier to establish, is independent of the Hilbert space on which the von Neumann is represented and enjoys remarkable stability properties. We will show in the next chapter that, for tracial von Neumann algebras, amenability implies hyperfiniteness which in turn is weaker than Property (P) (Exercise 10.10). So, finally, amenability is equivalent to Property (P). More generally, this is still true for any von Neumann algebra acting on a separable Hilbert space [**Con76**].

CHAPTER 11

Amenability and hyperfiniteness

In this chapter, two fundamental results are established. First, any amenable finite von Neumann algebra can be approximated by finite dimensional algebras, a deep fact due to Connes. Such algebras are said to be approximately finite dimensional (AFD) or hyperfinite. Second, we show the theorem due to Murray and von Neumann asserting that there is only one separable hyperfinite II_1 factor, up to isomorphism.

11.1. Every amenable finite von Neumann algebra is AFD

Definition 11.1.1. Let M be a finite von Neumann algebra. We say that M is *approximately finite dimensional* (AFD) or *hyperfinite* if for every finite subset $F = \{x_1, \dots, x_n\}$ of M, every normal tracial state τ and every $\varepsilon > 0$, there exist a finite dimensional $*$-subalgebra $Q \subset M$ with $1_M \in Q$ and $y_1, \dots, y_n$ in Q such that $\|x_i - y_i\|_{2,\tau} < \varepsilon$ for $i = 1, \dots, n$, where $\|x\|_{2,\tau} = \tau(x^*x)^{1/2}$ (although $\|\cdot\|_{2,\tau}$ needs not be a norm).

When M has a faithful normal tracial state τ, the above definition is equivalent to the next one, and does not depend on the choice of the faithful normal tracial state τ. We use the following notation. If (M, τ) is a tracial von Neumann algebra, the metric defined by the norm $\|\cdot\|_2$ is denoted by d_2. Given $\varepsilon > 0$ and two subsets C, D of M, we write $C \subset^{\varepsilon,2} D$ if for every $x \in C$ we have $d_2(x, D) < \varepsilon$.

Definition 11.1.2. We say that a tracial von Neumann algebra (M, τ) is *approximately finite dimensional* (AFD) or *hyperfinite* if for every finite subset F of M, there exists a finite dimensional $*$-subalgebra $Q \subset M$ with $1_M \in Q$, such that $F \subset^{\varepsilon,2} Q$.

The goal of this section is to show the following celebrated theorem.

Theorem 11.1.3. *Every amenable finite von Neumann algebra is* AFD.

In the rest of this chapter we limit ourself to the case of a von Neumann algebra M equipped with a normal faithful tracial state τ. The proof of the above theorem in the general case follows from the exercise 11.1. Moreover, for simplicity of presentation, we will assume that M is separable.

The hardest step is to prove Theorem 11.1.5 which states that a tracial amenable von Neumann algebra M has the local approximation property as defined below. Then a classical maximality argument will imply that M is AFD (Theorem 11.1.17).

Definition 11.1.4. We say that (M, τ) has the *local approximation property* if for every $\varepsilon > 0$, every non-zero projection $e \in M$ and every finite subset $F \subset \mathcal{U}(eMe)$, there exists a non-zero finite dimensional matrix algebra Q in eMe with unit q such that for $v \in F$,

(a) $\|[q, v]\|_2 < \varepsilon \|q\|_2$;
(b) $d_2(qvq, Q) < \varepsilon \|q\|_2$.

11.1.1. Amenability implies the local approximation property.

Theorem 11.1.5. *A tracial amenable von Neumann algebra (M, τ) has the local approximation property.*

Since eMe is still amenable for every $e \in \mathcal{P}(M)$, it is enough to take $e = 1$ and to prove the following claim.

Claim: given $\varepsilon > 0$ and a finite subset $\{v_1, \dots, v_l\}$ of unitaries in M there exists a non-zero finite dimensional matrix algebra Q with unit q such that for $1 \le k \le l$,

$$\|[q, v_k]\|_2 < \varepsilon \|q\|_2 \quad \text{and} \quad d_2(qv_kq, Q) < \varepsilon \|q\|_2. \tag{11.1}$$

Strategy of the proof. We choose a maximal abelian von Neumann subalgebra A of M. We will first apply to $\langle M, e_A \rangle$ the Connes' non-commutative version of the Day and Namioka arguments as in the previous chapter, in order to find a finite projection $p \in \langle M, e_A \rangle$ almost invariant under the unitaries v_k.[1] After a first approximation, we will show that this projection can be associated to a finite family $x_1, \dots, x_m$ of elements in M that are orthonormal with respect to the conditional expectation E_A. A local Rohlin type lemma followed by a technical deformation will allow to construct, from these elements $x_1, \dots, x_m$, a matrix units which generates a matrix algebra Q. Condition (11.1) will be a consequence of the almost invariance of p and of the local Rohlin type lemma. The maximality of A is only used in the proof of this local Rohlin type lemma.

We first state a few facts relative to the Jones' basic construction.

Two formulas in $\langle M, e_A \rangle$. Until Lemma 11.1.11, we only assume that A is an abelian von Neumann subalgebra of (M, τ). We recall that $\langle M, e_A \rangle$ is semi-finite and we endow it with the normal faithful semi-finite trace $\widehat{\tau}$ introduced in Section 8.4 (see also Section 9.4). Since $\langle M, e_A \rangle$ is the commutant of JAJ, we observe that JAJ is the center of $\langle M, e_A \rangle$.

Given a left A-bounded vector $\xi \in L^2(M)_A$, we denote, as in Section 8.4.2, by L_ξ the corresponding A-linear operator from $L^2(A)_A$ into $L^2(M)_A$. Let p_ξ be the orthogonal projection onto the A-submodule $\overline{\xi A}$ of $L^2(M)_A$. We will need to compute $\widehat{\tau}(JaJp_\xi)$ for $a \in A$. To that end, we consider

[1] Recall that $\langle M, e_A \rangle$ is the Jones' basic construction for $A \subset M$

the polar decomposition $\xi = \xi' \langle \xi, \xi \rangle_A^{1/2}$ of ξ (see Lemma 8.4.9). Then, with the notations of Section 8.4.3, we see that $p_\xi = p_{\xi'} = L_{\xi'} L_{\xi'}^*$. Using the fact that A is abelian and $L_{\xi'}$ is A-linear, we get $Ja^* J L_{\xi'} = L_{(\xi' a)}$ and thus $Ja^* J L_{\xi'} L_{\xi'}^* = L_{(\xi' a)} L_{\xi'}^*$. It follows from Proposition 8.4.17 that

$$\widehat{\tau}(JaJp_\xi) = \tau(\langle \xi', \xi' a^* \rangle_A).$$

But $\langle \xi', \xi' \rangle_A$ is the range projection of $\langle \xi, \xi \rangle_A^{1/2}$ (see Lemma 8.4.9), that is, the support $s(\langle \xi, \xi \rangle_A)$ of $\langle \xi, \xi \rangle_A$. Thus we get

$$\widehat{\tau}(JaJp_\xi) = \tau(a^* s(\langle \xi, \xi \rangle_A)). \tag{11.2}$$

We note that whenever $\xi \in M \subset L^2(M)$, then $\langle \xi, \xi \rangle_A = E_A(\xi^* \xi)$ (see Section 9.4.1).[2]

We will also need the following fact: given $x, y \in M$ such that $E_A(x^* x)$ and $E_A(y^* y)$ are projections in A, then

$$p_x p_y p_x = J E_A(x^* y) E_A(y^* x) J p_x. \tag{11.3}$$

This is a straightforward computation using the commutativity of A and the fact that $p_z(m) = z E_A(z^* m)$ for $m \in M$ and $z = x$ or $z = y$.

11.1.1.1. *A Følner type condition.*

Lemma 11.1.6. *Assume that M is amenable. Given $\varepsilon' > 0$, there exists a projection $p \in \langle M, e_A \rangle$ such that $\widehat{\tau}(p) < +\infty$ and*

$$\max_{1 \le k \le l} \|p - v_k p v_k^*\|_{2,\hat{\tau}} < \varepsilon' \|p\|_{2,\hat{\tau}}.$$

Proof. The proof is similar to that of Theorems 10.2.9 and 10.3.1. We set

$$C = \{(v_1 \varphi v_1^* - \varphi, \dots, v_l \varphi v_l^* - \varphi)\}$$

where φ runs over the normal states on $\langle M, e_A \rangle$ of the form $\widehat{\tau}(c\,\cdot)$ with $c \in \langle M, e_A \rangle_+$ and $\widehat{\tau}(c) = 1$. Using the existence of an hypertrace, we see that $(0, \dots, 0)$ is in the weak closure of the convex set C. Then, given $\delta > 0$, the same Day's convexity argument as in the proof of Theorem 10.2.9 provides a $c \in \langle M, e_A \rangle_+$ with $\widehat{\tau}(c) = 1$ and

$$\|\widehat{\tau}(v_k c v_k^* \cdot) - \widehat{\tau}(c\,\cdot)\| < \delta^2$$

for $1 \le k \le l$. We set $b = c^{1/2}$. By the Powers-Størmer inequality (see Theorem 10.2.8) we get

$$\|v_k b v_k^* - b\|_{2,\hat{\tau}}^2 \le \|\widehat{\tau}(v_k c v_k^* \cdot) - \widehat{\tau}(c\,\cdot)\| < \delta^2 \|b\|_{2,\hat{\tau}}^2.$$

Now using Theorem 10.3.6, we find a spectral projection p of b such that,

$$\max_{1 \le k \le l} \|p - v_k p v_k^*\|_{2,\hat{\tau}} < (3l\delta)^{1/2} \|p\|_{2,\hat{\tau}}.$$

We choose $\delta = (\varepsilon')^2/3l$. □

[2] We identify M to a subspace of $L^2(M)$.

In the next step, we show that we may take p of the form $\sum_{i=1}^m p_{x_i}$ where the $x_i \in M$ satisfy $E_A(x_i^* x_j) = \delta_{i,j} p_i \in \mathcal{P}(A)$.

11.1.1.2. *Approximation of finite-trace projections in $\langle M, e_A \rangle$.*

LEMMA 11.1.7. *Let p be a projection in $\langle M, e_A\rangle$ such that $\widehat{\tau}(p) < +\infty$. Given $\varepsilon' > 0$, there exist $x_1, \dots, x_m$ in M with $E_A(x_i^* x_j) = \delta_{i,j} p_i \in \mathcal{P}(A)$ for every i, j and*

$$\Big\| p - \sum_{i=1}^m p_{x_i} \Big\|_{2,\hat{\tau}} < \varepsilon'.$$

PROOF. We first observe that there exists an increasing sequence (z_n) of projections in the center of $\langle M, e_A\rangle$ such that $\lim_n z_n = 1$ in the s.o. topology and $p z_n L^2(M)$ is a finitely generated right A-module for every n (see Corollary 9.3.3). So, replacing p by $p z_n$ for n large enough, we may assume that $pL^2(M)$ is finitely generated. By Proposition 8.5.3, we know that $p = \sum_{i=1}^m p_{\xi_i}$ where $\xi_1, \dots, \xi_m$ is an orthonormal basis of the right A-module $pL^2(M)$. The elements ξ_i are left A-bounded, but are not necessarily in M. Our technical task is to show that we may replace $\xi_1, \dots, \xi_m$ by an orthonormal basis made of elements in M. We proceed by induction on m.

We set $\xi_0 = 0$, $x_0 = p_0 = 0$. Given $0 < \delta < 1/4$, we assume that we have found $x_0, \dots, x_{k-1}$ in M such that $E_A(x_i^* x_j) = \delta_{i,j} p_i \in \mathcal{P}(A)$ for $0 \le i, j \le k-1$ and $\Big\|\sum_{i=0}^{k-1} p_{\xi_i} - \sum_{i=0}^{k-1} p_{x_i}\Big\|_{2,\hat{\tau}} < \delta$. We want to show that there exists $x_k \in M$ such that $E_A(x_k^* x_k)$ is a projection, $E_A(x_k^* x_i) = 0$ for $i < k$ and $\Big\|\sum_{i=0}^{k} p_{\xi_i} - \sum_{i=0}^{k} p_{x_i}\Big\|_{2,\hat{\tau}} < 2\delta^{1/2}$. The lemma is then an easy consequence.

We view ξ_k as an operator affiliated to M. It follows that there exists an increasing sequence (q_n) of projections in M such that $q_n \xi_k \in M$ for every n and $\lim_n q_n = 1$ strongly. We have

$$\|p_{q_n\xi_k} - p_{\xi_k}\|_{2,\hat{\tau}}^2 \le 2\|p_{q_n\xi_k} - q_n p_{\xi_k} q_n\|_{2,\hat{\tau}}^2 + 2\|q_n p_{\xi_k} q_n - p_{\xi_k}\|_{2,\hat{\tau}}^2$$

and

$$\begin{aligned}\|p_{q_n\xi_k} - q_n p_{\xi_k} q_n\|_{2,\hat{\tau}}^2 &= \widehat{\tau}(p_{q_n\xi_k}) - 2\widehat{\tau}(q_n p_{\xi_k} q_n) + \widehat{\tau}(q_n p_{\xi_k} q_n p_{\xi_k} q_n)\\ &\le \widehat{\tau}(p_{q_n\xi_k}) - \widehat{\tau}(q_n p_{\xi_k} q_n),\end{aligned}$$

since $p_{q_n\xi_k} q_n p_{\xi_k} = q_n p_{\xi_k}$.

Moreover, by (11.2), we get

$$\widehat{\tau}(p_{q_n\xi_k}) = \tau(s(\langle q_n\xi_k, q_n\xi_k\rangle_A)) \le \tau(s(\langle \xi_k, \xi_k\rangle_A)) = \widehat{\tau}(p_{\xi_k}).$$

But $\lim_n \widehat{\tau}(q_n p_{\xi_k} q_n) = \widehat{\tau}(p_{\xi_k})$ and $\lim_n \|q_n p_{\xi_k} q_n - p_{\xi_k}\|_{2,\hat{\tau}} = 0$ and so, given $\delta' > 0$, we can choose n' such that $\big\|p_{q_{n'}\xi_k} - p_{\xi_k}\big\|_{2,\hat{\tau}} < \delta'$.

We set $y = q_{n'}\xi_k$, $y_0 = \sum_{i=0}^{k-1} p_{x_i}(y)$ and $y_1 = y - y_0$. We have $E_A(x_i^* y_1) = 0$ for $0 \le i \le k-1$, but $a = E_A(y_1^* y_1)$ may not be a projection. So, we will need later to modify slightly y_1. But before, we want to evaluate $\|p_{y_1} - p_y\|_{2,\hat\tau}$. We set $f = \sum_{i=0}^{k-1} p_{x_i} + p_{y_1}$ and $f_{k-1} = \sum_{i=0}^{k-1} p_{x_i}$. We have $p_y \leq f$ and so $f(f_{k-1} + p_y) = f_{k-1} + p_y$. Then

$$\begin{aligned}
&\|f - (f_{k-1} + p_y)\|_{2,\hat\tau}^2 = \widehat{\tau}(f) - \widehat{\tau}(f_{k-1} + p_y) + 2\widehat{\tau}(p_y f_{k-1})\\
&= \widehat{\tau}(f) - \widehat{\tau}(f_{k-1} + p_y) + 2\widehat{\tau}\big((p_y - p_{\xi_k})f_{k-1}\big) + 2\widehat{\tau}\big(p_{\xi_k}(\sum_{i=0}^{k-1} p_{x_i} - \sum_{i=0}^{k-1} p_{\xi_i})\big)\\
&< \widehat{\tau}(p_{y_1}) - \widehat{\tau}(p_y) + 2k^{1/2}\delta' + 2\delta,
\end{aligned}$$

where, to get the last inequality, we have used the Cauchy-Schwarz inequality and the fact that $\|f_{k-1}\|_{2,\hat\tau} \leq k^{1/2}$ and $\|p_{\xi_k}\|_{2,\hat\tau} \leq 1$. We remark that

$$\widehat{\tau}(p_{y_1}) = \tau\big(s(E_A(y_1^* y_1))\big) \leq \tau\big(s(E_A(y^* y))\big) = \widehat{\tau}(p_y).$$

It follows that

$$\|p_{y_1} - p_y\|_{2,\hat\tau}^2 = \|f - (f_{k-1} + p_y)\|_{2,\hat\tau}^2 \leq 2k^{1/2}\delta' + 2\delta.$$

Now, let us explain how we modify y_1. Let e_n be the spectral projection of $a = E_A(y_1^* y_1)$ corresponding to the interval $(1/n, +\infty)$. We put $x_k = y_1 e_n a^{-1/2}$. Then we still have $E_A(x_i^* x_k) = 0$ for $0 \leq i \leq k-1$ and morever $E_A(x_k^* x_k) = e_n$ is now a projection. Observe also that $p_{x_k} = p_{y_1 e_n}$. Using again (11.2), we see that

$$\begin{aligned}
\|p_{y_1} - p_{y_1 e_n}\|_{2,\hat\tau}^2 &= \|p_{y_1} - J e_n J p_{y_1}\|_{2,\hat\tau}^2\\
&= \widehat{\tau}(p_{y_1}) - \widehat{\tau}(J e_n J p_{y_1})\\
&= \tau(s(a)) - \tau(e_n s(a)),
\end{aligned}$$

where $s(a)$ is the support of a. With n sufficiently large, we can make $\|p_{y_1} - p_{x_k}\|_{2,\hat\tau}$ arbitrary small.

Finally we get that $\left\|\sum_{i=0}^{k} p_{\xi_i} - \sum_{i=0}^{k} p_{x_i}\right\|_{2,\hat\tau}$ is smaller than

$$\begin{aligned}
&\left\|\sum_{i=0}^{k-1} p_{\xi_i} - f_{k-1}\right\|_{2,\hat\tau} + \|p_{\xi_k} - p_y\|_{2,\hat\tau} + \|p_y - p_{y_1}\|_{2,\hat\tau} + \|p_{y_1} - p_{x_k}\|_{2,\hat\tau}\\
&< \delta + \delta' + (2k^{1/2}\delta' + 2\delta)^{1/2} + \|p_{y_1} - p_{x_k}\|_{2,\hat\tau}\\
&< \delta + \delta' + (2k^{1/2}\delta' + 2\delta)^{1/2}
\end{aligned}$$

for a sufficiently large n. Whenever $\delta < 1/4$, we can find δ' sufficiently small such that $\delta + \delta' + (2k^{1/2}\delta' + 2\delta)^{1/2} \leq 2\delta^{1/2}$. □

Our two last steps before proceeding to the proof of Theorem 11.1.5 are the local Rohlin type lemma 11.1.11 and the deformation lemma 11.1.12.

11.1.1.3. *A local Rohlin type lemma.* We need to show first the elementary fact that every abelian von Neumann algebra is AFD.

LEMMA 11.1.8. *Let (A, τ) be an abelian separable von Neumann algebra. There exists an increasing sequence (A_n) of finite dimensional von Neumann subalgebras of A such that $\overline{\cup_n A_n}^{wo} = A$.*

PROOF. Let (a_n) be a sequence in the unit ball $(A)_1$ of A which is dense in $(A)_1$ equipped with the $\|\cdot\|_2$-metric (see Proposition 2.6.7). We construct the algebras A_n by induction. First, there exist projections $e_1, \dots, e_k$ and scalars $\alpha_1, \dots, \alpha_k$ in $\mathbb{C}$ such that $\sum_{j=1}^k e_j = 1$ and $\left\| a_1 - \sum_{j=1}^k \alpha_j e_j \right\| < 1/2$. *A fortiori* we have the same inequality with respect to the $\|\cdot\|_2$-norm. We denote by A_1 the algebra generated by the projections $e_1, \dots, e_k$. Observe that $\|a_1 - E_{A_1}(a_1)\|_2 < 1/2$ and that $\|E_{A_1}(a_1)\| \leq 1$.

Assume that we have constructed finite dimensional algebras $A_1, \dots A_m$ such that $A_1 \subset A_2 \subset \cdots \subset A_m$ and

$$\left\| a_i - E_{A_j}(a_i) \right\|_2 < 2^{-j} \quad \text{for} \quad 1 \leq i \leq j \leq m. \tag{11.4}$$

As above, for $i = 1, \dots, m+1$ we can find projection e_j^i and scalars α_j^i such that

$$\left\| a_i - \sum_{j=1}^{k_i} \alpha_j^i e_j^i \right\| < 2^{-(m+1)}.$$

Let A_{m+1} be the algebra generated by A_m and the projections e_j^i. Then (11.4) is satisfied with $m+1$ instead of m.

We conclude that the unit ball of $\cup_n A_n$ is dense in the unit ball of A with respect to the $\|\cdot\|_2$-metric, that is, with respect to the s.o. topology. $\square$

We also need the following continuity property for conditional expectations.

LEMMA 11.1.9. *Let (M, τ) be a tracial von Neumann algebra and (B_n) a decreasing sequence of von Neumann subalgebras. We set $B = \cap_n B_n$. Then, for every $x \in M$ we have*

$$\lim_n \|E_{B_n}(x) - E_B(x)\|_2 = 0.$$

PROOF. The sequence (e_{B_n}) of orthogonal projections $e_{B_n} : L^2(M) \to L^2(B_n)$ is decreasing. We set $e = \wedge_n e_{B_n}$. We have obviously $e_B \leq e$, and it remains to show that $e \leq e_B$. Given $x \in M$, we have $\lim_n \|e_{B_n}(\widehat{x}) - e(\widehat{x})\|_2 = 0$. On the other hand, $(E_{B_n}(x))$ is a sequence bounded in norm by $\|x\|$. Therefore there is a subsequence $(E_{B_{n_k}}(x))_k$ which converges to some $x_0 \in M$ in the w.o. topology. It is easily seen that $x_0 \in B$ and that

$$\lim_k \left\langle e_{B_{n_k}}(\widehat{x}), \widehat{y} \right\rangle = \langle \widehat{x_0}, \widehat{y} \rangle$$

for every $y \in M$. It follows that $e(\widehat{x}) = \widehat{x_0} \in \widehat{B}$, hence $e \leq e_B$. $\square$

REMARK 11.1.10. The same result holds for an increasing sequence (B_n) and $B = \overline{\cup_n B_n}^{wo}$. We leave it as an exercise.

LEMMA 11.1.11 (**Local Rohlin type lemma**). *Let A be a maximal abelian subalgebra of (M, τ). Let $f \in A$ be a non-zero projection, $y_1, \dots, y_m \in M$, and $\varepsilon' > 0$. There exists a projection $e \in A$ with $e \leq f$ and*

$$\max_{1 \leq i \leq m} \|ey_i e - \lambda_i e\|_2 < \varepsilon' \|e\|_2,$$

where $\lambda_i = \tau(ey_i e)/\tau(e)$.

PROOF. Since A is abelian, using the spectral decomposition of the $E_A(y_i)$'s, we find a non-zero projection $f' \in A$ with $f' \leq f$, such that for all i,

$$\left\| E_A(y_i) f' - \lambda'_i f' \right\| < \varepsilon'/2$$

for some scalar $\lambda'_i \in \mathbb{C}$.

Let (A_n) be an increasing sequence of finite dimensional subalgebras of Af' such that $\overline{\cup_n A_n}^{wo} = Af'$. Note that

$$\cap_n (A'_n \cap f' M f') = A' f' \cap f' M f' = Af'$$

since Af' is a maximal abelian subalgebra of $f' M f'$. By Lemma 11.1.9 we have $\lim_n \left\| E_{A'_n \cap f' M f'}(x) - E_{Af'}(x) \right\|_2 = 0$ for every $x \in f' M f'$. It follows that there is an integer n_0 such that

$$\sum_{i=1}^m \left\| E_{A'_{n_0} \cap f' M f'}(f' y_i f') - E_{Af'}(f' y_i f') \right\|_2^2 < (\varepsilon'/2)^2 \|f'\|_2^2.$$

If we denote by $e_1, \dots, e_s$ the minimal projections of A_{n_0}, we get

$$\sum_{i=1}^m \left\| \sum_{j=1}^s (e_j y_i e_j - E_A(y_i) e_j \right\|_2^2 < (\varepsilon'/2)^2 \|f'\|_2^2$$

(see Example 9.1.2 (2) and since $E_{Af'}(f' y_i f') = E_A(y_i) f'$) and therefore, by Pythagoras' theorem,

$$\sum_{i=1}^m \sum_{j=1}^s \|e_j y_i e_j - E_A(y_i) e_j\|_2^2 < (\varepsilon'/2)^2 \sum_{j=1}^s \|e_j\|_2^2.$$

It follows that for some j, and for all i, we have

$$\|e_j y_i e_j - E_A(y_i) e_j\|_2 < (\varepsilon'/2) \|e_j\|_2$$

and thus $\|e_j y_i e_j - \lambda'_i e_j\|_2 < \varepsilon' \|e_j\|_2$.

Since $\big(\tau(e_j y_i e_j)/\tau(e_j)\big) e_j$ is the orthogonal projection of $e_j y_i e_j$ onto $\mathbb{C} e_j$, we see that $e = e_j$ satisfies the statement of the lemma. $\square$

11.1.1.4. *A deformation lemma.*

Lemma 11.1.12. *Let (M,τ) be a tracial von Neumann algebra. For η sufficiently small, $C>0$ and $m\in\mathbb{N}$, there exists $\delta(\eta,C,m)$ (that we simply write $\delta(\eta)$) with $\lim_{\eta\to 0}\delta(\eta)=0$ such that, given $y_1,\dots,y_m$ in M and $e\in\mathcal{P}(M)$ satisfying*

$$\|ey_i^*y_je-\delta_{i,j}e\|_2<\eta\|e\|_2$$

for all $1\le i,j\le m$ and $\max_i\|y_i\|\le C$, then there exist partial isometries $u_1,\dots,u_m$ and a projection e' in M with $e'\le e$, $u_i^*u_j=\delta_{i,j}e'$ and

$$\|e-e'\|_2<\delta(\eta)\|e'\|_2,$$
$$\forall i,\ \ \|y_ie'-u_i\|_2<\delta(\eta)\|e'\|_2.$$

Remark 11.1.13. Note that

$$\|y_ie-u_i\|_2\le\|y_i\|\|e-e'\|_2+\|y_ie'-u_i\|_2<(C+1)\delta(\eta)\|e'\|_2$$

so we will have (and indeed use) the same result with $\|y_ie-u_i\|_2$ in place of $\|y_ie'-u_i\|_2$ in the second inequality of the lemma.

The proof of this lemma is by induction on m and uses the next lemmas.

Lemma 11.1.14. *Let e be a projection in M and $x\in(eMe)_+$ such that $\|e-x\|_2<\varepsilon'\|e\|_2$ with $0<\varepsilon'<1$. Denote by e' the spectral projection of x corresponding to the interval $(1-\sqrt{\varepsilon'},1+\sqrt{\varepsilon'})$. Then we have*

$$e'\le e,\quad \|e-e'\|_2<\sqrt{\varepsilon'}\|e\|_2 \tag{11.5}$$
$$\|e'-xe'\|<\sqrt{\varepsilon'}. \tag{11.6}$$

Proof. Observe that if τ_e denote the trace $\tau/\tau(e)$ on eMe then, for $y\in eMe$, one has $\|y\|_2=\tau(e)^{1/2}\|y\|_{2,\tau_e}$. Therefore it suffices to consider the case $e=1$. The inequality (11.6) is obvious. To prove the second inequality of (11.5), we consider the spectral probability measure of x associated with the vector $\widehat{1}\in L^2(M)$. We have

$$\varepsilon'\|1-e'\|_2^2=\varepsilon'\mu\big(\mathbb{R}_+\setminus(1-\sqrt{\varepsilon'},1+\sqrt{\varepsilon'})\big)\le\int_{\mathbb{R}_+}|1-t|^2\,\mathrm{d}\mu(t)\le(\varepsilon')^2.$$

$\square$

Lemma 11.1.15. *Let e be a projection in M and $x\in M$ such that $x^*x\in eMe$ and $\|x^*x-e\|<\varepsilon'<1$. If $x=u|x|$ is the polar decomposition of x, then $\|x-u\|<1-(1-\varepsilon')^{1/2}<\sqrt{\varepsilon'}$ and $u^*u=e$.*

Proof. Again, it suffices to consider the case $e=1$. We observe that $\|x-u\|\le\||x|-1\|$ and that the spectrum of $|x|^2=x^*x$ is contained in the interval $(1-\varepsilon',1+\varepsilon')$. $\square$

Putting these two lemmas together we get:

LEMMA 11.1.16. *Let e be a projection in M and $x \in Me$ such that $\|x^*x - e\|_2 < \varepsilon'\|e\|_2$, where $\varepsilon' < 1$. There exist a projection $e' \leq e$ and a partial isometry u, namely the isometry given by the polar decomposition of xe' such that $u^*u = e'$ and*

$$\|e - e'\|_2 < \sqrt{\varepsilon'}\|e\|_2, \quad \|xe' - u\| < (\varepsilon')^{1/4},$$

from which it follows that

$$\|e\|_2 < (1 - \sqrt{\varepsilon'})^{-1}\|e'\|_2$$

and

$$\|xe' - u\|_2 \leq \|xe' - u\|\|e'\|_2 < (\varepsilon')^{1/4}\|e'\|_2.$$

Denoting by φ the function $t \mapsto t^{1/4}(1 - \sqrt{t})^{-1}$ we thus have

$$\|e - e'\|_2 < \varphi(\varepsilon')\|e'\|_2 \quad \text{and} \quad \|xe' - u\|_2 < \varphi(\varepsilon')\|e'\|_2.$$

PROOF OF LEMMA 11.1.12. As said, we proceed by induction on m. The step $m = 1$ follows from the previous lemma where we take $x = y_1 e$ and $\varepsilon' = \eta$.

Assume now that we have found a projection $e_k \in \mathcal{P}(M)$ and partial isometries $u_1, \dots, u_k$ such that

$$u_i^* u_j = \delta_{i,j} e_k \quad \text{for} \quad 1 \leq i, j \leq k, \tag{11.7}$$

$$\|y_i e_k - u_i\|_2 < \delta_k(\eta)\|e_k\|_2 \quad \text{for} \quad 1 \leq i \leq k, \tag{11.8}$$

$$\|e - e_k\|_2 < \delta_k(\eta)\|e_k\|_2 \quad \text{and} \quad e_k \leq e, \tag{11.9}$$

where δ_k only depends on η, C, k, and $\lim_{\eta\to 0} \delta_k(\eta) = 0$. We set $q = 1 - \sum_{i=1}^k u_i u_i^*$ and $y'_{k+1} = q y_{k+1}$. We have

$$\begin{aligned}
\|y'_{k+1} e_k - y_{k+1} e_k\|_2 &\leq \sum_{i=1}^k \|u_i^* y_{k+1} e_k\|_2 \\
&\leq \sum_{i=1}^k \|(u_i^* - e_k y_i^*) y_{k+1} e_k\|_2 + \sum_{i=1}^k \|e_k y_i^* y_{k+1} e_k\|_2 \\
&< k\delta_k(\eta)\|y_{k+1}\|\|e_k\|_2 + k\eta(1 + \delta_k(\eta))\|e_k\|_2 \\
&< k(C\delta_k(\eta) + \eta(1 + \delta_k(\eta)))\|e_k\|_2.
\end{aligned}$$

From this, straightforward computations show that

$$\|e_k (y'_{k+1})^* y'_{k+1} e_k - e_k\|_2 < \delta'_k(\eta)\|e_k\|_2$$

where again $\lim_{\eta\to 0} \delta'_k(\eta) = 0$.

Using anew Lemma 11.1.16 we find a projection $e_{k+1} \leq e_k$ and a partial isometry u_{k+1} such that $u_{k+1}^* u_{k+1} = e_{k+1}$, $\|e_k - e_{k+1}\|_2 < \varphi(\delta'_k(\eta))\|e_{k+1}\|_2$ and $\|y'_{k+1} e_{k+1} - u_{k+1}\|_2 < \varphi(\delta'_k(\eta))\|e_{k+1}\|_2$. We observe that $q u_{k+1} = u_{k+1}$ and so, replacing u_i by $u_i e_{k+1}$, we easily see that Condition (11.7) is fulfilled at the step $k + 1$, as well as Conditions (11.8), (11.9) with an appropriate function δ_{k+1}. ∎

We are now ready to prove Theorem 11.1.5

Proof of Theorem 11.1.5. We have to prove the claim which follows the statement of the theorem. We assume here that A is maximal abelian. Let $\delta > 0$ be given. By Lemmas 11.1.6 and 11.1.7, there exist $x_1, \dots, x_m$ in M with $E_A(x_i^* x_j) = \delta_{i,j} f_i \in \mathcal{P}(A)$ for every i, j and

$$\max_{1 \le k \le l} \|p - v_k p v_k^*\|_{2,\hat{\tau}}^2 < (\delta/l)\|p\|_{2,\hat{\tau}}^2,$$

where $p = \sum_{i=1}^m p_{x_i}$. For $y \in \langle M, e_A \rangle$ we have

$$\|y\|_{2,\hat{\tau}}^2 \ge \|py\|_{2,\hat{\tau}}^2 = \sum_{i=1}^m \|p_{x_i} y\|_{2,\hat{\tau}}^2 \ge \sum_{i=1}^m \|p_{x_i} y p_{x_i}\|_{2,\hat{\tau}}^2.$$

Using (11.2) and (11.3), it follows that

$$\begin{aligned}\|p - v_k p v_k^*\|_{2,\hat{\tau}}^2 &\ge \sum_{i=1}^m \|p_{x_i} - p_{x_i} v_k p v_k^* p_{x_i}\|_{2,\hat{\tau}}^2 \\ &\ge \sum_{i=1}^m \|p_{x_i} - \sum_{j=1}^m p_{x_i} p_{v_k x_j} p_{x_i}\|_{2,\hat{\tau}}^2 \\ &\ge \sum_{i=1}^m \|f_i - \sum_{j=1}^m E_A(x_i^* v_k x_j) E_A(x_j^* v_k^* x_i) f_i\|_2^2.\end{aligned}$$

Hence, we get

$$\sum_{k=1}^l \sum_{i=1}^m \tau\Big(\big(f_i - \sum_{j=1}^m E_A(x_i^* v_k x_j) E_A(x_j^* v_k^* x_i) f_i\big)^2\Big) < \delta \|p\|_{2,\hat{\tau}}^2 = \delta \sum_{i=1}^m \tau(f_i).$$

Since A is abelian, we have an inequality between integrals and therefore there exists a non-zero projection $f \in A$ such that, for every k,

$$\sum_{i=1}^m \big(f_i f - \sum_{j=1}^m E_A(x_i^* v_k x_j) E_A(x_j^* v_k^* x_i) f_i f\big)^2 < \delta \sum_{i=1}^m f_i f. \tag{11.10}$$

Moreover, again because A is abelian, there is a non-zero projection $f' \in A$, smaller than f, such that, for every j, either $f' f_j = f'$ or $f' f_j = 0$, with $f' f_i \neq 0$ for at least one i. Cutting (11.11) by f', and keeping only the indices i such that $f' f_i = f'$, we may assume that

$$\sum_{i=1}^m \big(f - \sum_{j=1}^m E_A(x_i^* v_k x_j) E_A(x_j^* v_k^* x_i) f\big)^2 < m\delta f, \tag{11.11}$$

with $f_i f = f$ for all i.

Now, we use the local Rohlin type lemma 11.1.11 to the family

$$\{x_i^* x_j, x_i^* v_k x_j : 1 \le i, j \le m, 1 \le k \le l\}.$$

Given $\eta > 0$, there exists a projection $e \in A$, with $e \le f$ and

$$\|ex_i^* x_j e - \delta_{i,j} e\|_2 < \eta \|e\|_2, \tag{11.12}$$

$$\|ex_i^* v_k x_j e - \lambda_{i,j,k} e\|_2 < \eta \|e\|_2. \tag{11.13}$$

The deformation lemma 11.1.12, the remark 11.1.13 and Equation (11.12) provide a projection $e' \in M$, $e' \le e$, partial isometries $u_1, \dots, u_m$ in M and a fonction $\delta : \mathbb{R}_+^* \to \mathbb{R}_+^*$ satisfying $\lim_{\eta \to 0} \delta(\eta) = 0$ such that for every i, j,

$$u_i^* u_j = \delta_{i,j} e', \quad \|e - e'\|_2 < \delta(\eta) \|e'\|_2 \quad \text{and} \quad \|x_i e - u_i\|_2 < \delta(\eta) \|e'\|_2.$$

We use Equation (11.13) to approximate in $\|\cdot\|_2$-norm

$$E_A(x_i^* v_k x_j) e = E_A(e x_i^* v_k x_j e)$$

by $e x_i^* v_k x_j e$ since it implies $\|E_A(e x_i^* v_k x_j e) - e x_i^* v_k x_j e\|_2 < 2\eta \|e\|_2$. In turn, $e x_i^* v_k x_j e$ is approximated by $u_i^* v_k u_j$.

Therefore, if η is chosen sufficiently small, we get from (11.11), where f is first replaced by e, and then e is approximated by e', that

$$\sum_{i=1}^m \tau\Big(\Big(e' - \sum_{j=1}^m u_i^* v_k u_j u_j^* v_k^* u_i\Big)^2\Big) < m\delta \tau(e'), \tag{11.14}$$

and moreover we get from (11.13) that

$$\|u_i^* v_k u_j - \lambda_{i,j,k} e'\|_2^2 < (\delta/m) \|e'\|_2^2. \tag{11.15}$$

We set $e_{i,j} = u_i u_j^*$, $q = \sum_{i=1}^m e_{i,i}$ and we denote by Q the matrix algebra generated by the matrix units $(u_{i,j})$.

By Pythagoras' theorem and (11.15), and since

$$\|e_{i,i} v_k e_{j,j} - \lambda_{i,j,k} e_{i,j}\|_2 = \|u_i(u_i^* v_k u_j - \lambda_{i,j,k} e') u_j^*\|_2 = \|u_i^* v_k u_j - \lambda_{i,j,k} e'\|_2,$$

we get

$$\begin{aligned} \Big\| q v_k q - \sum_{i,j} \lambda_{i,j,k} e_{i,j} \Big\|_2^2 &= \sum_{i,j} \|e_{i,i} v_k e_{j,j} - \lambda_{i,j,k} e_{i,j}\|_2^2 \\ &< m\delta \|e'\|_2^2 = \delta \|q\|_2^2, \end{aligned}$$

and so

$$d_2(q v_k q, Q) < \delta^{1/2} \|q\|_2. \tag{11.16}$$

It remains to estimate $\|q - v_k q v_k^*\|_2$. We have, by (11.14),

$$\Big\| q - \sum_i e_{i,i} v_k q v_k^* e_{i,i} \Big\|_2^2 = \sum_i \|e_{i,i} - e_{i,i} v_k q v_k^* e_{i,i}\|_2^2$$

and so

$$\Big\| q - \sum_i e_{i,i} v_k q v_k^* e_{i,i} \Big\|_2^2 = \sum_i \|e' - u_i^* v_k q v_k^* u_i\|_2^2 < \delta m \|e'\|_2^2 = \delta \|q\|_2^2. \tag{11.17}$$

An immediate computation shows that

$$\Big\|v_kqv_k^* - \sum_i e_{i,i}v_kqv_k^*e_{i,i}\Big\|_2^2 + \Big\|\sum_i e_{i,i}v_kqv_k^*e_{i,i}\Big\|_2^2 = \|v_kqv_k^*\|_2^2 = \|q\|_2^2$$

so that

$$\begin{aligned}\Big\|v_kqv_k^* &- \sum_i e_{i,i}v_kqv_k^*e_{i,i}\Big\|_2^2\\ &= \Big(\|q\|_2 + \Big\|\sum_i e_{i,i}v_kqv_k^*e_{i,i}\Big\|_2\Big)\Big(\|q\|_2 - \Big\|\sum_i e_{i,i}v_kqv_k^*e_{i,i}\Big\|_2\Big)\\ &< (2\|q\|_2)(\delta^{1/2}\|q\|_2),\end{aligned}$$

thanks to (11.17). It follows that

$$\|q - v_kqv_k^*\|_2 < (\delta^{1/2} + 2^{1/2}\delta^{1/4})\|q\|_2. \tag{11.18}$$

Chosing δ sufficiently small, (11.16) and (11.18) give our wanted inequalities (11.1). $\square$

11.1.2. The local approximation property implies the AFD property.

Theorem 11.1.17. *Let (M,τ) be a tracial von Neumann algebra which has the local approximation property. Then M is* AFD.

Proof. We fix $\varepsilon > 0$ and a finite subset F of the unit ball of M. We set $\delta = 3^{-1/2}\varepsilon$. Recall that every element of M is a linear combination of at most four unitary elements. So, since M has the local approximation property, there exists a finite matrix algebra Q with unit q such that

$$\|[q,x]\|_2 < \delta\|q\|_2 \quad \text{and} \quad d_2(qxq, Q) < \delta\|q\|_2 \tag{11.19}$$

for every $x \in F$.

We denote by E_Q the trace preserving conditional expectation from qMq onto Q. For $x \in F$, we deduce from (11.19) and from Pythagoras' theorem, first that

$$\|qxq - E_Q(qxq)\|_2 < \delta\|q\|_2,$$

and next, since $qx(1-q) + (1-q)xq$ is orthogonal to qMq, that

$$\begin{aligned}\|(x - (1-q)x(1-q)) &- E_Q(qxq)\|_2^2\\ &= \|qxq - E_Q(qxq)\|_2^2 + \|qx(1-q) + (1-q)xq\|_2^2\\ &= \|qxq - E_Q(qxq)\|_2^2 + \|q[q,x] + [x,q]q\|_2^2\\ &\le \|qxq - E_Q(qxq)\|_2^2 + 2\|[x,q]\|_2^2 \le \varepsilon^2\|q\|_2^2.\end{aligned}$$

Let us consider the set $\mathcal{S}$ of all families $\{Q_i\}_{i\in I}$ of matrix subalgebras Q_i whose units q_i are mutually orthogonal and are such that

$$\forall x \in F, \quad \big\|x - (1-q)x(1-q) - E_{\oplus_{i\in I}Q_i}(qxq)\big\|_2^2 \le \varepsilon^2\|q\|_2^2, \tag{11.20}$$

where $q = \sum_{i\in I} q_i$.

By the first part of the proof, $\mathcal{S}$ is not empty. Using Remark 11.1.10, a "passing to the limit argument" easily implies that $\mathcal{S}$ is inductively ordered by inclusion. We take a maximal element $\{Q_i\}_{i\in I}$ in $\mathcal{S}$ with corresponding set $\{q_i\}_{i\in I}$ of units.

We want to show that $q = \sum_{i\in I} q_i = 1$. Suppose, by contradiction, that $q \neq 1$. We apply the first part of the proof to $(1-q)M(1-q)$, which has the local approximation property, and to the set $\{(1-q)x(1-q) : x \in F\}$. There exists a non-zero finite dimensional algebra $P \subset (1-q)M(1-q)$, with unit p such that

$$\|(1-q)x(1-q) - (1-q-p)x(1-q-p) - E_P(pxp)\|_2^2 < \varepsilon^2\|p\|_2^2,$$

for all $x \in F$. Adding this inequality to (11.20) we get

$$\begin{aligned}\|x - (1-q-p)x(1-q-p) - &E_{(\oplus_i Q_i)\oplus P}((q+p)x(q+p))\|_2^2 \\ &\leq \varepsilon^2\|q+p\|_2^2\end{aligned}$$

after having observed that

$$E_{(\oplus_i Q_i)\oplus P}((q+p)v(q+p)) = E_{\oplus_i Q_i}(qxq) + E_P(pxp)$$

and using again Pythagoras' theorem.

This contradicts the maximality of $\{Q_i\}_{i\in I}$, and so we have $q = 1$.

Hence F is well approximated by elements of $\bigoplus_{i\in I} Q_i$, but $\bigoplus_{i\in I} Q_i$ is not finite dimensional when I is infinite. In this case, given $\varepsilon_1 < \varepsilon^2$, we choose a finite subset I_1 of I such that $\tau(1-\sum_{i\in I_1} q_i) < \varepsilon_1$ and $\|x - E_{\oplus_{i\in I_1} Q_i}(x))\|_2 \leq 2\varepsilon$ for $x \in F$. We set $f = 1 - \sum_{i\in I_1} q_i$ and

$$N = \mathbb{C}f \oplus \bigoplus_{i\in I_1} Q_i.$$

For $x \in F$, we have

$$\begin{aligned}\|x - E_N(x)\|_2 &\leq \|x - \sum_{i\in I_1} E_{Q_i}(q_i x q_i)\|_2 + \left\|\frac{\tau(fxf)}{\tau(f)} f\right\|_2 \\ &< 2\varepsilon + \sqrt{\varepsilon_1} < 3\varepsilon.\end{aligned}$$

It follows that $F \subset^{3\varepsilon} N$, where N is a finite dimensional unital subalgebra of M and this concludes the proof. $\square$

11.2. Uniqueness of separable AFD II_1 factors

When M is a II_1 factor, the following lemma shows that, in the definition of an AFD factor, we may assume that Q is a matrix algebra of type I_{2^n} for some n.

LEMMA 11.2.1. *Let M be an* AFD II_1 *factor. Given $\varepsilon > 0$ and a finite subset $F \subset \mathcal{U}(M)$, there exists, for some n, a type I_{2^n} subalgebra N of M, with $1_N = 1_M$ such that $F \subset^{\varepsilon} N$.*

PROOF. By definition, there exists a finite dimensional subalgebra Q with $1_Q = 1_M$ and $F \subset^{\varepsilon} Q$. We have $Q = \sum_{i=1}^{m} Q_i$ where each Q_i is isomorphic to some $k_i \times k_i$ matrix algebra (see Exercise 2.2). We denote by q_i the unit of Q_i and we choose mutually orthogonal minimal projections $e_i^1, \dots, e_i^{k_i}$ in Q_i, so that their sum is q_i. We fix an integer n. Dyadic approximations of the numbers $\tau(e_i^1)$, $i = 1, \dots, m$, allow to build mutually orthogonal projections f_i^j, $1 \le i \le m, 1 \le j \le n_i$, with $f_i^j \le e_i^1$ and $\tau(f_i^j) = 2^{-n}$ for all i, j, and which are such that

$$\tau(e_i^1 - \sum_{j=1}^{n_i} f_i^j) < 2^{-n}.$$

We select partial isometries $w_i^l \in Q_i$, $1 \le l \le k_i$, $1 \le i \le m$, such that $(w_i^l)^* w_i^l = e_i^1$ and $w_i^l (w_i^l)^* = e_i^l$. We consider the projections $w_i^l f_i^j (w_i^l)^*$ for all possible i, j, l and get in such a way a family of mutually orthogonal projections, each of trace 2^{-n}. We complete this family by appropriate orthogonal projections, of trace 2^{-n}, in such a way that the sum of all the projections of trace 2^{-n} we have built is 1. These projections are minimal projections of a subalgebra N of type I_{2^n}. In order to define N we have the freedom of the choice of the partial isometries relating these minimal projections. We do so that this N contains the partial isometries $w_i^l f_i^j$ for all i, j, l. We set $v_i^l = \sum_{j=1}^{n_i} w_i^l f_i^j$, $1 \le l \le k_i$, $1 \le i \le m$. Then $w_i^l - v_i^l$ is a partial isometry with $e_i^1 - \sum_{j=1}^{n_i} f_i^j$ as right support. It follows that

$$\left\| w_i^l - v_i^l \right\|_2^2 = \tau(e_i^1 - \sum_{j=1}^{n_i} f_i^j) < 2^{-n},$$

and so $d_2(w_i^l (w_j^l)^*, N) < 2^{1-n/2}$.

Since the $w_i^l (w_j^l)^*$ generate linearly Q, it is a routine exercise to see that if n is large enough, we have $F \subset^{2\varepsilon} N$. $\square$

THEOREM 11.2.2. *Let M be a separable* II_1 *factor. The following conditions are equivalent:*

(1) *M is amenable;*
(2) *M is* AFD*;*
(3) *there exists an increasing sequence (Q_n) of finite dimensional $*$-subalgebras of M, with the same unit as M, such that $(\cup Q_n)'' = M$;*
(4) *there exists in M an increasing sequence (Q_n) of matrix algebras M_n of type $\mathrm{I}_{2^{k_n}}$, with the same unit as M, such that $(\cup Q_n)'' = M$;*
(5) *M is isomorphic to the hyperfinite* II_1 *factor R.*

PROOF. (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) is immediate and (5) $\Rightarrow$ (1) has been proved in the theorem 10.2.4. Theorem 11.1.3 states that (1) $\Rightarrow$ (2). Let us show (2) $\Rightarrow$ (4). Let $\{x_n : n \ge 1\}$ be a countable, s.o. dense subset of the unit ball of M. For every n, Lemma 11.2.1 provides a $2^{k_n} \times 2^{k_n}$ matrix algebra Q_n with $1_M \in Q_n$ such that

$$\|x_i - E_{Q_n}(x_i)\|_2 < 2^{-n}, \quad \text{for} \quad 1 \leq i \leq n.$$

The main difficulty is that the matrix algebras Q_n obtained in such a way are not in increasing order. Therefore, we will construct them inductively.

Assume that we have constructed $Q_1 \subset \cdots \subset Q_n$ such that

$$\|x_i - E_{Q_k}(x_i)\|_2 < 2^{-k}, \quad \text{for} \quad 1 \leq i \leq k, \ k \leq n,$$

each Q_j being a matrix algebra of type $I_{2^{k_j}}$. We will construct Q_{n+1} of type $I_{2^{k_{n+1}}}$ with $Q_n \subset Q_{n+1}$ and

$$\|x_i - E_{Q_{n+1}}(x_i)\|_2 < 2^{-(n+1)}, \quad \text{for} \quad 1 \leq i \leq n+1. \tag{11.21}$$

Then, we will have

$$M = \overline{\cup_{n=1}^{\infty} Q_n}^{s.o}.$$

We consider a matrix units $(e_{i,j})$ of Q_n and we set $e = e_{1,1}$. Since eMe is amenable, hence AFD, given $\varepsilon > 0$, we can find a $2^k \times 2^k$ matrix algebra $N \subset eMe$ with $e \in N$, and elements $x_{i,j,k}$ in N such that

$$\|e_{1,i} x_j e_{k,1} - x_{i,j,k}\|_2 < \varepsilon\, 2^{-(n+1)}, \tag{11.22}$$

for $1 \leq j \leq n+1$ and $1 \leq i, k \leq 2^{k_n}$.

Let $(e^1_{i,j})_{1 \leq i,j \leq 2^k}$ be a matrix units of N. Then

$$\Big\{ e_{i,1} e^1_{k,l} e_{1,j} : \quad 1 \leq i, j \leq 2^{k_n}, \ 1 \leq k, l \leq 2^k \Big\}$$

is a matrix units which generates a $2^{k_{n+1}} \times 2^{k_{n+1}}$ matrix algebra Q_{n+1} with $k_{n+1} = k_n + k$. Obviously, Q_n is diagonally embedded into Q_{n+1}.

It remains to check Condition (11.21). Setting

$$y_j = \sum_{i,k=1}^{2^{k_n}} e_{i,1} x_{i,j,k} e_{1,k} \in Q_{n+1},$$

we have

$$\|x_j - y_j\|_2^2 = \sum_{i,k=1}^{2^{k_n}} \|e_{i,i} x_j e_{k,k} - e_{i,1} x_{i,j,k} e_{1,k}\|_2^2.$$

Since

$$\|e_{i,i} x_j e_{k,k} - e_{i,1} x_{i,j,k} e_{1,k}\|_2^2 \leq \|e_{1,i} x_j e_{k,1} - x_{i,j,k}\|_2^2,$$

we get, thanks to the inequalities (11.22),

$$\|x_j - y_j\|_2^2 < (2^{k_n} \varepsilon 2^{-(n+1)})^2.$$

So, if we choose $\varepsilon = 2^{-k_n}$, we obtain

$$\big\|x_j - E_{Q_{n+1}}(x_j)\big\|_2 < \|x_j - y_j\|_2 \leq 2^{-(n+1)} \quad \text{for} \quad 1 \leq i \leq n+1.$$

This completes the proof (2) $\Rightarrow$ (4).

To finish the proof of the theorem, let us now show that (4) $\Rightarrow$ (5). Note that M and R are of the form $M = (\cup P_n)''$ and $R = (\cup Q_n)''$ where

(P_n) and (Q_n) are increasing sequences with $P_n \simeq M_{2^n}(\mathbb{C}) \simeq Q_n$. Then there exists an isometric $*$-isomorphism Φ from the $*$-algebra $\mathcal{M} = \cup_n P_n$ onto $\mathcal{R} = \cup_n Q_n$ which preserves the traces: $\tau_R \circ \alpha = \tau_M|_{\mathcal{M}}$. Since $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{R}}$ are dense in $L^2(M)$ and $L^2(R)$ respectively, there is a unique unitary operator $U : L^2(M) \to L^2(R)$ such that $U\hat{x} = \widehat{\alpha(x)}$ for every $x \in \mathcal{M}$. Then $x \mapsto UxU^*$ defines an isomorphism from M onto R which extends α. $\square$

Remark 11.2.3. Every von Neumann subalgebra of an amenable tracial von Neumann algebra is amenable. Therefore, it follows from the theorem 11.2.2 that every infinite dimensional subfactor of R is isomorphic to R. Hence R is the smallest II_1 factor in the sense that every II_1 factor contains a subfactor isomorphic to R (see Proposition 4.2.6) and that every II_1 subfactor of R is isomorphic to R.

Similarly, by Theorem 11.2.2 we get that eRe is isomorphic to R for any non-zero projection of R. Hence, $\mathbb{R}_+^*$ is the fundamental group of R.

Finally, we observe that R appears in many ways, among them as:

- infinite tensor products of matrix algebras;
- $L(G)$ for every ICC amenable countable group G;
- $L^\infty(X) \rtimes G$ for every free ergodic p.m.p. action of an amenable countable group G (see Exercise 10.6).

We say that a separable factor M is *approximately finite dimensional* (AFD) if there exists an increasing sequence (Q_n) of finite dimensional $*$-subalgebras of M, with the same unit as M, such that $(\cup_n Q_n)'' = M$. Such an algebra is amenable (this can be shown as in Theorem 10.2.4, using a standard form of M).

Corollary 11.2.4. *There is a unique separable* AFD *type* II_∞ *factor, up to isomorphism.*

Proof. We observe first that $R\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N}))$ is a separable AFD type II_∞ factor. Now let M be such a factor, which is therefore amenable. By Exercise 8.1, we know that M is isomorphic to some $N\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N}))$ where N is a II_1 factor. Let p be any finite rank projection in $\mathcal{B}(\ell^2(\mathbb{N}))$. Then $(1\otimes p)(N\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N})))(1\otimes p) = N\overline{\otimes}\mathcal{B}(p\ell^2(\mathbb{N}))$ is amenable, and since there exists a norm-one projection from $N\overline{\otimes}\mathcal{B}(p\ell^2(\mathbb{N}))$ onto $N\otimes 1$ (for instance the trace preserving one), we get from Theorem 11.2.2 that N is isomorphic to R. $\square$

Exercise

Exercise 11.1. Let M be a finite von Neumann algebra and write $M = \sum_{i\in I}^{\oplus}(M_i, \tau_i)$ as a direct sum of tracial von Neumann algebras (see Exercise 6.2).

(i) Show that M is amenable if and only if each M_i is amenable.
(ii) Show that M is AFD if and only if each M_i is AFD.

Notes

The notion of approximately finite dimensional (AFD) II_1 factor was introduced by Murray and von Neumann[3] in [**MvN43**], therefore in the early '40s. The equivalence between conditions (2) to (5) in Theorem 11.2.2 was proved in this paper, where it is also shown that the crossed product associated with a free ergodic p.m.p. action of a locally finite group is AFD, as well as the group von Neumann algebra of any ICC locally finite group. In [**Dye63**], Dye established that free ergodic p.m.p. actions of groups with polynomial growth also give rise to this AFD factor. For abelian groups this had been stated by Murray and von Neumann.

Some 30 years after Murray and von Neumann breakthrough, Connes obtained ([**Con76**]) the other major achievement developped in this chapter by showing the remarkable fact that an injective (*i.e.*, amenable) von Neumann algebra is AFD (the converse being immediate). The simplified proof that we give in this chapter is borrowed from [**Pop86b**].

[3] They rather used the terminology of "approximately finite factor".

CHAPTER 12

Cartan subalgebras

A central problem in the theory of von Neumann algebras is the classification, up to isomorphism, of the group measure space von Neumann algebras $L^\infty(X) \rtimes G$ for free p.m.p. actions of countable groups, in terms of G and of the group action. The von Neumann subalgebra $L^\infty(X)$ plays a crucial role in the study of $L^\infty(X) \rtimes G$. It is a *Cartan subalgebra* (Definition 12.1.11), a notion that has attracted increasing attention over the years.

In the first section we study the abstract properties of Cartan inclusions $A \subset M$, where M is a tracial von Neumann algebra.

In the second section, we address the classification problem of group measure space von Neumann algebras, and more generally of von Neumann algebras of countable p.m.p. equivalence relations. We have already observed in Section 1.5.3 that the isomorphism class of $L^\infty(X) \rtimes G$ only depends on the equivalence relation given by the orbits of $G \curvearrowright X$. However we may have $L^\infty(X_1) \rtimes G_1 \simeq L^\infty(X_2) \rtimes G_2$ without the corresponding equivalence relations being isomorphic (see Section 17.3). One of the main result of Section 12.2 is that two free p.m.p. actions $G_1 \curvearrowright (X_1, \mu_1)$ and $G_2 \curvearrowright (X_2, \mu_2)$ are orbit equivalent if and only if the corresponding tracial Cartan inclusions $L^\infty(X_1) \subset L^\infty(X_1) \rtimes G_1$ and $L^\infty(X_2) \subset L^\infty(X_2) \rtimes G_2$ are isomorphic (Corollary 12.2.7).

In Section 12.3 we highlight an alternative to the notion of equivalence relation for the study of Cartan subalgebras, namely the notion of *full group*: there is a functorial bijective correspondence between the classes of tracial Cartan inclusions (up to isomorphism) and the classes of full groups equipped with 2-cocycles (Theorem 12.3.8). In particular, for $G \curvearrowright (X, \mu)$ the full group $[G]$ generated by G encodes all the information on the orbit equivalence class of the action (Corollary 12.3.10).

Finally, in the last section we use the background on Cartan subalgebras developped in the first section and techniques already applied in proving that amenable tracial von Neumann algebras are AFD (previous chapter) to give an operator algebraic proof of the fact that every amenable countable p.m.p. equivalence relation (hence every free p.m.p. action of any amenable countable group) is hyperfinite.

12.1. Normalizers and Cartan subalgebras

12.1.1. Preliminaries on normalizers of an abelian subalgebra. Given a von Neumann algebra M and a von Neumann subalgebra A, the *normalizer of A in M* is the group $\mathcal{N}_M(A)$ of unitary operators $u \in M$ such that $uAu^* = A$. Note that $\mathcal{N}_M(A)$ contains the unitary group $\mathcal{U}(A)$ as a normal subgroup.

When A is abelian, it is useful to introduce the notion of normalizing pseudo-group, more general than the group of normalizers.

Definition 12.1.1. Let M be a von Neumann algebra and A an abelian von Neumann subalgebra. The *normalizing pseudo-group of A in M* (or *quasi-normalizer*) is the set $\mathcal{GN}_M(A)$ of partial isometries $v \in M$ such that $vAv^* \subset A$ and $v^*Av \subset A$.

Note that if $v \in \mathcal{GN}_M(A)$, then vv^* and v^*v are two projections in A and $vAv^* = Avv^*$, $v^*Av = Av^*v$. The map $x \mapsto vxv^*$ is an isomorphism from Av^*v onto Avv^*. The set $\mathcal{GN}_M(A)$ is stable under product and adjoint. So, the linear span of $\mathcal{GN}_M(A)$ is an involutive subalgebra of M. The link between $\mathcal{N}_M(A)$ and $\mathcal{GN}_M(A)$ is described in the next lemma.

Lemma 12.1.2. *Let (M,τ) be a tracial von Neumann algebra and A an abelian von Neumann subalgebra. A partial isometry v belongs to $\mathcal{GN}_M(A)$ if and only if it is of the form uq with $u \in \mathcal{N}_M(A)$ and $q = v^*v \in A$.*

Proof. We may assume that $\mathcal{GN}_M(A)'' = M$. Given $v \in \mathcal{GN}_M(A)$, let $(v_i)_{i\in I}$ be a maximal family of elements in $\mathcal{GN}_M(A)$ such that $\{v_iv_i^* : i \in I\}$ and $\{v_i^*v_i : i \in I\}$ are both consisting of mutually orthogonal projections, with $v_{i_0} = v$ for some $i_0 \in I$. We set $e = 1 - \sum_i v_iv_i^*$ and $f = 1 - \sum_i v_i^*v_i$. Then e and f are two projections in A which are equivalent in the finite von Neumann algebra M since $1-e$ and $1-f$ are equivalent. We claim that $e = f = 0$. Otherwise, let w be a partial isometry with $ww^* = e$ and $w^*w = f$. We choose $\varepsilon > 0$ such that $\varepsilon < \|w\|_2$. Let $\lambda_1, \dots, \lambda_n$ in $\mathbb{C}$ and $u_1, \dots, u_n$ in $\mathcal{GN}_M(A)$ such that

$$\Big\|w - \sum_{j=1}^n \lambda_j u_j\Big\|_2 \le \varepsilon.$$

We have $\Big\|w - \sum_{j=1}^n \lambda_j e u_j f\Big\|_2 \le \varepsilon$, so that at least for one j we have $eu_jf \neq 0$. Then $w' = eu_jf$ belongs to $\mathcal{GN}_M(A)$ and is such that $ew' = w' = w'f$. But this contradicts the maximality of $(v_i)_{i\in I}$.

To conclude, we set $u = \sum_i v_i$. $\square$

The von Neumann generated by $\mathcal{N}_M(A)$ is the w.o. (or s.o.) closure of the linear span of $\mathcal{N}_M(A)$. It is also the closure of the linear span of $\mathcal{GN}_M(A)$.

12.1.2. Case of a maximal abelian von Neumann subalgebra. We begin by a property of maximal abelian $*$-subalgebras.

Proposition 12.1.3. *Let A be an abelian von Neumann subalgebra of a tracial von Neumann algebra (M,τ) and let (A_n) be an increasing sequence* *of von Neuman subalgebras of A such that $\cup_n A_n$ is w.o. dense in A. Then A is maximal abelian if and only if, for every $x \in M$,*

$$\lim_n \left\| E_{A_n' \cap M}(x) - E_{A_n}(x) \right\|_2 = 0.$$

Moreover, in this case we have, for $x \in M$,

$$\lim_n \left\| E_{A_n' \cap M}(x) - E_A(x) \right\|_2 = 0.$$

Proof. Observe first that $A_n' \cap M$ is a decreasing sequence of von Neumann algebras whose intersection is $A' \cap M$. Then by Lemma 11.1.9 and Remark 11.1.10 we have, for every $x \in M$,

$$\lim_n \left\| E_{A_n' \cap M}(x) - E_{A' \cap M}(x) \right\|_2 = 0 \text{ and } \lim_n \left\| E_{A_n}(x) - E_A(x) \right\|_2 = 0.$$

Then, the statement follows immediately since A is maximal abelian if and only if $A = A' \cap M$. □

Corollary 12.1.4. *Let A be a maximal abelian von Neumann subalgebra of a separable tracial von Neumann algebra (M, τ). Then, there is a sequence $(e_k^n)_{1 \leq k \leq m_n}$, $n \geq 1$, of partitions of the unit in A such that for every $x \in M$,*

$$\lim_n \left\| \sum_{k=1}^{m_n} e_k^n x e_k^n - E_A(x) \right\|_2 = 0$$

Proof. Let (A_n) be an increasing sequence of finite dimensional $*$-subalgebras of A whose union is w.o. dense in A (see Lemma 11.1.8). By the previous proposition we have

$$\lim_n \left\| E_{A_n' \cap M}(x) - E_A(x) \right\|_2 = 0.$$

We denote by e_k^n, $k = 1, \dots, m_n$, the minimal projections of A_n. It suffices to observe that $E_{A_n' \cap M}(x) = \sum_{k=1}^{m_n} e_k^n x e_k^n$. □

We now turn to the description of some useful properties of $\mathcal{GN}_M(A)$. *In the rest of this section (M, τ) will be a tracial von Neumann algebra and A will be a maximal abelian $*$-subalgebra (m.a.s.a.) of M. For simplicity of notation we assume the M is separable.* The reader will easily check that this assumption is not really needed.

Proposition 12.1.5. *Let $v \in \mathcal{GN}_M(A)$. Then there exists a non-zero projection $f \in A$ such that either $fvf = 0$ or fvf is a unitary element in Af.*

Proof. Assume first that $E_A(v) \neq 0$ and write $E_A(v) = ve$ as in Lemma 12.1.6. Then $eve = ve$ is a unitary in Ae and we take $f = e$.

Assume now that $E_A(v) = 0$. By the previous corollary, there exists a partition $(e_k)_{1 \leq k \leq m}$ of the unit in A such that

$$\sum_{k=1}^{m} \|e_k v e_k\|_2^2 = \left\| \sum_{k=1}^{m} e_k v e_k \right\|_2^2 \leq 1/2.$$

It follows that there is at least an index k such that $e_k v e_k$ is not a unitary in Ae_k. Then, $f = e_k - e_k v e_k v^* e_k$ is a non-zero projection in A such that $fvf = 0$. □

Recall that we denote by e_A the orthogonal projection from $L^2(M)$ onto $L^2(A)$. More generally, if $v \in \mathcal{GN}_M(A)$ the space $vL^2(A)$ is closed and we will denote by e_{vA} the corresponding orthogonal projection. We have $e_{vA} = ve_Av^* \in \langle M, e_A\rangle \cap A'$ and $vL^2(A) = L^2(A)v$.

Recall also that $\widehat{\tau}$ denotes the canonical normal faithful semi-finite trace on $\langle M, e_A\rangle$ and that $\widehat{\tau}(e_{vA}) = \tau(vv^*)$ (see Section 9.4.1).

The restriction of e_A to M is the conditional expectation E_A from M onto A (see Remark 9.1.3). Similarly, the restriction of e_{vA} to M is the map $x \mapsto vE_A(v^*x)$ since $x - vE_A(v^*x)$ is orthogonal to vA. We set $E_{vA}(x) = vE_A(v^*x)$. We establish below some features of these maps E_{vA} and e_{vA}.

Lemma 12.1.6. *Let $v_0, v \in \mathcal{GN}_M(A)$. There exists a unique projection $e \in A$ such that $E_{v_0A}(v) = ve$ with $e \leq v^*v$. In particular $E_{v_0A}(v)$ is a partial isometry.*

Proof. We consider first the case where $v_0 = 1$. Let $E_A(v) = wa$ be the polar decomposition of $E_A(v)$ and set $e = w^*w$. Then we have $e \leq v^*v$ since v^*v is greater that the right support of $E_A(v)$. For $b \in A$ we have $E_A(v)b = (vbv^*)E_A(v)$, that is $wab = (vbv^*)wa$. It follows that $wb = (vbv^*)w$ and therefore $v^*wb = bv^*w$. Thus v^*w is a partial isometry which belongs to A since A is maximal abelian. Moreover, we have

$$w^*v = E_A(w^*v) = w^*E_A(v) = w^*wa = a.$$

We see that a is a positive partial isometry, hence a projection. It follows that $a = e$ and $w = ve$.

Let us consider now the general case. By the first part, we have

$$E_{v_0A}(v) = v_0E_A(v_0^*v) = v_0(v_0^*v)f$$

where $f \in A$ is a projection such that $f \leq v^*v_0v_0^*v$. Thus

$$E_{v_0A}(v) = v(v^*v_0v_0^*v)f$$

and we set $e = (v^*v_0v_0^*v)f = f$.

The uniqueness of e is obvious. □

Lemma 12.1.7. *Let $v_1, v_2, v \in \mathcal{GN}_M(A)$ and suppose that v_1A and v_2A are orthogonal. Then the left (respectively right) supports of $E_{v_1A}(v)$ and $E_{v_2A}(v)$ are orthogonal.*

Proof. We have $E_{v_1A}(v) = ve_1$ and $E_{v_2A}(v) = ve_2$ with $e_1 \leq v^*v$ and $e_2 \leq v^*v$. Since v_1A and v_2A are orthogonal, we have

$$0 = \tau(e_1v^*ve_2) = \tau(e_1e_2),$$

so that $0 = e_1e_2$. □

Let $(v_i)_{i\in I}$ be a family of partial isometries in $\mathcal{GN}_M(A)$ such that the subspaces v_iA are mutually orthogonal. We set

$$\sum_{i\in I} v_iA = \Big\{ x \in M : x = \sum_{i\in I} v_ia_i, a_i \in A, \sum_{i\in I} \|v_ia_i\|_2^2 < +\infty \Big\}.$$

We observe that such an expression, if it exists, is unique as long as we require that $v_i^*v_ia_i = a_i$ for every i since $a_i = E_A(v_i^*x)$ in this case.

LEMMA 12.1.8. *The space* $\sum_{i\in I} v_iA$ *is closed in* M *equipped with the* $\|\cdot\|_2$*-norm. Moreover, for every* $v \in \mathcal{GN}_M(A)$ *there exists a unique projection* $e \in A$ *with* $e \leq v^*v$ *such that* $ve \in \sum_{i\in I} v_iA$ *and* $v - ve$ *is orthogonal to* $\sum_{i\in I} v_iA$*.*

PROOF. The closure of $\sum_{i\in I} v_iA$ in $L^2(M)$ is $\oplus_{i\in I} v_iL^2(A)$. Let $x \in M$. Its orthogonal projection on $\oplus_{i\in I} v_iL^2(A)$ is $\oplus_{i\in I} v_iE_A(v_i^*x)$. It follows that

$$M \cap \big(\oplus_{i\in I} v_iL^2(A)\big) = \sum_{i\in I} v_iA.$$

Let now $v \in \mathcal{GN}_M(A)$. Using the two previous lemmas, we get that $E_{v_iA}(v) = ve_i$ for a unique projection $e_i \in A$ such tha $e_i \leq v^*v$ and that these projections e_i are mutually orthogonal. Therefore $e = \sum_{i\in I} e_i$ is a projection in A. We have $ve = \sum_{i\in I} ve_i \in \sum_{i\in I} v_iA$ and $v - ve$ is orthogonal to $\sum_{i\in I} v_iA$. $\square$

Let $\mathcal{A}$ be the von Neumann algebra generated by $A \cup JAJ$. It is an abelian von Neumann subalgebra of $\langle M, e_A\rangle \cap A'$.

LEMMA 12.1.9. *Let* $v \in \mathcal{GN}_M(A)$*. Then* $e_{vA} \in \mathcal{A}$*. Moreover, we have*

$$Ae_{vA} = \mathcal{A}e_{vA} = \mathcal{A}'e_{vA}.$$

PROOF. Let us first recall a notation: given a von Neumann subalgebra N of $\mathcal{B}(L^2(M))$ and $\xi \in L^2(M)$, then $[N\xi] \in N'$ is the orthogonal projection on $\overline{N\xi}$. Thus $e_A = [A1]$ and for $v \in \mathcal{GN}_M(A)$ we have $[JAJv] = e_{vA}$.

Case $v = 1$. Let $x \in M$. Using Corollary 12.1.4, we see that there is a sequence $(e_k^n)_{1\leq k\leq m_n}$, $n \geq 1$, of partitions of the unit in A such that we have

$$\lim_n \Big\| \sum_{k=1}^{m_n} e_k^n x e_k^n - E_A(x) \Big\|_2 = 0.$$

If we set $P_n = \sum_{k=1}^{m_n} e_k^n J e_k^n J \in \mathcal{A}$, we get $\lim_n \|P_n(x) - e_A(x)\|_2 = 0$ for every $x \in M$. It follows that the sequence $(P_n)_n$ of projections converges to e_A in the s.o. topology and so $e_A \in \mathcal{A} \subset \mathcal{A}'$.

We have $e_A \leq [\mathcal{A}1] \leq [\mathcal{A}'1]$, and finally all three projections are the same since $[\mathcal{A}'1]$ is the smallest projection $p \in \mathcal{A}$ such that $p(1) = 1$.

Note that Ae_A is a maximal abelian subalgebra of $\mathcal{B}(L^2(A))$. Since $\mathcal{A}e_A$ is abelian and contains Ae_A we get that $\mathcal{A}e_A = Ae_A$ is maximal abelian and and therefore $\mathcal{A}e_A = \mathcal{A}'e_A$.

General case $v \in \mathcal{GN}_M(A)$. We set $e = v^*v$ and $f = vv^*$. Since v commutes with JAJ and belongs to the normalizing pseudo-group of A, we see that $x \mapsto vxv^*$ is an isomorphism from $\mathcal{A}e$ onto $\mathcal{A}f$. In particular, we have $e_{vA} = ve_Av^* \in \mathcal{A}$. The rest of the statement also follows by spatial isomorphism. $\square$

Remark 12.1.10. Given $b \in \mathcal{A}$ there exists a unique $a \in A$ such that $avv^* = a$ and $be_{vA} = ae_{vA}$. Moreover we have $\|a\| \leq \|b\|$. To see this inequality we take $a_1 \in A$ with $a_1e_A = (v^*bv)e_A$. We have $\|a_1\| \leq \|b\|$. To conclude, we observe that $a = va_1v^*$.

12.1.3. Cartan subalgebras.

Definition 12.1.11. Let (M, τ) be a tracial von Neumann algebra. A *Cartan subalgebra* is a maximal abelian von Neumann subalgebra A of M such that the normalizer $\mathcal{N}_M(A)$ generates M as a von Neumann algebra. Then we will also say that $A \subset M$ is a *tracial Cartan inclusion*, or simply a Cartan inclusion.

Note that in this case, the linear span of $\mathcal{N}_M(A)$ and the linear span of $\mathcal{GN}_M(A)$ are dense in M in the norm $\|\cdot\|_2$.

Proposition 12.1.12. *If A is a Cartan subalgebra of a tracial von Neumann algebra (M, τ), then $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{B}(L^2(M))$.*

Proof. Since A is a Cartan subalgebra of M the linear span N of $\mathcal{N}_M(A)$ is dense in $L^2(M)$. It follows that $L^2(M) = \bigvee_{u \in \mathcal{N}_M(A)} \overline{uA}^{\|\cdot\|_2}$ and so $1 = \bigvee_{u \in \mathcal{N}_M(A)} e_{uA}$. Since $\mathcal{A}'e_{uA} = \mathcal{A}e_{uA} \in \mathcal{A}$ we see that $\mathcal{A}' = \mathcal{A}$. $\square$

Remark 12.1.13. When A is a Cartan subalgebra of a separable tracial von Neumann algebra M, we get that the A-A-bimodule $L^2(M)$ has a cyclic vector (see Theorem 3.1.4).

We now show the existence of an orthonormal basis of M over A made of elements of $\mathcal{GN}_M(A)$ when A is a Cartan subalgebra.

Proposition 12.1.14. *Let A be a Cartan subalgebra of a tracial von Neuman algebra (M, τ). There is a family $(v_i)_{i \in I}$ of non-zero partial isometries in $\mathcal{GN}_M(A)$ such that the subspaces v_iA, $i \in I$, are mutually orthogonal and $M = \sum_{i \in I} v_iA$.*

Proof. Let $(v_i)_{i \in I}$ be a maximal family of non-zero partial isometries in $\mathcal{GN}_M(A)$ such that the subspaces v_iA, $i \in I$, are mutually orthogonal. Suppose that $\sum_{i \in I} v_iA \neq M$. Since the linear span of $\mathcal{GN}_M(A)$ is $\|\cdot\|_2$-dense in M, there exists $v \in \mathcal{GN}_M(A)$ such that $v \notin \sum_{i \in I} v_iA$. Let e be a projection in A such that $ve \in \sum_{i \in I} v_iA$ and $v - ve$ orthogonal to $\sum_{i \in I} v_iA$. Then $v - ve$ is a non-zero partial isometry in $\mathcal{GN}_M(A)$ and $(v - ve)A$ is orthogonal to v_iA for every $i \in I$. This contradicts the maximality of $(v_i)_{i \in I}$ $\square$

Note that $(v_i)_{i\in I}$ is an orthonormal basis of the right A-module $L^2(M)$, of course countable when M is separable. We will say that $(v_i)_{i\in I}$ is an *orthonormal basis over* A.

COROLLARY 12.1.15. *Let A be a Cartan subalgebra of a separable tracial von Neumann algebra M and let $(v_n)_{n\geq 1}$ be an orthonormal basis over A as in the previous proposition. Then*

$$\mathcal{A} = \Big\{ \sum_{n\geq 1} a_n e_{v_n A} : a_n \in A v_n v_n^*,\ \sup_n \|a_n\| < +\infty \Big\}.$$

Moreover, the above decomposition of every element of $\mathcal{A}$ is unique.

PROOF. This is immediate since $e_{v_n A} \in \mathcal{A}$ for every n, and $\sum_{n\geq 1} e_{v_n A} = 1$. It suffices to use Lemma 12.1.9 and Remark 12.1.10. $\square$

12.1.4. Basic examples of Cartan inclusions. In this section, (X,μ) will be a standard probability measure space and $A = L^\infty(X,\mu)$.

The typical example of a Cartan inclusion is provided by the group measure space construction. Let $M = L^\infty(X,\mu) \rtimes G$ where $G \curvearrowright (X,\mu)$ is a free p.m.p. action. As seen in Chapter 1, $L^\infty(X,\mu)$ is a maximal abelian von Neumann subalgebra of M and M is generated by $L^\infty(X,\mu)$ and the set $\{u_g : g\in G\}$ of canonical unitaries. Observe that these unitaries u_g normalize $L^\infty(X,\mu)$ and thus $L^\infty(X,\mu)$ is a Cartan subalgebra of M. Such Cartan subalgebras are called *group measure space Cartan subalgebras*. Observe that in this case, Proposition 12.1.14 is obvious: every $x\in M$ has a unique expression as $x = \sum_{g\in G} u_g a_g$ with $\sum_{g\in G} \|a_g\|_2^2 < +\infty$.

A more general example is given by p.m.p. equivalence relations. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X,μ). Then $A = L^\infty(X,\mu)$ is a Cartan subalgebra of $M = L(\mathcal{R})$. Indeed, we know that A is a maximal abelian subalgebra of M (Proposition 1.5.5). Moreover, $L(\mathcal{R})$ is generated, as a von Neumann algebra, by the partial isometries u_φ, where the φ's are partial isomorphisms between measurable subsets of X, whose graph is contained into $\mathcal{R}$ (see Section 1.5.2). Recall that u_φ is the partial isometry defined by $(u_\varphi\xi)(x,y) = \xi(\varphi^{-1}(x),y)$ if x is in the domain $D(\varphi^{-1})$ of φ^{-1} and $(u_\varphi\xi)(x,y) = 0$ otherwise. For $f\in A$, we have $u_\varphi f u_\varphi^* = \mathbf{1}_{D(\varphi^{-1})} f\circ\varphi^{-1}$ and therefore u_φ belongs to $\mathcal{GN}_M(A)$.

We keep the notation of Section 1.5.2. Recall that the elements of $L(\mathcal{R})$ may be viewed as elements of $L^2(\mathcal{R},\nu)$ via the identification $T \equiv T\mathbf{1}_\Delta$.

LEMMA 12.1.16. *Every $u \in \mathcal{N}_M(A)$ has a unique expression as fu_φ, where $f\in\mathcal{U}(A)$ and $\varphi \in \mathrm{Aut}\,(X,\mu)$ is such that $x \sim_{\mathcal{R}} \varphi(x)$ for a.e. $x\in X$.*

PROOF. Let φ be the automorphism of (X,μ) induced by the restriction of $\mathrm{Ad}\,(u)$ to A. Viewing u as an element of $L^2(\mathcal{R},\nu)$, we have, for every $a\in A$,

$$u(x,y)a(y) = a(\varphi^{-1}(x))u(x,y), \quad \text{for a.e. } (x,y)\in\mathcal{R}.$$

Since u is a unitary operator, it follows that for almost every $x \in X$ there is $y \sim_{\mathcal{R}} x$ with $u(x,y) \neq 0$ (by Exercise 1.16) and therefore we have $\varphi(x) \sim_{\mathcal{R}} x$.

Moreover, since $\mathrm{Ad}\,(u)_{|A} = \mathrm{Ad}\,(u_\varphi)_{|A}$ we see that $u_\varphi^* u \in \mathcal{U}(A)$. This concludes the proof, the uniqueness of the decomposition being obvious. $\square$

In the setting of equivalence relations, Proposition 12.1.12 has an easy proof, as shown below.

Proposition 12.1.17. *Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X,μ). We set $M = L(\mathcal{R})$ and $A = L^\infty(X,\mu)$. Then the von Neumann algebra $\mathcal{A}$ generated in $\mathcal{B}(L^2(M,\tau))$ by $A \cup JAJ$ is the von Neumann algebra of multiplication operators by the elements of $L^\infty(\mathcal{R},\nu)$ on $L^2(M,\tau) = L^2(\mathcal{R},\nu)$.*

Proof. We remark that for $a \in A$ and $\xi \in L^2(\mathcal{R},\nu)$ we have

$$(a\xi)(x,y) = a(x)\xi(x,y), \quad \text{and} \quad (JaJ\xi)(x,y) = a(y)\xi(x,y),$$

whence the inclusion $\mathcal{A} \subset L^\infty(\mathcal{R},\nu)$. We claim that $\mathcal{A}$ is a maximal abelian von Neumann subalgebra of $\mathcal{B}(L^2(\mathcal{R},\nu))$. To this end, by Theorem 3.1.4, it suffices to show that $\mathcal{A}$ has a cyclic vector. Let ξ_0 be a bounded strictly positive measurable function on $\mathcal{R}$ which belongs to $L^1(\mathcal{R},\nu)$ and therefore to $L^2(\mathcal{R},\nu)$ as well. Let $\eta \in L^2(\mathcal{R},\nu)$ be a function orthogonal to $\mathcal{A}\xi_0$. We may assume that $X = [0,1]$ with its canonical Borel structure. We have

$$\int_{[0,1]\times[0,1]} \eta(x,y) f(x) g(y) \xi_0(x,y) d\nu(x,y) = 0$$

for every continuous functions f, g on $[0,1]$, where we view $\xi_0 d\nu$ as a bounded measure on $[0,1]\times[0,1]$. It follows that $\eta = 0$ *a.e.* on $(\mathcal{R},\nu)$.

This shows our claim and consequently the lemma. $\square$

We leave it to the reader to translate Proposition 12.1.14 in the setting of equivalence relations.

12.2. Isomorphism of Cartan inclusions and orbit equivalence

Let us begin by recalling some definitions.

Definition 12.2.1. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two countable p.m.p. equivalence relations on (X_1,μ_1) and (X_2,μ_2) respectively.

(i) Let $\theta : (X_1,\mu_1) \to (X_2,\mu_2)$ be an isomorphism of probability measure spaces. We say that θ induces an isomorphism from $\mathcal{R}_1$ onto $\mathcal{R}_2$ (or by abuse of langage that θ is an *isomorphism from $\mathcal{R}_1$ onto $\mathcal{R}_2$*) if $(\theta\times\theta)(\mathcal{R}_1) = \mathcal{R}_2$ (up to null sets). Then we say that *$\mathcal{R}_1$ and $\mathcal{R}_2$ are isomorphic.*

(ii) Assume that $\mathcal{R}_1 = \mathcal{R}_{G_1 \curvearrowright X_1}$ and $\mathcal{R}_2 = \mathcal{R}_{G_2 \curvearrowright X_2}$ for p.m.p. actions $G_1 \curvearrowright (X_1,\mu_1)$ and $G_2 \curvearrowright (X_2,\mu_2)$ of countable groups. We say that the actions are *orbit equivalent* if there exists an isomorphism θ from $\mathcal{R}_1$ onto $\mathcal{R}_2$ (*i.e.*, such that for *a.e.* $x \in X_1$, we have $\theta(G_1 x) = G_2\theta(x)$. Then θ is called an *orbit equivalence.*

12.2.1. Isomorphisms. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two countable p.m.p. equivalence relations on (X_1, μ_1) and (X_2, μ_2) respectively. Let $\theta : (X_1, \mu_1) \to (X_2, \mu_2)$ be an isomorphism of probability measure spaces. We denote by θ_* the induced isomorphism $f \mapsto f \circ \theta^{-1}$ from $L^\infty(X_1)$ onto $L^\infty(X_2)$. We have observed in Section 1.5.3 that θ_* extends to an isomorphism from the von Neumann algebra $L(\mathcal{R}_1)$ onto $L(\mathcal{R}_2)$ whenever θ induces an isomorphism from $\mathcal{R}_1$ onto $\mathcal{R}_2$. The converse assertion holds true.

THEOREM 12.2.2. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be as above and let $\theta : (X_1, \mu_1) \to (X_2, \mu_2)$ be an isomorphism of probability measure spaces. The two following conditions are equivalent:*

(i) *θ induces an isomorphism from $\mathcal{R}_1$ onto $\mathcal{R}_2$;*
(ii) *θ_* extends to an isomorphism from the von Neumann algebra $L(\mathcal{R}_1)$ onto $L(\mathcal{R}_2)$.*

PROOF. It remains to prove that (ii) $\Rightarrow$ (i). We put $A_i = L^\infty(X_i)$ and $M_i = L(\mathcal{R}_i)$, $i = 1, 2$. We denote by τ_i the canonical tracial state on M_i. We recall from Section 7.1.3 (c) that we may identify $L^2(M_i, \tau_i)$ with $L^2(\mathcal{R}_i, \nu_i)$, where ν_i is the σ-finite measure on $\mathcal{R}_i$ defined by μ_i. For $\xi \in L^2(\mathcal{R}_i, \nu_i)$, the canonical conjugation operator J_i satisfies $J_i\xi(x, y) = \overline{\xi(y, x)}$.

Let α be an isomorphism from M_1 onto M_2 which extends θ_*. Let $U : L^2(M_1, \tau_1) \to L^2(M_2, \tau_2)$ be the unitary implementation of α: we have $UmU^* = \alpha(m)$ for every $m \in M_1$ and $U \circ J_1 = J_2 \circ U$ (see Remark 7.5.3).

We denote by $\mathcal{A}_i$ the von Neumann subalgebra of $\mathcal{B}(L^2(\mathcal{R}_i, \nu_i))$ generated by $A_i \cup J_i A_i J_i$. For $a \in A_1$, and $\xi \in L^2(M_2, \tau_2) = L^2(\mathcal{R}_2, \nu_2)$ we have

$$(UaU^*\xi)(x, y) = (\alpha(a)\xi)(x, y) = a(\theta^{-1}(x))\xi(x, y),$$

and

$$\begin{aligned}(UJ_1 a J_1 U^*\xi)(x, y) &= (J_2 U a U^* J_2 \xi)(x, y)\\ &= \overline{(UaU^* J_2\xi)(y, x)}\\ &= \overline{a(\theta^{-1}(y)}\xi(x, y) = (J_2\theta_* a J_2\xi)(x, y).\end{aligned}$$

By Proposition 12.1.17, we know that $\mathcal{A}_i = L^\infty(\mathcal{R}_i, \nu_i)$. Then, obviously we have $UL^\infty(\mathcal{R}_1, \nu_1)U^* = L^\infty(\mathcal{R}_2, \nu_2)$. Next, by Remark 3.3.2, we see that there is an isomorphism $\Theta : \mathcal{R}_1 \to \mathcal{R}_2$ with $\Theta_*\nu_1$ equivalent to ν_2 and $UM_F U^* = M_{F\circ\Theta^{-1}}$ for every $F \in L^\infty(\mathcal{R}_1, \nu_1)$, where M_F denotes the multiplication operator by F. Whenever $F(x, y) = a(x)b(y)$ with $a, b \in L^\infty(X_1, \mu_1)$ we have $F(\Theta^{-1}(x, y)) = F(\theta^{-1}(x), \theta^{-1}(y))$. Since $A_1 \cup B_1$ generates $L^\infty(\mathcal{R}_1, \nu_1)$ as a von Neumann algebra, we see that $\Theta^{-1}(x, y) = (\theta^{-1}(x), \theta^{-1}(y))$. Therefore, θ is an isomorphism from $\mathcal{R}_1$ onto $\mathcal{R}_2$. $\square$

DEFINITION 12.2.3. We say that *two tracial Cartan inclusions $A_1 \subset M_1$ and $A_2 \subset M_2$ are isomorphic* if there exists an isomorphism α from M_1 onto M_2 such that $\alpha(A_1) = A_2$ and $\tau_2 \circ \alpha = \tau_1$. Then we say that A_1 and A_2 are *conjugate*. If $M_1 = M_2$ and if α is an inner automorphism, we say that A_1, A_2 are *conjugate by an inner automorphism* or *unitarily conjugate*.

Corollary 12.2.4. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two countable p.m.p. equivalence relations on (X_1, μ_1) and (X_2, μ_2) respectively. The two following conditions are equivalent:*

(i) *the equivalence relations are isomorphic;*
(ii) *the tracial Cartan inclusions $L^\infty(X_i, \mu_i) \subset L(\mathcal{R}_i)$, $i = 1, 2$, are isomorphic.*

Proof. Let $\alpha : L(\mathcal{R}_1) \to L(\mathcal{R}_2)$ be a trace preserving isomorphism sending $L^\infty(X_1)$ onto $L^\infty(X_2)$. Since α is trace preserving its restriction to $L^\infty(X_1)$ is of the form $f \mapsto f \circ \theta^{-1}$ where $\theta : (X_1, \mu_1) \to (X_2, \mu_2)$ is a p.m.p. isomorphism. Then we apply the theorem 12.2.2. $\square$

Corollary 12.2.5. *Let $\mathcal{R}_1$ be a countable ergodic p.m.p. equivalence relation on (X_1, μ_1) such that $L^\infty(X_1)$ is the unique Cartan subalgebra of $L(\mathcal{R}_1)$, up to conjugacy. Then, for any countable ergodic p.m.p. equivalence relation $\mathcal{R}_2$ on some (X_2, μ_2), the von Neumann algebras $L(\mathcal{R}_1)$ and $L(\mathcal{R}_2)$ are isomorphic if and only if the equivalence relations are isomorphic.*

Proof. Let $\alpha : L(\mathcal{R}_2) \simeq L(\mathcal{R}_1)$ be an isomorphism (automatically trace preserving since the von Neumann algebras are factors). Then $\alpha(L^\infty(X_2))$ is a Cartan subalgebra of $L(\mathcal{R}_1)$ and therefore there is an automorphism β of $L(\mathcal{R}_1)$ such that $\beta \circ \alpha(L^\infty(X_2)) = L^\infty(X_1)$. Then the equivalence relations are isomorphic by Corollary 12.2.4. $\square$

We now state these results for group actions.

Corollary 12.2.6. *Let $G_1 \curvearrowright (X_1, \mu_1)$ and $G_2 \curvearrowright (X_2, \mu_2)$ be two free p.m.p. actions of countable groups, and let $\theta : (X_1, \mu_1) \to (X_2, \mu_2)$ be an isomorphism of probability measure spaces. The two following conditions are equivalent:*

(i) *θ is an orbit equivalence between the actions;*
(ii) *θ_* extends to an isomorphism from $L^\infty(X_1) \rtimes G_1$ onto $L^\infty(X_2) \rtimes G_2$.*

Proof. This follows immediately from Theorem 12.2.2, after having identified $L^\infty(X_i) \rtimes G_i$ with $L(\mathcal{R}_{G_i \curvearrowright X_i})$ (see Section 1.5.7). $\square$

Corollary 12.2.7. *Let $G_1 \curvearrowright (X_1, \mu_1)$ and $G_2 \curvearrowright (X_2, \mu_2)$ be two free p.m.p. actions of countable groups. The two following conditions are equivalent:*

(i) *the actions are orbit equivalent;*
(ii) *the tracial Cartan inclusions $L^\infty(X_i) \subset L^\infty(X_i) \rtimes G_i$, $i = 1, 2$, are isomorphic.*

Corollary 12.2.8. *Let $G_1 \curvearrowright (X_1, \mu_1)$ be a free ergodic p.m.p. action such that $L^\infty(X_1) \rtimes G_1$ has $L^\infty(X_1)$ as unique group measure space Cartan subalgebra, up to conjugacy. Then, for any free ergodic p.m.p. action*

$G_2 \curvearrowright (X_2, \mu_2)$ with G_2 countable, the corresponding crossed products are isomorphic if and only if the actions are orbit equivalent.

12.2.2. The automorphism group of an equivalence relation. Let $\mathcal{R}$ be a countable ergodic p.m.p. equivalence relation on (X, μ). We set $M = L(\mathcal{R})$ and $A = L^\infty(X)$ and we denote by $\mathrm{Aut}\,(M, A)$ the subgroup of automorphisms $\alpha \in \mathrm{Aut}\,(M)$ such that $\alpha(A) = A$. We denote by $\mathrm{Aut}\,(\mathcal{R})$ the subgroup of automorphisms $\theta \in \mathrm{Aut}\,(X, \mu)$ such that $(\theta \times \theta)(\mathcal{R}) = \mathcal{R}$. Let $\alpha \in \mathrm{Aut}\,(M, A)$. Its restriction to A induces an element of $\mathrm{Aut}\,(\mathcal{R})$ by Theorem 12.2.2. Moreover, this homomorphism π from $\mathrm{Aut}\,(M, A)$ into $\mathrm{Aut}\,(\mathcal{R})$ is surjective since every $\theta \in \mathrm{Aut}\,(\mathcal{R})$ comes from $\alpha_\theta : L_F \mapsto L_{F \circ (\theta^{-1} \times \theta^{-1})}$, where L_F is the convolution operator by F (defined in Section 1.5.2). On the other hand, π is not injective. Indeed, let c be a 1-*cocycle*, that is, a Borel function from $\mathcal{R}$ into $\mathbb{T}$ such that $c(x, x) = 1$ and $c(x, y) = c(x, z)c(z, y)$, up to null sets. Let U be the multiplication by c on $L^2(\mathcal{R}, \nu)$. Then $U L_F U^* = L_{cF}$ and therefore $L_F \mapsto L_{cF}$ is an element of $\mathrm{Aut}\,(M, A)$ whose restriction to A is trivial, and so an element of the kernel of π.

Note that $\mathrm{Aut}\,(\mathcal{R})$ acts on the abelian group $Z^1(\mathcal{R}, \mathbb{T})$ of those 1-cocycles by $\theta.c = c \circ (\theta^{-1} \times \theta^{-1})$. Exercise 12.2 shows that $\ker \pi$ is canonically identified to $Z^1(\mathcal{R}, \mathbb{T})$ and that $\mathrm{Aut}\,(M, A)$ is the semi-direct product $Z^1(\mathcal{R}, \mathbb{T}) \rtimes \mathrm{Aut}\,(\mathcal{R})$.

Let $\mathrm{Inn}\,(\mathcal{R})$ be the normal subgroup of $\mathrm{Aut}\,(\mathcal{R})$ consisting of all $\varphi \in \mathrm{Aut}\,(\mathcal{R})$ such that $\varphi(x) \sim_{\mathcal{R}} x$ for almost every $x \in X$. The *outer automorphism group* of $\mathcal{R}$ is $\mathrm{Out}\,(\mathcal{R}) = \mathrm{Aut}\,(\mathcal{R})/\mathrm{Inn}\,(\mathcal{R})$.

The group $\mathrm{Inn}\,(\mathcal{R})$ plays a key role in the study of $\mathcal{R}$ and of its von Neumann algebra. It is called the *full group* of $\mathcal{R}$ and is more usually denoted by $[\mathcal{R}]$. It follows from Lemma 12.1.16 that the group $\mathcal{N}_M(A)/\mathcal{U}(A)$ is canonically isomorphic to $[\mathcal{R}]$: to $u \in \mathcal{N}_M(A)$ we associate the unique $\varphi \in [\mathcal{R}]$ such that $\mathrm{Ad}\,(u) = \mathrm{Ad}\,(u_\varphi)$ when restricted to A, and then we pass to the quotient.

In the next section we introduce the abstract notion of full group of measure preserving automorphisms and apply it to the general construction of Cartan inclusions.

12.3. Cartan subalgebras and full groups

In this section (X, μ) is still a standard probability measure space and $A = L^\infty(X, \mu)$, equipped with the trace $\tau = \tau_\mu$.

12.3.1. Full groups of probability measure preserving automorphisms.

Definition 12.3.1. Let $G \subset \mathrm{Aut}\,(X, \mu)$ be a group of automorphisms of (X, μ). We say that G is a *full group* if whenever $\theta \in \mathrm{Aut}\,(X, \mu)$ is such that there exist a countable partition (X_n) of X into measurable subsets, and $\theta_n \in G$ such that the $\theta_n(X_n)$ are disjoint with $\theta_{|Xn} = \theta_{n|Xn}$, then $\theta \in G$.

When G is viewed as a subgroup of $\mathrm{Aut}\,(A, \tau)$, the notion of full group can be expressed in this setting: if $\theta \in \mathrm{Aut}\,(A, \tau)$ is such that there exist a countable partition of 1 by projections $p_n \in A$ and automorphisms $\theta_n \in G$ such that the $\theta_n(p_n)$ are mutually orthogonal with $\theta(a) = \sum_n \theta_n(ap_n)$ for $a \in A$, then $\theta \in G$. We will equally use the two formulations.

EXAMPLE 12.3.2. Assume that A is a Cartan subalgebra of a tracial von Neumann algebra M. Then $\{\mathrm{Ad}\,(u) : u \in \mathcal{N}_M(A)\}$ is a full group. It is denoted by $[\mathcal{N}_M(A)]$. Note that this group is canonically isomorphic to $\mathcal{N}_M(A)/\mathcal{U}(A)$.

In the case where $A = L^\infty(X) \subset M = L(\mathcal{R})$, then $[\mathcal{N}_M(A)]$ is canonically isomorphic to $[\mathcal{R}]$.

REMARK 12.3.3. There is also a natural notion of full pseudo-group of partial measure preserving isomorphisms of (X, μ). In the case of a countable p.m.p. equivalence relation $\mathcal{R}$ on (X, μ), the pseudo-group $[[\mathcal{R}]]$ of such isomorphisms whose graph is contained in $\mathcal{R}$ is an example of full pseudo-group.

LEMMA 12.3.4. *Let G be a subgroup of $\mathrm{Aut}\,(A, \tau)$. We denote by $[G]$ the set of automorphisms $\theta \in \mathrm{Aut}\,(A, \tau)$ with the property that there exist a countable partition of 1 by projections $p_n \in A$ and automorphisms $\theta_n \in G$ such that the $\theta_n(p_n)$ are mutually orthogonal with $\theta(a) = \sum_n \theta_n(ap_n)$ for $a \in A$. Then $[G]$ is a full group and it is the smallest full group that contains G.*

PROOF. Immediate. ☐

The group $[G]$ is called the *full group generated by G*.

REMARK 12.3.5. Let $M = L^\infty(X, \mu) \rtimes G$ where $G \curvearrowright (X, \mu)$ is a free p.m.p. action. Then $[G] = [\mathcal{N}_M(A)]$. We use the fact that every element of $[\mathcal{N}_M(A)]$ is of the form $\mathrm{Ad}\,(u_\theta)$ with $\theta \in [\mathcal{R}_{G \curvearrowright X}]$, in order to see that $[\mathcal{N}_M(A)] \subset [G]$.

12.3.2. Equivalence relations, full groups and Cartan inclusions. A natural problem is to understand what is the most general construction of tracial Cartan inclusions. There are two approaches of this problem.

12.3.2.1. *From equivalence relations to Cartan inclusions.* Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X, μ). The construction of $L(\mathcal{R})$ can be generalized by including a twist by a 2-cocycle. For $n \geq 1$, we denote by $\mathcal{R}^{(n)} \subset X^{n+1}$ the Borel space of all $(n+1)$-tuples $(x_0, \dots, x_n)$ of equivalent elements. We equip $\mathcal{R}^{(n)}$ with the σ-finite measure $\nu^{(n)}$ defined by

$$\nu^{(n)}(C) = \int_X |\pi_0^{-1}(x) \cap C| \,\mathrm{d}\mu(x),$$

where C is a Borel subset of $\mathcal{R}^{(n)}$ and $\pi_0(x_0, \dots, x_n) = x_0$.

A 2-*cocycle* for $(\mathcal{R},\mu)$ is a Borel map $c:\mathcal{R}^{(2)}\to\mathbb{T}$ such that

$$c(x_1,x_2,x_3)c(x_0,x_1,x_3)=c(x_0,x_2,x_3)c(x_0,x_1,x_2),\quad a.e. \tag{12.1}$$

We assume that c is normalized, that is, it takes value 1 as soon as two of its three variables are the same. Two 2-cocycles c,c' are *cohomologous* if there exists a Borel map $h:\mathcal{R}\to\mathbb{T}$ such that $h(x,x)=1$ *a.e.* and

$$c'(x,y,z)=h(x,y)h(x,z)^{-1}h(y,z)c(x,y,z),\quad a.e. \tag{12.2}$$

We fix a 2-cocycle c and keep the notations of Section 1.5.2. Then, given $F\in\mathcal{M}_b(\mathcal{R})$, we define a bounded operator L_F^c on $L^2(\mathcal{R},\nu)$ by

$$L_F^c(\xi)(x,y)=\sum_{z\mathcal{R}x}F(x,z)\xi(z,y)c(x,z,y).$$

It is straightforward to check that the von Neumann algebra $L(\mathcal{R},c)$ generated by these operators L_F^c, $F\in\mathcal{M}_b(\mathcal{R})$, retains exactly the same properties as $L(\mathcal{R})$. In particular, $A=L^\infty(X)$ is a Cartan subalgebra of $L(\mathcal{R},c)$. It is also immediately seen that, whenever c,c' are cohomologous as in (12.2), there is a spatial isomorphism from $L(\mathcal{R},c)$ onto $L(\mathcal{R},c')$, induced by the unitary $W:\xi\mapsto h\xi$, which preserves the Cartan subalgebra $L^\infty(X)$. We have $L(\mathcal{R},1)=L(\mathcal{R})$.

We get in this way the most general example of a pair (M,A) formed by a separable tracial von Neumann algebra and a Cartan subalgebra. We will sketch a proof by using the alternative construction via full groups.

12.3.2.2. *From full groups to Cartan inclusions.* Let us first translate the previous construction in terms of the full group $\mathcal{G}=[\mathcal{R}]$. We identify $\varphi\in\mathrm{Aut}\,(X,\mu)$ with $\varphi_*\in\mathrm{Aut}\,(A,\tau)$. For $\phi\in\mathcal{G}$, we denote by $p_\phi\in\mathcal{P}(A)$ the characteristic function of the set $\{x\in X:\phi(x)=x\}$. Given $a\in A$, we set $\phi(a)=a\circ\phi^{-1}$. For $\varphi,\psi\in\mathcal{G}$ and $x\in X$, we set $v_{\varphi,\psi}(x)=c(x,\varphi^{-1}(x),\psi^{-1}\varphi^{-1}(x))$.

Then v is a map from $\mathcal{G}\times\mathcal{G}$ to $\mathcal{U}(A)$ which satisfies the following properties, for all $\varphi,\psi,\phi,\varphi_1,\varphi_2\in\mathcal{G}$:

$$v_{\varphi,\psi}v_{\varphi\psi,\phi}=\varphi(v_{\psi,\phi})v_{\varphi,\psi\phi}, \tag{12.3}$$

$$p_{\varphi_1\varphi_2^{-1}}v_{\varphi_1,\psi}=p_{\varphi_1\varphi_2^{-1}}v_{\varphi_2,\psi},\ \ \psi(p_{\varphi_1\varphi_2^{-1}})v_{\psi,\varphi_1}=\psi(p_{\varphi_1\varphi_2^{-1}})v_{\psi,\varphi_2}, \tag{12.4}$$

$$p_\varphi v_{\varphi,\psi}=p_\varphi,\ \ \varphi(p_\psi)v_{\varphi,\psi}=\varphi(p_\psi),\ \ p_{\varphi\psi}v_{\varphi,\psi}=p_{\varphi\psi}. \tag{12.5}$$

The equations in (12.5) are the translation of the fact that c is normalized.

Moreover, if v, v' are associated with two cocycles c,c' then we easily check that c,c' are cohomologous if and only if there exists $w:\mathcal{G}\to\mathcal{U}(A)$ which satisfies the following conditions, for all $\varphi,\psi,\varphi_1,\varphi_2\in\mathcal{G}$:

$$p_{\varphi_1\varphi_2^{-1}}w_{\varphi_1}=p_{\varphi_1\varphi_2^{-1}}w_{\varphi_2}, \tag{12.6}$$

$$v'_{\varphi,\psi}=w_\varphi\varphi(w_\psi)v_{\varphi,\psi}w^*_{\varphi\psi}. \tag{12.7}$$

Definition 12.3.6. A 2-*cocycle* for a full group $\mathcal{G}$ on (A,τ) is a map $v : \mathcal{G}\times\mathcal{G}\to\mathcal{U}(A)$ which satisfies the conditions (12.3) and (12.4). The cocycle v is *normalized* if it satisfies the conditions (12.5). Two cocycles v, v' are *cohomologous* if there exists $w : \mathcal{G}\to\mathcal{U}(A)$ which satisfies the conditions (12.6) and (12.7), and then we write $v \sim v'$.

Every 2-cocycle is cohomologous to a normalized one.

Starting now from a general full group $\mathcal{G}$ of automorphisms of (A,τ) together with a normalized 2-cocycle v let us briefly describe the construction of the attached Cartan inclusion. We consider the vector space $\mathcal{M}$ of finite sums $\sum_\varphi a_\varphi u_\varphi$ with $a_\varphi \in A$ and $\varphi\in\mathcal{G}$. On $\mathcal{M}$ we define the product and involution by

$$(a_\varphi u_\varphi)(a_\phi u_\phi) = a_\varphi \varphi(a_\phi) v_{\varphi,\phi} u_{\varphi\phi},\ (a_\varphi u_\varphi)^* = \varphi^{-1}(a_\varphi^*) u_{\varphi^{-1}}.$$

We define a linear functional τ on $\mathcal{M}$ by $\tau(a_\varphi u_\varphi) = \tau(a_\varphi p_\varphi)$ and a sesquilinear form by $\langle x,y\rangle_\tau = \tau(x^*y)$. We easily see that this form is positive. We denote by $\mathcal{H}$ the Hilbert space completion of $\mathcal{M}/\mathcal{I}$ where

$$\mathcal{I} = \{x\in\mathcal{M} : \tau(x^*x) = 0\}.$$

Observe that $\mathcal{M}$ is represented on $\mathcal{H}$ by left multiplications.

Finally, we define $M = L(\mathcal{G}, v)$ to be the weak closure of $\mathcal{M}$ in this representation. Then τ defines a normal faithful tracial state on M and $A\subset M$ is a Cartan inclusion.

Remark 12.3.7. Assume that $\mathcal{G}$ is the full group generated by a countable group G of automorphisms of (A,τ). Let $\mathcal{R}$ be the equivalence relation implemented by the orbits of $\mathcal{G}$. Equivalently, we have $x\sim_{\mathcal{R}} y$ if and only if $Gx = Gy$. Then $\mathcal{R}$ is a countable p.m.p. equivalence relation. If c is the 2-cocycle for $\mathcal{R}$ such that $v_{\varphi,\psi}(x) = c(x,\varphi^{-1}(x),\psi^{-1}\varphi^{-1}(x))$ for *a.e.* x then we check that $A\subset L(\mathcal{G},v) = A\subset L(\mathcal{R},c)$.

12.3.2.3. *From Cartan inclusions to full groups.* We now start from a Cartan inclusion $A\subset M$ where M is a tracial von Neumann algebra. We set $\mathcal{G} = [\mathcal{N}_M(A)]$. The main problem is to choose a good section $\varphi\mapsto u_\varphi$ of the quotient map $\mathcal{N}_M(A)\to[\mathcal{N}_M(A)]$. We write $\mathcal{G}$ as a well ordered set $\{\varphi_i : i\in I\}$ with Id_A as a first element. We choose $u_{\mathrm{Id}_A} = 1$. Let J be an initial segment of I in the sense that whenever $j\in J$ then every smaller element is in J. We assume that we have chosen u_{φ_j} implementing φ_j for $j\in J$, in such a way that if $i,j\in J$ and $q\in\mathcal{P}(A)$ are so that φ_i and φ_j agree on Aq, then $u_{\varphi_i}q = u_{\varphi_j}q$. Let k be the first element of $I\setminus J$. There is a maximal projection $p\in\mathcal{P}(A)$ such that the restriction of φ_k to Ap does not agree with any φ_j, $j\in J$, on Aq for any $q\in\mathcal{P}(Ap)$. It follows that $\varphi_k = \bigoplus_{j\in J}\varphi_{j|Aq_j}\oplus\varphi_{k|Ap}$ for some mutually orthogonal projections $q_j\in\mathcal{P}(A)$ with $\sum_{j\in J} q_j = 1-p$. Then we set $u_{\varphi_k} = \sum_{j\in J} u_{\varphi_j} q_j + wp$ where $w\in\mathcal{N}_M(A)$ is any unitary that implements φ_k.

In this manner, we have obtained unitaries u_φ, $\varphi\in\mathcal{G}$ such that $u_{\mathrm{Id}_A} = 1$, $\mathrm{Ad}\,(u_\varphi) = \varphi$, and $u_\varphi q = u_\psi q$ for every projection $q\in\mathcal{P}(A)$ with φ,ψ

agreeing on Aq. We set $v_{\varphi,\psi} = u_\varphi u_\psi u_{\varphi\psi}^*$ for $\varphi, \psi \in \mathcal{G}$. It is straighforward to check that v is a 2-cocycle for the full group $\mathcal{G}$, due to the fact that our choice of the u_φ has been carried out in such a way that $p_{\varphi\psi^{-1}} u_\varphi u_\psi^{-1} = p_{\varphi\psi^{-1}}$. Moreover, the choice of v is unique up to the relation $\sim$.

Thus, we just have defined a functor from the category of tracial Cartan inclusions, with morphisms given by isomorphisms, into the category of pairs $(\mathcal{G} \curvearrowright (A,\tau), v/\!\sim)$ consisting of a full group $\mathcal{G}$ on an abelian von Neumann algebra (A,τ) and the class of a 2-cocycle $v : \mathcal{G}\times\mathcal{G} \to \mathcal{U}(A)$, with morphisms given trace preserving automorphisms of the abelian algebras that carries the full groups (resp. the class of 2-cocycles) onto each other.

The functor is one to one and onto, the inverse having been constructed in the subsection 12.3.2.2. This is summarised in the following theorem.

THEOREM 12.3.8. *To every tracial Cartan inclusion $A \subset M$ is associated the full group $\mathcal{G} = [\mathcal{N}_M(A)]$ and the class of a 2-cocycle $v : \mathcal{G} \times \mathcal{G} \to \mathcal{U}(A)$. Conversely every pair $(\mathcal{G} \curvearrowright (A,\tau), v/\!\sim)$ gives rise to a tracial Cartan inclusion. After passing to quotients, we get a functorial bijective correspondence between the set of isomorphism classes of tracial Cartan inclusions $A \subset M$ and the set of isomorphism classes of pairs $(\mathcal{G} \curvearrowright (A,\tau), v/\!\sim)$.*

We can now complete the theorem 12.2.2 as follows.

THEOREM 12.3.9. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two countable p.m.p. equivalence relations on (X_1,μ_1) and (X_2,μ_2) respectively, and let $\theta : (X_1,\mu_1) \to (X_2,\mu_2)$ be an isomorphism of probability measure spaces. The following conditions are equivalent:*

(i) *θ induces an isomorphism from $\mathcal{R}_1$ onto $\mathcal{R}_2$;*
(ii) *θ_* extends to an isomorphism from the von Neumann algebra $M_1 = L(\mathcal{R}_1)$ onto $M_2 = L(\mathcal{R}_2)$;*
(iii) *$\theta[\mathcal{R}_1]\theta^{-1} = [\mathcal{R}_2]$.*

PROOF. We apply Theorem 12.3.8 with trivial 2-cocycles. $\square$

Similarly, Corollary 12.2.6 is completed as follows.

COROLLARY 12.3.10. *Let $G_1 \curvearrowright (X_1,\mu_1)$ and $G_2 \curvearrowright (X_2,\mu_2)$ be two free p.m.p. actions of countable groups, and let $\theta : (X_1,\mu_1) \to (X_2,\mu_2)$ be an isomorphism of probability measure spaces. The following conditions are equivalent:*

(i) *θ is an orbit equivalence between the actions;*
(ii) *θ_* extends to an isomorphism from $L^\infty(X_1)\rtimes G_1$ onto $L^\infty(X_2)\rtimes G_2$;*
(iii) *$\theta[G_1]\theta^{-1} = [G_2]$.*

12.4. Amenable and AFD Cartan inclusions

Let A be a Cartan subalgebra of (M,τ) and let $\mathcal{A} = (A \cup JAJ)''$. We let $\mathcal{N}_M(A)$ act on $\mathcal{A}$ by $x \mapsto \mathrm{Ad}\,(u)(x) = uxu^*$. Recall that $\mathrm{Ad}\,(u)$ is an automorphism of A and fixes each element of JAJ.

Definition 12.4.1. We say that a tracial Cartan inclusion $A \subset M$ is *amenable* if there exists a state on $\mathcal{A}$ that is invariant under the action of $\mathcal{N}_M(A)$.

If (M,τ) is a tracial amenable von Neumann algebra (for instance the hyperfinite factor R), then every Cartan subalgebra A of M is amenable: it suffices to consider the restriction to $\mathcal{A}$ of an hypertrace.

We also observe that if e is a non-zero projection of A, then $eAe \subset eMe$ is an amenable Cartan inclusion when $A \subset M$ is so.

Remark 12.4.2. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X,μ) and take $M = L(\mathcal{R})$ and $A = L^\infty(X,\mu)$. We have $\mathcal{A} = L^\infty(\mathcal{R},\nu)$. For $f \in L^\infty(\mathcal{R},\nu)$ and $\varphi \in [\mathcal{R}]$ we set $f^\varphi(x,y) = f(\varphi^{-1}(x),y)$. Then $A \subset M$ is amenable if and only if there exists a state Φ on $L^\infty(\mathcal{R},\nu)$ such that $\Phi(f^\varphi) = \Phi(f)$ for every $f \in L^\infty(\mathcal{R},\nu)$ and $\varphi \in [\mathcal{R}]$. In this case we say that *the equivalence relation is amenable.*

Definition 12.4.3. We say that a tracial Cartan inclusion $A \subset M$ is *approximately finite dimensional* (AFD) if for every finite subset F of M and every $\varepsilon > 0$, there exist matrix units $(e^k_{i,j})_{1\leq i,j\leq n_k}$, $1 \leq k \leq m$, with $e^k_{i,j} \in \mathcal{GN}_M(A)$ for every i,j,k, where the m projections $\sum_{1\leq i\leq n_k} e^k_{i,i}$, $1 \leq k \leq m$, form a partition of the unit in A, such that if Q denotes the finite dimensional von Neumann algebra generated by the $e^k_{i,j}$ then $\|x - E_Q(x)\|_2 \leq \varepsilon$ for every $x \in F$.

Of course if the Cartan inclusion $A \subset M$ is AFD, then M is amenable and the inclusion is amenable. The aim of this section is to prove the converse.

Theorem 12.4.4. *Every amenable Cartan inclusion is AFD.*

We follow the main steps of the proof that an amenable finite von Neumann algebra is AFD (see Theorem 11.1.3). The principal step is to establish a local approximation property. For simplicity we assume that M is separable.

Definition 12.4.5. We say that a Cartan inclusion $A \subset M$ has the *local approximation property* if for every $\varepsilon > 0$, every non-zero projection $e \in A$ and every finite subset $F = \{u_1, \dots u_l\} \subset \mathcal{N}_{eMe}(eA)$ there exists a matrix units $(e_{i,j})_{1\leq i,j\leq m}$ in $\mathcal{GN}_{eMe}(eA)$, such that, if we set $q = \sum_i e_{i,i}$, and if N denotes the algebra generated by Aq and the $e_{i,j}$, then $qu_iq \in Q$ and

$$\sum_j \|[q,u_j\|_2^2 \leq \varepsilon \|q\|_2^2.$$

Remark 12.4.6. Let us keep the notation of the previous definition. Then every element x of Q has a unique expression of the form $x = \sum_{i,j} a_{i,j}e_{j,i}$ with $a_{i,j} \in Ae_{j,j}$.

Theorem 12.4.7. *An amenable Cartan inclusion has the local approximation property.*

We begin with the proof of a *Følner type condition.*

LEMMA 12.4.8. *Let $F = \{u_1, \dots u_l\}$ be a finite subset in $\mathcal{N}_M(A)$ and let $\varepsilon > 0$. There exists a projection $p \in \mathcal{A}$ such that $\widehat{\tau}(p) < +\infty$ and*

$$\sum_j \left\|p - u_j p u_j^*\right\|_{2,\hat{\tau}}^2 \leq \varepsilon \|p\|_{2,\hat{\tau}}^2.$$

PROOF. We proceed as in the proof of Lemma 11.1.6, replacing now $\langle M, e_A \rangle$ by $\mathcal{A}$, and using the existence of an invariant state under the action of $\mathcal{N}_M(A)$ on $\mathcal{A}$ instead of the existence of an hypertrace. □

PROOF OF THEOREM 12.4.7. Since $Ae \subset eMe$ is an amenable Cartan inclusion, it suffices to prove that the statement of Definition 12.4.5 holds with $e = 1$.

Let $p \in \mathcal{A}$ with $\widehat{\tau}(p) < +\infty$ such that

$$\sum_j \left\|p - u_j p u_j^*\right\|_{2,\widehat{\tau}}^2 \leq \varepsilon \|p\|_{2,\widehat{\tau}}^2.$$

We have $p = \sum_n a_n e_{v_n A}$ where $a_n \leq v_n v_n^*$ is a projection in A (by Corollary 12.1.15). Since $a_n e_{v_n A} = e_{a_n v_n A}$, if we set $w_n = a_n v_n$ we see that $p = \sum_n e_{w_n A}$ where (w_n) is an orthogonal system in $\mathcal{GN}_M(A)$, that is $E_A(w_i^* w_j) = 0$ if $i \neq j$. Since $\widehat{\tau}(p) < +\infty$, by approximation we may assume that p is a finite sum $\sum_{1 \leq k \leq m} e_{w_k A}$.

Thanks to Proposition 12.1.5, we see that there exists a partition of the unit $(s_n)_{n \geq 1}$ in A such that $s_n(w_i^* u_j w_k) s_n$ is either equal to 0 or is a unitary element in As_n for all i, j, k, n and such that the $s_n(w_i^* w_k) s_n$ have also the same property. Using Pythagoras' theorem, we get

$$\sum_n \Big(\sum_j \left\| (p - u_j p u_j^*) J s_n J \right\|_{2,\widehat{\tau}}^2 \Big) \leq \varepsilon \sum_n \|p J s_n J\|_{2,\widehat{\tau}}^2.$$

It follows that there exists at least an index n such that

$$\sum_j \left\| (p - u_j p u_j^*) J s_n J \right\|_{2,\widehat{\tau}}^2 \leq \varepsilon \|p J s_n J\|_{2,\widehat{\tau}}^2. \tag{12.8}$$

We choose such a n and set $s = s_n$. We observe that $e_A J s J = s e_A s$ and therefore

$$p J s J = \sum_k w_k s e_A s w_k^*.$$

Moreover, for every k we have either $w_k s = 0$ or $s(w_k^* w_k) s = s$. Thus, after a suitable relabeling, we may assume that p in (12.8) is of the form $p = \sum_k w_k e_A w_k^*$ with $w_j^* w_k = \delta_{j,k} s$ and $w_i^* u_j w_k \in As$ for all i, j, k. For this p we have

$$\sum_j \left\|p - u_j p u_j^*\right\|_{2,\widehat{\tau}}^2 \leq \varepsilon \|p\|_{2,\widehat{\tau}}^2.$$

We set $e_{k,l} = w_k w_l^*$ and $q = \sum e_{k,k}$, and we denote by Q the algebra generated by the matrix units $(e_{k,l})$ and Aq. Then, $q u_j q \in Aq$ for every j.

We have

$$\|p\|_{2,\widehat{\tau}}^2 = \widehat{\tau}(\sum w_k e_A w_k^*) = \sum \tau(w_k w_k^*) = \tau(q)$$

and

$$\begin{aligned}\left\|p - u_j p u_j^*\right\|_{2,\widehat{\tau}}^2 &= 2\sum \widehat{\tau}(p) - 2\sum_{i,k} \widehat{\tau}(w_i e_A w_i^* u_j w_k e_A w_k^* u_j^*) \\ &= 2\tau(q) - 2\sum_{i,k} \widehat{\tau}(w_i E_A(w_i^* u_j w_k) e_A w_k^* u_j^*) \\ &= 2\tau(q) - 2\sum_{i,k} \tau(w_i w_i^* u_j w_k w_k^* u_j^*) = \left\|q - u_j q u_j^*\right\|_2^2.\end{aligned}$$

Thus we have

$$\sum_j \left\|q - u_j q u_j^*\right\|_2^2 \leq \varepsilon \|q\|_2^2. \tag{12.9}$$

□

Proof of Theorem 12.4.4. Let $F = \{u_1, \dots u_l\}$ be a finite subset in $\mathcal{N}_M(A)$. We proceed as in the proof of the theorem 11.1.17. First, we observe that the inequality (12.9) implies the following one:

$$\sum_j \|u_j - (1-q)u_j(1-q) - qu_i q\|_2^2 \leq \varepsilon \|q\|_2^2.$$

Then, we consider the set $\mathcal{S}$ of all families $(Q_i)_{i\in I}$ of subalgebras with mutually orthogonal units q_i where each Q_i is generated by Aq_i and a matrix units $(e^i_{j,k})$ in $\mathcal{GN}_M(A)$, such that $q_i u_j q_i \in Q_i$ for all j, which satisfies

$$\sum_j \left\| u_j - (1-q)u_j(1-q) - \sum_i q_i u_j q_i \right\|_2^2 \leq \varepsilon \|q\|_2^2,$$

where $q = \sum_{i\in I} q_i$. This set $\mathcal{S}$ is not empty and it is inductively ordered by inclusion. We take a maximal element $(Q_i)_{i\in I}$ with corresponding set of units $(q_i)_{i\in I}$. We put $q = \sum_{i\in I} q_i$. Using the same arguments as in the proof of Theorem 11.1.17 we see that $q = 1$.

The set I needs not to be finite, but we can find a finite subset I_1 of I such that if $f = 1 - \sum_{i\in I_1} q_i$ and if $N = \mathbb{C}Af \oplus \bigoplus Q_i$ then $\|u_j - E_N(u_j)\|_2$ can be made small enough (see again the proof of Theorem 11.1.17).

Finally, we observe that each N_i (and $N_0 = Af$) is finite dimensional over A with an appropriate basis made of elements of $\mathcal{GN}_M(A)$ (see the remark 12.4.6). □

12.5. Amenable II_1 equivalence relations are hyperfinite

Definition 12.5.1. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on the Lebesgue probability measure space (X,μ).

(i) We say that $\mathcal{R}$ is *hyperfinite* if there exists an increasing sequence $(\mathcal{R}_n)_n$ of subequivalence relations, with finite orbits, such that $\cup_n \mathcal{R}_n = \mathcal{R}$, up to null sets.
(b) We say that $\mathcal{R}$ is of *type* II_1 if it is ergodic.

If $\mathcal{R}$ is a II_1 hyperfinite countable p.m.p. equivalence relation, then $L(\mathcal{R})$ is the hyperfinite II_1 factor, and therefore $\mathcal{R}$ is amenable (as defined in the remark 12.4.2). We will see that the converse is true, as a consequence of the following theorem.

THEOREM 12.5.2. *Let M be a separable II_1 factor.*

(i) *M is isomorphic to the hyperfinite factor R if and only if M has an amenable Cartan subalgebra.*
(ii) *If A_1 and A_2 are two Cartan subalgebras of R there exists an automorphism θ of R such that $\theta(A_1) = A_2$.*

PROOF. (i) Suppose that M has an amenable Cartan subalgebra A. Then the theorem 12.4.4 implies that A is AFD. By suitably modifying the matrix units $(e^k_{i,j})$ of Definition 12.4.3, with arguments similar to those that we used in the proofs of Lemma 11.2.1 and of (2) $\Rightarrow$ (4) in Theorem 11.2.2, we construct an increasing sequence of $2^{k_n} \times 2^{k_n}$ matrix algebras Q_n whose union is s.o. dense in M. It follows that M is isomorphic to R.

Moreover we can do so that each Q_n has a matrix units $(f^n_{i,j})_{1 \leq i,j \leq 2^{k_n}}$ satisfying:

(a) $f^n_{i,j} \in \mathcal{GN}_M(A)$ and $\sum_i f^n_{i,i} = 1$;
(b) every $f^n_{r,s}$ is the sum of some $f^{n+1}_{i,j}$ (property arising from a diagonal embedding of Q_n into Q_{n+1}).

(ii) Let A be a Cartan subalgebra of R and let us keep the above notation. We denote by A_0 the von Neumann subalgebra of R generated by the projections $f^n_{i,i}$ with $1 \leq i \leq 2^{k_n}$, $n \geq 1$. Then A_0 is abelian maximal in $\overline{\cup_n Q_n}^{so} = R$ and since $A_0 \subset A$ we get $A_0 = A$. This shows the uniqueness of A up to automorphism: there is an automorphism θ of R which sends A on $D^{\overline{\otimes}\infty}$ defined in the exercise 12.1. $\square$

COROLLARY 12.5.3. *Every II_1 amenable countable p.m.p. equivalence relation is hyperfinite. Moreover, there is only one II_1 hyperfinite countable p.m.p. equivalence relation, up to isomorphism.*

PROOF. We have already observed that R is a II_1 factor defined by a hyperfinite equivalence relation. Then the assertions of this corollary follow immediately from the previous theorem together with Corollary 12.2.4. $\square$

COROLLARY 12.5.4. *Any two ergodic p.m.p actions of countable amenable groups on Lebesgue probability measure spaces are orbit equivalent.*

PROOF. Let $G_i \curvearrowright (X_i, \mu_i)$, $i = 1, 2$, be two such actions. Then the equivalence relations $\mathcal{R}_{G_i \curvearrowright X_i}$ are amenable (Exercise 12.5) and ergodic, hence isomorphic. $\square$

Exercises

Exercise 12.1. Let (M_n, τ_n) be a sequence of tracial von Neumann algebras, and for every n, let A_n be a Cartan subalgebra of M_n. Show that $\overline{\otimes}_{n\in\mathbb{N}} A_n$ is a Cartan subalgebra of $\overline{\otimes}_{n\in\mathbb{N}} M_n$.

When $M_n = M$ for all n, we set $M^{\overline{\otimes}\infty} = \overline{\otimes}_{n\in\mathbb{N}} M_n$. In particular, if D denotes the diagonal subalgebra of $M_2(\mathbb{C})$, then $D^{\overline{\otimes}\infty}$ is a Cartan subalgebra of $M_2(\mathbb{C})^{\overline{\otimes}\infty}$.

Exercise 12.2. We keep the notation of Section 12.2.2.

(i) Show that $Z^1(\mathcal{R}, \mathbb{T})$ is the kernel of π (Hint: let $\alpha \in \ker \pi$ and let U be its unitary implementation. Show that U is the operator of multiplication by a Borel function $c : \mathcal{R} \to \mathbb{T}$. Show that $c(x, y) = c(y, x)^{-1}$ and that $c(x, z)c(z, y)$ does not depend on z. Conclude that c or $-c$ is a cocycle.)

(ii) For $c \in Z^1(\mathcal{R}, \mathbb{T})$ (resp. $\theta \in \mathrm{Aut}\,(\mathcal{R})$) we denote by α_c (resp. α_θ) the automorphism $L_F \mapsto L_{cF}$ (resp. $L_F \mapsto L_{F\circ(\theta^{-1}\times\theta^{-1})}$) of $L(\mathcal{R})$. Show that $(c, \theta) \mapsto \alpha_c \circ \alpha_\theta$ is an isomorphism from the group $Z^1(\mathcal{R}, \mathbb{T}) \rtimes \mathrm{Aut}\,(\mathcal{R})$ onto the group $\mathrm{Aut}\,(M, A)$.

Exercise 12.3. We keep the notation of the previous exercise. We let $[\mathcal{R}]$ act on $\mathcal{U}(A)$ by $\varphi(f) = f \circ \varphi^{-1}$ where $f : X \to \mathbb{T}$ is in $\mathcal{U}(A)$ and $\varphi \in [\mathcal{R}]$. Show that $(f, \varphi) \mapsto f u_\varphi$ is an isomorphism from the semi-direct product $\mathcal{U}(A) \rtimes [\mathcal{R}]$ onto $\mathcal{N}_M(A)$.

Exercise 12.4. We still set $M = L(\mathcal{R})$ and $A = L^\infty(X)$ as above. We denote by $\mathrm{Out}\,(M, A)$ the image of $\mathrm{Aut}\,(M, A)$ into $\mathrm{Out}\,(M)$. Let $B^1(\mathcal{R}, \mathbb{T})$ be the subgroup of $Z^1(\mathcal{R}, \mathbb{T})$ consisting of the function of the form $(x, y) \mapsto f(x)f(y)^{-1}$ where $f : X \to \mathbb{T}$ is Borel, and set $H^1(\mathcal{R}, \mathbb{T}) = Z^1(\mathcal{R}, \mathbb{T})/B^1(\mathcal{R}, \mathbb{T})$.

(i) Show that the action of $\mathrm{Aut}\,(\mathcal{R})$ on $Z^1(\mathcal{R}, \mathbb{T})$ gives by passing to the quotient an action of $\mathrm{Out}\,(\mathcal{R})$ on $H^1(\mathcal{R}, \mathbb{T})$.

(ii) Show that the isomorphism from $Z^1(\mathcal{R}, \mathbb{T}) \rtimes \mathrm{Aut}\,(\mathcal{R})$ onto $\mathrm{Aut}\,(M, A)$ gives, by passing to the quotient, an isomorphism from $H^1(\mathcal{R}, \mathbb{T}) \rtimes \mathrm{Out}\,(\mathcal{R})$ onto $\mathrm{Out}\,(M, A)$.

Exercise 12.5. Let $G \curvearrowright (X, \mu)$ be a measure preserving action of a countable amenable group. Show that the equivalence relation $\mathcal{R}_{G\curvearrowright X}$ is amenable.

Notes

The interest of maximal abelian subalgebras was emphasized in [**Dix54**], where Cartan subalgebras are called regular maximal abelian subalgebras. In the pioneering paper [**Sin55**], Singer highlighted the importance of Cartan subalgebras in the study of group measure space von Neumann algebras and obtained in particular the useful corollary 12.2.7. This led Dye to develop a comprehensive study of group actions up to orbit equivalence [**Dye59, Dye63**]. He emphasized the crucial role of the full group associated with a group action. In fact, he proved more than what is stated in Corollary 12.3.10: its condition (iii) is indeed equivalent to the algebraic isomorphism of the full groups. This result is known as *Dye's reconstruction theorem*. Dye's ideas, as well as the construction of Krieger [**Kri70**] of von Neumann algebras associated with non necessarily freely acting groups, were later carried on by Feldman and Moore [**FM77a, FM77b**] who provided an exhaustive study of countable non-singular equivalence relations and their von Neumann algebras. The notion of Cartan algebra was also considered by Vershik [**Ver71**].

The group $\mathrm{Aut}\,(M, A)$, where A is a Cartan subalgebra of a II_1 factor M was studied by Singer for group actions and by Feldman and Moore for equivalence relations in their above mentioned papers. The results obtained in the first section of this chapter come from [**Dye59, Dye63, Pop85**].

As mentioned above, the bases of orbit equivalence theory were laid by Dye. In his seminal paper [**Dye59**], he proved that a countable p.m.p. equivalence relation $\mathcal{R}$ is hyperfinite if and only if it is isomorphic to some $\mathcal{R}_{\mathbb{Z}\curvearrowright X}$ for a p.m.p. action of the group $\mathbb{Z}$ of integers. Moreover, he proved that two ergodic p.m.p. actions of $\mathbb{Z}$ are orbit equivalent, hence the uniqueness of the ergodic type II_1 hyperfinite equivalence relation up to isomorphism. In the paper [**Dye63**], Dye established that any p.m.p. action of an infinite abelian group (even of a group with polynomial growth) gives rise to a hyperfinite equivalence relation. This was extended much later by Ornstein and Weiss [**OW80**] to the case of actions of any countable amenable group. Therefore, any two ergodic p.m.p. actions of amenable groups are orbit equivalent.

As for the relations with operator algebras, in [**MvN43**] Murray and von Neumann established the hyperfiniteness of the group measure space II_1 factors arising from free ergodic p.m.p. actions of locally finite groups. In [**Dye63**], Dye proved that this is the case for any free ergodic action of any group giving rise to hyperfinite equivalence relations. It was known at the end of the '60s that for any free p.m.p. action $G \curvearrowright (X, \mu)$, the von Neumann algebra $L^\infty(X) \rtimes G$ has the Schwartz property (P) if and only if G is amenable [**Sch67, Gol71**]. Zimmer extended this study to the case of equivalence relations, for which he defined a notion of amenability. He showed that a countable p.m.p. equivalence relation $\mathcal{R}$ is amenable if and only if $L(\mathcal{R})$ is an injective von Neumann algebra [**Zim77a, Zim77b**][1]. He also observed [**Zim78**] that for a free p.m.p. action $G \curvearrowright X$, the group G is amenable if and only if $\mathcal{R}_{G\curvearrowright X}$ is amenable.

Finally, this circle of results was beautifully completed by Connes, Feldman and Weiss [**CFW81**] who proved that a countable p.m.p. equivalence relation is amenable if and only if it is hyperfinite[1]. As a consequence, for an

[1] In fact, these results are proved in the more general framework of non-singular equivalence relations.

ergodic p.m.p. equivalence relation $\mathcal{R}$, the II_1 factor $L(\mathcal{R})$ is hyperfinite if and only if $\mathcal{R}$ is hyperfinite, and the uniqueness of the hyperfinite II_1 factor is the operator algebra analogue of the uniqueness of the II_1 hyperfinite equivalence relation. The operator algebraic proof of the above Connes-Feldman-Weiss result presented in Section 12.4 is taken from [**Pop85, Pop07c**].

CHAPTER 13

Bimodules

As seen in Chapter 8, the study of M-modules gives few information on the tracial von Neumann algebra (M, τ). In contrast, the set of M-N-bimodules that we introduce now has a very rich structure. These modules play the role of generalized morphisms from M to N, in particular they are closely connected to completely positive maps from M to N as we will see in Section 13.1.2.

Particularly useful is the study of the set of (equivalence classes of) M-M-bimodules. It behaves in perfect analogy with the set of unitary representations of groups. We observe that to any unitary represention of a countable group G is associated a $L(G)$-$L(G)$-bimodule. The bimodules corresponding to the trivial and the regular representation are easily identified. The usual operations on representations have their analogues in the setting of bimodules, as well as the notion of weak containment. As a consequence, any property of G involving this notion has its counterpart for tracial von Neumann algebras. For instance, in the last section, the notion of amenable von Neumann algebra is interpreted in the setting of bimodules, as well as relative amenability. In Chapter 14, we similarly will use bimodules to define the very useful notion of relative property (T).

13.1. Bimodules, completely positive maps and representations

13.1.1. Definition and first examples.

Definition 13.1.1. Let M and N be two von Neumann algebras. A M-N-*bimodule* is a Hilbert space $\mathcal{H}$ which is both a left M-module and a right N-module, and is such that the left and right actions commute (see Definition 7.1.2). We will sometimes write ${}_M\mathcal{H}_N$ to make precise which von Neumann algebras are acting, and on which side, and denote by π_M, $\pi_{N^{op}}$ the corresponding representations, in case of ambiguity. Usually, for $x \in M, y \in N, \xi \in \mathcal{H}$, we write $x\xi y$ instead of $\pi_M(x)\pi_{N^{op}}(y)\xi$.

We say that two M-N-bimodules $\mathcal{H}_1$ and $\mathcal{H}_2$ are *isomorphic* (or *equivalent*) if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ which intertwines the representations.

A M-N-bimodule introduces a link between the von Neumann algebras M and N. It is also called a *correspondence* between M and N.

Example 13.1.2. Let us begin with the case $M = L^\infty(X_1, \mu_1)$ and $N = L^\infty(X_2, \mu_2)$, where (X_i, μ_i), $i = 1, 2$, are standard probability measure spaces. Let ν be a probability measure on $X_1 \times X_2$ whose projections are absolutely continuous with respect to μ_1 and μ_2. Let

$$n : X_1 \times X_2 \to \mathbb{N} \cup \{\infty\}$$

be a measurable function and denote by $\mathcal{H}(n)$ the $L^\infty(X_1 \times X_2, \nu)$-module with multiplicity n (see Section 8.1). Then $\mathcal{H}(n)$ becomes a M-N-bimodule by setting

$$f\xi g = (f \circ p_1)(g \circ p_2)\xi, \quad \forall f \in M, g \in N, \xi \in \mathcal{H}(n)$$

where p_1, p_2 denote the projections on X_1 and X_2 respectively.

This construction provides the most general example of a separable M-N-bimodule. Indeed, let $\mathcal{H}$ be a separable M-N-bimodule and denote by $L^\infty(X, \mu)$ the von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the images of M and N. A generalisation of results mentioned in Remark 3.3.2 implies that, for $i = 1, 2$, there exists a Borel map $q_i : X \to X_i$, such that $q_{i*}\mu$ is absolutely continuous with respect to μ_i, and which induces the (normal) canonical homomorphism from $L^\infty(X_i, \mu_i)$ into $L^\infty(X, \mu)$. To conclude, it suffices to introduce the image ν of μ under $q_1 \times q_2 : X \to X_1 \times X_2$ and to use Theorem 8.1.1. The measure ν represents the *graph* of the correspondence from M to N defined by the bimodule and n is its *multiplicity*[1].

In the rest of this chapter, M and N will always be tracial von Neumann algebras with trace denoted by τ, or τ_M, τ_N if necessary.

EXAMPLES 13.1.3. $L^2(M)$ is the most basic M-M-bimodule, called the *standard* or *identity* or *trivial M-M-bimodule*. As seen in Chapter 7, it is independent of the choice of the tracial normal faithful state on M, up to isomorphism. From it, many interesting bimodules can be built.

(a) The Hilbert tensor product $L^2(M) \otimes L^2(N)$ equipped with its obvious structure of M-N-bimodule is called the *coarse M-N-bimodule*:

$$x(\xi \otimes \eta)y = (x\xi) \otimes (\eta y), \quad \forall x \in M, y \in N, \xi \in L^2(M), \eta \in L^2(N).$$

When M and N are abelian as in the previous paragraph, this bimodule has multiplicity one and $\nu = \mu_1 \otimes \mu_2$.

Let us denote by $\mathcal{S}^2(L^2(N), L^2(M))$ the Hilbert space of the Hilbert-Schmidt operators from $L^2(N)$ into $L^2(M)$, that is, of the bounded operators $T : L^2(N) \to L^2(M)$ with $\mathrm{Tr}(T^*T) < +\infty$. It is a M-N-bimodule with respect to the actions by composition:

$$xTy = x \circ T \circ y, \ , \forall x \in M, y \in N, T \in \mathcal{S}^2(L^2(N), L^2(M)).$$

The map $\xi \otimes \eta \mapsto \theta_{J\eta,\xi}$, where $\theta_{J\eta,\xi}$ is the operator $\eta_1 \in L^2(N) \mapsto \langle J\eta, \eta_1\rangle \xi$, induces an equivalence between the M-N-bimodules $L^2(M) \otimes L^2(N)$ and $\mathcal{S}^2(L^2(N), L^2(M))$.

[1]See [**Con94**, V. Appendix B].

(b) To any normal homomorphism α from M into N is associated the M-N-bimodule $\mathcal{H}(\alpha)$, which is the Hilbert space $\alpha(1)L^2(N)$ endowed with the following actions:

$$\pi_M(x)\pi_{N^{op}}(y)\xi = \alpha(x)\xi y, \quad \forall x \in M, y \in N, \xi \in \alpha(1)L^2(N).$$

When M and N are abelian and $\alpha : L^\infty(X_1, \mu_1) \to L^\infty(X_2, \mu_2)$ is defined by $\theta : X_2 \to X_1$, then $\mathcal{H}(\alpha)$ can be viewed as $L^2(X_1 \times X_2, \nu)$, where ν is the image of μ_2 under the graph map $x_2 \to (\theta(x_2), x_2)$.

Going back to the general situation, it is easily seen that $\mathcal{H}(\alpha_1)$ and $\mathcal{H}(\alpha_2)$ are isomorphic if and only if there is a partial isometry $u \in N$ with $u^*u = \alpha_1(1)$, $uu^* = \alpha_2(1)$ and $u\alpha_1(x)u^* = \alpha_2(x)$ for all $x \in M$.

Specially important is the case where α belongs to the automorphism group $\mathrm{Aut}\,(M)$ of M. We get that the quotient group $\mathrm{Out}\,(M)$ of $\mathrm{Aut}\,(M)$ modulo the inner automorphisms embeds canonically into the space of (isomorphism classes of) M-M-bimodules.

(c) More generally, let p be a projection in $\mathcal{B}(\ell^2(I))\overline{\otimes}N$ and let α be a unital normal homomorphism from M into $p\big(\mathcal{B}(\ell^2(I))\overline{\otimes}N\big)p$. Then, $p\big(\ell^2(I) \otimes L^2(N)\big)$ is a M-N-bimodule, when equipped with the actions

$$x\xi y = \alpha(x)\xi y \quad \forall x \in M, y \in N,$$

the right action being the restriction of the diagonal one. We denote by $\mathcal{H}(\alpha)$ this bimodule.

Proposition 8.2.2, applied to right N-modules instead of left M-modules, implies that this is the most general example of M-N-bimodule.

Definition 13.1.4. A M-N-bimodule $\mathcal{H}$ is said to be of *finite (Jones') index*, or *bifinite*, if it is both a finite left M-module and a finite right N-module, *i.e.,* $\dim({}_M\mathcal{H}) < +\infty$ and $\dim(\mathcal{H}_N) < +\infty$.

The terminology comes from the fact that the M-N-bimodule ${}_M L^2(M)_N$ (or equivalently the N-M-bimodule ${}_N L^2(M)_M$) has finite index if and only if $[M : N] < +\infty$ when $N \subset M$ is a pair of separable II_1 factors. More generally, we have:

Proposition 13.1.5. *Let M, N be II_1 separable factors. Then a separable M-N-bimodule $\mathcal{H}$ is of finite index if and only if it is isomorphic to $\mathcal{H}(\alpha)$ for some normal unital homomorphism $\alpha : M \to p\big(\mathcal{B}(\ell^2_n) \otimes N\big)p = pM_n(N)p$, some n and some projection $p \in M_n(N)$, such that $[pM_n(N)p : \alpha(M)] < +\infty$. Moreover, in this case we have*

$$\dim(\mathcal{H}_N) = (Tr \otimes \tau_N)(p) \quad \text{and} \quad \dim({}_M\mathcal{H}) = \frac{[pM_n(N)p : \alpha(M)]}{(Tr \otimes \tau_N)(p)},$$

where Tr is the usual trace on $M_n(\mathbb{C})$.

Proof. It follows Propositions 8.5.3 and 8.6.1 that the M-N-bimodule $\mathcal{H}$ is finite as right N-module if and only if it is of the form $\mathcal{H}(\alpha)$ for some normal unital homomorphism $\alpha : M \to pM_n(N)p$, some n and some projection $p \in M_n(N)$. By definition, we have $\dim(\mathcal{H}_N) = (\mathrm{Tr} \otimes \tau_N)(p)$. Let us prove the other equality. We apply the result of Exercise 9.15 to the left $pM_n(N)p$-module $p(\ell^2_n \otimes L^2(N))$ and the subfactor $\alpha(M)$ of $pM_n(N)p$. This gives

$$\begin{aligned}\dim({}_M\mathcal{H}) &= \dim\big({}_{\alpha(M)}p(\ell^2_n \otimes L^2(N))\big)\\ &= [pM_n(N)p : \alpha(M)] \dim\big({}_{pM_n(N)p}p(\ell^2_n \otimes L^2(N))\big).\end{aligned}$$

Finally, we have $\dim\big({}_{pM_n(N)p}p(\ell^2_n \otimes L^2(N))\big) = 1/(\mathrm{Tr}\otimes\tau_N)(p)$ by Exercises 8.9 and 8.7. □

13.1.2. Bimodules and completely positive maps. The previous examples already give an indication that bimodules may be viewed as generalized morphisms between von Neumann algebras. This will be made more precise now.

– *From completely positive maps to bimodules.* Let (M, τ_M), (N, τ_N) be two tracial von Neumann algebras. Let $\phi : M \to N$ be a normal completely positive map[2]. We define on the algebraic tensor product $\mathcal{H}_0 = M \odot N$ a sesquilinear functional by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_\phi = \tau_N(y_1^*\phi(x_1^*x_2)y_2), \quad \forall x_1, x_2 \in M, y_1, y_2 \in N.$$

The complete positivity of ϕ implies the positivity of this functional. We denote by $\mathcal{H}(\phi)$ the completion of the quotient of $\mathcal{H}_0$ modulo the null space of the sesquilinear functional.

We let M and N act on $\mathcal{H}_0$ by

$$x(\sum_{i=1}^n x_i \otimes y_i)y = \sum_{i=1}^n xx_i \otimes y_iy.$$

Using again the complete positivity of ϕ, we easily obtain, for $\xi \in \mathcal{H}_0$, and $x \in M$, $y \in N$, that

$$\langle x\xi, x\xi\rangle_\phi \le \|x\|^2\langle\xi,\xi\rangle_\phi, \quad \langle \xi y, \xi y\rangle_\phi \le \|y\|^2\langle\xi,\xi\rangle_\phi.$$

For instance, the first inequality is a consequence of the fact that in $M_n(M)$ we have $[x_i^*x^*xx_i]_{i,j} \le \|x\|^2[x_i^*x_j]_{i,j}$ for every $x_1, \dots, x_n, x \in M$.

It follows that the actions of M and N pass to the quotient and extend to representations on $\mathcal{H}(\phi)$. Moreover, these representations are normal, thanks to the fact that ϕ is normal.

Let $\alpha : M \to N$ be a normal homomorphism. The reader will check that the map $x \otimes y \to \alpha(x)y$ gives an identification of the bimodule $\mathcal{H}(\alpha)$ we have just constructed with the bimodule constructed in Example 13.1.3 (b), so that the notations are compatible.

Another important particular case is when ϕ is the trace preserving conditional expectation E_N from M onto a von Neumann subalgebra N. Then the map sending $x \otimes y$ to $\widehat{xy}$ gives rise to an isomorphism between the M-N-bimodules $\mathcal{H}(E_N)$ and ${}_ML^2(M)_N$.

[2] For basic facts related to completely positive maps see Section A.3 in the appendix.

Let us go back to the general case $\mathcal{H}(\phi)$. This bimodule comes equipped with a special vector, namely the class of $1_M \otimes 1_N$, denoted ξ_ϕ. Note that

$$\|x\xi_\phi\|_\phi^2 = \tau_N(\phi(x^*x)) \text{ and } \|\xi_\phi y\|_\phi^2 = \tau_N(y^*\phi(1)y) \leq \|\phi(1)\|\tau_N(y^*y).$$

In particular, ξ_ϕ is left N-bounded. In addition this vector is *cyclic* in the sense that $\overline{\operatorname{span} M\xi_\phi N} = \mathcal{H}_\phi$. We also observe that $\phi(x) = (L_{\xi_\phi})^* x L_{\xi_\phi}$ where $L_{\xi_\phi} : L^2(N) \to \mathcal{H}(\phi)$ is the operator defined by the bounded vector ξ_ϕ.

A pair $(\mathcal{H}, \xi_0)$ consisting of a bimodule $\mathcal{H}$ and a non-zero (cyclic) vector ξ_0 is called a *pointed (cyclic) bimodule.*

– *From bimodules to completely positive maps.* Conversely, let us start from a pointed M-N-bimodule $(\mathcal{H}, \xi_0)$ where ξ_0 is left N-bounded. Let $T = L_{\xi_0} : L^2(N) \to \mathcal{H}$ be the bounded N-linear operator associated with ξ_0. Then $\phi : x \mapsto T^*\pi_M(x)T$ is a completely positive normal map from M into N.

In case ξ_0 is cyclic, the pair $(\mathcal{H}(\phi), \xi_\phi)$ constructed from this ϕ is isomorphic to $(\mathcal{H}, \xi_0)$ under the unitary operator $U : \mathcal{H}(\phi) \to \mathcal{H}$ sending $x \otimes y$ onto $x\xi_0 y$.

Observe also that if we had started from $(\mathcal{H}, \xi_0) = (\mathcal{H}(\phi), \xi_\phi)$ for some ϕ, then we would have retrieved ϕ from this latter construction.

DEFINITION 13.1.6. Let $\mathcal{H}$ be a M-N-bimodule. A (right) *coefficient* of $\mathcal{H}$ is a completely positive map $\phi : M \to N$ of the form $x \mapsto L_\xi^* x L_\xi$ where ξ is a left N-bounded vector[3].

– *Subtracial and subunital completely positive maps. Subtracial vectors.* These notions will be very useful later, in the study of Property (T) and of the Haagerup property for tracial von Neumann algebras.

DEFINITION 13.1.7. Let $\phi : M \to N$ be a completely positive map. We say that ϕ is *subtracial* if $\tau_N \circ \phi \leq \tau_M$ and that ϕ is *subunital* if $\phi(1) \leq 1$.

We say that ϕ is *tracial* if $\tau_N \circ \phi = \tau_M$. Whenever $\phi(1) = 1$, then ϕ is *unital.*

Note that a subtracial completely positive map is normal (by Proposition 2.5.11) and that a subunital completely positive map is equivalently defined by $\|\phi\| = \|\phi(1)\| \leq 1$. The proof of the following lemma is straightforward.

LEMMA 13.1.8. *Let $(\mathcal{H}, \xi)$ be a pointed M-N-bimodule with ξ left N-bounded. Denote by $\phi : M \to N$ the corresponding coefficient, i.e., $\phi(x) = L_\xi^* x L_\xi$ for $x \in M$.*

(i) *ϕ is subtracial if and only if $\langle \xi, x\xi \rangle \leq \tau_M(x)$ for every $x \in M_+$.*
(ii) *ϕ is subunital if and only if $\langle \xi, \xi y \rangle \leq \tau_N(y)$ for every $y \in N_+$.*

When (i) and (ii) are statisfied, we will say that the vector ξ is *subtracial.* A *tracial vector* is a vector ξ such that $\langle \xi, x\xi \rangle = \tau_M(x)$ for every $x \in M$ and

[3]Similarly, we could consider left coefficients $\psi : N \to M$.

$\langle \xi, \xi y\rangle = \tau_N(y)$ for every $y \in N$. Note that ξ is tracial if and only if the corresponding completely positive map is tracial and unital.

The following result is a generalisation of the Radon-Nikodým type Lemma 2.5.3.

LEMMA 13.1.9. *Let ϕ and ψ be two completely positive maps from M to N such that $\psi - \phi$ is still a completely positive map. We assume that $\psi(x) = L_\xi^* x L_\xi$ where ξ is a left N-bounded vector in some M-N-bimodule $\mathcal{H}$. Then there is a left N-bounded vector $\eta \in \mathcal{H}$ such that $\phi(x) = L_\eta^* x L_\eta$ for every $x \in M$. More precisely, $\eta = S\xi$ where $S : \mathcal{H} \to \mathcal{H}$ is a M-N-linear contraction.*

PROOF. We may assume that ξ is a cyclic vector in the M-N-bimodule $\mathcal{H}$. We observe that ψ, and hence ϕ, are normal. We define a M-N-linear contraction T from $\mathcal{H}$ onto $\mathcal{H}(\phi)$ by

$$T(\sum_i m_i \xi n_i) = \sum_i m_i \xi_\phi n_i.$$

Indeed, we have, since $\psi - \phi$ is completely positive,

$$\begin{aligned}
\Big\|\sum_i m_i \xi_\phi n_i\Big\|^2 &= \sum_{i,j} \langle m_i \otimes n_i, m_j \otimes n_j\rangle \\
&= \sum_{i,j} \tau_N(n_i^* \phi(m_i^* m_j) n_j) \leq \sum_{i,j} \tau_N(n_i^* L_\xi^*(m_i^* m_j) L_\xi n_j) \\
&\leq \sum_{i,j} \tau_N((L_{m_i\xi n_i})^* L_{m_j \xi n_j}) = \Big\|\sum_i m_i \xi n_i\Big\|^2.
\end{aligned}$$

We set $S = |T|$ and $\eta = S\xi$. This vector is left N-bounded and we have, for $x \in M$,

$$L_\eta^* x L_\eta = L_\xi^* S x S L_\xi = L_\xi^* T^* x T L_\xi = L_{\xi_\phi}^* x L_{\xi_\phi} = \phi(x),$$

since S and T are M-N linear and $T\xi = \xi_\phi$.

$\square$

COROLLARY 13.1.10. *Let C be the convex set of all subunital completely positive maps from M to M. Endowed with the topology of pointwise weak operator convergence, it is compact. Moreover, Id_M is an extremal point of this compact convex set.*

PROOF. The first assertion is immediate. Now, suppose that $\mathrm{Id}_M = \lambda\phi_1 + (1-\lambda)\phi_2$ with $\phi_1, \phi_2 \in C$ and $\lambda \in]0,1[$. We note first that $\phi_1(1) = 1 = \phi_2(1)$. Next, we apply the previous lemma with $\mathcal{H} = L^2(M)$, $N = M$, $\psi = \mathrm{Id}_M$ and therefore $\xi = 1 \in L^2(M)$. We get $\lambda\phi_1(x) = L_\eta^* x L_\eta$ for every $x \in M$, where η belongs to the center of M. Thus, we see that $\lambda\phi_1(x) = x z_1$, where z_1 is in the center of M. But then $\lambda = z_1$ and $\phi_1 = \mathrm{Id}_M$. $\square$

The next lemma allows to approximate vectors by subtracial ones.

LEMMA 13.1.11. *Let $\mathcal{H}$ be a M-N-bimodule and $\xi \in \mathcal{H}$. Let $T_0 \in L^1(M, \tau_M)_+$ such that $\langle \xi, x\xi \rangle = \tau_M(xT_0)$ for every $x \in M$ and let $S_0 \in L^1(N, \tau_N)_+$ such that $\langle \xi, \xi y \rangle = \tau_N(yS_0)$ for every $y \in N$. We set $T = f(T_0)$ and $S = f(S_0)$ where $f(t) = \min(1, t^{-1/2})$ for $t \geq 0$. Then the vector $\xi' = T\xi S$ is subtracial and we have*

$$\left\|\xi - \xi'\right\|^2 \leq 2\|T_0 - 1\|_1 + 2\|S_0 - 1\|_1.$$

PROOF. Given $x \in M_+$, we have, since $S \leq 1$ and $T_0T^2 \leq 1$,

$$\left\langle \xi', x\xi' \right\rangle \leq \langle \xi, TxT\xi \rangle = \tau_M(TxTT_0) \leq \tau_M(x).$$

Similarly, we get the other condition for ξ' to be subtracial.

Moreover, we have

$$\begin{aligned} \left\|\xi - \xi'\right\|^2 &\leq 2\|\xi - T\xi\|^2 + 2\|\xi - \xi S\|^2 \\ &\leq 2\tau_M((1-T)^2T_0) + 2\tau_N((1-S)^2S_0) \\ &\leq 2\|T_0 - 1\|_1 + 2\|S_0 - 1\|_1. \end{aligned}$$

□

REMARK. Note that if ξ belongs to a M-N-submodule of $\mathcal{H}$, the same holds for ξ'.

13.1.3. Bimodules from representations of groups. As shown above, bimodules may be seen as generalized morphisms between von Neumann algebras. We now point out that they also play the same role as unitary representations in group theory. We remind the reader that a unitary representation $(\pi, \mathcal{H})$ of a group G is a group homomorphism from G into the unitary group $\mathcal{U}(\mathcal{B}(\mathcal{H}))$. The *trivial representation* ι_G is the homomorphism $s \mapsto \iota_G(s) = 1 \in \mathbb{C}$.

– *Bimodules and representations.* Let G be a countable group and let $M = L(G)$ be the corresponding tracial von Neumann algebra. Recall that $\ell^2(G) = L^2(M)$, where the M-M-bimodule structure of $L^2(M)$ comes from the left and right regular representations of G: given $s, t \in G$, we have

$$u_s f u_t = \lambda(s)\rho(t^{-1})f, \quad \forall f \in \ell^2(G).$$

Let π be a unitary representation in a Hilbert space $\mathcal{K}_\pi$. The Hilbert space $\mathcal{H}(\pi) = \ell^2(G) \otimes \mathcal{K}_\pi$ is equipped with two commuting actions of G defined by

$$u_s(f \otimes \xi)u_t = (u_s f u_t) \otimes \pi(s)\xi, \quad \forall s, t \in G, f \in \ell^2(G), \xi \in \mathcal{K}_\pi.$$

These actions extend to $L(G)$ and give to $\mathcal{H}(\pi)$ a structure of M-M-bimodule. This is clear for the right G-action. On the other hand, the left G-action is equivalent to a multiple of the left regular representation since the unitary operator defined on $\ell^2(G) \otimes \mathcal{K}_\pi$ by

$$U(\delta_t \otimes \xi) = \delta_t \otimes \pi(t)^*\xi, \quad \forall t \in G, \xi \in \mathcal{K}_\pi,$$

satisfies

$$U(u_s \otimes \pi(s))U^* = u_s \otimes \mathrm{Id}_{\mathcal{K}_\pi}, \quad \forall s \in G,$$

Obviously, the equivalence class of $\mathcal{H}(\pi)$ only depends on the equivalence class of the representation π. The trivial M-M-bimodule $L^2(M)$ is associated with the trivial representation of G and the coarse bimodule $L^2(M) \otimes L^2(M)$ corresponds to the left regular representation (Exercise 13.12).

This construction can be easily extended to crossed products. Let (B, τ) be a tracial von Neumann algebra and let σ be a group homomorphism from G into $\mathrm{Aut}\,(B, \tau)$. Now we set $M = B \rtimes G$ and keep the notations of Section 5.2. We have $L^2(M) = \ell^2(G) \otimes L^2(B)$. Consider again a representation π of G. This time, the corresponding M-M-bimodule is

$$\mathcal{H}(\pi) = \ell^2(G) \otimes L^2(B) \otimes \mathcal{K}_\pi,$$

equipped with the commuting actions that are well-defined by

$$(\hat{x} \otimes \xi)y = \hat{x}y \otimes \xi = \widehat{xy} \otimes \xi, \quad \forall x \in M, \xi \in \mathcal{K}_\pi, y \in M;$$
$$b(\hat{x} \otimes \xi) = b\hat{x} \otimes \xi = \widehat{bx} \otimes \xi, \quad \forall x \in M, \xi \in \mathcal{K}_\pi, b \in B;$$
$$u_s(\hat{x} \otimes \xi) = u_s\hat{x} \otimes \pi(s)\xi = \widehat{u_s x} \otimes \pi(s)\xi, \quad \forall x \in M, \xi \in \mathcal{K}_\pi, s \in G.$$

– *Completely positive maps and positive definite functions.* We have seen in the previous section that cyclic pointed M-N-bimodules $(\mathcal{H}, \xi_0)$ (with ξ_0 left N-bounded) are in bijective correspondence with normal completely positive maps from M to N (up to isomorphism). This is analogous to the well known fact that positive definite functions on groups correspond to equivalence classes of unitary representations equipped with a cyclic vector (*i.e.*, pointed cyclic unitary representations).

Recall that a a complex-valued function φ defined on a countable group G is *positive definite* (or of *positive type*) if, for every finite subset $\{s_1, \dots, s_n\}$ of G, the $n \times n$ matrix $[\varphi(s_i^{-1}s_j)]$ is positive, that is, for every $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, we have $\sum_{i,j=1}^n \overline{\lambda_i}\lambda_j \varphi(s_i^{-1}s_j) \geq 0$. Obviously, given a unitary representation π in a Hilbert space $\mathcal{H}$, any *coefficient* of π, that is any function $s \mapsto \langle \xi, \pi(s)\xi \rangle$ with $\xi \in \mathcal{H}$, is positive definite. Conversely, given a positive definite function φ on G there is a unique (up to isomorphism) triple $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ (called the GNS construction) composed of a unitary representation and a cyclic vector, such that $\varphi(s) = \langle \xi_\varphi, \pi_\varphi(s)\xi_\varphi \rangle$ for all $s \in G$. These two constructions are inverse from each other[4].

Completely positive maps on group von Neumann algebras, and more generally on crossed products, are closely related to positive definite functions.

Proposition 13.1.12. *Let G be a countable group, (B, τ) a tracial von Neumann algebra and $\sigma : G \curvearrowright (B, \tau)$ a trace preserving action. We set $M = B \rtimes G$.*

[4]See for instance [**BdlHV08**, Theorem C.4.10].

(i) *Let $\phi : M \to M$ be a completely positive map. Then $\varphi : s \mapsto \tau(\phi(u_s)u_s^*)$ is positive definite.*

(ii) *Let φ be a positive definite function on G. There is a unique normal completely positive map $\phi : M \to M$ such that $\phi(bu_s) = \varphi(s)bu_s$ for every $b \in B$ and $s \in G$. More precisely, let $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ be the GNS construction associated with φ and let $\mathcal{H}(\pi_\varphi)$ be the corresponding M-M-bimodule. The vector $\xi_0 = \widehat{1_M} \otimes \xi_\varphi$ is left bounded, and the completely positive map $\phi : M \to M$ associated with $(\mathcal{H}(\pi_\varphi), \xi_0)$ satisfies $\phi(bu_s) = \varphi(s)bu_s$ for every $b \in B$ and $s \in G$.*

(iii) *Let $\mathcal{H}$ be a M-M-bimodule and ξ_0 a left bounded vector. We associate the unitary representation π of G in $\mathcal{K} = \overline{\mathrm{span}}\{u_s\xi_0 u_s^* : s \in G\}$ defined by*

$$\pi(s)\eta = u_s \eta u_s^*, \quad \forall \eta \in \mathcal{K}.$$

If φ is the positive definite function on G defined by $(\mathcal{K}, \pi, \xi_0)$ and if $\phi : M \to M$ is the completely positive map defined by $(\mathcal{H}, \xi_0)$ we have $\varphi(s) = \tau_M(\phi(u_s)u_s^*)$ for every $g \in G$.

PROOF. We leave the easy verifications as an exercise. □

13.2. Composition (or tensor product) of bimodules

The parallel between group representations and bimodules can be carried on further. The classical operations on representations have their analogues for bimodules. First, the *addition* (or *direct sum*) of M-N-bimodules is defined in an obvious way. Second, given a M-N-bimodule $\mathcal{H}$, the *contragredient bimodule* is the conjugate Hilbert space $\overline{\mathcal{H}}$ equipped with the actions

$$y \cdot \overline{\xi} \cdot x = \overline{x^* \xi y^*} \quad \forall x \in M, y \in N.$$

The most interesting operation is the *composition*, or *tensor product* of bimodules, which corresponds to the tensor product of representations. For the notations and properties of bounded vectors used below, we refer to Section 8.4.2.

13.2.1. Definition of the tensor product.

PROPOSITION 13.2.1. *Let M be a tracial von Neumann algebra, let $\mathcal{H}$ be a right M-module and let $\mathcal{K}$ be a left M-module. The formula*

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \langle \xi_1, \xi_2 \rangle_M \eta_2 \rangle_{\mathcal{K}} \tag{13.1}$$

defines a positive sesquilinear form on the algebraic tensor product $\mathcal{H}^0 \odot \mathcal{K}$, where $\mathcal{H}^0$ is the subspace of left M-bounded vectors.

PROOF. We have to show that, for $\sum_{i=1}^n \xi_i \otimes \eta_i \in \mathcal{H}^0 \odot \mathcal{K}$, the quantity

$$\left\langle \sum_{i=1}^n \xi_i \otimes \eta_i, \sum_{j=1}^n \xi_j \otimes \eta_j \right\rangle = \sum_{i,j=1}^n \left\langle \eta_i, \langle \xi_i, \xi_j \rangle_M \eta_j \right\rangle_{\mathcal{K}}$$

is non-negative, or equivalently that the matrix $[\langle \xi_i, \xi_j \rangle_M]_{1\leq i,j\leq n} \in M_n(M)$ is positive. Viewing $[\langle \xi_i, \xi_j \rangle_M]_{1\leq i,j\leq n}$ as an operator acting on $L^2(M)^{\oplus n}$ it is enough to check that

$$\sum_{i,j=1}^{n} \left\langle \widehat{x_i}, \langle \xi_i, \xi_j \rangle_M \widehat{x_j} \right\rangle \geq 0$$

for $x_1, \dots, x_n \in M$. But this is immediate because

$$\sum_{i,j=1}^{n} \left\langle \widehat{x_i}, \langle \xi_i, \xi_j \rangle_M \widehat{x_j} \right\rangle = \left\langle \sum_{i=1}^{n} L_{\xi_i} \widehat{x_i}, \sum_{j=1}^{n} L_{\xi_j} \widehat{x_j} \right\rangle_{\mathcal{K}}.$$

$\square$

We denote by $\mathcal{H} \otimes_M \mathcal{K}$ the Hilbert space deduced from $\mathcal{H}^0 \odot \mathcal{K}$ by separation and completion relative to the sesquilinear form defined in (13.1). The image of $\xi \otimes \eta \in \mathcal{H} \odot \mathcal{K}$ in $\mathcal{H} \otimes_M \mathcal{K}$ will be denoted by $\xi \otimes_M \eta$.

We may perform the analogous construction, starting from $\mathcal{H} \odot {}^0\mathcal{K}$. As such, we obtain Hilbert spaces that are canonically isomorphic, so there is no ambiguity in the definition.

Proposition 13.2.2. *Let M be a tracial von Neumann algebra, let $\mathcal{H}$ be a right M-module and let $\mathcal{K}$ be a left M-module. The restrictions to $\mathcal{H}^0 \odot {}^0\mathcal{K}$ of the sesquilinear forms defined on $\mathcal{H}^0 \odot \mathcal{K}$ and $\mathcal{H} \odot {}^0\mathcal{K}$ coincide. Moreover, the three Hilbert spaces obtained by separation and completion from these spaces are the same.*

Proof. It suffices to show that for $\xi_1, \xi_2 \in \mathcal{H}^0$ and $\eta_1, \eta_2 \in {}^0\mathcal{K}$, we have

$$\langle \eta_1, \langle \xi_1, \xi_2 \rangle_M \eta_2 \rangle_{\mathcal{K}} = \langle \xi_1 \, {}_M\langle \eta_1, \eta_2 \rangle, \xi_2 \rangle_{\mathcal{H}}.$$

Using Lemmas 8.4.6 and 8.4.5, we get

$$\begin{aligned}\langle \eta_1, \langle \xi_1, \xi_2 \rangle_M \eta_2 \rangle_{\mathcal{K}} &= \tau({}_M\langle \langle \xi_1, \xi_2 \rangle_M \eta_2, \eta_1 \rangle) = \tau(\langle \xi_1, \xi_2 \rangle_M \, {}_M\langle \eta_2, \eta_1 \rangle) \\ &= \langle \xi_1, \xi_2 \, {}_M\langle \eta_2, \eta_1 \rangle \rangle_{\mathcal{H}} = \langle \xi_1 \, {}_M\langle \eta_1, \eta_2 \rangle, \xi_2 \rangle_{\mathcal{H}}.\end{aligned}$$

This proves our claim. The second part of the statement follows from the density of $\mathcal{H}^0$ and ${}^0\mathcal{K}$ in $\mathcal{H}$ and $\mathcal{K}$ respectively. $\square$

Proposition 13.2.3. *Let M, N, P be three tracial von Neumann algebras and let $\mathcal{H}$ be a M-N-bimodule and $\mathcal{K}$ a N-P-bimodule. Then $\mathcal{H} \otimes_N \mathcal{K}$ is a M-P-bimodule with respect to the actions given by*

$$x(\xi \otimes_N \eta) = (x\xi) \otimes_N \eta, \quad (\xi \otimes_N \eta)y = \xi \otimes_N (\eta y),$$

for $x \in M, y \in P, \xi \in \mathcal{H}^0, \eta \in {}^0\mathcal{K}$.

Proof. We only consider the left action, the other case being dealt with similarly. Given $\xi_1, \dots, \xi_n \in \mathcal{H}^0$ and $\eta_1 \dots, \eta_n \in {}^0\mathcal{K}$, we have

$$\Big\| \sum_{i=1}^n (x\xi_i) \otimes_N \eta_i \Big\|^2 = \sum_{i,j=1}^n \big\langle \eta_i, \langle x\xi_i, x\xi_j \rangle_N \eta_j \big\rangle_{\mathcal{K}}$$
$$\leq \|x\|^2 \Big\| \sum_{i=1}^n \xi_i \otimes_N \eta_i \Big\|^2,$$

since $[\langle x\xi_i, x\xi_j\rangle_N] \leq \|x\|^2 [\langle \xi_i, \xi_j \rangle_N]$ in $M_n(N)_+$. It follows that the left multiplication by x extends to a bounded operator on $\mathcal{H} \otimes_N \mathcal{K}$. To see that the representation of M is normal, we remark that, for $\xi, \eta \in \mathcal{H}^0$, the map $x \mapsto \langle \xi, x\eta \rangle_N = L_\xi^* x L_\eta$ from M into N is w.o. continuous. □

Definition 13.2.4. The M-P-bimodule $\mathcal{H} \otimes_N \mathcal{K}$ is called the *composition* or *(Connes) tensor product* of the bimodules ${}_M\mathcal{H}_N$ and ${}_N\mathcal{K}_P$.

13.2.2. Properties of the tensor product. Below, M, N, P, Q are tracial von Neumann algebras.

Proposition 13.2.5 (Associativity). *Let $\mathcal{H}$ be a M-N-bimodule, $\mathcal{K}$ a N-P-bimodule and $\mathcal{L}$ a P-Q-bimodule. The M-Q-bimodules $(\mathcal{H} \otimes_N \mathcal{K}) \otimes_P \mathcal{L}$ and $\mathcal{H} \otimes_N (\mathcal{K} \otimes_P \mathcal{L})$are canonically isomorphic.*

Proof. One easily shows that the map $U : (\xi \otimes \eta) \otimes \mu \to \xi \otimes (\eta \otimes \mu)$, with $\xi \in \mathcal{H}^0, \eta \in \mathcal{K}, \mu \in {}^0\mathcal{L}$, extends to an isomorphim of the above mentioned M-Q-bimodules □

The distributivity of the tensor product with respect to the direct sum is easy to establish, as well as the canonical isomorphisms

$${}_M(\mathcal{H} \otimes_N L^2(N))_N \simeq {}_M\mathcal{H}_N \simeq {}_M(L^2(M) \otimes_M \mathcal{H})_N.$$

Proposition 13.2.6. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two right N-modules and $\mathcal{K}_1, \mathcal{K}_2$ be two left N-modules. Let $S : \mathcal{H}_1 \to \mathcal{H}_2$ and $T : \mathcal{K}_1 \to \mathcal{K}_2$ be two bounded N-linear maps. There exists a unique bounded operator $S \otimes_N T : \mathcal{H}_1 \otimes_N \mathcal{H}_2 \to \mathcal{K}_1 \otimes_N \mathcal{K}_2$ such that $(S \otimes_N T)(\xi \otimes_N \eta) = S\xi \otimes_N T\eta$ for every $\xi \in \mathcal{H}_1^0$ and $\eta \in {}^0\mathcal{H}_2$. Moreover, if $\mathcal{H}_1, \mathcal{K}_1$ are M-N-bimodules, $\mathcal{H}_2, \mathcal{K}_2$ are N-P-bimodules, and if S, T intertwine the actions, then $S \otimes_N T$ is M-P-linear.*

Proof. The straightforward proof is left to the reader. □

Proposition 13.2.7. *Let $\mathcal{H}$ be a M-N-bimodule and $\mathcal{K}$ a N-P-bimodule. Then the map $\overline{\eta} \otimes \overline{\xi} \mapsto \overline{\xi \otimes \eta}$ defines a linear application from $\overline{\mathcal{K}}^0 \otimes_N {}^0\overline{\mathcal{H}}$ into $\overline{\mathcal{H} \otimes_N \mathcal{K}}$ which extends to an isomorphism of P-M-bimodule from $\overline{\mathcal{K}} \otimes_N \overline{\mathcal{H}}$ onto $\overline{\mathcal{H} \otimes_N \mathcal{K}}$.*

Proof. Again, the proof is a straightforward computation. □

13.3. Weak containment

Given a group G, assumed to be countable for simplicity, the set $\mathcal{R}ep(G)$ of equivalence classes of unitary representations of G in separable Hilbert spaces contains a lot of informations relative to G. Therefore, given two tracial von Neumann algebras M and N, it is tempting to study the space $\mathrm{Bimod}\,(M,N)$ of equivalence classes of M-N-bimodules[5], which plays the same role as $\mathcal{R}ep(G)$. We develop further the similarities between these two spaces and describe below the analogue for bimodules of the notion of weak containment of group representations. We first give a quick survey for the case of group representations.

13.3.1. Weak containment for group representations.

Definition 13.3.1. Let $(\pi,\mathcal{H})$ and $(\rho,\mathcal{K})$ be two unitary representations of G. We say that π is *weakly contained in* ρ, and we write $\pi \prec \rho$, if every coefficient of π can be approximated by finite sums of coefficients of ρ. More precisely, $\pi \prec \rho$ if for every $\xi \in \mathcal{H}$, every finite subset F of G and every $\varepsilon > 0$, there exist $\eta_1,\dots,\eta_n \in \mathcal{K}$ such that

$$\Big|\langle \xi, \pi(g)\xi\rangle - \sum_{i=1}^{n} \langle \eta_i, \rho(g)\eta_i\rangle\Big| \leq \varepsilon$$

for all $g \in F$.

If $\pi \prec \rho$ and $\rho \prec \pi$, we say that π and ρ are *weakly equivalent* and denote this by $\pi \sim \rho$.

Note that $\pi \prec \rho$ if and only if every normalized[6] coefficient φ of π is the pointwise limit of a sequence of convex combinations of normalized coefficients of ρ.

Of course, any subrepresentation of ρ is weakly contained in ρ. We also observe that a representation is weakly equivalent to any of its multiples.

Remark 13.3.2. Although we will not need this result, we mention the following equivalent formulation of the notion of weak containment. Remark first that every unitary representation $(\pi,\mathcal{H})$ gives rise to a representation of the involutive Banach algebra $\ell^1(G)$ by

$$\forall f \in \ell^1(G), \quad \pi(f) = \sum_{g\in G} f(g)\pi(g).$$

It is a straightforward exercise to deduce from the definition that, whenever $\pi \prec \rho$, we have

$$\forall f \in \ell^1(G), \quad \|\pi(f)\| \leq \|\rho(f)\|.$$

The converse is true. See for instance [**Dix77**, Chapter 18] about this fact.

[5]Strictly speaking, we have to fix a huge cardinal and consider bimodules on Hilbert spaces whose dimension does not exceed it, in order to avoid paradoxically large sets. This restriction will be implicit.

[6]$\varphi : g \mapsto \langle \xi, \pi(g)\xi\rangle$ is said to be normalized if $\varphi(e) = \|\xi\|^2 = 1$.

When π is irreducible, the definition may be spelled out in the following simpler way.

PROPOSITION 13.3.3. *An irreducible representation π of G is weakly contained in ρ if and only if every coefficient of π is the pointwise limit of a sequence of coefficients of ρ.*

SKETCH OF PROOF. First, we observe that on the unit ball of $\ell^\infty(G)$ the weak* topology coincides with the topology of pointwise convergence. Denote by Q the set of normalized coefficients of ρ and let C be the closure of the convex hull of Q in the weak* topology. Note that C is a compact convex set.

Assume that $\pi \prec \rho$ and let $\varphi : g \mapsto \langle \xi, \pi(g)\xi \rangle$ be a normalized coefficient of π. Then φ belongs to C. Since π is irreducible, its normalized coefficients are extremal points of the convex set of all normalized positive definite functions[7]. In particular, φ is an extremal point of C. To conclude, it suffices to use the classical result in functional analysis which tells us that the weak* closure of the generating set Q of C contains the extremal points of C.[8] $\square$

Observe that the trivial representation ι_G of G is contained in $(\rho, \mathcal{K})$ if and only if ρ has a non-zero invariant vector. Similarly, we have the following simple description of the weak containment of ι_G in $(\rho, \mathcal{K})$ based on the notion of almost having invariant vectors, that we recall first.

DEFINITION 13.3.4. Let $(\pi, \mathcal{H})$ be a unitary representation of a group G.

(i) Given a finite subset F of G and $\varepsilon > 0$, a vector $\xi \in \mathcal{H}$ is (F, ε)-*invariant* if $\max_{g \in F} \|\pi(g)\xi - \xi\| < \varepsilon \|\xi\|$.

(ii) We say that $(\pi, \mathcal{H})$ *almost has invariant vectors* if π has (F, ε)-invariant vectors for every finite subset $F \subset G$ and every $\varepsilon > 0$.

PROPOSITION 13.3.5. *The following conditions are equivalent:*

(i) *ι_G is weakly contained in ρ, i.e., $\iota_G \prec \rho$;*

(ii) *there exists a net of coefficients of ρ converging to* 1 *pointwise;*

(iii) *$(\rho, \mathcal{K})$ almost has invariant vectors.*

PROOF. Obviously, we have (ii) $\Rightarrow$ (i). The equivalence between (ii) and (iii) is a consequence of the two following classical inequalities: for every unit vector ξ,

$$\|\rho(g)\xi - \xi\|^2 = 2|1 - \Re\langle \xi, \rho(g)\xi \rangle| \leq 2|1 - \langle \xi, \rho(g)\xi \rangle|, \tag{13.2}$$

$$|1 - \langle \xi, \rho(g)\xi \rangle| = |\langle \xi, \xi - \rho(g)\xi \rangle| \leq \|\rho(g)\xi - \xi\|. \tag{13.3}$$

[7] See for instance [**BdlHV08**, Theorem C.5.2].

[8] See [**Con90**, Theorem 7.8, Chapter V] for instance.

Let us show that (i) ⇒ (iii) (without using Proposition 13.3.3). Assume that (iii) does not hold. There exist $\varepsilon > 0$ and a finite subset F of G such that

$$\sum_{g \in F} \|\rho(g)\xi - \xi\|^2 \geq \varepsilon \|\xi\|^2$$

for every $\xi \in \mathcal{K}$. This inequality is still valid when the representation ρ is replaced by any of its multiple. It follows that there is no sequence of coefficients of a countable multiple of ρ which converges to 1 pointwise, and thus ι_G is not weakly contained in ρ □

– *The Fell topology on $\mathcal{R}ep(G)$.* The notion of weak containment is closely related to the Fell topology on $\mathcal{R}ep(G)$, defined as follows. Let $(\pi, \mathcal{H})$ be a unitary representation of G. Given $\varepsilon > 0$, a finite subset F of G, and $\xi_1, \dots, \xi_n \in \mathcal{H}$, let $V(\pi; \varepsilon, F, \xi_1, \dots, \xi_n)$ be the set of $(\rho, \mathcal{K}) \in \mathcal{R}ep(G)$ such that there exist $\eta_1, \dots, \eta_n \in \mathcal{K}$ with,

$$\forall i, j, \forall g \in F, \quad |\langle \xi_i, \pi(g)\xi_j \rangle - \langle \eta_i, \rho(g)\eta_j \rangle| < \varepsilon.$$

These sets $V(\pi; \varepsilon, F, \xi_1, \dots, \xi_n)$ form a basis of neighbourhoods of π for a topology on $\mathcal{R}ep(G)$, called the *Fell topology*.

When $(\pi, \mathcal{H})$ has a cyclic vector ξ, meaning that the linear span of $\pi(G)\xi$ is dense in $\mathcal{H}$, then π has a basis of neighbourhoods of the form $V(\pi; \varepsilon, F, \xi)$. Indeed, any $\xi_i \in \mathcal{H}$ is as close as we wish to a linear combination $\sum_k \lambda_k \pi(g_k)\xi$. Hence, given $V(\pi; \varepsilon, F, \xi_1, \dots, \xi_n)$, we easily see that it contains some $V(\pi; \varepsilon', F', \xi)$.

Obviously, a representation π is weakly contained in ρ whenever it belongs to the closure of $\{\rho\}$. In fact, we have $\pi \prec \rho$ if and only if π is in the closure of the infinite (countable) multiple $\rho^{\oplus\infty}$ of ρ.[9] This is obvious if π has a cyclic vector ξ and in the general case, one uses the fact that π is a sum of representations having a cyclic vector.

However, we note that whenever π is irreducible[10], we do have $\pi \prec \rho$ if and only if $\pi \in \overline{\{\rho\}}$. This follows from Proposition 13.3.3 and from the fact that ξ has cyclic vectors.

This last observation also shows that the trivial representation ι has a base of neighbourhoods of the form $W'(\iota; \varepsilon, F)$ with $\varepsilon > 0$, F finite subset of G, where $W'(\iota; \varepsilon, F)$ is the set of representations $(\pi, \mathcal{H})$ such that there exists a unit vector $\xi \in \mathcal{H}$ satisfying

$$\max_{g \in F} \|\pi(g)\xi - \xi\| \leq \varepsilon.$$

13.3.2. Weak containment for bimodules. Among several equivalent definitions, we choose to introduce this notion via the Fell topology on Bimod (M, N).

[9]See [**Fel62**, Theorem 1.1].

[10]This applies in particular to the trivial representation ι_G.

– *The Fell topology on* Bimod(M,N). This topology is defined by the assignement of the following basis of neighbourhoods

$$V(\mathcal{H};\varepsilon,E,F,S)$$

of each $\mathcal{H}\in$ Bimod(M,N), with $\varepsilon>0$ and where E, F, $S=\{\xi_1,\dots,\xi_n\}$ range over all finite subsets of M, N and $\mathcal{H}$ respectively: $V(\mathcal{H};\varepsilon,E,F,S)$ is the set of all M-N-bimodules $\mathcal{K}$ such that there exist $\eta_1,\dots,\eta_n\in\mathcal{K}$ with

$$|\langle\xi_i,x\xi_j y\rangle-\langle\eta_i,x\eta_j y\rangle|<\varepsilon$$

for every $x\in E$, $y\in F$, $1\le i,j\le n$.

– *Neighbourhoods of the trivial bimodule.* When $\mathcal{H}$ has a cyclic vector ξ, it is easily seen that it has a basis of neighbourhoods of the form $V(\mathcal{H};\varepsilon,E,F,\{\xi\})$. This applies in particular, when $M=N$ to the trivial M-M-bimodule $L^2(M)$, where we take $\xi=\widehat{1}$. In this case we may take $E=F$ and we set

$$V(L^2(M);\varepsilon,F)=V(L^2(M);\varepsilon,F,F,\{\hat{1}\}).$$

Note that $V(L^2(M);\varepsilon,F)$ is the set of M-M bimodules $\mathcal{K}$ such that there exists $\eta\in\mathcal{K}$ with

$$\max_{x,y\in F}|\tau(xy)-\langle\eta,x\eta y\rangle|<\varepsilon.$$

Moreover, by taking F with $1_M\in F$, we may assume that $\|\eta\|=1$.

As we will see in the sequel, it is very important to understand what it means that $L^2(M)$ is adherent to a given bimodule, with respect to the Fell topology. So it may be useful to have at hand different kinds of neighbourhoods of $L^2(M)$. We now describe another basis.

Given $\varepsilon>0$ and a finite subset F of M, we define

$$W(L^2(M);\varepsilon,F)$$

to be the set of M-M-bimodules $\mathcal{H}$ such that there exists $\xi\in\mathcal{H}$ with $\|\xi\|=1$ and

$$\|x\xi-\xi x\|<\varepsilon,\quad |\langle\xi,x\xi\rangle-\tau(x)|<\varepsilon.$$

for every $x\in F$.

Lemma 13.3.6. *The family of sets $W(L^2(M);\varepsilon,F)$, where F ranges over the finite subsets of M, and $\varepsilon>0$, forms a basis of neighbourhoods of $L^2(M)$.*

Proof. Setting $E=\{x,x^*,x^*x,xx^*:x\in F\}\cup\{1_M\}$, we check that (with $\varepsilon<1$),

$$V(L^2(M);\varepsilon^2/4,E)\subset W(L^2(M);\varepsilon,F).$$

Therefore, $W(L^2(M);\varepsilon,F)$ is a neighbourhood of $L^2(M)$.

Conversely, given $\varepsilon>0$ and a finite subset E of the unit ball of M, we have

$$W(L^2(M);\varepsilon/2,F)\subset V(L^2(M);\varepsilon,E)$$

with $F=E\cup E^2$, due to the inequality, for $\|\xi\|\le 1$,

$$|\langle\xi,x\xi y\rangle-\tau(xy)|\le|\langle\xi,xy\xi\rangle-\tau(xy)|+\|x\|\|y\xi-\xi y\|.$$

□

Now, let

$$W'(L^2(M);\varepsilon,F)$$

be the set of M-M bimodules $\mathcal{H}$ such that there exists $\xi\in\mathcal{H}$ with $\|\xi\|=1$ and $\|x\xi-\xi x\|<\varepsilon$ for all $x\in F$. When M is a factor this family of sets forms a simpler basis of neighbourhoods of $L^2(M)$.

PROPOSITION 13.3.7. *Let M be a II_1 factor. The family of sets*

$$W'(L^2(M);\varepsilon,F),$$

where F ranges over the finite subsets of M, and $\varepsilon>0$, forms a basis of neighbourhoods of $L^2(M)$.

PROOF. It suffices to show that, given $\varepsilon>0$ and a finite subset F of the unit ball of M, there exist $\varepsilon_1>0$ and a finite subset F_1 of M such that $W'(L^2(M);\varepsilon_1,F_1)\subset W(L^2(M);\varepsilon,F)$.

Let $\mathcal{H}$ be a M-M-bimodule and $\xi\in\mathcal{H}$ with $\|\xi\|=1$. We first observe that for $x_1,x_2\in M$ we have

$$|\langle\xi,x_1x_2\xi\rangle-\langle\xi,x_2x_1\xi\rangle|\leq\|x_1\|(\|x_2\xi-\xi x_2\|+\|x_2^*\xi-\xi x_2^*\|). \tag{13.4}$$

Second, using the Dixmier averaging theorem (see Theorem 6.4.1 and Exercise 6.3), given $\delta>0$, there are unitary elements $u_1,\dots,u_n$ in M such that

$$\left\|\frac{1}{n}\sum_{i=1}^n u_ixu_i^*-\tau(x)1_M\right\|<\delta \tag{13.5}$$

for all $x\in F$. We set

$$F_1=F\cup\{u_1,\dots,u_n\}.$$

If $\xi\in\mathcal{H}$ with $\|\xi\|=1$ is such that $\|x\xi-\xi x\|<\delta$ for every $x\in F_1$, we have, for $x\in F$,

$$|\langle\xi,u_ixu_i^*\xi\rangle-\langle\xi,x\xi\rangle|<2\delta,$$

by taking $x_1=u_ix$ and $x_2=u_i^*$ in (13.4).

Then, it follows from (13.5) that, for $x\in F$,

$$|\tau(x)-\langle\xi,x\xi\rangle|<3\delta,$$

and therefore

$$W'(L^2(M);\varepsilon/3,F_1)\subset W(L^2(M);\varepsilon,F).$$

□

– *Weak containment.* Given a M-N-bimodule $\mathcal{K}$, we denote by $\mathcal{K}^{\oplus\infty}$ the Hilbert direct sum of countably many copies of $\mathcal{K}$.

DEFINITION 13.3.8. Let $\mathcal{H}$ and $\mathcal{K}$ be two M-N-bimodules. We say that $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$, and we write $\mathcal{H} \prec \mathcal{K}$, if $\mathcal{H}$ belongs to the closure of $\mathcal{K}^{\oplus\infty}$ in $\mathrm{Bimod}\,(M,N)$.

REMARK 13.3.9. A M-N-bimodule $\mathcal{H}$ gives rise to a representation $\pi_{\mathcal{H}}$ of the involutive algebra $M \odot N^{op}$ on $\mathcal{H}$. The analogue of the result mentioned in Remark 13.3.2 holds: one has $\mathcal{H} \prec \mathcal{K}$ if and only if $\|\pi_{\mathcal{H}}(x)\| \leq \|\pi_{\mathcal{K}}(x)\|$ for every $x \in M \odot N^{op}$ (see [**AD95**] for instance).

When $\mathcal{K}$ is a M-M-bimodule, then $L^2(M) \prec \mathcal{K}$ is equivalent to the following property: for every $\varepsilon > 0$ and every finite subset F of M, there exists a vector $\eta \in \mathcal{K}^{\oplus\infty}$ with

$$|\tau(xy) - \langle \eta, x\eta y\rangle| < \varepsilon,$$

for every $x, y \in F$ (and by Lemma 13.3.6 we may, if we wish, require that $\max_{x\in F} \|x\eta - \eta x\| < \varepsilon$).

The weak containment property of $L^2(M)$ can also be read as follows.

PROPOSITION 13.3.10. *Let $\mathcal{K}$ be a M-M-bimodule. The following conditions are equivalent:*

(i) *$L^2(M)$ is weakly contained in $\mathcal{K}$;*
(ii) (*resp.* (ii′)) *for every $\varepsilon > 0$ and every finite subset F of M there exists a subtracial and subunital completely positive* (*resp. a completely positive*) *map $\phi : M \to M$ such that*
  (a) $\max_{x\in F} \|\phi(x) - x\|_2 \leq \varepsilon$*;*
  (b) *ϕ is a finite sum of coefficients of $\mathcal{K}$.*

Let η be a left M-bounded vector of a M-M-bimodule $\mathcal{H}$ (for instance a subtracial vector) and denote by $\phi : x \mapsto L_\eta^* x L_\eta$ the associated completely positive map. In the proof of the above proposition, we will use repeatedly the following equality (see Lemma 8.4.5)

$$\forall x, y \in M, \langle \eta, x\eta y\rangle = \tau(\phi(x)y). \tag{13.6}$$

PROOF. Obviously (ii) $\Rightarrow$ (ii'). Let us show that (ii') $\Rightarrow$ (i). Let $\varepsilon > 0$ and F be given and let $\phi : M \to M$ be a completely positive map satisfying the conditions (a) and (b). There exists a left bounded vector $\eta = (\eta_1, \dots, \eta_n)$ in $\mathcal{K}^{\oplus n}$, for some integer n, such that $\phi(x) = L_\eta^* x L_\eta$ for every $x \in M$. Then we have, for $x, y \in F$,

$$|\tau(xy) - \langle \eta, x\eta y)\rangle| = |\tau(xy) - \tau(\phi(x)y)| \leq \|x - \phi(x)\|_2 \|y\|_2 \leq \max_{y\in F} \|y\|_2 \varepsilon.$$

Therefore, we have $L^2(M) \prec \mathcal{K}$.

(i) $\Rightarrow$ (ii) Assume that $L^2(M) \prec \mathcal{K}$ and let $\varepsilon > 0$ and F be given. We set $F_1 = F \cup F^* \cup \{x^*x : x \in F \cup F^*\}$. Given $\varepsilon_1 > 0$ and using the next lemma, we find a subtracial vector η in some finite direct sum of copies of $\mathcal{K}$ such that

$$|\tau(x) - \langle \eta, x\eta\rangle| \leq \varepsilon_1 \quad \text{and} \quad \|x\eta - \eta x\| \leq \varepsilon_1$$

for $x \in F_1$.

Let ϕ be the completely positive map defined by η. For $x \in F$, we have

$$\begin{aligned}
\|\phi(x) - x\|_2^2 &= \tau(\phi(x)^*\phi(x)) + \tau(x^*x) - \tau(\phi(x)^*x) - \tau(x^*\phi(x)) \\
&\leq 2\tau(x^*x) - \tau(\phi(x)^*x) - \tau(x^*\phi(x)) \\
&\leq |\tau(x^*x) - \langle \eta, x^*\eta x\rangle| + |\tau(x^*x) - \langle \eta, x\eta x^*\rangle| \\
&\leq |\tau(x^*x) - \langle \eta, x^*x\eta\rangle| + \|x\|\|x\eta - \eta x\| \\
&\qquad\qquad + |\tau(xx^*) - \langle \eta, xx^*\eta\rangle| + \|x\|\|x^*\eta - \eta x^*\| \\
&\leq 2\varepsilon_1(1 + \max_{x\in F}\|x\|).
\end{aligned}$$

It suffices to take ε_1 such that $2\varepsilon_1(1 + \max_{x\in F}\|x\|) \leq \varepsilon$ to get (ii). $\square$

In the lemma below, we use the following notation: given a M-M-bimodule $\mathcal{H}$ and $\eta \in \mathcal{H}$, we denote by ω_η^l the functional $x \mapsto \langle \eta, x\eta\rangle$ and by ω_η^r the functional $x \mapsto \langle \eta, \eta x\rangle$.

Lemma 13.3.11. *Let $\mathcal{K}$ be a M-M-bimodule and let (η_i) be a net in $\mathcal{K}^{\oplus\infty}$ such that for all $x \in M$,*

$$\lim_i \omega_{\eta_i}^l(x) = \tau(x) = \lim_i \omega_{\eta_i}^r(x), \tag{13.7}$$

$$\lim_i \|x\eta_i - \eta_i x\| = 0. \tag{13.8}$$

Then there exists a net (η_i') of subtracial vectors in $\mathcal{K}^{\oplus\infty}$ such that

$$\lim_i \left\|\omega_{\eta_i'}^l - \tau\right\| = 0 = \lim_i \left\|\omega_{\eta_i'}^r - \tau\right\|, \tag{13.9}$$

$$\lim_i \left\|x\eta_i' - \eta_i' x\right\| = 0, \quad \forall x \in M.$$

Moreover, if $\eta_i \in \mathcal{K}_0^{\oplus\infty}$ for all i, where $\mathcal{K}_0$ is a subbimodule of $\mathcal{K}$, then the η_i' may still be taken in $\mathcal{K}_0^{\oplus\infty}$.

If M is separable, we may replace nets by sequences.

Proof. We first claim that in (13.7), we may replace the weak convergence of the nets $(\omega_{\eta_i}^l)_i$ and $(\omega_{\eta_i}^r)_i$ by the $\|\cdot\|$-convergence. Indeed, given a finite subset F of M and $\varepsilon > 0$, our assumption implies that $(0,0)$ belongs to the $\sigma(M_*^2, M^2)$ closure of the set of $(\tau - \omega_\eta^l, \tau - \omega_\eta^r)$ where η runs over the set of elements such that $\max_{x\in F}\|x\eta - \eta x\| \leq \varepsilon$. By a classical convexity argument, we see that there is a convex combination $(\sum_{k=1}^n \lambda_k \omega_{\eta_{i_k}}^l, \sum_{k=1}^n \lambda_k \omega_{\eta_{i_k}}^r)$

such that

$$\left\|\tau - \sum_{k=1}^{n} \lambda_k \omega^l_{\eta_{i_k}}\right\| \leq \varepsilon, \quad \left\|\tau - \sum_{k=1}^{n} \lambda_k \omega^r_{\eta_{i_k}}\right\| \leq \varepsilon,$$
$$\max_{x\in F} \|x\eta_{i_k} - \eta_{i_k} x\| \leq \varepsilon, \, \forall k.$$

We denote by $\eta_{F,\varepsilon}$ the element $(\lambda_1^{1/2}\eta_{i_1}, \dots, \lambda_n^{1/2}\eta_{i_n})$, viewed as a vector in $\mathcal{K}^{\oplus\infty}$. Then we have

$$\left\|\tau - \omega^l_{\eta_{F,\varepsilon}}\right\| \leq \varepsilon, \quad \left\|\tau - \omega^r_{\eta_{F,\varepsilon}}\right\| \leq \varepsilon,$$
$$\max_{x\in F} \|x\eta_{F,\varepsilon} - \eta_{F,\varepsilon} x\| \leq \varepsilon.$$

So, we may now assume, for (η_i), the $\|\cdot\|$-convergence as in (13.9). We set $\omega^l_{\eta_i} = \tau(\cdot T_i)$ and $\omega^r_{\eta_i} = \tau(\cdot S_i)$ with $T_i, S_i \in L^1(M,\tau)_+$. Then we have $\lim_i \|T_i - 1\|_1 = 0 = \lim_i \|S_i - 1\|_1$. Therefore, by Lemma 13.1.11, we may assume that the η_i's are subtracial. The second assertion is a consequence of the remark following Lemma 13.1.11.

Finally, let us assume that M is separable. Let D be a countable subset of the unit ball $(M)_1$, dense in this ball with respect to the $\|\cdot\|_2$-topology. Since the η_i's are subtracial, it is easily seen that Condition (13.8) holds for all $x \in M$ if and only if it holds for all $x \in D$. It follows that we may replace nets by sequences. □

13.4. Back to amenable tracial von Neumann algebras

13.4.1. Amenability and asymptotically central nets. Recall that a countable group G is amenable if and only if its left regular representation λ_G almost has invariant vectors, or in other terms if and only if its trivial representation ι_G is weakly contained in λ_G. In the dictionary translating group representations into bimodules, ι_G corresponds to the trivial bimodule and λ_G corresponds to the coarse bimodule (see Section 13.1.3). We can reformulate Theorem 10.2.9 as follows.

Theorem 13.4.1. *Let (M,τ) be a tracial von Neumann algebra. The following conditions are equivalent:*

(i) *M is amenable;*
(ii) *${}_M L^2(M)_M$ belongs to the closure of ${}_M L^2(M) \otimes L^2(M)_M$ with respect to the Fell topology;*
(iii) *${}_M L^2(M)_M \prec {}_M L^2(M) \otimes L^2(M)_M$.*

Proof. (i) $\Leftrightarrow$ (ii) We identify the coarse bimodule $L^2(M) \otimes L^2(M)$ to the M-M-bimodule $\mathcal{S}^2(L^2(M))$ of Hilbert-Schmidt operators on $L^2(M)$. Then $L^2(M)$ belongs to the closure of $\mathcal{S}^2(L^2(M))$ with respect to the Fell topology if and only if there exists a net $(T_i)_i$ of Hilbert-Schmidt operators with $\|T_i\|_{2,\mathrm{Tr}} = 1$ for every i, such that $\lim_i \|xT_i - T_i x\|_{2,\mathrm{Tr}} = 0$ and $\lim_i \mathrm{Tr}(T_i^* x T_i) = \tau(x)$ for every $x \in M$. This is exactly the content of Property (3) in Theorem 10.2.9.

(ii) $\Rightarrow$ (iii) is obvious. Let us show that (iii) $\Rightarrow$ (i). Denote by $\mathcal{H}$ the Hilbert direct sum of countably many copies of ${}_M L^2(M) \otimes L^2(M)_M$, that we view as a left $\mathcal{B}(L^2(M))$-module. Assume that (iii) is satisfied. There exists a net $(\eta_i)_{i\in I}$ of unit vectors in $\mathcal{H}$ such that $\lim_i \|x\eta_i - \eta_i x\| = 0$ and $\lim_i \langle \eta_i, x\eta_i\rangle = \tau(x)$ for all $x \in M$. Let $\psi \in \mathcal{B}(L^2(M))^*$ be a cluster point of the net of states $T \mapsto \langle \eta_i, T\eta_i\rangle$ in the weak* topology. We have $\psi(x) = \tau(x)$ for $x \in M$. Moreover, for every unitary element $u \in M$ and $T \in \mathcal{B}(L^2(M))$ we have

$$\begin{aligned}\psi(uTu^*) &= \lim_j \left\langle \eta_{i_j}, uTu^*\eta_{i_j}\right\rangle = \lim_j \left\langle u^*\eta_{i_j}, Tu^*\eta_{i_j}\right\rangle \\ &= \lim_j \left\langle \eta_{i_j}u^*, (T\eta_{i_j})u^*\right\rangle = \lim_j \left\langle \eta_{i_j}, T\eta_{i_j}\right\rangle = \psi(T).\end{aligned}$$

It follows that ψ is a hypertrace for (M,τ) and therefore M is amenable. $\square$

13.4.2. Amenability and approximation of the identity. Another very useful characterisation of amenability for a group G, recalled in Proposition 10.1.2, is the following one: G is amenable if and only if there exists a net (φ_i) of finitely supported positive definite functions on G which converges pointwise to 1. We now state the analogue for tracial von Neumann algebras.

THEOREM 13.4.2. *Let (M,τ) be a tracial von Neumann algebra. The following conditions are equivalent:*

(i) *M is amenable;*
(ii) *there exists a net (ϕ_i) (or a sequence if M is separable) of subunital and subtracial finite rank completely positive maps from M to M such that for all $x \in M$,*

$$\lim_i \|\phi_i(x) - x\|_2 = 0.$$

PROOF. (i) $\Rightarrow$ (ii). Assume that M is amenable. By Theorem 13.4.1 and Proposition 13.3.10, there exists a net (ϕ_i) of subtracial elements completely positive maps $\phi_i : M \to M$, which are finite sums of coefficients of the coarse bimodule $L^2(M) \otimes L^2(M)$, such that $\lim_i \|\phi_i(x) - x\|_2 = 0$ for all $x \in M$. Moreover, by density of $L^2(M) \odot M$ in $L^2(M) \otimes L^2(M)$, we may take these coefficients associated to vectors in $L^2(M)\odot M$, as observed in the statement of Proposition 13.3.10.

Therefore, we have to show is that a completely positive map ϕ defined by a vector $\eta = \sum_{k=1}^n \xi_k \otimes m_k \in L^2(M) \odot M$ has a finite rank. A straightforward computation shows that $\phi : x \mapsto L_\eta^* x L_\eta$ is given by

$$\phi(x) = \sum_{i,j=1}^n \langle \xi_i, x\xi_j\rangle m_i^* m_j.$$

So, we see that ϕ is the composition of the completely positive map $x \in M \mapsto [\langle \xi_i, x\xi_j\rangle] \in M_n(\mathbb{C})$ and of the completely positive map $[a_{i,j}] \in M_n(\mathbb{C}) \mapsto \sum_{i,j} a_{i,j} m_i^* m_j \in M$.

To prove the converse, it suffices to show that every subunital and subtracial finite rank completely positive map $\phi : M \to M$ is a coefficient of $L^2(M) \otimes L^2(M)$. This is proved in the next lemma. $\square$

LEMMA 13.4.3. *Every subunital and subtracial finite rank completely positive map $\phi : M \to M$ is a coefficient of $L^2(M) \otimes L^2(M)$.*

PROOF. Let $\{y_1, \dots, y_n\}$ be an orthonormal basis of $\phi(M) \subset L^2(M)$. We have, for $x \in M$,

$$\phi(x) = \sum_{i=1}^{n} y_i \tau(y_i^* \phi(x)).$$

Observe that $x \mapsto \tau(y_i^* \phi(x))$ is $\|\cdot\|_2$-continuous since

$$|\tau(y_i^* \phi(x))|^2 \le \tau(y_i^* y_i) \tau(\phi(x)^* \phi(x)) \le \|x\|_2^2,$$

where the last inequality uses the fact that ϕ is subunital and subtracial.

Let $\eta_i \in L^2(M)$ such that $\tau(y_i^* \phi(x)) = \langle \eta_i, x \rangle$ for every $x \in M$. We set $\eta = \sum_i \eta_i \otimes y_i^*$ and $\xi = 1 \otimes 1$. These vectors are left bounded and $\phi(x) = L_\eta^* x L_\xi$. But $\phi(x^*) = \phi(x)^*$ and therefore $L_\eta^* x L_\xi = L_\xi^* x L_\eta$. It follows that

$$L_{\xi+\eta}^* x L_{\xi+\eta} = L_\xi^* x L_\xi + L_\eta^* x L_\eta + 2\phi(x).$$

Now, we deduce from Lemma 13.1.9 that ϕ is a coefficient of $L^2(M) \otimes L^2(M)$. $\square$

13.4.3. Relative amenability. Let H be a subgroup of a group G. The *quasi-regular representation* $\lambda_{G/H}$ is the unitary representation of G by translations in $\ell^2(G/H)$. We say that H is *co-amenable* in G if ι_G is weakly contained in $\lambda_{G/H}$. Again, there are many other equivalent definitions, among them the existence of a G-invariant mean on $\ell^\infty(G/H)$.[11] If we set $Q = L(H)$ and $M = L(G)$ we note that the M-M-bimodule corresponding to the representation $\lambda_{G/H}$ is $L^2(M) \otimes_Q L^2(M)$ (Exercise 13.12). It is not difficult to show that H is co-amenable in G if and only if the trivial M-M-bimodule $L^2(M)$ is weakly contained in $L^2(M) \otimes_Q L^2(M)$.[12]

More generally, in the setting of tracial von Neumann algebras, there is a useful notion of relative amenability that is characterized below by equivalent conditions similar to the characterisations of an amenable tracial von Neumann algebra given in Chapter 10 and in Theorem 13.4.1. For an inclusion $P \subset N$ of von Neumann algebras, we say that a state ψ on N is *P-central* if $\psi(xy) = \psi(yx)$ for every $x \in P$ and $y \in N$.

THEOREM 13.4.4. *Let P, Q be two von Neumann subalgebras of a tracial von Neumann algebra (M, τ). The following conditions are equivalent:*

(i) *there exists a conditional expectation from $\langle M, e_Q \rangle$ onto P whose restriction to M is E_P^M;*

[11] This notion is studied in detail in [**Eym72**].

[12] See [**AD95**, Prop. 3.5].

(ii) *there is a P-central state ψ on $\langle M, e_Q\rangle$ such that $\psi_{|_M} = \tau$;*
(iii) *there is a P-central state ψ on $\langle M, e_Q\rangle$ such that ψ is normal on M and faithful on $\mathcal{Z}(P' \cap M)$;*
(iv) *there is a net (ξ_i) of norm-one vectors in $L^2(\langle M, e_Q\rangle)$ such that $\lim_i \|x\xi_i - \xi_i x\| = 0$ for every $x \in P$ and $\lim_i \langle \xi_i, x\xi_i\rangle = \tau(x)$ for every $x \in M$;*
(v) *${}_M L^2(M)_P$ is weakly contained in ${}_M L^2(M) \otimes_Q L^2(M)_P$.*

PROOF. The equivalence between (i) and (ii) is a straightforward generalisation of Proposition 10.2.5, which is the particular case where $Q = \mathbb{C}$ and $P = M$.

The implication (ii) $\Rightarrow$ (iii) is obvious. Conversely, assume that (iii) holds. Let a be the unique element in $L^1(M)_+$ such that $\psi(x) = \tau(xa)$ for every $x \in M$. Since ψ is P-central, we have $uau^* = a$ for every $u \in \mathcal{U}(P)$ and therefore $a \in L^1(P' \cap M)_+$. Let I be the directed set of finite subsets of $\mathcal{U}(P' \cap M)$. For $i = \{u_1, \dots, u_n\} \in I$, we set

$$a_i = \frac{1}{n}\sum_{k=1}^{n} u_k a u_k^* \in L^1(P' \cap M)_+,$$

and for $m \in \mathbb{N}$, we set $a_{i,m} = E^c_{1/m}(a_i) a_i^{-1/2} \in P' \cap M$, where $E^c_{1/m}(a_i)$ is the spectral projection of a_i relative to the interval $(1/m, +\infty)$. Next, we introduce the positive linear functionals $\varphi_{i,m}$ on $\langle M, e_Q\rangle$ defined by

$$\varphi_{i,m}(x) = \frac{1}{n}\sum_{k=1}^{n} \psi(u_k^* a_{i,m} x a_{i,m} u_k)$$

for $x \in \langle M, e_Q\rangle$. Since $a_{i,m}u_k \in P' \cap M$, the functional $\varphi_{i,m}$ is still P-central. Moreover, for $x \in M$, we have $\varphi_{i,m}(x) = \tau(x E^c_{1/m}(a_i))$. Now, let us observe that, in the s.o. topology,

$$\lim_i \lim_m E^c_{1/m}(a_i) = \lim_i s(a_i) = \lim_i \bigvee_{u \in i} s(uau^*) = z$$

where z is the smallest projection in $\mathcal{Z}(P' \cap M)$ such that $az = a$. Since $\psi(1 - z) = \tau((1 - z)a) = 0$, and ψ is faithful on $\mathcal{Z}(P' \cap M)$, we see that $z = 1$. Hence, the state $\varphi = \lim_i \lim_m \varphi_{i,m}$ is P-central and τ is its restriction to M. Thus (ii) is satisfied.

The M-M bimodules $L^2(M) \otimes_Q L^2(M)$ and $L^2(\langle M, e_Q\rangle)$ are canonically isomorphic (see Exercise 13.13). The proof of (iv) $\Leftrightarrow$ (ii) is similar to the proof of the equivalence of (3) with the existence of a hypertrace in Theorem 10.2.9. We just replace the semi-finite von Neumann algebra $\mathcal{B}(L^2(M))$ by the semi-finite von Neumann algebra $\langle M, e_Q\rangle$ and we observe that $L^2(\langle M, e_Q\rangle)$ plays the role of Hilbert-Schmidt operators. Moreover, we only consider the right P-action on $L^2(\langle M, e_Q\rangle)$.

Let us prove the equivalence between (v) and the four other conditions. By Exercise 13.14, ${}_M L^2(M)_P$ has a basis of neighbourhoods of the form

$W(\varepsilon, E, F)$, where E, F are finite subsets of M and P respectively and $W(\varepsilon, E, F)$ is the set of M-P-bimodules $\mathcal{H}$ such that there exists $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ and

$$\|x\xi - \xi x\| < \varepsilon \text{ for all } x \in P, \quad |\langle \xi, x\xi\rangle - \tau(x)| < \varepsilon \text{ for all } x \in M.$$

It follows that (iv) $\Rightarrow$ (v). Assume now that (v) holds. Then there is a net (ξ_i) in an infinite multiple of ${}_M L^2(M) \otimes_Q L^2(M)_P$ such that

$$\lim_i \|x\xi_i - \xi_i x\| = 0 \text{ for all } x \in P, \quad \lim_i \langle \xi_i, x\xi_i\rangle = \tau(x) \text{ for all } x \in M.$$

We let $\langle M, e_Q\rangle$ act on $L^2(M) \otimes_Q L^2(M)$ by $x(\eta \otimes_Q \eta') = (x\eta) \otimes_Q \eta'$. Let $\psi \in \langle M, e_Q\rangle^*$ be a cluster point of the net of states $x \mapsto \langle \xi_i, x\xi_i\rangle$ in the weak* topology. Then ψ satisfies the conditions of (ii). $\square$

Definition 13.4.5. Let P, Q be two von Neumann subalgebras of a tracial von Neumann algebra (M, τ). We say that P is *amenable relative to Q inside M* if the equivalent assertions of Theorem 13.4.4 are satisfied.

If M is amenable relative to Q inside M (case $P = M$), one says that M is *amenable relative to Q*, or that Q is *co-amenable in M*. In particular, M is amenable if and only if it is amenable relative to $Q = \mathbb{C}1$.

Remark 13.4.6. If M is amenable, then P is amenable relative to Q inside M for every pair (P, Q) of von Neumann subalgebras, since there exists an hypertrace.

As an example, consider a trace preserving action of a group G on a tracial von Neumann algebra (Q, τ). Then Q is co-amenable in $M = Q \rtimes G$ if an only if G is amenable. Indeed, we have observed in Subsection 9.4.3 that $\langle M, e_Q\rangle$ is $Q\overline{\otimes}\mathcal{B}(\ell^2(G))$ and it is shown in [**AD79**, Proposition 4.1] that the existence of a conditional expectation from $Q\overline{\otimes}\mathcal{B}(\ell^2(G))$ onto $Q \rtimes G$ is equivalent to the amenability of G (see [**Pop86a**, Theorem. 3.2.4. (3)] for another proof).

Remark 13.4.7. We will need later a slightly more general version of relative amenability where P is only a von Neumann subalgebra of pMp for some non-zero projection p of M, that is, p is the unit of P. In this situation, we say that P is amenable relative to Q inside M if $P \oplus \mathbb{C}(1-p)$ is *amenable relative to Q inside M*. The version of Theorem 13.4.4 in this setting is easily spelled out. In particular this relative amenability property can be expressed by the following equivalent properties:

(ii) *there exists a P-central state ψ on $p\langle M, e_Q\rangle p$ such that $\psi(pxp) = \tau(pxp)/\tau(p)$ for every $x \in M$;*
(iii) *there is a P-central state ψ on $p\langle M, e_Q\rangle p$ such that ψ is normal on pMp and faithful on $\mathcal{Z}(P' \cap pMp)$;*
(iv) *there is a net (ξ_i) of norm-one vectors in $L^2(p\langle M, e_Q\rangle p)$ such that $\lim_i \|x\xi_i - \xi_i x\| = 0$ for every $x \in P$ and $\lim_i \langle \xi_i, x\xi_i\rangle = \tau(x)$ for every $x \in pMp$.*

Whenever Q is an amenable subalgebra of M, the amenability of a von Neumann subalgebra P of pMp is equivalent to its amenability relative to Q inside M (see Exercise 13.16). It is sometimes a handy way to show that P is amenable. We will see an illustration of this observation in Chapter 19 where we will use the following lemma which provides a useful criterion for relative amenability.

LEMMA 13.4.8. *Let P, Q be two von Neumann subalgebras of a tracial von Neumann algebra (M,τ). We assume that there is a Q-M-bimodule $\mathcal{K}$ and a net $(\xi_i)_{i\in I}$ of elements ξ_i in a multiple $\mathcal{H}$ of $L^2(M)\otimes_Q \mathcal{K}$ such that:*

(a) $\limsup_i \|x\xi_i\| \le \|x\|_2$ *for all* $x \in M$*;*
(b) $\limsup_i \|\xi_i\| > 0$*;*
(c) $\lim_i \|y\xi_i - \xi_i y\| = 0$ *for all* $y \in P$*.*

Then, there exists a non-zero projection $p' \in \mathcal{Z}(P' \cap M)$ such that Pp' is amenable relative to Q inside M.

PROOF. We first claim that we may assume, in addition, that

$$\lim_i \|\xi_i\| > 0.$$

Indeed, let us introduce the set J of all triples $j = (E, F, \varepsilon)$, where $E \subset M$, $F \subset P$ are finite sets and $\varepsilon > 0$. This set J is directed by

$$(E, F, \varepsilon) \le (E', F', \varepsilon') \text{ if } E \subset E',\ F \subset F',\ \varepsilon' \le \varepsilon.$$

Let us fix $j = (E, F, \varepsilon)$. Using Conditions (a) and (c) we find $i_0 \in I$ such that $\|x\xi_i\| \le \|x\|_2 + \varepsilon$ and $\|y\xi_i - \xi_i y\| \le \varepsilon$ for $x \in E$, $y \in F$ and $i \ge i_0$. We set $\delta = \limsup_i \|\xi_i\| > 0$. Let $i \ge i_0$ be such that $\|\xi_i\| \ge \delta/2$. Then we define $\eta_j = \xi_i$ for this choice of i. A straightforward verification shows that $\limsup_j \|x\eta_j\| \le \|x\|_2$ for all $x \in M$, $\liminf_j \|\eta_j\| > 0$, and $\lim_j \|y\eta_j - \eta_j y\| = 0$ for $y \in P$. By taking an appropriate subnet we may assume that $\lim_j \|\eta_j\| > 0$. This proves our claim.

Observe that $L^2(M)\otimes_Q\mathcal{K}$ is a left $\langle M, e_Q\rangle$-module. Let ψ_j be the normal state on $\langle M, e_Q\rangle$ defined by

$$\psi_j(x) = \frac{1}{\|\eta_j\|^2}\langle \eta_j, x\eta_j\rangle$$

and let $\psi : \langle M, e_Q\rangle \to \mathbb{C}$ be a weak* limit of a subnet of $(\psi_j)_{j\in J}$. Then ψ is P-central and its restriction to M is normal since it is majorized by a multiple of τ, thanks to Condition (a) and the fact that $\lim_j \|\eta_j\| > 0$. Finally, if p' denote the minimal projection in $\mathcal{Z}(P'\cap M)$ such that $\psi(p') = 1$, we get that the restriction of ψ to $\mathcal{Z}((Pp')' \cap p'Mp')$ is faithful and it follows that Pp' is amenable relative to Q inside M. $\square$

Exercises

EXERCISE 13.1. (i) Let M be a tracial von Neumann algebra. Let $\mathcal{H}$ be a left M-module and let $\xi \in \mathcal{H}$. Show that there exists a sequence (p_n) of projections in M which converges to the identity in the s.o. topology and such that $p_n\xi$ is right bounded for every n (Hint: consider the functional $x \mapsto \langle \xi, x\xi \rangle$ and use the Radon-Nikodým theorem).

(ii) Let M, N be two tracial von Neumann algebras and let $\mathcal{H}$ be a M-N-bimodule. Show that the subspace of vectors which are both right M-bounded and left N-bounded is dense in $\mathcal{K}$.

EXERCISE 13.2. Let $\mathcal{H}_N$ and ${}_N\mathcal{K}$ be two modules on a tracial von Neumann algebra N. For any $n \in N$ and for either $\xi \in \mathcal{H}^0$ and $\eta \in \mathcal{K}$ or $\xi \in \mathcal{H}$ and $\eta \in {}^0\mathcal{K}$, show that $\xi \otimes_N n\eta = \xi n \otimes_N \eta$.

EXERCISE 13.3. Let $\mathcal{H}$ be a M-N-bimodule and $\mathcal{K}$ be a N-P-bimodule.

(i) For $\xi \in \mathcal{H}^0$, show that the map $L_{\mathcal{K}}(\xi) : \eta \mapsto \xi \otimes_N \eta$ is bounded and P-linear from $\mathcal{K}_P$ into $(\mathcal{H} \otimes_N \mathcal{K})_P$. If in addition, $\eta \in \mathcal{K}^0$, show that $\xi \otimes_N \eta$ is left P-bounded and that $L_{\xi \otimes_N \eta} = L_{\mathcal{K}}(\xi) \circ L_\eta$.

(ii) Prove the similar statement for right bounded vectors.

EXERCISE 13.4. Let $\mathcal{H}$ and $\mathcal{K}$ as above. Let (ξ_i) (resp. (η_j)) be a family of left bounded vectors in $\mathcal{H}$ (resp. $\mathcal{K}$) such that $\sum_i L_{\xi_i} L_{\xi_i}^* = \mathrm{Id}_{\mathcal{H}}$ (resp. $\sum_i L_{\eta_i} L_{\eta_i}^* = \mathrm{Id}_{\mathcal{K}}$). Show that $\sum_{i,j} L_{\xi_i \otimes_N \eta_j} (L_{\xi_i \otimes_N \eta_j})^* = \mathrm{Id}_{\mathcal{H} \otimes_N \mathcal{K}}$.

EXERCISE 13.5. Let $\mathcal{H}$ be a M-N-bimodule, where M, N are II_1 factors. Assume that $\dim(\mathcal{H}_N) < +\infty$.

(i) Show that every left N-bounded vector is also right M-bounded.

(ii) Let $(\xi_i)_{1 \leq i \leq n}$ be an orthonormal basis of $\mathcal{H}_N$. Show that the operator $\sum_{i=1}^n {}_M\langle \xi_i, \xi_i \rangle$ is scalar, equal to $\dim(\mathcal{H}_N) 1_M$.

EXERCISE 13.6. Let N, P be II_1 factors, $\mathcal{H}$ a right N-module and $\mathcal{K}$ a N-P-bimodule. Assume that $\dim(\mathcal{H}_N) < +\infty$ and $\dim(\mathcal{K}_P) < +\infty$. Show that $\dim((\mathcal{H} \otimes_N \mathcal{K})_P) = \dim(\mathcal{H}_N) \dim(\mathcal{K}_P)$.

We define the *Jones' index of a M-N-bimodule* $\mathcal{H}$ to be $\dim({}_M\mathcal{H}) \dim(\mathcal{H}_N)$. It is tempting to consider its square root $[{}_M\mathcal{H}_N]$ as the dimension of the bimodule, but this quantity does not have the expected properties: for bimodules over II_1 factors, we have well $[{}_M(\mathcal{H} \otimes_N \mathcal{K})_P] = [{}_M\mathcal{H}_N][{}_N\mathcal{K}_P]$ as shown in the next exercise, but only $[{}_M(\mathcal{H} \oplus \mathcal{K})_N] \geq [{}_M\mathcal{H}_N] + [{}_M\mathcal{K}_N]$. However, for a II_1 factor M, there is a good dimension function (*i.e.,* both additive and multiplicative) on the set of (equivalence classes) of finite Jones' index M-M-bimodules, whose value is $[{}_M\mathcal{H}_M]$ when $\mathcal{H}$ is irreducible, and is extended by additivity on any finite Jones' index M-M-bimodule (see [**Rob95**], [**LR97**]).

EXERCISE 13.7. Let M, N, P be II_1 factors.

(i) Let $\mathcal{H}$ be a M-N-bimodule. Show that $[{}_M\mathcal{H}_N] = [{}_N\overline{\mathcal{H}}_M]$.

(ii) Let $\mathcal{H}$ and $\mathcal{K}$ be two M-N-bimodules of finite index. Show that

$$[{}_M\mathcal{H}_N \oplus_M \mathcal{K}_N] \geq [{}_M\mathcal{H}_N] + [{}_M\mathcal{K}_N].$$

(iii) Let $\mathcal{H}$ be a M-N-module and let $\mathcal{K}$ be a N-P-bimodule, both of finite index. Show that

$$[{}_M\mathcal{H} \otimes_N \mathcal{K}_P] = [{}_M\mathcal{H}_N][{}_N\mathcal{K}_P].$$

EXERCISE 13.8. Let $\alpha \in \mathrm{Aut}(M)$. Show that $\mathcal{K}(\alpha) = \overline{\mathcal{H}(\alpha)}$ is canonically isomorphic to the M-M-bimodule $L^2(M)$ equipped with the actions $x\xi y = x\xi\alpha(y)$.

EXERCISE 13.9. Let α, β be two automorphisms of M. Show that the M-M-bimodules $\mathcal{H}(\beta \circ \alpha)$ and $\mathcal{H}(\alpha) \otimes_M \mathcal{H}(\beta)$ are canonically isomorphic (and similarly for $\mathcal{K}(\beta \circ \alpha)$ and $\mathcal{K}(\beta) \otimes_M \mathcal{K}(\alpha)$).[13]

EXERCISE 13.10. Let M, N, p and α as in Proposition 13.1.5 and let ${}_N\mathcal{K}_P$ be a N-P-bimodule, where P is a II_1 factor.

(i) Show that the M-P-bimodules ${}_{\alpha(M)}\big(p(\ell^2_n \otimes L^2(N))\big) \otimes_N \mathcal{K}_P$ and ${}_{\alpha(M)}\big(p(\ell^2_n \otimes \mathcal{K})\big)_P$ are isomorphic.

(ii) We now consider a finite index inclusion $\beta : N \to qM_m(P)q$ and take ${}_N\mathcal{K}_P = {}_N\mathcal{H}(\beta)_P$. Show that ${}_M\mathcal{H}(\alpha) \otimes_N \mathcal{H}(\beta)_P$ is isomorphic to ${}_M\mathcal{H}((\mathrm{Id}_{\mathcal{B}(\ell^2_n)} \otimes \beta) \circ \alpha)_P$, thus extending the previous exercise.

EXERCISE 13.11. Let M be a II_1 factor and denote by $\mathrm{Bimod}_1(M)$ the set of equivalence classes of M-M-bimodules of index 1.

(i) Show that $\mathrm{Bimod}_1(M)$ equipped with the tensor product and the contragredient map is a group.

(ii Show that $\mathrm{Bimod}_1(M)$ is the set of equivalence classes of bimodules $\mathcal{H}(\psi)$ where ψ ranges over the isomorphisms from M onto some $p(M_n(\mathbb{C}) \otimes M)p$.

(iii) Show that $\mathcal{H}(\psi) \mapsto (\mathrm{Tr}\otimes\tau)(p)$ induces a well-define homomorphism from $\mathrm{Bimod}_1(M)$ onto $\mathfrak{F}(M)$ whose kernel is $\mathrm{Out}(M)$, so that we have a short exact sequence of groups

$$1 \to \mathrm{Out}(M) \to \mathrm{Bimod}_1(M) \to \mathcal{F}(M) \to 1.$$

EXERCISE 13.12. Let G be a group and $M = L(G)$. Prove that the following M-M-bimodules are canonically isomorphic:

(i) $\mathcal{H}(\overline{\pi})$ and $\overline{\mathcal{H}(\pi)}$ for every representation π of G;

(ii) $\mathcal{H}(\pi_1 \otimes \pi_2)$ and $\mathcal{H}(\pi_1) \otimes_M \mathcal{H}(\pi_2)$ for every representations π_1, π_2 of G;

(iii) $\mathcal{H}(\lambda_{G/H})$ and $\ell^2(G) \otimes_{L(H)} \ell^2(G)$, where $\lambda_{G/H}$ is the quasi-regular representation of G associated with a subgroup H.

EXERCISE 13.13. Let Q be a von Neumann subalgebra of a tracial von Neumann algebra (M, τ). Show that the map $me_Qm_1 \mapsto \widehat{m} \otimes_Q \widehat{m_1}$ induces an isomorphism from the $\langle M, e_Q\rangle$-$\langle M, e_Q\rangle$-bimodule $L^2(\langle M, e_Q\rangle, \widehat{\tau})$ onto the $\langle M, e_Q\rangle$-$\langle M, e_Q\rangle$-bimodule $L^2(M,\tau) \otimes_Q L^2(M,\tau)$.

EXERCISE 13.14. Let P be a von Neumann subalgebras of a tracial von Neumann algebra (M, τ). Show ${}_ML^2(M)_P$ has a basis of neighbourhoods of the form $W(\varepsilon, E, F)$, where E, F are finite subsets of M and P respectively

[13]Therefore, $\alpha \mapsto \mathcal{K}(\alpha)$ behaves better with respect to the composition, and is often taken as the right definition of the bimodule defined by α.

and $W(\varepsilon, E, F)$ is the set of M-P-bimodules $\mathcal{H}$ such that there exists $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ and

$$\|x\xi - \xi x\| < \varepsilon \text{ for all } x \in F, \quad |\langle \xi, x\xi \rangle - \tau(x)| < \varepsilon \text{ for all } x \in M.$$

EXERCISE 13.15. Let M, N, P be tracial von Neumann algebras and let ${}_M\mathcal{H}_N$, ${}_M\mathcal{K}_N$, ${}_N\mathcal{L}_P$ be bimodules.

(i) Show that ${}_M\mathcal{H}_N$ belongs to the closure of ${}_M\mathcal{K}_N$ in the Fell topology if and only if for every finite subsets E of M, F of N and $S = \{\xi_1, \dots, \xi_k\}$ of $\mathcal{H}^0$, and for every neighbourhood $\mathcal{V}$ of 0 in M with respect to the w.o. topology, there exist $\eta_1, \dots, \eta_k$ in $\mathcal{K}^0$ such that

$$\langle \xi_i, x\xi_j y \rangle_M - \langle \eta_i, x\eta_j y \rangle_M \in \mathcal{V}$$

for every i, j, every $x \in E$ and every $y \in N$.

(ii) Assume that ${}_M\mathcal{H}_N \prec {}_M\mathcal{K}_N$. Show that

$${}_M\mathcal{H}_N \otimes_N \mathcal{L}_P \prec {}_M\mathcal{K}_N \otimes_N \mathcal{L}_P.$$

EXERCISE 13.16. Let Q be an amenable von Neumann subalgebra of a tracial von Neumann algebra (M, τ) and let P be a von Neumann subalgebra of pMp for some non-zero projection of M. Show that P is amenable with respect to Q inside M if and only if P is amenable

Notes

The notion of bimodule was introduced by Connes in the beginning of the eighties. He was motivated by the need of developing the right framework in order to define property (T) for II_1 factors [**Con82**] (see the next chapter). The content of his unpublished manuscript notes that were circulated at that time may be found in his book [**Con94**, V. Appendix B] where bimodules are called *correspondences*. The Fell topology on the space of bimodules is described in [**CJ85**].

The subject was further developed in [**Pop86a**] and nowadays proves to be unvaluable for the study of the structure of von Neumann algebras and in particular to translate, in the setting of von Neumann algebras, properties of groups that are expressed in terms of representations.

Assuming that M is a II_1 factor, a good point of view is to consider M-M-bimodules of finite index as generalized symmetries. For instance, the set $\mathrm{Bimod}_f(M)$ of equivalence classes of such bimodules, equipped with the direct sum and the tensor product, gives informations on the groups $\mathrm{Out}\,(M)$ and $\mathfrak{F}(M)$ (see Exercise 13.11). We have $\mathbb{N}^* \subset \mathrm{Bimod}_f(M)$ where an integer n is identified with the multiple $\ell^2_n \otimes L^2(M)$ of the trivial M-M-bimodule $L^2(M)$. Explicit computations of $\mathrm{Bimod}_f(M)$ have been achieved in [**Vae08**], with in particular explicit examples for II_1 factors M for which $\mathrm{Bimod}_f(M)$ is trivial, that is, reduced to $\mathbb{N}^*$. In this striking case the groups $\mathrm{Out}\,(M)$ and $\mathfrak{F}(M)$ are trivial and M has no other finite index subfactor than the trivial ones of index n^2, $n \geq 1$.

The notion of amenability relative to a von Neumann subalgebra was introduced in [**Pop86a**]. The more general notion described in Definition 13.4.5 is due to Ozawa and Popa [**OP10a**] and is very useful, in particular to study group measure space II_1 factors (see [**OP10a**], [**Ioa15**], [**PV14a**], [**PV14b**]). Section 13.4.3 comes from [**OP10a**]. For the lemma 13.4.8, we have followed the slight modification exposed in [**Ioa15**, Lemma 2.3].

CHAPTER 14

Kazhdan property (T)

In analogy with Kazhdan property (T) and relative property (T) in group theory, we introduce and study in this chapter the notions of property (T) for a tracial von Neumann agebra (M, τ) and more generally of relative property (T) for an inclusion $B \subset M$. These notions are defined in terms of rigid behaviours of completely positive maps or bimodules, which correspond respectively to rigid behaviours of positive definite functions and unitary represention in case of groups. A crucial and expected feature of II_1 factors with Property (T) is their lack of flexibility. For instance, we will see that their groups of outer automorphisms and their fundamental groups are countable. Separability arguments and rigidity are also used to show that the functor $G \mapsto L(G)$ defined on the ICC groups with Property (T) is at most countable to one and that there is no separable universal II_1 factor.

14.1. Kazhdan property (T) for groups

We first briefly recall some facts in the group case.

Definition 14.1.1. Let H be a subgroup of a group G. We say that the pair (G, H) has the *relative property* (T) (or that $H \subset G$ is a *rigid embedding*, or that H is a *relatively rigid subgroup* of G) if every unitary representation π of G which almost has G-invariant vectors (*i.e.*, $\iota_G \prec \pi$) has a non-zero H-invariant vector.

We say that G has the (Kazdhan) property (T), if the pair (G, G) has the *relative property* (T).

We list below a few other characterisations of relative property (T).

Proposition 14.1.2. *The following conditions are equivalent:*

(a) *the pair (G, H) has the relative property* (T)*;*
(b) *there exist a finite subset F of G and $\delta > 0$ such that if $(\pi, \mathcal{H})$ is a unitary representation of G and $\xi \in \mathcal{H}$ is a unit vector satisfying $\max_{g \in F} \|\pi(g)\xi - \xi\| \leq \delta$, then there is a non-zero H-invariant vector $\eta \in \mathcal{H}$;*
(c) *for every $\varepsilon > 0$, there exist a finite subset F of G and $\delta > 0$ such that if $(\pi, \mathcal{H})$ is a unitary representation of G and $\xi \in \mathcal{H}$ is a unit vector satisfying $\max_{g \in F} \|\pi(g)\xi - \xi\| \leq \delta$, then there is a H-invariant vector $\eta \in \mathcal{H}$ with $\|\xi - \eta\| \leq \varepsilon$;*
(d) *every net of positive definite functions on G that converges pointwise to the constant function 1 converges uniformly on H;*

(e) *for every $\varepsilon > 0$, there exist a finite subset F of G and $\delta > 0$ such that if φ is a positive definite function satisfying* $\max_{g\in F} |\varphi(g) - 1| \le \delta$, *then one has* $\sup_{h\in H} |\varphi(h) - 1| \le \varepsilon$.

The equivalence between (a) and (b) as well as the equivalence between (c), (d) and (e) are easily proved and (c) $\Rightarrow$ (b) is obvious. The hardest part (when H is not a normal subgroup of G) is (b) $\Rightarrow$ (c).[1] Condition (b) means that there exists a neighbourhood in $\mathcal{R}ep(G)$ of the trivial representation of G such that every representation in this neighbourhood has a non-zero H-invariant vector.

Examples of groups with Property (T) are plentiful. For instance, lattices in higher rank semi-simple Lie groups and in $\mathrm{Sp}(1, n)$ are such groups. The pair $(\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}), \mathbb{Z}^2)$ has the relative property (T) and more generally, for every non amenable subgroup G of $\mathrm{SL}(2, \mathbb{Z})$ (*e.g.* $\mathbb{F}_2$), the pair $(\mathbb{Z}^2 \rtimes G, \mathbb{Z}^2)$ has the relative property (T). References for all these facts are given in the notes at the end of the chapter.

Let us point out that a discrete amenable group has property (T) only when finite.

14.2. Relative property (T) for von Neumann algebras

Let (M, τ) be a tracial von Neumann algebra and B a von Neumann subalgebra of M. Our definition of rigidity for the inclusion (M, B) is modelled on Condition (e) in Proposition 14.1.2. Unlike what we often do, in this section we do not assume that M is separable. In fact, we will see in the next section that a II_1 factor which has the property (T) is automatically separable.

Definition 14.2.1. We say that the pair (M, B) has the *relative property* (T) (or that $B \subset M$ is a *rigid embedding*, or that B is a *relatively rigid von Neumann subalgebra* of M) if for every $\varepsilon > 0$, there exist a finite subset F of M and $\delta > 0$ such that whenever $\phi : M \to M$ is a subunital and subtracial completely positive map[2] satisfying $\max_{x\in F} \|\phi(x) - x\|_2 \le \delta$, then one has $\|\phi(b) - b\|_2 \le \varepsilon$ for every $b \in B$ with $\|b\|_\infty \le 1$. Whenever the pair (M, M) has the relative property (T), we say that M has the *property* (T).

Remark 14.2.2. We have dropped τ in the definition since it can be shown that it is in fact independent of the choice of the normal faithful trace (see [**Pop06a**, Proposition 4.1]).

In the definition, we may limit ourself to tracial and unital completely positive maps by replacing, if necessary ϕ by the unital tracial completely

[1] See [**Jol05**].

[2] Recall that ϕ is subunital if $\phi(1) \le 1$ and subtracial if $\tau \circ \phi \le \tau$. In particular, ϕ is normal by Proposition 2.5.11.

positive map $\widetilde{\phi}$ defined as

$$\widetilde{\phi}(x) = \phi(x) + \frac{(\tau - \tau \circ \phi)(x)}{(\tau - \tau \circ \phi)(1)}\big(1 - \phi(1)\big)$$

($\widetilde{\phi} = \phi + (\tau - \tau \circ \phi)1$ if $\phi(1) = 1$). We leave the details to the reader.

We now prove a characterisation of rigidity analogous to (c) in Proposition 14.1.2. We will use the correspondence between completely positive maps and bimodules described in Section 13.1.2 and in particular the two following inequalities.

LEMMA 14.2.3. *Let $\mathcal{H}$ be a M-M-bimodule, $\xi \in \mathcal{H}$ a tracial vector*[3] *and ϕ the corresponding tracial and unital completely positive map. Then, for every $x \in M$, we have*

$$\|x\xi - \xi x\|^2 \leq 2\|\phi(x) - x\|_2 \|x\|_2, \tag{14.1}$$

$$\|\phi(x) - x\|_2 \leq \|x\xi - \xi x\|. \tag{14.2}$$

PROOF. Recall that $\phi(x) = L_\xi^* x L_\xi$ (where $L_\xi : L^2(M) \to \mathcal{H}$ is the right M-linear map defined by ξ), so that

$$\langle \phi(x), y \rangle_{L^2(M)} = \tau(\phi(x)^* y) = \langle x\xi, \xi y \rangle_{\mathcal{H}}$$

for every $x, y \in M$. Then we have

$$\begin{aligned} \|x\xi - \xi x\|^2 &= \|x\xi\|^2 + \|\xi x\|^2 - 2\Re\langle x\xi, \xi x\rangle \\ &= 2\|x\|_2^2 - 2\Re\tau(\phi(x)x^*) \leq 2\|\phi(x) - x\|_2 \|x\|_2. \end{aligned}$$

The second inequality is given by

$$\begin{aligned} \|\phi(x) - x\|_2^2 &= \|\phi(x)\|_2^2 + \|x\|_2^2 - 2\Re\langle \phi(x), x\rangle \\ &\leq 2\|x\|_2^2 - 2\Re\langle x\xi, \xi x\rangle = \|x\xi - \xi x\|^2. \end{aligned}$$

$\square$

PROPOSITION 14.2.4. *Let (M, τ) be a tracial von Neumann algebra and B a von Neumann subalgebra of M. The following conditions are equivalent:*

(i) *$B \subset M$ is a rigid embedding;*
(ii) *for every $\varepsilon' > 0$, there exist a finite subset F of M and $\delta > 0$ such that for any M-M-bimodule $\mathcal{H}$ and any tracial vector $\xi \in \mathcal{H}$ satisfying $\max_{x \in F} \|x\xi - \xi x\| \leq \delta$, there exists a B-central vector $\eta \in \mathcal{H}$ with $\|\xi - \eta\| \leq \varepsilon'$.*

PROOF. (i) $\Rightarrow$ (ii). Let $\varepsilon' > 0$ be given and set $\varepsilon = (\varepsilon')^2/2$. For this ε, consider F and δ as in Definition 14.2.1. Let $\mathcal{H}$ be a M-M-bimodule and $\xi \in \mathcal{H}$ a tracial vector such that $\|x\xi - \xi x\| \leq \delta$ for every $x \in F$. Let ϕ be the corresponding tracial and unital completely positive map. We have

$$\|\phi(x) - x\|_2^2 \leq \|x\xi - \xi x\|^2 \leq \delta^2.$$

[3] *i.e.*, $\langle \xi, x\xi \rangle = \tau(x) = \langle \xi, \xi x \rangle$ for every $x \in M$.

It follows that $\|\phi(b) - b\|_2 \le \varepsilon$ for b in the unit ball of B. Then, for $u \in \mathcal{U}(B)$ we get

$$\begin{aligned}\|\xi - u\xi u^*\|^2 &= 2 - 2\Re\tau(\phi(u)u^*)\\ &= 2\Re\tau((u-\phi(u))u^*) \le 2\|u - \phi(u)\|_2 \le 2\varepsilon.\end{aligned}$$

Let $\eta \in \mathcal{H}$ be the element of smallest norm in the closed convex hull of

$$\{u\xi u^* : u \in \mathcal{U}(B)\}.$$

We have $u\eta u^* = \eta$ for every $u \in \mathcal{U}(B)$, and so η is B-central. Morever, we see that $\|\xi - \eta\| \le (2\varepsilon)^{1/2} = \varepsilon'$.

(ii) $\Rightarrow$ (i). Let $\varepsilon > 0$ be given and set $\varepsilon' = \varepsilon/2$. For this ε', consider F and δ satisfying condition (ii). Let $\phi : M \to M$ be a tracial and unital completely positive map such that, for $x \in F$,

$$\|\phi(x) - x\|_2 \le \delta',$$

with $2\delta' \max_{x\in F} \|x\|_2 = \delta^2$. Let $(\mathcal{H}, \xi)$ be the pointed M-M-bimodule associated with ϕ. For $x \in F$, we have

$$\|x\xi - \xi x\|^2 \le 2\|\phi(x) - x\|_2 \|x\|_2 \le \delta^2.$$

Therefore, there exists a B-central vector $\eta \in \mathcal{H}$ with $\|\xi - \eta\| \le \varepsilon'$. Then, for b in the unit ball of B we get

$$\begin{aligned}\|\phi(b) - b\|_2 &\le \|b\xi - \xi b\|\\ &\le \|b(\xi - \eta) - (\xi - \eta)b\| \le 2\varepsilon' = \varepsilon.\end{aligned}$$

$\square$

Corollary 14.2.5. *Let (M, τ) be a tracial von Neumann algebra. For every von Neumann subalgebra B having the property* (T), *the inclusion $B \subset M$ is rigid.*

Remark 14.2.6. The statement of the previous proposition is still true when Condition (ii) is replaced by the following condition:

(ii') *for every $\varepsilon' > 0$, there exist a finite subset F of M and $\delta' > 0$ such that whenever $\mathcal{H}$ is a M-M-bimodule which admits a vector ξ satisfying the conditions*

$$\left\|\omega_\xi^l - \tau\right\| \le \delta',\ \left\|\omega_\xi^r - \tau\right\| \le \delta',\ \max_{x\in F} \|x\xi - \xi x\| \le \delta', \tag{14.3}$$

there exists a B-central vector $\eta \in \mathcal{H}$ with $\|\xi - \eta\| \le \varepsilon'$.

Recall that ω_ξ^l and ω_ξ^r are respectively the states $x \mapsto \langle \xi, x\xi\rangle$ and $x \mapsto \langle \xi, \xi x\rangle$.

For the proof that Condition (i) (of the previous proposition) implies (ii'), we observe that we may add in (ii') the subtraciality of ξ, by using Lemma 13.1.11. We keep the outline of the proof of (i) $\Rightarrow$ (ii) in Proposition 14.2.4, with the following changes. Given $\varepsilon' > 0$ we keep considering ε, δ and F as before, and we take $\delta' > 0$ such that $(\delta')^2 + 2\delta'\|x\|_\infty^2 \le \delta^2$ for every $x \in F$. Let $\xi \in \mathcal{H}$ be a subtracial vector satisfying Condition (14.3). Let

ϕ be still defined by ξ. Then ϕ is subtracial and subunital, and we get, for $x \in F$,

$$\begin{aligned}\|\phi(x) - x\|_2^2 &\leq 2\tau(x^*x) - \langle x\xi, \xi x\rangle - \langle \xi x, x\xi\rangle \\ &\leq \|x\xi - \xi x\|^2 + (\tau(x^*x) - \omega_\xi^l(x^*x)) + (\tau(xx^*) - \omega_\xi^r(xx^*)) \\ &\leq (\delta')^2 + 2\delta'\|x\|_\infty^2 \leq \delta^2.\end{aligned}$$

It follows that $\|\phi(b) - b\|_2 \leq \varepsilon$ for b in the unit ball of B. Then, for $u \in \mathcal{U}(B)$ we get

$$\|\xi - u\xi u^*\|^2 = 2\Re\tau((u - \phi(u))u^*) + 2(\|\xi\|^2 - 1) \leq 2\varepsilon.$$

We end as in the proof of (i) $\Rightarrow$ (ii).

Proposition 14.2.7. *Let $H \subset G$ be an inclusion of groups. We set $B = L(H)$ and $M = L(G)$. The following conditions are equivalent:*

(i) *the pair (G, H) has the relative property* (T)*;*
(ii) *the pair (M, B) has the relative property* (T).

Proof. (i) $\Rightarrow$ (ii). Given $\varepsilon > 0$, we choose F and δ as in Condition (c) of Proposition 14.1.2. Let $\mathcal{H}$ be a M-M-bimodule with a unit vector ξ such that $\|u_g\xi - \xi u_g\| \leq \delta$ for $g \in F$. This vector is (F, δ)-invariant for the representation π defined by $\pi(g)(\eta) = u_g\eta u_g^*$ for $g \in G$, $\eta \in \mathcal{H}$. Therefore, there is a H-invariant vector η with $\|\xi - \eta\| \leq \varepsilon$ and η is obviously B-central. Observe that here we do not need the traciality of ξ.

(ii) $\Rightarrow$ (i). We assume that $B \subset M$ is a rigid embedding and we claim that Condition (d) of Proposition 14.1.2 holds. Let (φ_i) be a net of positive definite functions on G, normalized by $\varphi_i(e) = 1$, converging to 1 pointwise. Let ϕ_i be the completely positive map associated with φ_i. Recall that $\phi_i(u_g) = \varphi_i(g)u_g$ for $g \in G$. We get a net of tracial and unital completely positive maps such that $\lim_i \|\phi_i(x) - x\|_2 = 0$ for every x of the form u_g and so for every $x \in M$. It follows that $\lim_i \sup_{\{b\in B:\|b\|\leq 1\}} \|\phi_i(b) - b\|_2 = 0$, from which we immediately deduce the uniform convergence to 1 on H of the net (φ_i). $\square$

Example 14.2.8. As already said, the pair $(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}), \mathbb{Z}^2)$ has the relative property (T). The action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2$ yields a trace preserving action on the von Neumann algebra $L(\mathbb{Z}^2)$, and by Fourier transform, on $L^\infty(\mathbb{T}^2)$. We get the following canonical isomorphisms of pairs of von Neumann algebras:

$$\begin{aligned}\big(L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})), L(\mathbb{Z}^2)\big) &\simeq \big(L(\mathbb{Z}^2) \rtimes \mathrm{SL}(2,\mathbb{Z}), L(\mathbb{Z}^2)\big) \\ &\simeq \big(L^\infty(\mathbb{T}^2) \rtimes \mathrm{SL}(2,\mathbb{Z}), L^\infty(\mathbb{T}^2)\big).\end{aligned}$$

So, the pair $\big(L^\infty(\mathbb{T}^2) \rtimes \mathrm{SL}(2,\mathbb{Z}), L^\infty(\mathbb{T}^2)\big)$ has the relative property (T).

More generally, for any non amenable subgroup G of $\mathrm{SL}(2,\mathbb{Z})$, the pair $\big(L^\infty(\mathbb{T}^2) \rtimes G, L^\infty(\mathbb{T}^2)\big)$ has the relative property (T) since it is the case for $(\mathbb{Z}^2 \rtimes G, \mathbb{Z}^2)$.

EXAMPLE 14.2.9. The free group $\mathbb{F}_n$ does not have the property (T). Indeed, $\mathbb{F}_n$ has an infinite abelian quotient, namely $\mathbb{Z}^n$. But, using for instance the condition (d) in Proposition 14.1.2, we see that the property (T) is stable by passing to the quotient, and $\mathbb{Z}^n$ does not have this property.

It follows that $L(\mathbb{F}_n)$, $n \geq 2$, is not isomorphic with any factor $L(G)$ whenever G is ICC and has the property (T), for instance $\mathrm{SL}(3, \mathbb{Z})$.

REMARK 14.2.10. Let M be a II_1 factor and B a von Neumann subalgebra of M. In this context, there is another natural notion of relative property (T) stated as follows: we say that M has the *property* (T) *relative to* B, or that B is *co-rigid* in M if there exist a finite subset F of M and $\delta > 0$ such that every M-M-bimodule $\mathcal{H}$ with a B-central unit vector ξ satisfying $\max_{x \in F} \|x\xi - \xi x\| \leq \delta$ contains a non-zero M-central vector. In particular, M has the property (T) if and only if it has the property (T) relative to $B = \mathbb{C}1$ (see Theorem 14.5.2).

It is easy to see that whenever $M = B \rtimes G$, then B is co-rigid in M if and only if G has the property (T). For a normal subgroup H of G, the co-rigidity of $L(H)$ into $L(G)$ is equivalent to the property (T) of the quotient group G/H (see [**Pop86a, AD87**]).

Rigid embeddings have several natural stability properties. We will only need the following one for factors.

PROPOSITION 14.2.11. *Let $B \subset M$ be a rigid embedding. Let p be a non-zero projection in B. Then $pBp \subset pMp$ is a rigid embedding.*

PROOF. We only consider the case where M is a II_1 factor. We denote by τ_p the tracial state on pMp. Observe that for $y \in pMp$, we have $\|y\|_{2,\tau} = \tau(p)^{1/2}\|y\|_{2,\tau_p}$.

Using the comparison theorem about projections, we find partial isometries $v_1 = p, v_2, \dots, v_n$ such that $\sum_{i=1}^n v_i v_i^* = 1$ and $v_i^* v_i \leq p$ for every i. We fix $\varepsilon > 0$ and set $\varepsilon_0 = \varepsilon\tau(p)^{1/2}$. Let F_0 be a finite subset of M and $\delta_0 > 0$ such that whenever $\phi_0 : M \to M$ is a subunital and subtracial completely positive map satisfying $\max_{x \in F_0} \|\phi_0(x) - x\|_{2,\tau} \leq \delta_0$, then $\|\phi_0(b) - b\|_{2,\tau} \leq \varepsilon_0$ for every b in the unit ball of B.

We set $F = \{v_i^* x v_j : x \in F_0, 1 \leq i, j \leq n\}$ and $\delta = \delta_0 n^{-2}\tau(p)^{-1/2}$. Let $\phi : pMp \to pMp$ be a subunital and subtracial completely positive map, *i.e.*, $\phi(p) \leq p$ and $\tau_p \circ \phi \leq \tau_p$, such that

$$\max_{y \in F} \|\phi(y) - y\|_{2,\tau_p} \leq \delta.$$

We define $\phi_0 : M \to M$ by

$$\phi_0(x) = \sum_{i,j=1}^n v_i \phi(v_i^* x v_j) v_j^*.$$

It is easily checked that ϕ_0 is a subunital and subtracial completely positive map. Moreover we have, for $x \in F_0$,

$$\begin{aligned}\|\phi_0(x) - x\|_{2,\tau} &\leq \sum_{i,j} \left\|v_i \phi(v_i^* x v_j) v_j^* - v_i v_i^* x v_j v_j^*\right\|_{2,\tau} \\ &\leq \sum_{i,j} \|\phi(v_i^* x v_j) - v_i^* x v_j\|_{2,\tau} \\ &\leq n^2 \tau(p)^{1/2} \delta = \delta_0.\end{aligned}$$

It follows that $\|\phi_0(b) - b\|_{2,\tau} \leq \varepsilon_0$ for b in the unit ball $(B)_1$ of B and therefore we have $\|\phi(pbp) - pbp\|_{2,\tau_p} \leq \varepsilon$ for every $b \in (pBp)_1$. $\square$

14.3. Consequences of property (T) for II_1 factors

In this section, M denotes a II_1 factor which has the property (T).

14.3.1. Separability. We are going to show the following result.

Proposition 14.3.1. *Every II_1 factor which has the property (T) is separable.*

For the proof, we will use the lemma below.

Lemma 14.3.2. *Let (M, τ) be a tracial von Neumann algebra and Q a von Neumann subalgebra. Let $0 < \varepsilon < 2^{-1/2}$ be such that*

$$\|x - E_Q(x)\|_2 \leq \varepsilon$$

for x in the unit ball $(M)_1$ of M. Then there exists a non-zero projection $q \in \langle M, e_Q \rangle \cap M'$ such that the right Q-module $qL^2(M)$ has a finite dimension.

Proof. Recall that $\widehat{\tau}$ denotes the canonical trace on the basic construction $\langle M, e_Q \rangle$. For $u \in \mathcal{U}(M)$, we have

$$\begin{aligned}\|e_Q - u e_Q u^*\|_{2,\widehat{\tau}}^2 &= 2\widehat{\tau}(e_Q - e_Q u e_Q u^* e_Q) \\ &= 2(1 - \tau(E_Q(u) E_Q(u)^*)) \\ &= 2\|u - E_Q(u)\|_{2,\tau}^2 \leq 2\varepsilon^2.\end{aligned}$$

Using the averaging lemma 14.3.3 below in the semi-finite von Neumann algebra $(\langle M, e_Q \rangle, \widehat{\tau})$ with $c = e_Q$ and with the unitary group of M as G, we get a positive element $h \in \langle M, e_Q \rangle \cap M'$ such that $\widehat{\tau}(h) \leq \widehat{\tau}(e_Q) = 1$ and $\|e_Q - h\|_{2,\widehat{\tau}} \leq \sqrt{2}\varepsilon$. In particular, we have $h \neq 0$ and for q it suffices to take a non-zero spectral projection of h. Then $\dim(qL^2(M)_Q) = \widehat{\tau}(q) < +\infty$. $\square$

Lemma 14.3.3. *Let M be a von Neumann algebra equipped with a normal faithful semi-finite trace Tr. Let $c \in M_+$ be such that $Tr(c) < +\infty$ and let G be a unitary subgroup of M. The w.o. closed convex hull $\mathcal{C} \subset M_+$ of $\{ucu^* : u \in G\}$ contains a unique element h of minimal $\|\cdot\|_{2,Tr}$-norm. Moreover, we have $uhu^* = u$ for every $u \in G$ and $Tr(h) \leq Tr(c)$.*

Proof. For every $y \in \mathcal{C}$, we have obviously $\|y\|_\infty \leq \|c\|_\infty$ and since Tr is lower semi-continuous[4], we also have $\mathrm{Tr}(y) \leq \mathrm{Tr}(c)$. The inclusion of $\mathcal{C}$ into the Hilbert space $L^2(M, \mathrm{Tr})$ is continuous when $\mathcal{C}$ and the Hilbert space are respectively equipped with the w.o. topology and the weak topology. Indeed, for x_1, x_2 such $\mathrm{Tr}(x_i^* x_i) < +\infty$, $i = 1, 2$, the linear functional $y \mapsto \mathrm{Tr}(x_1 x_2 y) = \langle (x_1 x_2)^*, y \rangle_{L^2(M)}$ is w.o. continuous on $\mathcal{C}$ and these $x_1 x_2$ generate linearly a dense subspace of $L^2(M, \mathrm{Tr})$. Since $\mathcal{C}$ is w.o. compact, it is closed in $L^2(M, \mathrm{Tr})$ with respect to the weak topology. Being convex it is also $\|\cdot\|_{2,\mathrm{Tr}}$-closed. It follows that $\mathcal{C}$ contains a unique element h of minimal $\|\cdot\|_{2,\mathrm{Tr}}$-norm. Since $uhu^* \in \mathcal{C}$ for every $u \in G$ and since $\|uhu^*\|_{2,\mathrm{Tr}} = \|h\|_{2,\mathrm{Tr}}$ we see that $uhu^* = h$. □

Proof of Proposition 14.3.1. We fix $\varepsilon < 2^{-1/2}$ and we choose (F, δ) as in Definition 14.2.1: whenever $\phi : M \to M$ is a subunital and subtracial completely positive map satisfying $\max_{x \in F} \|\phi(x) - x\|_2 \leq \delta$ then $\|\phi(x) - x\|_2 \leq \varepsilon$ for every x in the unit ball $(M)_1$. Let Q be the von Neumann subalgebra of M generated by F. Since $E_Q(x) = x$ for $x \in F$, we get $\|x - E_Q(x)\|_2 \leq \varepsilon$ for every $x \in (M)_1$. Lemma 14.3.2 gives a non-zero projection q in $\langle M, e_Q \rangle \cap M'$ with $\widehat{\tau}(q) < +\infty$. Then $qL^2(M)$ is a M-Q-bimodule with $\dim(qL^2(M)_Q) < +\infty$. Cutting down, if necessary, $qL^2(M)$ by a projection of the center of Q, we may assume by Corollary 9.3.3 that $qL^2(M)$ is finitely generated as a right Q-module. Then, by Proposition 8.5.3 there is an integer $n > 0$ and a projection $p \in M_n(\mathbb{C}) \otimes Q$ such that $qL^2(M)$ is isomorphic, as a right Q-module, to $p\big(\ell_n^2 \otimes L^2(Q)\big)$. The structure of left M-module of $qL^2(M)$ gives a normal unital homomorphism from M into $p\big(M_n(\mathbb{C}) \otimes Q\big)p$. This homomorphism is an embedding since M is a factor. But $p\big(M_n(\mathbb{C}) \otimes Q\big)p$ is separable and so M is also separable. □

14.3.2. M **is a full factor and the outer automorphism group** $\mathrm{Out}\,(M)$ **is countable.** Recall that we denote by $\mathrm{Inn}\,(M)$ the normal subgroup of $\mathrm{Aut}\,(M) = \mathrm{Aut}\,(M, \tau)$ formed by the inner automorphisms. We equip $\mathrm{Aut}\,(M)$ with the topology for which a net (α_i) converges to α if for every $x \in M$ we have $\lim_i \|\alpha_i(x) - \alpha(x)\|_2 = 0$ (see Section 7.5.3) The outer automorphism group $\mathrm{Out}\,(M) = \mathrm{Aut}\,(M)/\mathrm{Inn}\,(M)$ will be endowed with the quotient topology.

Proposition 14.3.4. *Let M be a* II_1 *factor having the property* (T). *Then* $\mathrm{Inn}\,(M)$ *is an open sugroup of* $\mathrm{Aut}\,(M)$.

Proof. We will only use the fact that since M has the property (T), there exist a finite subset F of M and $\delta > 0$ such that if $\mathcal{H}$ is a M-M-bimodule with a tracial vector ξ satisfying

$$\max_{x \in F} \|x\xi - \xi x\| \leq \delta,$$

[4]For some of the properties of Tr used in this proof, see Exercises 8.3, 8.4 and 8.5.

then $\mathcal{H}$ contains a non-zero M-central vector. We claim that

$$V = \left\{ \alpha \in \mathrm{Aut}\,(M) : \max_{x \in F} \|\alpha(x) - x\|_2 \leq \delta \right\}$$

is a neighbourhood of Id_M contained in $\mathrm{Inn}\,(M)$. Indeed, for $\alpha \in V$ we have $\max_{x\in F} \left\| \alpha(x)\widehat{1} - \widehat{1}x \right\|_2 \leq \delta$. Applying our assumption to the M-M-bimodule $\mathcal{H}(\alpha)$ (see Example 13.1.3 (b)), we obtain the existence of a non-zero vector $\eta \in L^2(M)$ such that $\alpha(x)\eta = \eta x$ for every $x \in M$. It follows that $\eta^*\eta$ is in the center of M, thus is a scalar operator. Hence $\alpha(x)u = ux$, where u is the unitary operator of the polar decomposition of η. □

In particular, $\mathrm{Inn}\,(M)$ is also a closed subgroup of $\mathrm{Aut}\,(M)$. A factor with such a property is said to be *full*.

Proposition 14.3.5. *Let M be a II_1 factor which has the property* (T)*. Then the group* $\mathrm{Out}\,(M) = \mathrm{Aut}\,(M)/\mathrm{Inn}\,(M)$ *of outer automorphisms is countable.*

Proof. We have seen in Section 7.5.3 that $\mathrm{Aut}\,(M)$ is a Polish group, due to the separability of M. Since $\mathrm{Inn}\,(M)$ is open, the group $\mathrm{Out}\,(M)$ is discrete, and being Polish it is countable. □

14.3.3. The fundamental group $\mathfrak{F}(M)$ is countable. Recall that $\mathfrak{F}(M)$ is the subgroup of $\mathbb{R}_+^*$ consisting of the positive numbers t such that the amplification M^t is isomorphic to M.

Proposition 14.3.6. *If the II_1 factor M has the property* (T)*, its fundamental group $\mathfrak{F}(M)$ is countable.*

Proof. Assume that $\mathfrak{F}(M)$ is not countable and choose $c \in]0,1[$ such that $\mathfrak{F}(M) \cap [c,1]$ is not countable. For every $t \in \mathcal{F}(M) \cap [c,1]$, we choose a projection p_t with $\tau(p_t) = t$, and an isomorphism θ_t from M onto p_tMp_t. We may choose these projections such that $p_s < p_t$ whenever $s < t$ (see Exercise 3.3). Note also that

$$\tau \geq \tau \circ \theta_t = \tau(p_t)\tau \geq c\tau.$$

We take (F,δ) corresponding to $\varepsilon = 1/2$ in Definition 14.2.1 and write $F = \{x_1, \dots, x_n\}$. We set $\xi_t = (\theta_t(x_1), \dots, \theta_t(x_n)) \in \mathcal{H} = L^2(M)^{\oplus n}$. Since $\mathcal{H}$ is separable, there are two elements $s < t$ in $\mathfrak{F}(M) \cap [c,1]$ with $\|\xi_s - \xi_t\|_{\mathcal{H}} \leq \delta c^{1/2}$ and therefore

$$\max_{x\in F} \|\theta_s(x) - \theta_t(x)\|_2 \leq \delta c^{1/2}.$$

We set $q = \theta_t^{-1}(p_s)$ and $\theta = \theta_t^{-1} \circ \theta_s$. We observe that θ is an isomorphism from M onto qMq and that $\tau \circ \theta = \tau(q)\tau \leq \tau$.

We have $\max_{x\in F} \|\theta(x) - x\|_2 \leq \delta$, and therefore $\|\theta(x) - x\|_2 \leq 1/2$ for every x in the unit ball of M. Then for $u \in \mathcal{U}(M)$ we have $\|\theta(u)u^* - 1\|_2 \leq 1/2$. By the usual convexity argument, we deduce the existence of a non-zero element $\xi \in L^2(M)$ such that $\theta(u)\xi u^* = \xi$ for every $u \in \mathcal{U}(M)$. It follows that $\xi^*\xi$ is a scalar operator and that θ is an inner automorphism, in contradiction with $q < 1$. $\square$

Remark 14.3.7. The separability argument used in the above proof can be applied in many situations. We will give two other illustrations in the next section, showing the power of this method.

For the original proof of the proposition 14.3.6 due to Connes [**Con80a**], see Exercise 14.3

14.4. Rigidity results from separability arguments

A nice feature of groups having Property (T) is that all their representations in a II_1-factor are isolated. More precisely, we have the following result.

Proposition 14.4.1. *Let G be a group having the property (T) and let (M, τ) be a tracial von Neumann algebra. Let (F, δ) satisfying the condition (b) of Proposition 14.1.2 with $H = G$. Let p_1, p_2 be two projections in M and let $\pi_i : G \to \mathcal{U}(p_i M p_i)$ be two group homomorphisms such that $\|\pi_1(g) - \pi_2(g)\|_2 \le \delta \|p_1 p_2\|_2$ for every $g \in F$. There exists a non-zero partial isometry $v \in M$ such that $\pi_1(g)v = v\pi_2(g)$ for every $g \in G$. Moreover, if M is II_1 factor and if, for instance, $\pi_2(G)$ generates M, then v is unitary and therefore the representations π_1 and π_2 are equivalent.*

Proof. Consider the representation $g \mapsto \pi_1(g) J \pi_2(g) J$ on the Hilbert space $\mathcal{H} = p_1 L^2(M) p_2$ and the vector $\xi = p_1 p_2 \in \mathcal{H}$. Then we have

$$\|\pi(g)\xi - \xi\|_2 = \|\pi_1(g) p_1 p_2 - p_1 p_2 \pi_2(g)\|_2 \le \|\pi_1(g) - \pi_2(g)\|_2.$$

It follows that $\max_{g \in F} \|\pi(g)\xi - \xi\|_2 \le \delta \|\xi\|_2$ and therefore there exists a non-zero vector $\eta \in \mathcal{H}$ such that $\pi(g)\eta = \eta$ for every $g \in G$, that is, $\pi_1(g)\eta = \eta\pi_2(g)$ for every $g \in G$. Let $\eta = v|\eta|$ be the polar decomposition of η. Then $|\eta|$ commutes with $\pi_2(G)$ and so we get $\pi_1(g)v = v\pi_2(g)$ for every $g \in G$.

The last statement of the proposition follows from the fact that v^*v commutes with $\pi_2(G)$. $\square$

14.4.1. Connes' rigidity conjecture is true, up to countable classes. Connes conjectured[5] that if G_1, G_2 are two ICC groups with Property (T), then the II_1 factors $L(G_1)$ and $L(G_2)$ are isomorphic if and only if the groups are isomorphic. This conjecture is still out of reach, but we have the following result.

Theorem 14.4.2. *The functor $G \mapsto L(G)$ defined on the* ICC *groups with Property* (T) *is at most countable to one.*

The proof relies on the following theorem[6].

[5] in [**Con82**]

[6] due to Shalom [**Sha00**]

THEOREM 14.4.3. *Every Property* (T) *group is a quotient of a finitely presented group with Property* (T).

PROOF OF THEOREM 14.4.2. Assume that $M = L(G_i)$ for uncountably many non isomorphic ICC groups with Property (T). Since the set of finitely presented groups is countable, we may assume that all these groups are quotient of the same group G with Property (T). Let $u_i : G_i \to \mathcal{U}(M)$ be the canonical embedding. We denote by q_i the quotient homomorphism $G \to G_i$, and we set $\pi_i = u_i \circ q_i$.

We represent M in standard form on $L^2(M)$ and ξ is the canonical tracial vector in $L^2(M)$. Let (F, δ) satisfying the condition (b) of Proposition 14.1.2 with $H = G$. Since $L^2(M)$ is separable and I is uncountable, there exist distinct i, j in I such that $\max_{g\in F} \|\pi_i(g)\xi - \pi_j(g)\xi\|_2 < \delta$. It follows from Proposition 14.4.1 that the representations π_i and π_j are unitarily equivalent: there exists $v \in \mathcal{U}(M)$ such that $\big(u_i \circ q_i(g)\big)v = v\big(u_j \circ q_j(g)\big)$ for $g \in G$. This contradicts the fact that the groups G_i and G_j are not isomorphic. □

14.4.2. There is no separable universal II_1 factor.

THEOREM 14.4.4. *There is no separable* II_1 *factor* M *such that every separable factor is isomorphic to a subfactor of* M.

This time the proof uses a deep result of Gromov-Olshanskii[7].

THEOREM 14.4.5. *There exists a countable group* G *with Property* (T), *which has uncountably many pairwise non isomorphic quotient groups* G_i, $i \in I$, *all of which are simple and* ICC.

PROOF OF THEOREM 14.4.4. Assume that there is a separable II_1 factor M which contains $L(G_i)$ for $i \in I$. As in the proof of Theorem 14.4.2, we introduce the representations $\pi_i = u_i \circ q_i$ of G. We still represent M in standard form on $L^2(M)$ and ξ is the canonical tracial vector in $L^2(M)$. Since π_i is a non-trivial representation and ξ is separating for M we have $\sup_{g\in G} \|\pi_i(g)\xi - \xi\| > 0$. It follows that there is an integer $n > 0$ and an uncountable subset I_1 of I such that $\sup_{g\in G} \|\pi_i(g)\xi - \xi\| > 1/n$ for all $i \in I_1$.

We choose $\varepsilon < 1/2n$. There exist a finite subset F of G and $\delta > 0$ such that if $(\pi, \mathcal{H})$ is a unitary representation of G and $\zeta \in \mathcal{H}$ is a unit vector satisfying $\max_{g\in F} \|\pi(g)\zeta - \zeta\| < \delta$, then there is a G-invariant vector η such that $\|\zeta - \eta\| < \varepsilon$.

Since $L^2(M)$ is separable and I_1 is uncountable, there exist distinct i, j in I_1 such that $\max_{g\in F} \|\pi_i(g)\xi - \pi_j(g)\xi\|_2 < \delta$. It follows that there exists a non-zero vector $\eta \in L^2(M)$ such that $\pi_i(g)\eta = \eta\pi_j(g)$ for all $g \in G$ with $\|\xi - \eta\|_2 < \varepsilon$. Let $H = \{g \in G : \pi_i(g)\eta = \eta\}$. Then H is a subgroup of G which contains the normal subgroups $\ker \pi_i$ and $\ker \pi_j$. Since these normal subgroups are distinct and since the groups G_i and G_j are simple,

[7] revisited by Ozawa in [**Oza04b**].

we see that $H = G$. It follows that $\sup_{g\in G} \|\pi_i(g)\xi - \xi\| \leq 2\varepsilon < 1/n$, a contradiction. $\square$

14.5. Some remarks about the definition of relative property (T)

We have defined the relative property (T) by translating Conditions (c) and (d) of Proposition 14.1.2 in the setting of tracial von Neumann algebras, where in particular a formulation in terms of completely positive maps proves to be very convenient. The analogue of Condition (b) of Proposition 14.1.2 for a pair (M, B), where M is equipped with a normal faithful trace τ, would be the following:

- *there exists a neighbourhood W of the trivial bimodule $L^2(M)$ such that every M-M-bimodule in W has a non-zero B-central vector.*

Using the basis of neighbourhoods of the form $W(L^2(M); \varepsilon, F)$ described in Section 13.3.2, the above condition reads as

(iii) *there exist a finite subset F of M and $\delta' > 0$ such that whenever $\mathcal{H}$ is a M-M-bimodule which admits a unit vector ξ satisfying*

$$\forall x \in F, \quad \|x\xi - \xi x\| < \delta', \quad |\langle \xi, x\xi\rangle - \tau(x)| < \delta', \quad |\langle \xi, \xi x\rangle - \tau(x)| < \delta',$$

then $\mathcal{H}$ has a non-zero B-central vector.

The reader is invited to compare this condition (iii) with Condition (ii') in Remark 14.2.6. In (iii), ξ is not required to be tracial or "almost tracial" in the sense of Remark 14.2.6. On the other hand, the existing B-central vector is not required to be close to ξ, that is, no "continuity constants" are involved.

In case M is a II_1 factor, the tracial condition is not a serious issue.

Proposition 14.5.1. *Let B be a von Neumann subalgebra of a* II_1 *factor M. The following conditions are equivalent:*

(i) *the pair (M, B) has the relative property* (T)*;*

(ii") *for every $\varepsilon > 0$ there exist a finite subset F of M and $\delta > 0$ such that for any M-M-bimodule $\mathcal{H}$ and any unit vector $\xi \in \mathcal{H}$, with* $\max_{x\in F} \|x\xi - \xi x\| \leq \delta$*, there is a B-central vector η with $\|\xi - \eta\| \leq \varepsilon$.*

Proof. Assume that (ii") does not hold whereas (i) is satisfied. There exist $c > 0$, a M-M-bimodule $\mathcal{H}$ and a net (ξ_i) of unit vectors in $\mathcal{H}$ such that $\lim_i \|x\xi_i - \xi_i x\| = 0$ for every $x \in M$ and $\left\|\xi_i - P^B\xi_i\right\| \geq c$ for every i, where P^B denotes the orthogonal projection on the subspace of B-central vectors. Then any weak* limit of the net $(\omega^l_{\xi_i})$ of vector states on M is a trace, and thanks to the uniqueness of the tracial state τ on M we see that $\lim_i \omega^l_{\xi_i} = \tau$ in the weak* topology. Similarly, we have $\lim_i \omega^r_{\xi_i} = \tau$.

Let F be a finite subset of M and $\delta' > 0$ satisfying Condition (ii') of Remark 14.2.6 with respect to $\varepsilon' = c/2$. Using a convexity argument as in the proof of Lemma 13.3.11, we find a unit vector $\eta = (\lambda_1^{1/2}\xi_{i_1}, \dots, \lambda_n^{1/2}\xi_{i_n})$, viewed as an element of $\mathcal{H}^{\oplus\infty}$, with $\lambda_j > 0$, $\sum_{j=1}^n \lambda_j = 1$, such that

$$\begin{aligned}\left\|\tau - \omega_\eta^l\right\| \le \delta', \quad \left\|\tau - \omega_\eta^r\right\| \le \delta',\\ \max_{x\in F}\left\|x\eta - \eta x\right\| \le \delta'.\end{aligned}$$

It follows that there exists a B-central vector $\zeta \in \mathcal{H}^{\oplus\infty}$ such that $\|\eta - \zeta\| \le c/2$. Therefore we have

$$\sum_{j=1}^n \lambda_j \left\|\xi_{i_j} - P^B\xi_{i_j}\right\|^2 \le c^2/4,$$

in contradiction with the fact that $\left\|\xi_i - P^B\xi_i\right\| \ge c$ for every i. □

Equivalence with definitions of Property (T) without "continuity constants" is much more difficult to obtain. Concerning II_1 factors, this makes use of a delicate argument as we will see in the proof of the next theorem.

Theorem 14.5.2. *Let M be a II_1 factor. The following conditions are equivalent:*

(a) *M has the property* (T)*;*
(b) *there exists a neighbourhood V of $L^2(M)$ such that every element of V has a non-zero central vector, that is (in the factor case), there exist a finite subset F of M and $\delta > 0$ such that if $\mathcal{H}$ is a M-M-bimodule with a unit vector ξ satisfying*

$$\max_{x\in F}\|x\xi - \xi x\| \le \delta,$$

then $\mathcal{H}$ contains a non-zero M-central vector.

Note that the existence of a non-zero M-central vector is equivalent to the fact that $L^2(M)$ is a M-M-submodule of $\mathcal{H}$. So, Condition (b) means that there exists a neighbourhood V of $L^2(M)$ in $\mathrm{Bimod}\,(M)$ such that every element of V contains $L^2(M)$ as a M-M-subbimodule. It is the original Connes-Jones definition of Property (T).

Proof of Theorem 14.5.2. Since Condition (ii") of Proposition 14.5.1 with $B = M$ implies (b), we only have to show that (b) $\Rightarrow$ (ii") stated with $B = M$. If (b) holds, as seen in Proposition 14.3.4, $\mathrm{Inn}\,(M)$ is open and thus closed in $\mathrm{Aut}\,(M)$. By Theorem 15.3.2 to be proved in the next chapter, M has spectral gap, that is, there exist a finite subset F_0 of $\mathcal{U}(M)$ and $c > 0$ such that $c\max_{u\in F_0}\|u\xi - \xi u\|_2 \ge \|\xi\|_2$ for every $\xi \in L^2(M)$ with $\langle \xi, \widehat{1}\rangle = 0$.

Let $\mathcal{H}$ be a M-M-bimodule, that we write as $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ where $\mathcal{H}_0$ is a multiple of $L^2(M)$ and $\mathcal{H}_1$ has no non-zero central vector. Let F, δ be given by condition (b). We set $F' = F \cup F_0$. Let $\xi \in \mathcal{H}$ be a unit vector and put $\alpha = \max_{x\in F'}\|x\xi - \xi x\|$. We write $\xi = \xi_0 + \xi_1$ with $\xi_i \in \mathcal{H}_i$, $i = 0, 1$. Observe that $\max_{x\in F}\|x\xi_1 - \xi_1 x\| \le \alpha$. So, since $\mathcal{H}_1$ has no non-zero central vector we see that $\|\xi_1\| \le \alpha/\delta$. Now, we write $\xi_0 = \xi_0' + \xi_0''$ where ξ_0' is a central vector and ξ_0'' is orthogonal to the space of central vectors. In $\mathcal{H}_0$, which is a multiple of the trivial M-M-bimodule, we have

$$\left\|\xi_0''\right\| \leq c \max_{u\in F_0} \left\|u\xi_0'' - \xi_0'' u\right\| = c \max_{u\in F_0} \|u\xi_0 - \xi_0 u\| \leq c\alpha.$$

It follows that

$$\left\|\xi - \xi_0'\right\|^2 = \left\|\xi_1 + \xi_0''\right\|^2 \leq (\alpha/\delta)^2 + c^2\alpha^2.$$

To conclude, given $\varepsilon > 0$, we set $\delta' = \varepsilon/k$ with $k = \sqrt{c^2 + 1/\delta^2}$. We have found a pair (F', δ') such that whenever $\mathcal{H}$ is a M-M-bimodule with a unit vector ξ satisfying $\max_{x\in F'} \|x\xi - \xi x\| \leq \delta'$, then there is a central vector ξ_0' with $\|\xi - \xi_0'\|^2 \leq \varepsilon$. ☐

REMARKS 14.5.3. **(a)** Whenever M is the von Neumann algebra of a group G, not necessarily ICC (so M is possibly not a factor), and $B = L(H)$ where H is a subgroup of G, it is still true that (M, B) has the relative property (T) if and only if there exists a neighbourhood V of $L^2(M)$ such that every element of V has a non-zero B-central vector [**Jol93, Bek06**].

(b) More results in the relative case have been obtained in [**Pop06a**, Theorem 4.3] and [**PP05**] involving, as in the Connes-Jones result 14.5.2, rather subtle arguments. Let us only mention for example the following theorem without "continuity constants". We denote by $\mathcal{N}_M(B)$ is the group of unitaries $u \in M$ such that $uBu^* = B$.

THEOREM 14.5.4. *Let M be a II_1 factor and B a von Neumann subalgebra such that $B' \cap M \subset B$ and $\mathcal{N}_M(B)' \cap M = \mathbb{C}$. The following conditions are equivalent:*

(a) *the pair (M, B) has the relative property* (T) *;*
(b) *there exist a finite subset F of M and $\delta > 0$ such that if $\mathcal{H}$ is a M-M-bimodule with a unit vector ξ satisfying*

$$\max_{x\in F} \|x\xi - \xi x\| \leq \delta, \ \left\|\omega_\xi^l - \tau\right\| \leq \delta, \ \left\|\omega_\xi^r - \tau\right\| \leq \delta,$$

then $\mathcal{H}$ contains a non-zero B-central vector.

Exercises

EXERCISE 14.1. Let B be a von Neumann subalgebra of a tracial von Neumann algebra M. Show that the pair (M, B) has the relative property (T) if and only if for every net $(\phi_i)_{i\in I}$ of tracial and unital completely positive maps from M to M such that $\lim_i \|\phi_i(x) - x\|_2 = 0$ for $x \in M$ (such a net is called a *deformation of the identity*), then $\lim_i \sup_{b\in(B)_1} \|\phi_i(b) - b\|_2 = 0$. Moreover, if M is separable, it suffices to consider sequences.

EXERCISE 14.2. Let B be a von Neumann subalgebra of a tracial von Neumann algebra M.

(i) Show that for every projection z in the center of B, then $zL^2(B)$ is a B-B-subbimodule of $L^2(B)$, and that every B-B-subbimodule of $L^2(B)$ is of this form.

(ii) Let $\mathcal{H}$ be a M-M-bimodule. Show that $\mathcal{H}$ contains a non-zero B-central vector if and only if, as a B-B-bimodule, $\mathcal{H}$ contains a non-zero B-B-subbimodule of $L^2(B)$.

Exercise 14.3. Let M be a II_1 factor having the property (T). Let $t \in \mathcal{F}(M)$ and let $\theta_t \in \mathrm{Aut}\,(M\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N}))$ such that $\mathrm{mod}(\theta_t) = t$ (see Exercise 8.15). Let α_t be the unique element of $\mathrm{Out}\,(M\overline{\otimes}M)$ corresponding to $\theta_t \otimes \theta_t^{-1}$ (see Exercise 8.16). Show that the map $t \mapsto \alpha_t$ is injective and conclude that $\mathfrak{F}(M)$ is countable.

Notes

Property (T) for groups was introduced by Kazhdan in [**Kaž67**]. Such discrete groups being finitely generated, this was a handy way to show that the (Poincaré) fundamental group of some locally symmetric Riemannian manifolds are finitely generated. The notion of relative property (T) is implicit in Kazhdan's work and was made explicit by Margulis, in particular the fact that $(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}), \mathbb{Z}^2)$ has the relative property (T) [**Mar82**] .

Since then, Property (T) proved to be very fruitful in diverse domains, in particular ergodic theory and operator algebras.

For an exhaustive study of the notion of property (T) for groups and its applications we recommend the book [**BdlHV08**]. A proof, not using Lie group theory, that some countable groups such as $\mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$, have Property (T) was given by Shalom at the end of the 90's [**Sha99**]. The fact that $(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}), \mathbb{Z}^2)$ has the relative property (T) is a key result for his proof, combined with the fact that $\mathrm{SL}(n,\mathbb{Z})$ has the so-called *bounded generation property*. A nice simpler proof, based on Shalom's ideas, but using a weaker bounded generation property is provided in [**BO08**, Section 12.1]. The result stating that $(\mathbb{Z}^2 \rtimes G, \mathbb{Z}^2)$ has the relative property (T) for any non-amenable subgroup of $\mathrm{SL}(n,\mathbb{Z})$ is due to Burger [**Bur91**].

In [**Con80a**], Connes discovered that II_1 factors of the form $L(G)$, where G is an ICC group with Property (T), have the remarkable properties shown in Section 14.3. This provided the first examples of II_1 factors with countable (Murray-von Neumann) fundamental groups, yet without explicit computation. Later, Connes [**Con82**] defined Property (T) for any II_1 factor in such a way that $L(G)$ has Property (T) if and only if G has this property. This was developed by Connes and Jones in [**CJ85**]. In addition to the fact that Property (T) factors have countable fundamental groups and outer automorphisms groups, they showed that they are non-amenable in a strong sense. For instance they cannot be embedded in a von Neumann algebra of a free group, as we will see in Chapter 16. Theorem 14.5.2 is also taken from [**CJ85**]. Further results on Property (T) for von Neumann algebras are contained in [**Pop86a**].

The notion of relative property (T) for pairs of finite von Neumann algebras was introduced and studied in [**Pop06a**]. This property was used in a crucial way to give the first example of a II_1 factor M with a fundamental group $\mathfrak{F}(M) \neq \mathbb{R}_+^*$ and explicitly computed. Precisely, it is shown in [**Pop06a**] that $\mathfrak{F}(M) = \{1\}$ for $M = L(\mathbb{Z}^2 \rtimes \mathbb{F}_n)$ (see Corollary 18.3.2). This breakthrough was followed by the discovery of many other remarkable applications of rigidity properties in operator algebras and ergodic theory. The versions with and without "continuity constants" as explained in Section 14.5 have both their interest. In particular, the version with "continuity constant" is well adapted to show the stability of Property (T) under various operations such as tensor products, reduction or induction (see [**Pop86a, Pop06a**]). Theorem 14.5.4 is Corollary 2 in [**PP05**].

The method of deriving rigidity statements "up to countable classes" applied in Section 14.4 grew out of Connes' initial rigidity paper [**Con80a**]. It was developed in [**Pop86a**] and gave rise to many applications not present in this monograph. For instance, it is shown in [**Pop86a**, Corollary 4.5.2] that if a II_1 factor has Property (T) then the set $\mathfrak{I}(M)$ of index values of the subfactors of M (and thus $\mathfrak{F}(M)$ is countable. Such arguments were revived at the beginning of the 2000's, leading to new applications, among them Theorem 14.4.4, due to Ozawa [**Oza04b**], and Theorem 14.4.2 appearing in the survey [**Pop07b**] along with other examples. The non existence of a separable universal II_1 factor solves Problem 4.4.29 in [**Sak98**]. An explicit construction of uncountably many separable II_1 factors that cannot be embedded in a fixed separable II_1 factor is provided in [**NPS07**].

CHAPTER 15

Spectral gap and Property Gamma

By definition, unitary representations of a Property (T) group G have the spectral gap property, in the sense that they do not have almost invariant vectors as soon as they do not have invariant vectors. In particular, every ergodic trace preserving action of G on a tracial von Neumann algebra (M, τ) has spectral gap, meaning that the Koopman representation of G on the orthogonal in $L^2(M)$ of the one-dimensional space of G-invariant vectors does not have almost invariant vectors. However, even in the absence of Property (T), any non-amenable group G has such actions, for instance Bernoulli actions, and therefore displays some weak form of rigidity.

We study in this chapter the spectral gap property for the action of the unitary group $\mathcal{U}(M)$ by inner automorphisms on a II_1 factor M. The class of II_1 factors having this property plays an important role in the subject. We show that a factor in this class admits the following remarkable equivalent characterisations:

(1) by not having the property Gamma of von Neumann (Theorem 15.2.4);
(2) by the property for its group of inner automorphisms to be closed in its automorphism group, in which case we say that the factor is full (Theorem 15.3.2).

Historically, Property Gamma was introduced at first to show that the hyperfinite factor R and the group von Neumann factors relative to the free groups $\mathbb{F}_n$, $n \geq 2$, are not isomorphic. Indeed, R has Property Gamma but not $L(\mathbb{F}_n)$, as shown in the last part of this chapter.

In this chapter, unless explicitly mentioned, the groups are not assumed to be countable.

15.1. Actions with spectral gap

15.1.1. First definitions. We begin by introducing the notion of spectral gap for unitary representations of a group G.

Definition 15.1.1. Let $(\pi, \mathcal{H})$ be a unitary representation of G. We will say that π has *spectral gap* if it does not weakly contain the trivial representation ι_G, that is, if it satisfies the following (obviously equivalent) conditions:

(i) there exist a finite subset F of G and $c > 0$ such that
$$\max_{g\in F} \|\pi(g)\xi - \xi\| \geq c\|\xi\|$$
for every $\xi \in \mathcal{H}$;

(ii) for every bounded net (ξ_i) in $\mathcal{H}$, one has $\lim_i \|\pi(g)\xi_i - \xi_i\| = 0$ for every g if and only if $\lim_i \|\xi_i\| = 0$. (When G is countable, it suffices to consider sequences instead of nets.)

(iii) for every $\varepsilon > 0$, there exist a finite subset F of G and $\delta > 0$ such that, if $\xi \in \mathcal{H}$ satisfies $\max_{g\in F} \|\pi(g)\xi - \xi\| \leq \delta$, then $\|\xi\| \leq \varepsilon$.

Remark 15.1.2. One often finds in the literature a different definition, where it is asked the trivial representation not to be weakly contained in the restriction of π to the orthogonal of the subspace of G-invariant vectors. Our definition is more convenient for our purpose in this chapter. In particular, a trace preserving action with spectral gap (see Definition 15.1.5) is automatically ergodic.

Of course, in Condition (i) we may take F symmetric, that is, $F = F^{-1}$. Then, this condition is easily translated into a property of the spectrum $\mathrm{Sp}(h)$ of the self-adjoint contraction $h = (1/n)\sum_{g\in F} \pi(g)$, where n denote the cardinal of F. More precisely, $\mathrm{Sp}(h)$ is always contained in $[-1, 1]$ and we have the following results.

Lemma 15.1.3. *We keep the above notation.*

(i) *Assume that* $\mathrm{Sp}(h) \subset [-1, 1-\delta]$ *for some* $\delta < 1$. *Then for every* $\xi \in \mathcal{H}$, *we have* $\max_{g\in F} \|\pi(g)\xi - \xi\| \geq \sqrt{2\delta}\|\xi\|$.

(ii) *Assume that there exists* $c > 0$ *such that*
$$\max_{g\in F} \|\pi(g)\xi - \xi\| \geq c\|\xi\|$$
for every $\xi \in \mathcal{H}$. *Then, we have* $\mathrm{Sp}(h) \subset [-1, 1 - c^2/(2n)]$.

Proof. (i) Assume that $\mathrm{Sp}(h) \subset [-1, 1-\delta]$. Then for $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, we have $\langle \xi, h\xi\rangle \leq 1 - \delta$. So, there exists $g \in F$ with
$$\Re\langle \xi, \pi(g)\xi\rangle \leq 1 - \delta,$$
from which we deduce immediately that $\|\pi(g)\xi - \xi\| \geq \sqrt{2\delta}$.

(ii) Assume that $\max_{g\in F} \|\pi(g)\xi - \xi\| \geq c$ for every norm-one vector $\xi \in \mathcal{H}$. Take $g \in F$ with $\|\pi(g)\xi - \xi\| \geq c$. It follows that $\Re\langle \xi, \pi(g)\xi\rangle \leq 1 - c^2/2$ and then
$$\begin{aligned}\langle \xi, h\xi\rangle &\leq \frac{n-1}{n} + (1/n)\Re\langle \xi, \pi(g)\xi\rangle \\ &\leq 1 - c^2/(2n).\end{aligned}$$
Thus
$$\|h\xi - (1-\lambda)\xi\| \geq \langle \xi, (1-\lambda)\xi - h\xi\rangle \geq c^2/(2n) - \lambda$$
for every $\lambda \in \mathbb{R}$. It follows that the operator $h - (1-\lambda)\mathrm{Id}_{\mathcal{H}}$ is invertible whenever $0 < \lambda < c^2/(2n)$, hence the conclusion. $\square$

COROLLARY 15.1.4. *A representation π of G has spectral gap if and only if there exists a symmetric finite subset F of G and $\delta < 1$ such that* $\mathrm{Sp}(h) \subset [-1, 1-\delta]$, *where* $h = (1/|F|)\sum_{g\in F} \pi(g)$.[1]

15.1.2. Spectral gap for trace preserving group actions on tracial von Neumann algebras. Let $\sigma : G \curvearrowright (M,\tau)$ be a trace preserving action of a group G on a tracial von Neumann algebra (M,τ). Recall that σ extends to a unitary representation (sometimes called the Koopman representation associated with the action) of G on $L^2(M)$, well defined by

$$\sigma_g(\widehat{x}) = \widehat{\sigma_g(x)}$$

for $g \in G$ and $x \in M$. This representation has $\widehat{1}$ as invariant vector. We denote by $L^2_0(M)$ the orthogonal complement of $\mathbb{C}\widehat{1}$ in $L^2(M)$ and we identify, as usual M to the corresponding dense subspace of $L^2(M)$.

DEFINITION 15.1.5. We say that $\sigma : G \curvearrowright (M,\tau)$ has *spectral gap* if the restriction of σ to $L^2_0(M)$ has spectral gap. A probability measure preserving action $G \curvearrowright (X,\mu)$ is said to have *spectral gap* if the corresponding action $G \curvearrowright (L^\infty(X,\mu),\tau_\mu)$ has this property.

Note that an action with spectral gap is ergodic, that is, the space $L^2(M)^G$ of fixed points in $L^2(M)$ (or equivalently (see Exercise 15.3), the space M^G of fixed points in M) is reduced to the scalars.

DEFINITION 15.1.6. Let $\sigma : G \curvearrowright (M,\tau)$ be a trace preserving action.

(i) We say that a net (ξ_i) in $L^2(M)$ is *asymptotically G-invariant* if for every $g \in G$,

$$\lim_i \|\sigma_g(\xi_i) - \xi_i\|_2 = 0.$$

(ii) We say that (ξ_i) is *asymptotically trivial* if

$$\lim_i \|\xi_i - \langle \xi_i, \hat{1}\rangle \hat{1}\|_2 = 0.$$

In the context of trace preserving actions, the equivalent properties characterizing spectral gap in Definition 15.1.1 can be expressed as follows:

(i) there exist a finite subset F of G and $c > 0$ such that

$$\forall x \in M, \quad \max_{g\in F} \|\sigma_g(x) - x\|_2 \geq c\|x - \tau(x)1\|_2;$$

(ii) every $\|\cdot\|_2$-bounded asymptotically G- invariant net (or sequence if G is countable) in M is asymptotically trivial;

(iii) for every $\varepsilon > 0$, there exist a finite subset F of G and $\delta > 0$ such that if $x \in M$ satisfies $\max_{g\in F} \|\sigma_g(x) - x\|_2 \leq \delta$, then $\|x - \tau(x)1\|_2 \leq \varepsilon$.

[1]In other term, the spectrum of h does not meet some neighbourhood of 1, hence the terminology of spectral gap.

REMARKS 15.1.7. **(a)** Let us mention that no ergodic probability measure preserving action of an amenable countable group has spectral gap and that in fact this property is another characterisation of amenability ([**Sch81**]).

(b) On the other hand, obviously every unitary representation of a Property (T) group has spectral gap. In this case, we even have a *uniform spectral gap property* in the sense that there exist a finite subset F of G and $c > 0$ such that for *every* unitary representation $(\pi, \mathcal{H})$ of G without non-zero invariant vector, then $\max_{g\in F} \|\pi(g)\xi - \xi\| \geq c\|\xi\|$ for $\xi \in \mathcal{H}$. For a group G, the fact that all its ergodic probability measure preserving actions have spectral gap implies that it has the property (T) ([**CW80, Sch81**]).

Bernoulli actions of non-amenable groups are the most basic examples of actions with spectral gap.

PROPOSITION 15.1.8. *Let (Y, ν) be a standard probability measure space and let G be a non-amenable countable group. We set $X = Y^G$ (the product of copies of Y indexed by G) and $\mu = \nu^{\otimes G}$. Then the Bernoulli action $\sigma : G \curvearrowright (X, \mu)$ has spectral gap.*

PROOF. For simplicity, we assume that ν is diffuse and we take $Y = \mathbb{T}$ with its Haar probability measure. Via Fourier transform, we identify $L^2(X, \mu)$ to $\ell^2(\mathbb{Z}^{(G)})$ where $\mathbb{Z}^{(G)}$ is the group of finitely supported functions from G to $\mathbb{Z}$. We set $\Gamma = \mathbb{Z}^{(G)}$. We let G act by left translations on Γ. Under the identification we have made, the unitary representation of G on $\ell^2(\Gamma)$ is defined, for $g \in G$ and $\gamma \in \Gamma$, by

$$\sigma_g \delta_\gamma = \delta_{g\gamma}.$$

This representation preserves globally the orthonormal set $\{\delta_\gamma : \gamma \in \Gamma, \gamma \neq 0\}$. Choose a set E of representatives of the orbits of the left G-action on $\Gamma \setminus \{0\}$. For $\gamma \in E$, let G_γ be the stabilizer of γ under the G-action. Note that G_γ is a finite subgroup of G since, except for finitely many ones, the components of γ are equal to 0. We now observe that the unitary representation σ restricted to $L^2_0(X, \mu)$ is equivalent to $\oplus_{\gamma\in E}\lambda_{G/G_\gamma}$. Moreover, since G_γ is finite the representation the quasi-regular representation λ_{G/G_γ} is equivalent to a subrepresentation of λ_G.

Assume that ι_G is weakly contained in $\oplus_{\gamma\in E}\lambda_{G/G_\gamma}$. Then it is weakly contained in a multiple of λ_G. But since any such multiple is weakly equivalent to λ_G, we see that ι_G is weakly contained in λ_G. This contradicts the fact that G is not amenable.

When (Y, ν) is not diffuse, we start with any orthonormal basis of $L^2(Y, \nu)$. It yields an orthonormal basis of $L^2(X, \mu)$ and we proceed in a way similar to what we did above with the basis $(\delta_\gamma)_{\gamma\in\Gamma}$. $\square$

More generally, ergodic generalized Bernoulli actions $G \curvearrowright (Y^Z, \nu^{\otimes Z})$ (see Definition 1.4.7) of a non-amenable countable group G have a spectral gap under the assumption that the stabilizers of the action $G \curvearrowright Z$ are amenable. The proof follows the same lines as the previous one for classical Bernoulli actions. We leave it as an exercise (Exercise 15.1). It needs the following result from group representations theory[2].

LEMMA 15.1.9. *Let G be a non-amenable countable group and $(H_i)_{i\in I}$ a family of amenable subgroups of G. We denote by λ_{G/H_i} the quasi-regular representation of G in $\ell^2(G/H_i)$. Then the trivial representation ι_G of G is not weakly contained in the Hilbert direct sum $\oplus_{i\in I}\lambda_{G/H_i}$.*

PROOF. Our assumption is that the trivial representation ι_{H_i} of H_i is weakly contained into its regular representation λ_{H_i}. The corresponding representations of G obtained by induction are respectively λ_{G/H_i} and λ_G. By continuity of induction of representations, we get that each λ_{G/H_i} is weakly contained into λ_G.

Assume that ι_G is weakly contained in $\oplus_{i\in I}\lambda_{G/H_i}$. Then it is weakly contained in a multiple of λ_G, hence in λ_G, in contradiction with the fact that G is not amenable. $\square$

15.1.3. Spectral gap for II_1 factors. This notion concerns the case where M is a II_1 factor and $G = \mathcal{U}(M)$ is the unitary group of M when we let $\mathcal{U}(M)$ act on M by $(u,x) \mapsto \mathrm{Ad}(u)(x) = uxu^*$.

DEFINITION 15.1.10. We say that the II_1 factor M has *spectral gap* if the action of $\mathcal{U}(M)$ on M has spectral gap, which is expressed by the three equivalent conditions:

(i) there exist a finite subset F of $\mathcal{U}(M)$ (or of M) and $c > 0$ such that
$$\forall x \in M, \quad \max_{u\in F} \|[u,x]\|_2 \geq c\|x - \tau(x)1\|_2;$$
(ii) every $\|\cdot\|_2$-bounded net (x_i) in M which asymptotically commutes with M (*i.e.*, is such that $\lim_i \|[y,x_i]\|_2 = 0$ for every $y \in M$) is asymptotically trivial;
(iii) for every $\varepsilon > 0$, there exist a finite subset F of $\mathcal{U}(M)$ (or of M) and $\delta > 0$ such that if $x \in M$ satisfies $\max_{u\in F} \|[u,x]\|_2 \leq \delta$, then $\|x - \tau(x)1\|_2 < \varepsilon$.

15.2. Spectral gap and Property Gamma

15.2.1. Property Gamma. This property is the first invariant that was introduced to show the existence of non hyperfinite II_1 factors.

Let M be a II_1 factor. A $\|\cdot\|_\infty$-bounded net (x_i) in M such that $\lim_i \|[y,x_i]\|_2 = 0$ for every $y \in M$ is said to be *central*.

DEFINITION 15.2.1. We say that M has *Property Gamma* if there exists a central net (x_i) in M which is not asymptotically trivial.

[2]Concerning the classical notions of induced representation and weak containment used in its proof, we refer for instance to [**BdlHV08**, Appendix F] and [**BdlHV08**, Appendix E] respectively (see also Section 13.3.1 for the latter).

REMARKS 15.2.2. **(a)** Since $\|x\|_2 \leq \|x\|_\infty$ for every $x \in M$, we see that Property Gamma is stronger than the property of not having spectral gap.

(b) The above definition is equivalent to the following one: there exists $c > 0$ such that for every finite subset F of M and every $\delta > 0$ there exists $x \in (M)_1$ with $\max_{x\in F} \|[y,x]\|_2 \leq \delta$ and $\|x - \tau(x)1\|_2 \geq c$.

(c) When M is separable, Property Gamma is equivalent to the existence of a central sequence (x_n) which is not asymptotically trivial. Indeed, take $c > 0$ as above let denote by $x_{F,\delta}$ an element of $(M)_1$ satisfying the above conditions relative to (F, δ). Since M is separable there exists an increasing sequence (F_n) of finite subsets of $(M)_1$ such that $\cup F_n$ is dense in $((M)_1, \|\cdot\|_2)$. For each integer n we set $x_n = x_{F_n,1/n}$. Then the sequence (x_n) is central and not asymptotically trivial.

Property Gamma is easily characterized by the following theorem when M is separable. Given a free ultrafilter ω on $\mathbb{N}$, we recall that the ultrapower M^ω has been defined in Section 5.4. We see M as a von Neumann subalgebra of M^ω, in an obvious way.

THEOREM 15.2.3. *Let M be a separable II_1 factor and let ω be free ultrafilter on $\mathbb{N}$. The following conditions are equivalent:*

(i) *M has Property Gamma;*
(ii) *$M' \cap M^\omega \neq \mathbb{C}1$;*
(iii) *$M' \cap M^\omega$ is diffuse;*
(iv) *there exists a central sequence (v_n) in the unitary group of M such that $\tau(v_n) = 0$ for all n.*

PROOF. (i) $\Rightarrow$ (ii). Let (x_n) be a central sequence such that for some $c > 0$ and for every n we have $\|x_n - \tau(x_n)1\|_2 \geq c$. Then obviously $(x_n)_\omega \in M' \cap M^\omega$ is not scalar.

(ii) $\Rightarrow$ (iii). Let $p \in M^\omega \cap M'$ be a non trivial projection and set $\tau_\omega(p) = \lambda \in]0,1[$. Let (p_n) be a representative of p which consists of projections such that $\tau(p_n) = \lambda$ for every n (see Lemma 5.4.2). The functional $x \in M \mapsto \tau_\omega(xp)$ is a trace and therefore we have $\lim_\omega \tau(xp_n) = \lambda\tau(x)$ for $x \in M$.

Let (F_n) be an increasing sequence of finite subsets of $(M)_1$ such that $\cup_n F_n$ is s.o. dense in $(M)_1$. We can choose a subsequence (p_{k_n}) of (p_n) such that for $n \geq 1$ we have

(a) $\|[p_n, p_{k_n}]\|_2 \leq 1/n$ and $\left|\tau(p_n p_{k_n}) - \lambda^2\right| \leq 1/n$;
(b) $\max_{x\in F_n} \|[p_{k_n}, x]\|_2 \leq 1/n$.

It follows that $(p_n p_{k_n})_\omega$ is a non-zero projection in $M^\omega \cap M'$ which is strictly smaller than p. Therefore, p is not a minimal projection.

(iii) $\Rightarrow$ (iv). Since the von Neumann algebra $M' \cap M^\omega$ is diffuse, it contains a projection p such that $\tau_\omega(p) = 1/2$ (see Exercise 3.3). Let (p_n) be a representative of p consisting of projections p_n in M with $\tau(p_n) = 1/2$ for every n (see Lemma 5.4.2). We set $v_n = 2p_n - 1$. Then (v_n) is a sequence of unitaries in M with $\tau(v_n) = 0$ and $\lim_\omega \|[x, v_n]\|_2 = 0$ for every $x \in M$.

(iv) $\Rightarrow$ (i) is obvious. $\square$

15.2.2. Property Gamma and spectral gap. We now state the main result of this section.

THEOREM 15.2.4. *Let M be a separable* II_1 *factor. Then M has Property Gamma if and only if it does not have spectral gap.*

To prove this theorem, we need some preliminaries about ultrapowers. Even if M is assumed to be separable, we cannot limit ourself to sequences in Definition 15.1.10 (ii) and we will have to deal with directed sets. Given such a set I, a filter $\mathcal{F}$ of subsets of I is said to be *cofinal* if for every $i_0 \in I$ the set $\{i \in I : i \geq i_0\}$ belongs to $\mathcal{F}$. When $I = \mathbb{N}$, the cofinal ultrafilters are the free ultrafilters.

Let ω be a cofinal ultrafilter on the directed set I and let M be a II_1 factor. The ultrapower M^ω along ω is defined exactly as we did in Section 5.4 for free ultrafilters on $\mathbb{N}$. We will need the following immediate characterisation of Property Gamma in terms of such ultrapowers.

LEMMA 15.2.5. *Let M be a separable* II_1 *factor. The following conditions are equivalent:*

(i) *M has Property Gamma;*
(ii) *there exists a directed set I and a cofinal ultrafilter ω on I such that $M^\omega \cap M' \neq \mathbb{C}$.*

Given a Hilbert space $\mathcal{H}$ and a cofinal ultrafilter ω on a directed set I we define the ultrapower $\mathcal{H}^\omega$ as the quotient of $\ell^\infty(I, \mathcal{H})$ by the subspace of all nets (ξ_i) such that $\lim_\omega \|\xi_i\|_{\mathcal{H}} = 0$. For $(\xi_i) \in \ell^\infty(I, \mathcal{H})$ we denote by $(\xi_i)_\omega$ its class in $\mathcal{H}^\omega$. We easily see that $\mathcal{H}^\omega$, endowed with the scalar product $\langle (\xi_i)_\omega, (\eta_i)_\omega \rangle = \lim_\omega \langle \xi_i, \eta_i \rangle_{\mathcal{H}}$ is a Hilbert space.

Let M be a II_1 factor and I, ω as above. Then M acts to the left in a natural way on $L^2(M)^\omega$, by setting

$$\forall x \in M,\ \forall (\xi_i)_\omega \in L^2(M)^\omega, \quad x(\xi_i)_\omega = (x\xi_i)_\omega.$$

Similarly, M acts to the right on $L^2(M)^\omega$.

The Hilbert space $L^2(M^\omega)$ is a closed subspace of $L^2(M)^\omega$, stable under the above actions of M. The orthogonal of $L^2(M^\omega)$ in $L^2(M)^\omega$ is denoted by $L^2(M)^\omega \ominus L^2(M^\omega)$. We will denote by $L^2(M)^\omega \cap M'$ the space of elements $(\xi_i)_\omega$ in $L^2(M)^\omega$ such that $\lim_\omega \|y\xi_i - \xi_i y\|_2 = 0$ for every $y \in M$, and similarly we will use the notation $L^2(M^\omega) \cap M'$.

LEMMA 15.2.6. *Let M be a separable* II_1 *factor and let $(\xi_i)_\omega$ be a self-adjoint (i.e., $\xi_i = \xi_i^*$ for all i) element of $L^2(M)^\omega \ominus L^2(M^\omega)$. Then $(|\xi_i|)_\omega$ is also orthogonal to $L^2(M^\omega)$.*

PROOF. For every interval J in $\mathbb{R}$ we denote by $E_J(\xi_i)$ the spectral projection of ξ_i relative to J. Since $(E_J(\xi_i))_\omega \in M^\omega$, we see that

$$\begin{aligned}\langle (\xi_i E_J(\xi_i))_\omega, (\eta_i)_\omega \rangle &= \lim_\omega \langle \xi_i E_J(\xi_i), \eta_i \rangle_{L^2(M)} \\ &= \lim_\omega \langle \xi_i, E_J(\xi_i)\eta_i \rangle_{L^2(M)} = \langle (\xi_i)_\omega, (E_J(\xi_i))_\omega (\eta_i)_\omega \rangle = 0\end{aligned}$$

for every $(\eta_i)_\omega \in L^2(M^\omega)$. Thus $(\xi_i E_J(\xi_i))_\omega$ is orthogonal to $L^2(M^\omega)$.

We have

$$(|\xi_i|)_\omega = (\xi_i E_{[0+\infty[}(\xi_i))_\omega - (\xi_i E_{]-\infty,0[}(\xi_i))_\omega$$

and therefore $(|\xi_i|)_\omega$ is orthogonal to $L^2(M^\omega)$. □

The following lemma is crucial for the proof of Theorem 15.2.4.

LEMMA 15.2.7. *Assume that* $L^2(M)^\omega \ominus L^2(M^\omega)$ *contains a non-zero element* $(\xi_i)_\omega$ *which commutes with* M. *Then for every finite subset* F *of* $\mathcal{U}(M)$ *and every* $\varepsilon > 0$ *there exists a non-zero projection* $e \in M$ *such that* $\tau(e) \leq \varepsilon$ *and* $\max_{u\in F} \|[u,e]\|_2 \leq \varepsilon \|e\|_2$.

PROOF. We may assume that $(\xi_i)_\omega$ is self-adjoint with $\|\xi_i\|_2 \leq 1$ for every i, and by the previous lemma we may even assume that $\xi_i \in L^2(M)_+$ for every i. Morever, we may take $(\xi_i)_\omega$ with a support as small as we wish. Indeed, for $a > 0$ and $\eta \in L^2(M)_+$ let us denote by $E_a(\eta)$ the spectral projection of η corresponding to the interval $[0,a]$ and by $E_a^c(\eta)$ its spectral projection corresponding to $]a,+\infty[$. We have $\lim_\omega \langle \xi_i, \xi_i E_a(\xi_i)\rangle_{L^2(M)} = 0$ and therefore $\lim_\omega \|\xi_i - \xi_i E_a^c(\xi_i)\|_2 = 0$. The support of $\xi_i E_a^c(\xi_i)$ is smaller than $E_a^c(\xi_i)$ with $\tau(E_a^c(\xi_i)) \leq a\tau(\xi_i) \leq a$.

For $a > 0$, we set $\xi_{i,a} = \xi_i E_a^c(\xi_i)$. For $u \in F$, we have $\lim_\omega \|[u,\xi_{i,a}]\|_2 = 0$ and $\lim_\omega \|\xi_{i,a}\|_2 > 0$. Therefore, given $\varepsilon' > 0$, there exists i such that

$$\max_{u\in F} \|[u,\xi_{i,a}]\|_2 \leq \varepsilon' \|\xi_{i,a}\|_2.$$

Now, by Theorem 10.3.6 there exists $t_0 > 0$ such that

$$\max_{u\in F} \big\|[u, E_{t_0}^c(\xi_{i,a})]\big\|_2 < (3n\varepsilon')^{1/2} \big\|E_{t_0}^c(\xi_{i,a})\big\|_2,$$

where n is the cardinal of F. Observe that $\tau(E_{t_0}^c(\xi_{i,a})) \leq \tau(E_a^c(\xi_i)) \leq a$. To conclude, we first choose $a < \varepsilon$ and ε' with $(3n\varepsilon')^{1/2} < \varepsilon$, and then we get i and t_0 and set $e = E_{t_0}^c(\xi_{i,a})$. □

PROOF OF THEOREM 15.2.4. We assume that M does not have spectral gap and want to show that M has Property Gamma. There exists a $\|\cdot\|_2$-bounded net $(x_i)_{i\in I}$ of self-adjoint elements of M which asymptotically commutes with M but is not asymptotically trivial. Therefore there is a cofinal ultrafilter ω on I such that $\lim_\omega \|[y,x_i]\|_2 = 0$ for every $y \in M$ and $\lim_\omega \|x_i - \tau(x_i)1\| = c > 0$. Replacing x_i by $(x_i - \tau(x_i)1)/\|x_i - \tau(x_i)1\|_2^{-1}$ for i large enough we get a net (x_i) of elements of M, with $\|x_i\|_2 = 1$ for every i, such that $\lim_\omega \|[y,x_i]\|_2 = 0$ for every $y \in M$ and which satisfies $\tau(x_i) = 0$ for every i.

We have $\|(x_i)_\omega\|_2 = 1$ and $(x_i)_\omega y = y(x_i)_\omega$ for every $y \in M$. Moreover, if $\widehat{1}_\omega$ denotes the unit of M viewed as an element of $L^2(M^\omega) \subset L^2(M)^\omega$, we have $\langle (x_i)_\omega, \widehat{1}_\omega\rangle_{L^2(M)^\omega} = 0$.

Assume by contradiction that M does not have Property Gamma and therefore

$$L^2(M^\omega) \cap M' = \mathbb{C}\widehat{1}_\omega.$$

It follows that $(x_i)_\omega$ does not belong to $L^2(M^\omega)$. Let f be the orthogonal projection from $L^2(M)^\omega$ onto $L^2(M^\omega)$. Observe that f commutes with M. Replacing $(x_i)_\omega$ by $(x_i)_\omega - f(x_i)_\omega$ we may assume that $(x_i)_\omega$ is orthogonal to $L^2(M^\omega)$. Using Lemma 15.2.6 we see that there exists a non-zero element, that we still denote $(x_i)_\omega$, with $x_i \in M_+$ and $\|x_i\|_2 \leq 1$ for every i, which commutes with M and is orthogonal to $L^2(M^\omega)$.

Let us show that given any non-zero projection q in M, for every finite subset F of qMq and every $\varepsilon > 0$ we may find a non-zero projection e in M with $e \leq q$, $\tau(e) \leq \varepsilon$ and $\max_{y\in F} \|[y,e]\|_2 \leq \varepsilon \|e\|_2$. To that purpose we consider $q(x_i)_\omega = q(x_i)_\omega q$. It is a positive element in $L^2(qMq)^\omega \cap (qMq)'$ which is orthogonal to $L^2((qMq)^\omega)$. Moreover we have $q(x_i)_\omega q \neq 0$ since $\langle q(x_i)_\omega, (x_i)_\omega\rangle = \|(x_i)_\omega\|^2 \tau(q)$, due to the fact that $y \mapsto \langle y(x_i)_\omega, (x_i)_\omega\rangle$ is a trace on M. Then it suffices to apply Lemma 15.2.7 to $q(x_i)_\omega q$ instead of $(x_i)_\omega$ and qMq instead of M.

Using this fact and a maximality argument, we now show that given a finite subset F of $M_{s.a}$ and $\varepsilon > 0$ there exists a projection $q \in M$ such that $\max_{y\in F} \|[y,q]\|_2 \leq \varepsilon$ and $\tau(q) = 1/2$. Let $\mathcal{E}$ be the set of projections e such that $\max_{y\in F} \|[y,e]\|_2 \leq \varepsilon\|e\|_2$ and $\tau(e) \leq 1/2$. With its usual order, this set is inductive and therefore has a maximal element q. We claim that $\tau(q) = 1/2$. Otherwise, we set $\varepsilon_1 = 1/2 - \tau(q)$, $q_1 = 1 - q$ and $F_1 = \{q_1 y q_1 : y \in F\}$. Then there exists a non-zero projection $p \in q_1 M q_1$ such that $\max_{y\in F_1} \|[y,p]\|_2 \leq \varepsilon\|p\|_2$ and $\tau(p) \leq \varepsilon_1$. We set $q' = q + p$.

Straightforward computations, using Pythagoras' theorem, show that whenever y is self-adjoint then

$$\begin{aligned}\big\|[y,q']\big\|_2^2 &= 2\big\|q'y(1-q')\big\|_2^2 = 2\big\|qy(1-q')\big\|_2^2 + 2\big\|py(1-q')\big\|_2^2\\ &= \|[q,(1-p)y(1-p)]\|_2^2 + \|[p,(1-q)y(1-q)]\|_2^2.\end{aligned}$$

Moreover, since

$$\|[q,(1-p)y(1-p)]\|_2^2 = 2\tau(qy(1-p)yq) - 2\tau(yqyq)$$

we see that $\|[q,(1-p)y(1-p)]\|_2^2 \leq \|[q,y]\|_2^2$. It follows that

$$\big\|[y,q']\big\|_2^2 \leq \varepsilon^2\tau(q) + \varepsilon^2\tau(p),$$

and therefore $\|[y,q']\|_2 \leq \varepsilon\|q'\|_2$, with $\tau(q') \leq 1/2$ and this contradicts the maximality of q in $\mathcal{E}$.

In conclusion, we get a net (q_i) of projections in M which asymptotically commutes with M and is such that $\tau(q_i) = 1/2$ for all i. This is impossible since M does not have Property Gamma. $\square$

Remark 15.2.8. The result stated in Theorem 15.2.4 is remarkable. It does not extend to the following classical situation where one considers a trace preserving action $G \curvearrowright (M,\tau)$ of a countable group on a tracial von Neumann algebra. In this setting, the analogue of not having Property Gamma is known in the literature as *strong ergodicity*. It means that every $\|\cdot\|_\infty$-bounded asymptotically G-invariant sequence (x_n) in M (*i.e.*, such that, for all $g \in G$, $\lim_n \|\sigma_g(x_n) - x_n\|_2 = 0$) is asymptotically trivial (*i.e.*, $\lim_n \|x_n - \tau(x_n)1\|_2 = 0$).

Every action with spectral gap is strongly ergodic and every strongly ergodic action is ergodic. However, there are strongly ergodic actions that do not have the spectral gap property. An example of such a probability measure preserving action of the free group $\mathbb{F}_3$ is given in [**Sch81**, Example 2.7].

15.3. Spectral gap and full II_1 factors

In this section, M will still be a separable II_1 factor.

15.3.1. Fullness and spectral gap. We show below the noteworthy fact that the spectral gap property can be expressed as a topological property of the group $\mathrm{Aut}\,(M)$ of automorphisms[3] of M. Recall that $\mathrm{Aut}\,(M)$ is endowed with the topology for which a net (α_i) converges to α if for every $x \in M$ we have $\lim_i \|\alpha_i(x) - \alpha(x)\|_2 = 0$. It is a Polish group (see Section 7.5.3).

Definition 15.3.1. We say that M is *full* if the subgroup $\mathrm{Inn}\,(M)$ of inner automorphisms of M is closed in $\mathrm{Aut}\,(M)$ (and so the group $\mathrm{Out}\,(M) = \mathrm{Aut}\,(M)/\mathrm{Inn}\,(M)$ is a Polish group).

The terminology is explained by the fact that M is full if and only if $\mathrm{Inn}\,(M)$ is complete.

Theorem 15.3.2. *Let M be a separable II_1 factor. The following conditions are equivalent:*

(i) *M is a full factor;*
(ii) *M has spectral gap.*

Proof. We denote by θ the homomorphism $u \mapsto \mathrm{Ad}\,(u)$ from $\mathcal{U}(M)$ onto $\mathrm{Inn}\,(M)$. The s.o. topology on $\mathcal{U}(M)$ is defined by the metric $d(u,v) = \|u-v\|_2$ which makes it a complete metric space. Endowed with the quotient topology, the group $\mathcal{U}(M)/\mathbb{T}1$, quotient of $\mathcal{U}(M)$ by the group $\mathbb{T}1$ of scalar unitaries, is a Polish group. If $[u]$ denotes the class of u, the quotient metric is $d'([u],[v]) = \inf_{\lambda\in\mathbb{T}} d(\lambda u, v)$. The homomorphism θ gives, by passing to the quotient, a continuous isomorphism θ' from the Polish group $\mathcal{U}(M)/\mathbb{T}1$ onto the topological group $\mathrm{Inn}\,(M)$.

Let us prove that (i) $\Rightarrow$ (ii). Assume that $\mathrm{Inn}\,(M)$ is closed in the Polish group $\mathrm{Aut}\,(M)$. Then $\mathrm{Inn}\,(M)$ is itself a Polish group and the open mapping theorem (see B.4 in the appendix) implies that θ' is a homeomorphism. Let ω be a free ultrafilter on $\mathbb{N}$. Using the theorems 15.2.4 and 15.2.3, in

[3] They are automatically trace preserving.

order to prove (ii) it suffices to show that $M^\omega \cap M' = \mathbb{C}1$. It is enough to show that every unitary in M^ω which commutes with M is scalar. Every unitary in M^ω is of the form $(u_n)_\omega$ where (u_n) is a sequence of unitaries in M (see Exercise 5.12). If $(u_n)_\omega$ commutes with M then $\lim_\omega \mathrm{Ad}\,(u_n) = \mathrm{Id}_M$ in $\mathrm{Inn}\,(M)$. Since $(\theta')^{-1}$ is continuous we see that $\lim_\omega [u_n] = [1]$ in $\mathcal{U}(M)/\mathbb{T}1$ and therefore there exists a sequence (λ_n) of elements of $\mathbb{T}$ such that $\lim_\omega \|u_n - \lambda_n 1\|_2 = 0$.

Let us prove now that (ii) $\Rightarrow$ (i). Assume that M has spectral gap. For every integer $n \geq 1$ there exist a finite subset F_n of M and $\delta_n > 0$ such that if $u \in \mathcal{U}(M)$ satisfies $\max_{x\in F_n} \|[u,x]\|_2 \leq \delta_n$ then

$$d(u, \mathbb{T}1) = \inf_{\lambda\in\mathbb{T}} \|u - \lambda 1\|_2 \leq 1/2^n.$$

Let $\alpha \in \overline{\mathrm{Inn}\,(M)}$ and let (v_n) be a sequence of unitaries such that

$$\lim_n \mathrm{Ad}\,(v_n) = \alpha.$$

We choose this sequence in such a way that for every n we have

$$\max_{x\in F_n} \left\|[v_{n+1}^{-1}v_n, x]\right\|_2 = \max_{x\in F_n} \left\|\big(\mathrm{Ad}\,(v_{n+1}^{-1}v_n)\big)(x)\right\|_2 \leq \delta_n$$

and so $d(v_{n+1}^{-1}v_n, \mathbb{T}1) \leq 1/2^n$. Then for each n we choose $u_n \in \mathcal{U}(M)$ such that $\mathrm{Ad}\,(u_n) = \mathrm{Ad}\,(v_n)$ and $\|u_{n+1} - u_n\|_2 \leq 1/2^n$. It follows that the sequence (u_n) converges in the s.o. topology to an unitary u, and we have $\mathrm{Ad}\,(u) = \alpha$. $\square$

15.4. Property Gamma and inner amenability

Let G be a countable group. For $f \in \ell^\infty(G)$ and $s \in G$ we denote by $\mathrm{ad}_s(f)$ the function $t \mapsto f(s^{-1}ts)$. A mean m on $\ell^\infty(G)$ is said to be *inner invariant* if $m \circ \mathrm{ad}_s = m$ for every $s \in G$. Of course the Dirac measure at e is such a mean. We say that m is *non-trivial* if it is supported on $G \setminus \{e\}$.

Definition 15.4.1. We say that G is *inner amenable* if it carries a non-trivial inner invariant mean.

Every amenable group G is inner amenable. Indeed, starting from respectively left and right invariant means m_l and m_r we build a mean m as follows. Given $f \in \ell^\infty(G)$ we denote by F the function $s \mapsto m_l(f_s)$, where $(f_s)(x) = f(xs)$. We set $m(f) = m_r(F)$. Then m is easily seen to be a non-trivial inner invariant mean.

The free groups $\mathbb{F}_n$ with $n \geq 2$ generators are not inner amenable. Indeed, let $a_1, \dots, a_k, \dots$ be the generators of $\mathbb{F}_n$. Let S be the subset of $\mathbb{F}_n$ consisting of the elements which, in reduced form, end by a non-zero power of a_1. We make the following easy observation:

(i) $S \cup a_1 S a_1^{-1} = \mathbb{F}_n \setminus \{e\}$,
(ii) $S, a_2 S a_2^{-1}, a_2^{-1} S a_2$ are pairwise disjoint.

This immediately implies the non-inner amenability of $\mathbb{F}_n$.

Proposition 15.4.2. *Let G be a countable group and $(L(G),\tau)$ the corresponding tracial von Neumann algebra. Assume that $(L(G),\tau)$ has Property Gamma. Then the group G is inner amenable.*

Proof. Let (v_n) be a sequence of unitary operators in $L(G)$ with $\tau(v_n) = 0$ for every n and such that $\lim_n \|xv_n - v_n x\|_2 = 0$ for every $x \in L(G)$. In particular, for every $s \in G$ we have

$$\lim_n \|v_n - u_s v_n u_s^*\|_2 = 0,$$

where the u_s are the canonical unitaries in $L(G)$. We view v_n as an element of $\ell^2(G)$. Note that $\widehat{v_n}(e) = \tau(v_n) = 0$, where e is the identity of G. We set $\xi_n = |\widehat{v_n}|^2$. Then (ξ_n) is a sequence of elements in $\ell^1(G)_+$ with $\xi_n(e) = 0$, $\|\xi_n\|_1 = 1$. Since, by the Powers-Størmer inequality and Lemma 7.4.10, we have

$$\begin{aligned}\|\xi_n - \mathrm{ad}_s(\xi_n)\|_1 &\leq \left\|\xi_n^{1/2} - \mathrm{ad}_s(\xi_n^{1/2})\right\|_2 \left\|\xi_n^{1/2} + \mathrm{ad}_s(\xi_n^{1/2})\right\|_2\\ &\leq 2\left\|\xi_n^{1/2} - \mathrm{ad}_s(\xi_n^{1/2})\right\|_2 \leq 2\sqrt{2}\|v_n - u_s v_n u_s^*\|_2^{1/2},\end{aligned}$$

we see that $\lim_n \|\xi_n - \mathrm{ad}_s \xi_n\|_1 = 0$. It follows immediately that any weak* cluster point in $\ell^\infty(G)^*$ of the sequence (ξ_n) is a non-trivial inner invariant mean. □

Examples 15.4.3. **(a)** *The hyperfinite factor R is amenable and has Property Gamma.*

Indeed, let us write $R = \overline{\cup_n Q_n}^{w.o}$ where $Q_n = M_{2^n}(\mathbb{C})$. Given $\varepsilon > 0$ and $x_1, \dots, x_m$ in M, there exists an integer n and $y_1, \dots, y_m$ in Q_n such that $\|x_i - y_i\|_2 \leq \varepsilon/2$ for $i = 1, \dots, m$. Since $Q_{n+1} = Q_n \otimes M_2(\mathbb{C})$, setting $U = 1 \otimes u$ where u is a unitary in $M_2(\mathbb{C})$ with trace equal to zero, we get a unitary $U \in M$ with $\tau(U) = 0$ and which commutes with the y_i. It follows immediately that $\|Ux_i - x_i U\|_2 \leq \varepsilon$ for all i.

(b) *The factor $L(\mathbb{F}_n)$, $n \geq 2$, does not have Property Gamma.*

This follows from Proposition 15.4.2 since $\mathbb{F}_n$ is not inner amenable.

(c) *$R\overline{\otimes}L(\mathbb{F}_n)$ has Property Gamma and is not amenable.*

Indeed is easily checked that every II_1 factor of the form $M \otimes N$ where M has Property Gamma retains this property. Therefore the II_1 factor $R\overline{\otimes}L(\mathbb{F}_n)$, $n \geq 2$, has Property Gamma. It is not amenable. Otherwise, because of the existence of a conditional expectation from $R\overline{\otimes}L(\mathbb{F}_n)$ onto $L(\mathbb{F}_n)$ (the trace preserving one for instance), $L(\mathbb{F}_n)$ would be amenable, which is not the case since the group $\mathbb{F}_n$ is not amenable.

(d) *Every* II_1 *factor that has Property* (T) *is full.*

This follows from the fact that $\mathrm{Inn}\,(M)$ is an open subgroup of $\mathrm{Aut}\,(M)$ (see Proposition 14.3.4).

As a consequence, the factors R, $L(\mathbb{F}_n)$, $R\overline{\otimes}L(\mathbb{F}_n)$ are not isomorphic. A factor with Property (T) is isomorphic neither to R nor to $R\overline{\otimes}L(\mathbb{F}_n)$. We will see in the next chapter that it is not isomorphic to any factor with the Haagerup property like $L(\mathbb{F}_n)$.

Exercises

Exercise 15.1. Let $G \curvearrowright Z$ be an action of a countable group G on a set Z. We assume that G is non-amenable, that the stabilizers of this action are amenable and that the G-orbits are infinite. Let (Y, ν) be a standard probability measure space. Show that the generalized Bernoulli action $G \curvearrowright (Y^Z, \nu^{\otimes Z})$ has spectral gap.

Exercise 15.2. Show that the stabilizers of the natural action of $SL(2, \mathbb{Z})$ on $\mathbb{Z}^2 \setminus \{0, 0\}$ are amenable and conclude that the action $SL(2, \mathbb{Z}) \curvearrowright (\mathbb{T}^2, \lambda)$ has spectral gap.

Exercise 15.3. Let $G \curvearrowright (M, \tau)$ be a trace preserving action.

(i) Show that the algebra M^G of G-invariant elements in M is dense in the Hilbert space $L^2(M)^G$ of G-invariant vectors in $L^2(M)$.
(ii) Conclude that $G \curvearrowright (M, \tau)$ is ergodic if and only if $L^2(M)^G = \mathbb{C}\hat{1}$.

Exercise 15.4. Let (φ_n) be a sequence of states on a tracial von Neumann algebra (M, τ). We assume the existence of a sequence (e_n) of projections in M and of $c > 0$ such that $\lim_n \tau(e_n) = 0$ and $\varphi_n(e_n) \geq c$ for every n. Show that (φ_n) has a weak* cluster point in M^* which is a non-normal state.

Exercise 15.5. Let $G \curvearrowright (M, \tau)$ be a trace preserving action. We assume the existence of a sequence (p_n) of non-zero projections in M such that $\lim_n \|p_n\|_1 = 0$ and $\lim_n \|\sigma_g(p_n) - p_n\|_1 / \|p_n\|_1 = 0$ for every $g \in G$. Show that M has a non-normal G-invariant state.

Exercise 15.6. Let $\sigma : G \curvearrowright (M, \tau)$ be a trace preserving action and let ω be a free ultrafilter on $\mathbb{N}$ such that $(M^\omega)^G$ is diffuse (or more generally contains non-zero projections of trace as small as we wish).

(i) Show that for every $\delta > 0$, every $\varepsilon > 0$ and every finite subset F of G, there exists a non-zero projection $p \in M$ such that $\tau(p) < \delta$ and $\max_{g \in F} \|\sigma_g(p) - p\|_1 \leq \varepsilon \tau(p) = \varepsilon \|p\|_1$.
(ii) If in addition the group G is countable, show that there is a non-normal G-invariant state on M.
(iii) Show that (ii) does not necessarily hold when G is not countable.

Exercise 15.7. Let $\sigma : G \curvearrowright (M, \tau)$ be an ergodic action of a non necessarily countable group. Show that if the action has spectral gap, there exists a countable subgroup G_0 of G for which τ is the only G_0-invariant state (Hint: take F as in Definition 15.1.1 (i) and let G_0 be the subgroup generated by F. Consider a G_0-invariant state ψ on M and use a Day's convexity argument to fing a net (ϕ_i) of normal states on M that converges to ψ in the weak* topology and is such that $\lim_i \|\phi_i \circ \sigma_t - \phi_i\| = 0$ for $t \in F$. Use the Powers-Størmer inequality to show that $\lim_i \phi_i = \tau$).

Exercise 15.8. Let M be a separable II_1 factor. Show that M has Property Gamma if and only if for every countable subgroup G_0 of $\mathcal{U}(M)$ there exists a non-normal G_0-invariant state on M under the Ad-action (equivalently, if and only if for every unital separable (with respect to the norm $\|\cdot\|_\infty$) C^*-subalgebra B_0 of M there is a non-normal state φ on M such that $\varphi(xy) = \varphi(yx)$ for every $x \in M$ and every $y \in B_0$).

Exercise 15.9. Let G be an ICC group with the property (T) and let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action. Let $M = L^\infty(X) \rtimes G$ be the corresponding crossed product.

(i) We denote by E the trace preserving conditional expectation from M onto $L^\infty(X)$ and by u_s, $s \in G$, the canonical unitaries. Let (x_n) be a $\|\cdot\|_2$-bounded sequence in M such that $\lim_n \|[x_n, u_s]\|_2 = 0$ for every $s \in G$. Show that $\lim_n \|x_n - E(x_n)\|_2 = 0$.

(ii) Show that the crossed product $M = L^\infty(X) \rtimes G$ does not have the property Gamma

Exercise 15.10. Let R be the hyperfinite II_1 factor and (N_n) be an increasing sequence of subfactors of type I_{2^n} such that $(\cup N_n)'' = R$. Let $\alpha \in \mathrm{Aut}\,(R)$ and for each n chose a unitary element $u_n \in R$ such that $\alpha(x) = u_n x u_n^*$ for every $x \in N_n$ (see Exercice 2.7). Show that for every $x \in R$, we have $\lim_n \|\alpha(x) - u_n x u_n^*\|_2 = 0$ and conclude that $\mathrm{Inn}\,(R)$ is dense in $\mathrm{Aut}\,(R)$.[4]

Exercise 15.11. Let M and N be two II_1 factors.

(i) We assume that M is full. Show that there exist a finite subset $F_1 \in \mathcal{U}(M)$ and $c_1 > 0$ such that for every $z \in M \odot N$ we have

$$\sum_{u \in F_1} \|[u \otimes 1, z]\|_2^2 \geq c_1 \|z - E_N(z)\|_2^2.$$

(Hint: write z as $z = \sum_{i=1}^n x_i \otimes y_i \in M \overline{\otimes} N$ where (y_i) is orthonormal in $(N, \|\cdot\|_2)$).

(ii) Assume that N is full. Show that there exist a finite subset $F_2 \in \mathcal{U}(N)$ and $c_2 > 0$ such that for every $z \in M \odot N$ we have

$$\sum_{u \in F_2} \|[u, E_N(z)]\|_2^2 \geq c_2 \|E_N(z) - \tau(z)\|_2^2.$$

(iii) Conclude that $M \overline{\otimes} N$ is full whenever M and N are full.

Notes

The idea of studying the dynamics of a group action by spectral tools dates back to Koopman's paper [**Koo31**] for $\mathbb{Z}$-actions. Since then, it proved to be a very fruitful technique in ergodic theory.

[4]The group $\mathrm{Out}\,(M)$ is very big: it contains a copy of every separable locally compact group [**Bla58**].

The notion of spectral gap is seen as a rigidity property. Recall that every ergodic p.m.p. action of a group satisfying property (T) has spectral gap. However, there exist very interesting examples of spectral gap property under actions of groups which do not have property (T) as pointed out in [**Sch80, Sch81, Jon83a, Pop06b**] (see Proposition 15.1.8, Exercises 15.1, 15.2). Thanks to this notion of spectral gap, surprisingly general rigidity results can be obtained for group actions and their crossed products (see [**Pop08**]).

Ultrapowers and central sequences are very useful tools to investigate the structure of II_1 factors. They have led to the discovery of new II_1 factors in the sixties [**Sak69, Chi69, DL69, ZM69**], culminating in the final remarkable proof around 1969 of the existence of uncountably many non isomorphic separable such factors [**McD69a, McD69b, Sak70**]. The importance of central sequences was also revealed in another paper of McDuff [**McD70**]. For a II_1 factor M, she proved that $M' \cap M^\omega$ is either abelian or a II_1 factor and that the latter case occurs if and only if M and $M \overline{\otimes} R$ are isomorphic. In this case, M is now called a *McDuff factor*.

Later, Connes made further very deep uses of these notions and most of the main ideas of this chapter are due to him. Theorem 15.2.3 essentially comes from [**Con74**], as well as Theorem 15.3.2 (see also [**Sak74**] for the latter). Theorem 15.2.4 is borrowed from [**Con76**]. We have chosen to express these results in terms of the more recent terminology of spectral gap (see [**Pop08, Pop12**]).

Property Gamma is one of the invariants for II_1 factors introduced by Murray and von Neumann in [**MvN43**]. They showed there that R has property Gamma while $L(\mathbb{F}_n)$, $n \geq 2$, has not. The proof we give in Section 15.4 uses the more recent notion of inner amenability due to Effros [**Eff75**]. Proposition 15.4.2 is taken from Effros' paper. The question of whether the inner amenability of the group G implies that $L(G)$ has property Gamma remained open since then and has been solved only recently in the negative by Vaes [**Vae12**].

The fact that $R \overline{\otimes} L(\mathbb{F}_n)$ is not isomorphic to R was proved to J.T. Schwartz [**Sch63**], by using his so-called property (P) instead of Connes' notion of injectivity.

The notion of strong ergodicity for a probability measure preserving action of a countable group was coined in [**Sch81**]. Connes and Weiss [**CW80**] proved that a countable group G has property (T) if and only if every ergodic p.m.p. action of G on a probability measure space is strongly ergodic, and Schmidt has shown that this is also equivalent to the fact that every ergodic p.m.p. action of G has spectral gap. On the other hand, a countable group G is amenable if and only if no ergodic p.m.p. action of G is strongly ergodic, and also if and only if no ergodic p.m.p. action of G has spectral gap [**CFW81, Sch81**]. This is to be compared with the fact that the group $\mathbb{F}_3$ has an action which is strongly ergodic but without spectral gap ([**Sch81**, Example 2.7]).

CHAPTER 16

Haagerup property (H)

We extend to tracial von Neumann algebras the Haagerup property which, in the setting of groups, was first detected for free groups. Here, we make use of the similarities between positive definite functions on groups and completely positive maps on von Neumann algebras. For further applications to the study of II_1 factors, we also discuss the notion of relative Haagerup property.

16.1. Haagerup property for groups

DEFINITION 16.1.1. A group G is said to have the *Haagerup property* (or *property* (H)) if there exists a net (sequence if G is countable) (φ_i) of positive definite functions on G such that $\lim_i \varphi_i = 1$ pointwise and $\varphi_i \in c_0(G)$ for every i.[1]

EXAMPLES 16.1.2. Obviously, amenable groups have property (H).

Free groups $\mathbb{F}_k$ with $k \geq 2$ generators are the most basic examples of non amenable groups with Property (H). This follows from the fact, that the word length function $g \mapsto |g|$ is conditionally negative definite on $\mathbb{F}_k$. Recall that $\psi : G \to \mathbb{R}$ is *conditionally negative definite* if

(a) $\psi(e) = 0$, $\psi(g) = \psi(g^{-1})$ for every $g \in G$,
(b) for any integer n, any $g_1, \dots, g_n \in G$ and any real numbers $c_1, \dots, c_n$ with $\sum_{i=1}^n c_i = 0$, we have $\sum_{i,j=1}^n c_i c_j \psi(g_i^{-1} g_j) = 0$.

By Schoenberg's theorem, this property holds if and only if for every $t > 0$, the function $\exp(-t\psi)$ is positive definite[2]. Since the length function on $\mathbb{F}_k$ is proper, we see that $\varphi_n : g \mapsto \exp(-|g|/n)$ vanishes to infinity. Thus (φ_n) so defined is a sequence of positive definite functions on $\mathbb{F}_k$ such that $\lim_n \varphi_n = 1$ pointwise and $\varphi_n \in c_0(\mathbb{F}_k)$ for every n.

Note that $\mathbb{F}_k$ acts properly on its Cayley graph, which is a tree. More generally, every group that acts properly on a tree has the property (H). Moreover, Property (H) is stable under taking subgroups, direct or free products. A group that contains a subgroup of finite index with the property (H) has this property. Therefore, $\mathrm{SL}(2, \mathbb{Z})$ has this property since it contains $\mathbb{F}_2$ as a subgroup of index 12.[3]

[1] $c_0(G)$ denotes the algebra of complex-valued functions on G, vanishing to 0 at infinity.

[2] See [**BdlHV08**, Corollary C.4.19] for instance.

[3] For a comprehensive treatment of these questions, we refer to the book [**CCJ$^+$01**].

Remark 16.1.3. Let G be a group with Property (H). Then it is clear from the definition and Proposition 14.1.2 (d) that G cannot contain any infinite relatively rigid subgroup. However, this fact does not characterize Property (H) as shown in [**dC06**].

16.2. Haagerup property for von Neumann algebras

We saw that amenability and Property (T) for groups have their analogues for finite von Neumann algebras. Similarly, Property (H) can be translated in terms of operator algebras.

Let (M,τ) be a tracial von Neumann algebra. Let $\phi : M \to M$ be a subtracial completely positive map. For $x \in M$, using Schwarz inequality, we get $\|\phi(x)\|_2 \leq \|\phi\|^{1/2}\|x\|_2$. It follows that there is a bounded operator T_ϕ on $L^2(M)$ such that $T_\phi(x) = \phi(x)$ for $x \in M$. We observe that $\|T_\phi\| \leq 1$ when ϕ is moreover subunital.

Definition 16.2.1. We say that (M,τ) has the *Haagerup property* or *property* (H) if there exists a net (ϕ_i) of subtracial and subunital completely positive maps $\phi_i : M \to M$ such that

(a) $\lim_i \|T_{\phi_i}(x) - x\|_2 = 0$ for every $x \in M$;
(b) T_{ϕ_i} is a compact operator on $L^2(M)$ for every i.

Of course, when M is separable, one may replace nets by sequences in this definition.

By Theorem 13.4.2, we see that every amenable tracial von Neumann algebra has Property (H). Group von Neumann algebras provide examples of von Neumann algebras with Property (H) as shown by the following proposition.

Proposition 16.2.2. *A group G has the Haagerup property if and only if $L(G)$ has the Haagerup property.*

Proof. Assume first that G has the Haagerup property. Let (φ_i) be a sequence of positive definite functions as in Definition 16.1.1.We may assume that $\varphi_i(e) = 1$ for every i. Let $\phi_i : L(G) \to L(G)$ be the completely positive map such that $\phi_i(u_g) = \varphi_i(g)u_g$ for $g \in G$ (see Proposition 13.1.12). It is straightforward to check that ϕ_i is tracial and unital. Moreover, since $\lim_{g\to\infty} \varphi_i(g) = 0$, the diagonal operator T_{ϕ_i} is compact. That condition (a) of the previous definition is satisfied is immediate too.

Conversely, let (ϕ_i) be as in the previous definition. For each i we introduce $\varphi_i : g \mapsto \tau(\phi_i(u_g)u_g^*)$. We get a net (φ_i) of positive definite functions which converges to 1 pointwise. Moreover, for every i, since $\varphi_i(g) = \langle u_g, T_{\phi_i}(u_g)\rangle$, where T_{ϕ_i} is a compact operator and $(u_g)_{g\in G}$ is an orthonormal basis of $L^2(L(G)) = \ell^2(G)$, we see that $\lim_{g\to\infty} \varphi_i(g) = 0$. $\square$

Proposition 16.2.3. *Let M be a* II_1 *factor with the property* (H). *Then M contains no diffuse relatively rigid subalgebra B.*

Proof. Let (ϕ_i) be a net of completely positive maps as in Definition 16.2.1. Assume the existence of a diffuse relatively rigid von Neumann subalgebra B. Then there exists i such that $\|\phi_i(u) - u\|_2 \leq 1/2$ for every $u \in \mathcal{U}(B)$. Since B is diffuse, any maximal von Neumann subalgebra A of B is diffuse. In particular, there exists a unitary operator $v \in A$ such that $\tau(v^n) = 0$ for $n \neq 0$. The family $(v^n)_{n \geq 0}$ is orthonormal in $L^2(M)$, and therefore $\lim_{n\to\infty} v^n = 0$ in the weak topology of $L^2(M)$. Using the fact that T_{ϕ_i} is compact, we get $\lim_n \|\phi_i(v^n)\|_2 = \lim_n \|T_{\phi_i}(v^n)\|_2 = 0$. Hence, we have

$$1/2 \geq \lim_n \|\phi_i(v^n) - v^n\|_2 = \lim_n \|v^n\|_2 = 1,$$

a contradiction. $\square$

16.3. Relative property (H)

Let (M, τ) be a tracial von Neumann algebra and B a von Neumann subalgebra of M. We have defined the notions of amenability of M relative to B and of property (T) relative to B (see Definition 13.4.5 and Remark 14.2.10). Similarly, we introduce in this section the notion of property (H) relative to B so that property (H) for M is retrieved when $B = \mathbb{C}1$. To that purpose, we need some preliminaries.

When we replace the algebra of scalar operators by a von Neumann subalgebra B of M and consider the right B-module $L^2(M)_B$, we have to replace the semi-finite factor $\mathcal{B}(L^2(M))$ by the commutant of JBJ, (*i.e.*, the commutant of the right B-action[4]), which is the semi-finite von Neumann algebra $\langle M, e_B \rangle$, endowed with its canonical normal faithful semi-finite trace $\widehat{\tau}$. There are several natural notions of compact operator in $\langle M, e_B \rangle$, which differ slightly and coincide with the usual notion of compact operator in the case $B = \mathbb{C}1$ (see Exercise 9.9). Here, we favor the two following ones.

The first one is defined for any semi-finite von Neumann algebra N. It is the norm-closed two-sided ideal $\mathcal{I}(N)$ of N generated by the finite projections of N (which play the role of the finite rank projections in the usual case). We have $T \in \mathcal{I}(N)$ if and only if the spectral projections $e_t(|T|)$ of $|T|$ relative to every interval $[t, +\infty[$, $t > 0$, are finite (see Exercise 9.8). The second one is the norm-closed two-sided ideal $\mathcal{I}_0(\langle M, e_B \rangle)$ of $\langle M, e_B \rangle$ generated by e_B (see Proposition 9.4.3). Note that since $\widehat{\tau}(e_B) = 1$, the projection e_B is finite and so $\mathcal{I}_0(\langle M, e_B \rangle) \subset \mathcal{I}(\langle M, e_B \rangle)$. The slight difference between these two ideals is made precise in the next proposition.

Proposition 16.3.1. *Let $T \in \mathcal{I}(\langle M, e_B \rangle)$. For every $\varepsilon > 0$, there exists a projection $z \in \mathcal{Z}(B)$ such that $\tau(1 - z) \leq \varepsilon$ and $TJzJ \in \mathcal{I}_0(\langle M, e_B \rangle)$.*

Note that $\langle M, e_B \rangle = JB'J$ and therefore $J\mathcal{Z}(B)J$ is the center of $\langle M, e_B \rangle$. For the proof of the proposition, we use the following lemma.

[4]Recall that J is the canonical conjugation operator on $L^2(M)$.

LEMMA 16.3.2. *Let N be a semi-finite von Neumann algebra. Let $f \in N$ be a finite projection and let $e \in N$ be a projection whose central support is* 1. *Then, for any non-zero projection $z \in \mathcal{Z}(N)$, there is a non-zero projection $z' \in \mathcal{Z}(N)$ and finitely many mutually orthogonal projections $e_1, \dots, e_n$ in N such that $z' \leq z$, $fz' = \oplus_{i=1}^n e_i z'$ and $e_i z' \preceq ez'$ for every i.*

PROOF. Note that $ez \neq 0$ whenever $z \neq 0$. Without loss of generality, we may take $z = 1$. By Theorem 2.4.8, there exists $z_1 \in \mathcal{Z}(N)$ with $fz_1 \preceq ez_1$ and $e(1-z_1) \preceq f(1-z_1)$. If $z_1 \neq 0$ the proof is finished. Assume that $z_1 = 0$, so that $e \preceq f$. Let $\{e_i : i \in I\}$ be a maximal family of mutually orthogonal projections, equivalent to e, such that $\sum_{i\in I} e_i \leq f$ and set $e_0 = f - \sum_{i \in I} e_i$. We take a projection $z_2 \in \mathcal{Z}(N)$ such that $e_0 z_2 \preceq ez_2$ and $e(1 - z_2) \preceq e_0(1 - z_2)$. We claim that $ez_2 \neq 0$ and therefore $z_2 \neq 0$. Otherwise, we would have $e \preceq e_0$ contradicting the maximality of $\{e_i : i \in I\}$. We obtain $fz_2 = \sum_{i\in I} e_i z_2 \oplus e_0 z_2$. Moreover, since fz_2 is a finite projection, and since the projections $e_i z_2$, $i \in I$, are equivalent, we see that the set I is finite. $\square$

PROOF OF PROPOSITION 16.3.1. We use the previous lemma with $N = \langle M, e_B \rangle$ and $e = e_B$. Given any finite projection f in $\langle M, e_B\rangle$, we deduce immediately that there exists a increasing net (z_i) of projections in the center $J\mathcal{Z}(B)J$ of $\langle M, e_B\rangle$ with $\sup_i z_i = 1$ and $fJz_iJ \in \mathcal{I}_0\langle M, e_B\rangle$ for every n. In other terms, for every $\delta > 0$ there exists a projection $z \in \mathcal{Z}(B)$ such that $\tau(1 - z) \leq \delta$ and $fJzJ \in \mathcal{I}_0\langle M, e_B\rangle$.

Let $T \in \mathcal{I}\langle M, e_B\rangle$. For every integer $n \geq 1$ there is a linear combination T_n of finite projections in $\langle M, e_B\rangle$ such that $\|T - T_n\| \leq 2^{-n}$. By the claim of the first paragraph, there is a projection $z_n \in \mathcal{Z}(B)$ such that $\tau(1 - z_n) \leq 2^{-n}\varepsilon$ and $T_nJz_nJ \in \mathcal{I}_0\langle M, e_B\rangle$. We set $z = \wedge z_n$. Then we have $\tau(1 - z) \leq \sum_n 2^{-n}\varepsilon \leq \varepsilon$ and $T_nJzJ \in \mathcal{I}_0\langle M, e_B\rangle$. Since $\|(T - T_n)JzJ\| \leq \|T - T_n\| \leq 2^{-n}$ for every n, we get $TJzJ \in \mathcal{I}_0\langle M, e_B\rangle$. $\square$

We say that a completely positive map $\phi : M \to M$ is B-*bimodular* if $\phi(bxb') = b\phi(x)b'$ for every $b, b' \in B$ and $x \in M$. If ϕ is subtracial in addition, then it is clear that T_ϕ commutes with the right and left B-actions on $L^2(M)$, and so $T_\phi \in \langle M, e_B\rangle \cap B'$.

DEFINITION 16.3.3. We say that M has the *property* (H) *relative to* B if there exists a net of subtracial and subunital B-bimodular completely positive maps $\phi_i : M \to M$ such that

(i) $T_{\phi_i} \in \mathcal{I}(\langle M, e_B\rangle)$ for every i;
(ii) $\lim_i \|\phi_i(x) - x\|_2 = 0$ for every $x \in M$.

REMARK 16.3.4. In this definition, we may assume as well that $T_{\phi_i} \in \mathcal{I}_0(\langle M, e_B\rangle)$. Indeed, given a subtracial, B-bimodular completely positive map $\phi : M \to M$, by Proposition 16.3.1, there is an increasing sequence (z_n) of projections in $\mathcal{Z}(B)$ such that $\lim_n z_n = 1$ in the w.o. topology and $T_\phi Jz_nJ \in \mathcal{I}_0(\langle M, e_B\rangle)$ for every n. If we set $\psi_n = \phi(z_n \cdot z_n) = z_n\phi z_n$, we get $T_{\psi_n} = z_nT_\phi Jz_nJ \in \mathcal{I}_0(\langle M, e_B\rangle)$. So it suffices to approximate ϕ by the sequence (ψ_n).

PROPOSITION 16.3.5. *Let $\sigma : G \curvearrowright (B, \tau)$ be a trace preserving action of a group G on a tracial von Neumann algebra (B, τ). Then $M = B \rtimes G$ has the property* (H) *relative to B if and only if G has the property* (H).

PROOF. Assume first that G has the Haagerup property. Let (φ_i) be a net of positive definite functions in $c_0(G)$ such that $\varphi_i(e) = 1$ for every i and $\lim_i \varphi_i = 1$ pointwise. We denote by ϕ_i the completely positive map from M to M such that $\phi_i(bu_g) = \varphi_i(g)bu_g$ for every $b \in B$ and $g \in G$ (see Proposition 13.1.12). Obviously, ϕ_i is a trace-preserving and unital B-bimodular completely positive map. We recall that $(u_g)_{g\in G}$ is an orthornormal basis of the right B-module $L^2(M)$ and we write $L^2(M) = \oplus_{g\in G} u_g L^2(B)$ (see Example 9.4.3). Moreover, $u_g e_B u_g^*$ is the orthogonal projection from $L^2(M)$ onto $u_g L^2(B)$. For $b \in B$ and $g \in G$, we have

$$T_{\phi_i}(u_g\widehat{b}) = T_{\phi_i}(\widehat{\sigma_g(b)u_g}) = \varphi_i(g)\widehat{\sigma_g(b)u_g} = \varphi_i(g)u_g\widehat{b}.$$

It follows that T_{ϕ_i} is the diagonal operator $\sum_{g\in G}\varphi_i(g)u_g e_B u_g^*$. It belongs to $\mathcal{I}_0(\langle M, e_B\rangle)$ since $\lim_{g\to\infty}\varphi_i(g) = 0$. Finally, we observe that $\|\phi_i(u_g) - u_g\|_2 = |1 - \varphi_i(g)|$ and thus $\lim_i \|\phi_i(x) - x\|_2 = 0$ when x is a finite linear combination of elements of the form bu_g, $b \in B, g \in G$. This still holds for every $x \in M$, thanks to the Kaplansky density theorem and the fact that on the unit ball of M the s.o. topology is induced by the $\|\cdot\|_2$-norm.

Conversely, assume that M has the property (H) relative to B an let (ϕ_i) be a net of subtracial and subunital B-bimodular completely positive maps satisfying condition (ii) of Definition 16.3.3 and $T_{\phi_i} \in \mathcal{I}_0(\langle M, e_B\rangle)$ for every i. We set $\varphi_i(g) = \tau(\phi_i(u_g)u_g^*)$ for $g \in G$. Since $|\varphi_i(g) - 1| = \left|\tau((\phi_i(u_g) - u_g)u_g^*)\right|$, by using the Cauchy-Schwarz inequality we see that the net (φ_i) of positive definite functions converges to 1 pointwise. It remains to check that each φ_i vanishes to infinity. We have

$$|\varphi_i(g)| = |\langle u_g, T_{\phi_i}(u_g)\rangle| \leq \|T_{\phi_i}(u_g)\|_2.$$

The elements of the form $L_\eta L_\xi^*$, where ξ, η are left B-bounded in $L^2(M)$, linearly generate a norm dense subspace of $\mathcal{I}_0(\langle M, e_B\rangle)$ (see Proposition 9.4.3). Therefore it suffices to show that $\lim_{g\to\infty}\left\|L_\eta L_\xi^*(u_g)\right\|_2 = 0$. An easy exercise (Exercice 9.12) shows that $\xi = \sum_{g\in G} u_g b_g$ where $\sum_{g\in G} b_g^* b_g$ converges in B for the s.o. topology. Moreover, we have $b_g = (L_{u_g})^* L_\xi$, and so $L_\xi^* u_g = (b_g)^*$. It follows that

$$\left\|L_\eta L_\xi^*(u_g)\right\|_2 \leq \|L_\eta\|\|b_g\|_2,$$

and we conclude by observing that obviously $\lim_{g\in G}\|b_g\|_2 = 0$. $\square$

REMARK 16.3.6. Relative property (H) is not a weakening of relative amenability: there exist pairs (M, B) such that M is amenable relative to B whilst it does not have the relative property (H). For instance, consider any non-trivial group Q. Set $H = \oplus_{n\geq 0} Q$ and let G be the wreath product

$(\oplus_{n\in\mathbb{Z}}Q) \rtimes \mathbb{Z}$. Then $L(G)$ is amenable relative to $L(H)$ (see [**MP03**]), but it does not have the relative property (H), since H is not almost normal in G (see Exercise 16.1). A stronger notion of relative amenability, sometimes called *s-amenability relative to B* has been considered in [**Pop99**], which is indeed a strengthening of the relative property (H). For more comments in this subject see [**Pop06a**, Remarks 3.5].

Exercise

EXERCISE 16.1. Let G be a countable group and H a subgroup. We assume that $M = L(G)$ has the property (H) relative to $B = L(H)$. Let (ϕ_i) be a net of subtracial and subunital B-bimodular completely positive maps satisfying condition (ii) of Definition 16.3.3 and $T_{\phi_i} \in \mathcal{I}_0(\langle M, e_B\rangle)$ for every i. We set $\varphi_i(g) = \tau(\phi_i(u_g)u_g^*)$ for $g \in G$.

(i) Show that (φ_i) is a net of H-bi-invariant positive definite functions which converges to 1 pointwise.
(ii) Show that, viewed as a function on $H \setminus G$, each φ_i belongs to $c_0(H \setminus G)$.
(iii) Conclude that H is *almost normal* in G, that is, for every $g \in G$, HgH is a finite union of left, and also of right, H-cosets.

Notes

In order to prove that the reduced C^*-algebras of the free groups $\mathbb{F}_n$, $n \geq 2$, have the Grothendieck metric approximation property, Haagerup showed in [**Haa79**] that these groups satisfy the condition introduced in Definition 16.1.1. The crucial step was to establish that the word length function on these groups is conditionally negative definite.

In the context of II_1 factors, the Haagerup property was defined by Connes [**Con80b**] and Choda [**Cho83**] who proved that a group von Neumann algebra $L(G)$ has the Haagerup property if and only if the group G has the Haagerup property. In [**CJ85**], Connes and Jones proved, among other results, that a II_1 factor with the Haagerup property cannot contain any II_1 factor having the property (T).

In [**Jol02**] Jolissaint studied in detail the Haagerup property for tracial von Neumann algebras and established in particular that the definition 16.2.1 does not depend on the choice of the faithful normal tracial state. A relative Haagerup property was introduced by Boca [**Boc93**] in order to construct irreducible inclusions of II_1 factors with the Haagerup property, of any index $s > 4$. In [**Pop06a**], Popa has provided a detailed study of the relative Haagerup property as defined in 16.3.3. Again, the definition does not depend on the choice of the faithful normal tracial state. The results of Section 16.3 are taken from this paper.

Since $\mathrm{SL}(2,\mathbb{Z})$ has the Haagerup property, $L(\mathbb{Z}^2) \subset L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})) = L(\mathbb{Z}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$ has the relative Haagerup property. In addition it is a rigid inclusion. This also holds for any non amenable subgroup of $\mathrm{SL}(2,\mathbb{R})$ instead of $\mathrm{SL}(2,\mathbb{Z})$, for instance $\mathbb{F}_2$. These two features are the main ingredients in Popa's proof that the fundamental group of $L(\mathbb{Z}^2 \rtimes \mathbb{F}_2)$ is the same as the fundamental group of the orbit equivalence relation defined by the corresponding action of $\mathbb{F}_2$ on the dual $\mathbb{T}^2$ of $\mathbb{Z}^2$ (see Section 18.3).

CHAPTER 17

Intertwining-by-bimodules technique

In this chapter we introduce a new powerful tool, called the intertwining-by-bimodules technique. This technique provides very tractable conditions allowing to detect whether two subalgebras are intertwined via a partial isometry. Under suitable conditions, for instance if the two subalgebras are Cartan subalgebras, it happens even more: the two subalgebras are unitarily conjugate, that is, conjugate by an inner automorphism.

This method has many applications. As a simple first illustration, in Section 17.3 we present a family of examples of II_1 factors with two Cartan subalgebras for which the intertwining-by-bimodules technique provides a quick proof of their non-conjugacy by an inner automorphism.

Next, at the end of this chapter, it will be applied to show that the hyperfinite factor R has uncountably many non-unitarily conjugate Cartan subalgebras (although they are conjugate by automorphisms as shown in Theorem 12.5.2).

In the next chapter, we will exploit this method in the course of the study of a II_1 factor whose fundamental group is trivial. In Chapter 19 it will be used to show that the factors associated with non-abelian free groups are prime.

17.1. The intertwining theorem

Let (M, τ) be a tracial von Neumann algebra, and P, Q two von Neumann subalgebras such that there exists $u \in \mathcal{U}(M)$ with $u^*Pu \subset Q$. Obviously, $\mathcal{H} = uL^2(Q)$ is a P-Q-subbimodule of ${}_PL^2(M)_Q$ with $\dim(\mathcal{H}_Q) = 1 < +\infty$.

More generally, the existence of a P-Q-subbimodule of ${}_PL^2(M)_Q$ with $\dim(\mathcal{H}_Q) < +\infty$ is characterized as follows.

Theorem 17.1.1. *Let (M, τ) be a tracial von Neumann algebra, $f \in M$ a non-zero projection, and let P, Q be two von Neumann subalgebras of fMf and M respectively. The following conditions are equivalent:*

(i) *there is no net[1] (u_i) of unitary elements in P such that, for every $x, y \in M$, $\lim_i \|E_Q(x^*u_iy)\|_2 = 0$;*

(ii) *there exists a non-zero element $h \in (f\langle M, e_Q\rangle f)_+ \cap P'$ with $\widehat{\tau}(h) < +\infty$;*[2]

[1]If M is separable, it suffices to consider sequences.

[2]We recall that $\widehat{\tau}$ denotes the canonical normal, faithful, semi-finite trace on $\langle M, e_Q\rangle$.

(iii) *there exists a non-zero P-Q-subbimodule $\mathcal{H}$ of $fL^2(M)$ such that* $\dim(\mathcal{H}_Q) < +\infty$*;*

(iv) *there exist an integer $n \geq 1$, a projection $q \in M_n(\mathbb{C}) \overline{\otimes} Q$, a non-zero partial isometry $v \in M_{1,n}(\mathbb{C}) \otimes fM$ and a normal unital homomorphism $\theta : P \to q(M_n(\mathbb{C}) \otimes Q)q$ such that $v^*v \leq q$ and $xv = v\theta(x)$ for every $x \in P$.*[3]

PROOF. (i) $\Rightarrow$ (ii). If (i) holds, there exist $\varepsilon > 0$ and a finite subset $F \subset M$ such that

$$\forall u \in \mathcal{U}(P), \quad \max_{x,y \in F} \|E_Q(x^*uy)\|_2 \geq \varepsilon.$$

We may assume that $x = fx$ for every $x \in F$. We set $c = \sum_{x \in F} xe_Qx^*$ and we denote by $\mathcal{C}$ the w.o. closed convex hull of $\{ucu^* : u \in \mathcal{U}(P)\}$ in $\langle M, e_Q \rangle$. Let $h \in (f\langle M, e_Q\rangle f)_+ \cap P'$ be its element of minimal $\|\cdot\|_{2,\widehat{\tau}}$-norm (see Lemma 14.3.3). We have $\widehat{\tau}(h) \leq \widehat{\tau}(c) == \sum_{x \in F} \tau(xx^*) < +\infty$.

It remains to show that $h \neq 0$. For $m \in M$, we have

$$\widehat{\tau}(e_Q m^* e_Q m e_Q) = \tau(E_Q(m)^* E_Q(m)) = \|E_Q(m)\|_2^2,$$

and so, for $u \in \mathcal{U}(P)$,

$$\sum_{y \in F} \widehat{\tau}(e_Q y^* ucu^* y e_Q) = \sum_{x,y \in F} \|E_Q(x^*u^*y)\|_2^2 \geq \varepsilon^2.$$

Since $\widehat{\tau}(e_Q \cdot e_Q)$ is a normal state on $\langle M, e_Q \rangle$, we get

$$\sum_{y \in F} \widehat{\tau}(e_Q y^* h y e_Q) \geq \varepsilon^2,$$

whence $h \neq 0$.

(ii) $\Rightarrow$ (iii) is obvious: take a non-zero spectral projection q of h such that $\widehat{\tau}(q) < +\infty$ and consider the bimodule $qL^2(M)$.

(iii) $\Rightarrow$ (iv). Cutting down the bimodule $\mathcal{H} \subset fL^2(M)$ by an appropriate central projection of Q, we may assume that $\mathcal{H}$ is finitely generated as a right Q-module (see Corollary 9.3.3). By Proposition 8.5.3, there is an integer $n \geq 1$ and a Q-linear isometry $W : \mathcal{H} \to L^2(Q)^{\oplus n}$. We set $q = WW^* \in M_n(Q) = M_n(\mathbb{C}) \otimes Q$. Since $\mathcal{H}$ is a left P-module, we get a unital homomorphism $\theta : P \to q(M_n(Q))q$ defined by $\theta(x)W = Wx$ for $x \in P$. We define $\varepsilon_k \in L^2(Q)^{\oplus n}$ as $\varepsilon_k = (0, \dots, \widehat{1_Q}, \dots, 0)$ ($\widehat{1_Q}$ in the k-th coordinate) and set $\xi_k = W^*\varepsilon_k$, for $k = 1, \dots, n$. Let $\xi \in \mathcal{H}^{\oplus n} \subset (fL^2(M))^{\oplus n}$ be the row vector $(\xi_1, \dots, \xi_n)$. For $x \in P$, we write $\theta(x)$ as the matrix $[\theta_{i,j}(x)]_{i,j} \in q(M_n(Q))q$. We have

$$\begin{aligned} x\xi_j = xW^*\varepsilon_j = W^*\theta(x)\varepsilon_j &= \sum_i W^*\theta_{i,j}(x)\varepsilon_i \\ &= \sum_i W^*(\varepsilon_i \theta_{i,j}(x)) = \sum_i (W^*\varepsilon_i)\theta_{i,j}(x) = \sum_i \xi_i \theta_{i,j}(x), \end{aligned}$$

[3] $M_{1,n}(\mathbb{C})$ denotes the space of $1 \times n$ matrices with complex entries.

since W^* is Q-linear. Hence, we get $x\xi = \xi\theta(x) \in L^2(M)^{\oplus n}$.

The Hilbert space $L^2(M_n(M))$ is canonically isomorphic to the Hilbert space $M_n(L^2(M))$ with scalar product

$$\langle [\eta_{i,j}], [\eta'_{i,j}] \rangle = n^{-2} \sum_{i,j} \langle \eta_{i,j}, \eta'_{i,j} \rangle_{L^2(M)}$$

and its obvious structure of $M_n(M)$-$M_n(M)$-bimodule. For $x \in P$ we denote by $\widetilde{x}$ the element of $M_n(M)$ whose entries are equal to zero, except the first diagonal one which is x. Define $\widetilde{\xi} \in L^2(M_n(M))$ as

$$\widetilde{\xi} = \begin{bmatrix} \xi \\ 0_{n-1,n} \end{bmatrix},$$

where $0_{n-1,n}$ is the 0 matrix with $n-1$ rows and n columns. We have $\widetilde{x}\widetilde{\xi} = \widetilde{\xi}\theta(x)$ for every $x \in P$. We view $\widetilde{\xi}$ as a closed operator affiliated with $M_n(M)$. Let $\widetilde{\xi} = V|\widetilde{\xi}|$ be its polar decomposition. It is straightforward to check that $|\widetilde{\xi}|$ commutes with $\theta(x)$ and that $V = \begin{bmatrix} v \\ 0_{n-1,n} \end{bmatrix}$ with $v = (v_1, \dots, v_n) \in (M_{1,n}(\mathbb{C}) \otimes fM)q$. Moreover, we have

$$\widetilde{x}V = V\theta(x),$$

and therefore $xv = v\theta(x)$ for every $x \in P$.

(iv) $\Rightarrow$ (i). Assume that (iv) holds and let (u_i) be a net in $\mathcal{U}(P)$. We have

$$(\mathrm{Id}_n \otimes E_Q)(v^* u_i v) = (\mathrm{Id}_n \otimes E_Q)\big(v^* v)\theta(u_i)\big) = \big((\mathrm{Id}_n \otimes E_Q)(v^* v)\big)\theta(u_i),$$

and so

$$\|(\mathrm{Id}_n \otimes E_Q)(v^* u_i v)\|_2 = \|(\mathrm{Id}_n \otimes E_Q)(v^* v)\|_2 \neq 0.$$

This shows (i). □

With additional technical tools, we may assume that $n = 1$ in the statement (iv), provided P is replaced by one of its corners.

Theorem 17.1.2. *The four conditions of Theorem 17.1.1 are equivalent to*

(v) *there exist non-zero projections $p \in P$ and $q \in Q$, a unital normal homomorphism $\theta : pPp \to qQq$ and a non-zero partial isometry $v \in pMq$ such that $xv = v\theta(x)$ for every $x \in pPp$. Moreover we have $vv^* \in (pPp)' \cap pMp$ and $v^*v \in \theta(pPp)' \cap qMq$.*

For the proof of (iii) in Theorem 17.1.1 implies (v), we will need the following lemma.

Lemma 17.1.3. *Let P, Q be two tracial von Neumann algebras and $\mathcal{H}$ a P-Q-bimodule such that $\dim(\mathcal{H}_Q) < +\infty$. There exist a non-zero projection $p \in P$, a non-zero pPp-Q-subbimodule $\mathcal{K}$ of $\mathcal{H}$ and a projection $q_0 \in Q$ such that $\mathcal{K}$ is isomorphic to $q_0 L^2(Q)$ as a right Q-module.*

Proof. As seen in the proof of (iii) ⇒ (iv) in Theorem 17.1.1 we may assume that $\mathcal{H}$ is isomorphic, as a right Q-module, to some $e\big(L^2(Q)^{\oplus n}\big)$ with $e \in M_n(Q)$. Moreover, by Proposition 8.5.3 we may even assume that e is a diagonal projection, say $e = \mathrm{Diag}(e_1, \dots, e_n)$ with $e_i \in \mathcal{P}(Q)$, $i = 1, \dots, n$. We write e as $\sum_{i=1}^n \overline{e}_i$, where $\overline{e}_i$ is the diagonal projection in $M_n(Q)$ whose only non-zero entry is e_i at the i-th row and column. Let $\theta : P \to eM_n(Q)e$ be the unital homomorphism deduced from the left P-module structure of $\mathcal{H}$.

First case: P has no abelian projection. We choose an integer k such that $2^k \geq n$. By Proposition 5.5.8, there exist 2^k equivalent orthogonal projections $p_1, \dots, p_{2^k}$ in P such that $\sum_{i=1}^{2^k} p_i = 1_P$. Let $E_Z : Q \otimes M_n(\mathbb{C}) \to \mathcal{Z}(Q) \otimes 1$ be the normal faithful center-valued trace. Then we have

$$2^k E_Z(\theta(p_1)) \leq \sum_{j=1}^{n} E_Z(1_Q \otimes e_{j,j}) = nE_Z(1_Q \otimes e_{1,1}),$$

where $(e_{i,j})$ is the canonical matrix units of $M_n(\mathbb{C})$. It follows from Proposition 9.1.8 that $\theta(p_1) \precsim 1_Q \otimes e_{1,1}$ in $M_n(Q)$. Therefore, $\theta(p_1)(L^2(Q)^{\oplus n})$ is a $p_1 P p_1$-Q-subbimodule of $\mathcal{H}$ which is isomorphic, as a right Q-module, to some submodule of $L^2(Q)$.

Second case: P has an abelian projection p, that is, $p \neq 0$ and pPp is abelian. We choose i such that $\theta(p)\overline{e}_i \neq 0$. Let l be the left support of $\theta(p)\,\overline{e}_i$. We have $l \leq \theta(p)$ and $l \precsim \overline{e}_i$ in $eM_n(Q)e$. Let A be any maximal abelian von Neumann subalgebra of $\theta(p)M_n(Q)\theta(p)$ which contains $\theta(pPp)$. By Lemma 17.2.1 below, we see that l is equivalent to a projection $l' \in A$. Obviously, we have $l' \in \theta(pPp)' \cap eM_n(Q)e$. Observe that $l' \precsim \overline{e}_i$ in $eM_n(Q)e$ and so the pPp-Q-bimodule $l'(L^2(Q)^{\oplus n})$ is isomorphic, as a right Q-module, to $q_0 L^2(Q)$ where $e_i \geq q_0 \in \mathcal{P}(Q)$. □

Proof of Theorem 17.1.2. We show that (iii) in Theorem 17.1.1 implies (v). We consider projections $p \in P$, $q_0 \in Q$ and $\mathcal{K}$ as in Lemma 17.1.3 and we introduce the unital homomorphism $\theta : pPp \to q_0 Q q_0$ induced by the structure of pPp-module of $\mathcal{K}$.

Then, to conclude, it suffices to follow the proof of (iii) ⇒ (iv) in Theorem 17.1.1, with $n = 1$ and pPp instead of P. The last statement of (v) is obvious.

Let us show now that (v) implies the condition (ii) of Theorem 17.1.1. Let v and θ be as in (v). Then pPp commutes with ve_Qv^*. Let (f_i) be a maximal family of mutually orthogonal projections in P such that $f_i \precsim p$ in P for every i. Then $\sum_i f_i$ is the central support of p in P (see Exercise 2.5).

For each i, let v_i be a partial isometry in P such that $v_i^* v_i \leq p$ and $v_i v_i^* = f_i$. We set $h = \sum_i v_i(ve_Qv^*)v_i^*$. Then $h \in \langle M, e_Q \rangle$ commutes with

P. Indeed, let $y \in P$. We have

$$\begin{aligned}\big(\sum_i v_i(ve_Qv^*)v_i^*\big)y &= \big(\sum_i v_i(ve_Qv^*)v_i^*\big)y\big(\sum_j v_jv_j^*\big)\\ &= \sum_{i,j} v_i(ve_Qv^*)(v_i^*yv_j)v_j^*\\ &= \sum_{i,j} v_i(v_i^*yv_j)(ve_Qv^*)v_j^* = \sum_j yv_j(ve_Qv^*)v_j^*.\end{aligned}$$

Moreover, we have

$$\widehat{\tau}(h) = \sum_i \widehat{\tau}(v_i(ve_Qv^*)v_i^*) = \sum_i \tau(v_ivv^*v_i^*) \leq \tau(z(p)) < +\infty.$$

$\square$

Definition 17.1.4. When one of the equivalent conditions of Theorem 17.1.1 is satisfied, we say that *a corner of P can be intertwined into Q inside* M, or simply that P *embeds into* Q *inside* M, and we write $P \prec_M Q$.

17.2. Unitary conjugacy of Cartan subalgebras

We will see now that for Cartan subalgebras, the above embedding property is equivalent to unitary conjugacy. We need the two following lemmas.

Lemma 17.2.1. *Let A be a maximal abelian von Neumann subalgebra of a tracial von Neumann algebra (M,τ). Every projection in M is equivalent to a projection in A.*

Proof. The key point is to show that for any non-zero projection $p \in M$ there is a non-zero projection $q \in A$ with $q \precsim p$. Once this is established, let us show how to conclude. Let e a projection in M be given. We take a maximal projection $f \in A$ such that $f \precsim e$ (see Exercise 9.4). Let $u \in \mathcal{U}(M)$ such that $ufu^* \leq e$. If $p = e - ufu^* \neq 0$, we apply the key point to $u^*pu \in (1-f)M(1-f)$: there exists a non-zero projection $q \in A(1-f)$ such that $q \precsim u^*pu$. Then we have $f + q \precsim f + u^*pu = u^*eu \sim e$, in contradiction with the maximality of f. Therefore f and e are equivalent.

Let us sketch the proof of the key point. Recall that E_Z denotes the center-valued trace on M. We choose $c > 0$ such the spectral projection z of $E_Z(p)$ relative to the interval $[c,1]$ is non-zero. Truncating everything by z we may assume that $E_Z(p) \geq c1_M$. In particular, the central support $z(p)$ is equal to 1_M.

Suppose first that A contains a non-zero projection q which is abelian in M. Then, we immediately get $q \precsim p$ (see Proposition 5.5.2).

Whenever A does not contain any non-zero abelian projection, we use several times the Exercise 17.1 to construct non-zero projections $1 = q_0 \geq q_1 \geq \cdots \geq q_n$ in A such that $E_Z(q_n) \leq 2^{-n}1_M \leq c1_M$. Then, by Proposition 9.1.8 we get $q_n \precsim p$. $\square$

LEMMA 17.2.2. *Let A be a Cartan subalgebra of a II_1 factor M and let p, q be two non-zero projections of A such that $\tau(p) = \tau(q)$. There exists $u \in \mathcal{N}_M(A)$ such that $upu^* = q$.*

PROOF. We first show that there exist a non-zero projection $e \in A$ and $u \in \mathcal{N}_M(A)$ such that $e \leq p$ and $ueu^* \leq q$. Indeed, the projection $\bigvee_{u\in\mathcal{N}_M(A)} upu^*$ commutes with $\mathcal{N}_M(A)$ and therefore with M, and so is the unit of M. In particular, there exists $u \in \mathcal{N}_M(A)$ such that $upu^*q \neq 0$. We set $e = pu^*qu$.

Now, we consider a maximal family $\{(e_i, u_i)\}$ of pairs $(e_i, u_i) \in \mathcal{P}(A) \times \mathcal{N}_M(A)$ such that each of the two families of projections (e_i) and $(u_ie_iu_i^*)$ is made of mutually orthogonal projections, with $\sum e_i \leq p$ and $\sum u_ie_iu_i^* \leq q$. Then we have $\sum e_i = p$. Otherwise, we consider the projections $p' = p - \sum e_i$ and $q' = q - \sum u_ie_iu_i^*$. They have the same trace, and applying the first part of the proof to them, we contradict the maximality of the family.

Next, we set $v = \sum u_ie_i$. This partial isometry satisfies $v^*v = p$, $vv^* = q$ and $vAv^* = Aq$. The same argument applied to $1-p$ and $1-q$ gives a partial isometry w such that $w^*w = 1 - p$, $ww^* = 1 - q$ and $wAw^* = A(1 - q)$. The operator $u = v + w$ is in $\mathcal{N}_M(A)$ and we have $upu^* = q$. $\square$

THEOREM 17.2.3. *Let A and B be two Cartan subalgebras of a II_1 factor M such that $A \prec_M B$. Then there exists a unitary element $u \in M$ such that $u^*Au = B$.*

PROOF. By Theorem 17.1.2, there exist non-zero projections $p \in A$, $q \in B$, a non-zero partial isometry $v \in pMq$ and a normal unital homomorphism $\theta : Ap \to Bq$ such that $av = v\theta(a)$ for every $a \in Ap$. Moreover we have

$$vv^* \in (Ap)' \cap pMp = Ap \quad \text{and} \quad v^*v \in \theta(Ap)' \cap qMq.$$

The crucial step is to construct a partial isometry $w \in pMq$ such that

$$\forall a \in Ap,\ aw = w\theta(a), \quad ww^* \in Ap, \quad w^*w \in Bq. \tag{17.1}$$

Indeed, assume for the moment that such a w exists. By cutting down w to the left by an appropriate projection of A we may assume that $\tau(ww^*) = 1/n$ for some integer n. We set $e_1 = ww^*$ and $f_1 = w^*w$. We take projections $e_2, \dots, e_n$ in A and $f_2, \dots, f_n$ in B, all having the same trace $1/n$ and such that $\sum_{i=1}^n e_i = 1 = \sum_{i=1}^n f_i$. By Lemma 17.2.2, for $i \in \{1, \dots, n\}$, there exist $u_i \in \mathcal{N}_M(A)$ with $u_ie_1u_i^* = e_i$ and $v_i \in \mathcal{N}_M(B)$ with $v_if_1v_i^* = f_i$.

We set $u = \sum_{i=1}^n u_iwv_i^*$. Then u is a unitary element of M. Moreover, we have, for $a \in A$,

$$\begin{aligned} u^*au &= \sum_{i,j} v_iw^*u_i^*au_jwv_j^* = \sum_i v_iw^*(u_i^*au_ip)wv_i^* \\ &= \sum_i v_iw^*w\theta(u_i^*au_ip)v_i^* \in B, \end{aligned}$$

whence $u^*Au \subset B$, and so $u^*Au = B$ since u^*Au is maximal abelian.

Let us show how to construct w satisfying (17.1). We set $N = \theta(Ap)' \cap qMq$ and $f = v^*v$. By Lemma 17.2.1, f is equivalent in N to a projection in Bq. So, let $v' \in N$ be a partial isometry such that $v'(v')^* = v^*v = f$ and $(v')^*v' \in Bq$. We set $w = vv'$. A straightforward computation shows that the conditions in (17.1) are fulfilled. □

17.3. II_1 factors with two non-conjugate Cartan subalgebras

A Cartan subalgebra A of a II_1 factor M is called a *group measure space Cartan subalgebra* if there exists a free ergodic p.m.p. action $G \curvearrowright (X,\mu)$ such that $A \subset M$ is isomorphic to $L^\infty(X) \subset L^\infty(X) \rtimes G$, that is, there exists an isomorphism α from M onto $L^\infty(X) \rtimes G$ such that $\alpha(A) = L^\infty(X)$. In this section, we provide a family of examples of II_1 factors with at least two non unitarily conjugate group measure space Cartan subalgebras.

Let H be a countable abelian group and $H \hookrightarrow K$ a dense and injective homomorphism from H into a compact abelian group K. We are given an action $\alpha : G \curvearrowright H$ of a countable group G by automorphisms and a free ergodic p.m.p. action $G \curvearrowright (X,\mu)$. We assume that α extends to an action by homeomorphisms on K that we still denote by α. Then the semi-direct product $H \rtimes G$ acts on $K \times X$ by

$$h.(k,x) = (h+k,x), \qquad g.(k,x) = (\alpha_g(k), g.x),$$

for all $h \in H, g \in G, k \in K, x \in X$. This action is free, ergodic and p.m.p.

Dualizing the embedding $H \hookrightarrow K$, we get the embedding $\widehat{K} \hookrightarrow \widehat{H}$ of the dual groups and an action of G by automorphisms of $\widehat{K}$ and $\widehat{H}$. Then the semi-direct product $\widehat{K} \rtimes G$ acts freely and ergodically on $\widehat{H} \times X$ by

$$\kappa.(\chi,x) = (\kappa+\chi,x), \qquad g.(\chi,x) = (\chi \circ \alpha_{g^{-1}}, g.x),$$

for all $\kappa \in \widehat{K}, g \in G, \chi \in \widehat{H}, x \in X$.

The Fourier transforms on the first and third component of

$$L^2(K) \otimes L^2(X) \otimes \ell^2(H) \otimes \ell^2(G)$$

induce (after permutation between $\ell^2(\widehat{K})$ and $L^2(\widehat{H})$) a unitary operator from this Hilbert space onto $L^2(\widehat{H}) \otimes L^2(X) \otimes \ell^2(\widehat{K}) \otimes \ell^2(G)$ which implements a canonical isomorphism

$$M = L^\infty(K \times X) \rtimes (H \rtimes G) \simeq L^\infty(\widehat{H} \times X) \rtimes (\widehat{K} \rtimes G).$$

The Cartan subalgebras $L^\infty(K \times X)$ and $L^\infty(\widehat{H} \times X)$ are not unitarily conjugate. Indeed, let (h_n) be a sequence of elements in H going to infinity. Denote, as usual, by u_h and $u_{(h,g)}$ the canonical unitaries in $L(H)$ and $L^\infty(K \times X) \rtimes (H \rtimes G)$. We have $u_{h_n} u_{(h,g)} = u_{h_n+h,g}$ and so $E_{L^\infty(K\times X)}(u_{h_n} u_{(h,g)}) = 0$ whenever n is sufficiently large. It follows that

$$L^\infty(\widehat{H}) = L(H) \not\prec_M L^\infty(K \times X)$$

(see Exercise 17.3).

A fortiori, $L^\infty(\widehat{H} \times X)$ cannot be unitarily conjugate to $L^\infty(K \times X)$.

EXAMPLE 17.3.1. As an example of this situation, we can take $G = \mathrm{SL}(n, \mathbb{Z})$, $n \geq 2$, acting on $H = \mathbb{Z}^n$ and embed $\mathbb{Z}^n$ into $K = \mathbb{Z}_p^n$, where $\mathbb{Z}_p$ is the ring of p-adic integers for some prime number p. In this example, one can show that the two Cartan subalgebras $L^\infty(K \times X)$ and $L^\infty(\widehat{H} \times X)$ are not even conjugate by an automorphism of the ambient crossed product, and therefore the corresponding equivalence relations are not isomorphic.

Indeed, we first observe that $\widehat{K} \rtimes \mathrm{SL}(n, \mathbb{Z}) = \varinjlim \big(\widehat{\mathbb{Z}/p^k\mathbb{Z}}\big)^n \rtimes \mathrm{SL}(n, \mathbb{Z})$. Then

- for $n = 2$, $\widehat{K} \rtimes \mathrm{SL}(2, \mathbb{Z})$ has the Haagerup property since it is the direct limit of groups with the Haagerup property (see [**CCJ$^+$01**, Proposition 6.1.1]), whereas $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ does not have the Haagerup property, because $\mathbb{Z}^2$ is an infinite subgroup rigidly embedded into $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$;
- for $n > 3$, $\mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z})$ has Property (T), whereas $\widehat{K} \rtimes \mathrm{SL}(2, \mathbb{Z})$ does not have this property, as the direct limit of a strictly increasing sequence of groups (see [**dlHV89**, page 10]).

But, if two groups have free ergodic p.m.p. orbit equivalent actions and if one of them has the property (T), the second one also has the property (T) ([**Pop86a**], [**AD87**], [**Fur99a**], independently). Similarly, the Haagerup property is stable under such orbit equivalence [**Pop06a**, Remark 3.5.6°]. It follows that the group actions $(H \rtimes G) \curvearrowright (K \times X)$ and $(\widehat{K} \rtimes G) \curvearrowright (\widehat{H} \times X)$ are not orbit equivalent and therefore, by Corollary 12.2.7, the corresponding Cartan algebras are not conjugate by an automorphism.

REMARK 17.3.2. In some other examples, the two Cartan subalgebras $L^\infty(K \times X)$ and $L^\infty(\widehat{H} \times X)$ may be conjugate by an automorphism. This happens, for instance, when the action $G \curvearrowright H$ is trivial. Then $L^\infty(K \times X) \rtimes (H \rtimes G)$ is isomorphic to the tensor product of $L^\infty(X) \rtimes G$ and $L^\infty(K) \rtimes H \simeq L^\infty(\widehat{H}) \rtimes \widehat{K}$. This latter algebra is the hyperfinite II_1 factor since the abelian group H acts freely and ergodically on K. It follows that the Cartan subalgebras $L^\infty(K)$ and $L^\infty(\widehat{H})$ are conjugate by an automorphism (see Theorem 12.5.2) and therefore $L^\infty(K \times X)$ and $L^\infty(\widehat{H} \times X)$ are also conjugate by an automorphism.

17.4. Cartan subalgebras of the hyperfinite factor R

PROPOSITION 17.4.1. *The hyperfinite factor R has uncountably many Cartan subalgebras that are not unitarily conjugate.*

PROOF. We write R as the infinite tensor product $M_2(\mathbb{C})^{\overline{\otimes}\infty}$ of 2×2-matrix algebras. We denote by D the diagonal subalgebra of $M_2(\mathbb{C})$. Let $u(\theta) \in M_2(\mathbb{C})$ be the rotation of angle θ and set $D_\theta = u(\theta) D u(\theta)^*$. Then $\mathcal{D} = D^{\overline{\otimes}\infty}$ and $\mathcal{D}_\theta = D_\theta^{\overline{\otimes}\infty}$ are Cartan subalgebras of R (see Exercise 12.1) and we claim that they are not unitarily conjugate whenever $\theta \notin \mathbb{Z}(\pi/2)$. To this end, we will construct a sequence (v_n) of unitaries in $\mathcal{D}_\theta$ such that for every $x, y \in R$, $\lim_n \|E_{\mathcal{D}}(xv_n y)\|_2 = 0$ and then use Theorem 17.1.1. Since $\mathcal{N}_M(\mathcal{D})$ generates R it suffices to show that $\lim_n \|E_{\mathcal{D}}(xv_n)\|_2 = 0$ for every $x \in M$ and even, by approximation it is enough to take $x = x_k \overline{\otimes} 1 \in M_2(\mathbb{C})^{\otimes k} \overline{\otimes} 1$.

We consider the unitary $v = u(\theta)\mathrm{Diag}(1, i)u(\theta)^* \in D_\theta$ where $\mathrm{Diag}(1, i)$ is the diagonal 2×2 matrix with entries 1 and i. A straightforward computation shows that

$$E_D(v) = \mathrm{Diag}\big(\cos^2(\theta) + i\sin^2(\theta), \sin^2(\theta) + i\cos^2(\theta)\big)$$

and so $\|E_D(v)\|_2^2 = \cos^4(\theta) + \sin^4(\theta)$.

We set $v_n = v^{\otimes n} \overline{\otimes} 1 \in \mathcal{D}_\theta$ and $R_k = M_2(\mathbb{C})^{\otimes k}$. For $n > k$ we have

$$E_{\mathcal{D}}(xv_n) = E_{R_k}(x_k v^{\otimes k})\, \overbrace{E_D(v) \otimes \cdots \otimes E_D(v)}^{n-k}\, \overline{\otimes} 1$$

and therefore

$$\|E_{\mathcal{D}}(xv_n)\|_2 = \left\|E_{R_k}(x_k v^{\otimes k})\right\|_2 \big(\cos^4(\theta) + \sin^4(\theta)\big)^{(n-k)/2}.$$

It follows that $\lim_n \|E_{\mathcal{D}}(xv_n)\|_2 = 0$ whenever $\theta \notin \mathbb{Z}(\pi/2)$. $\square$

Exercises

Exercise 17.1. Let (M, τ) be a tracial von Neumann algebra with center $Z \neq M$.

(i) Let p be a projection in M and let z be the spectral projection of $E_Z(p)$ corresponding to the interval $]0, 1/2]$. Observe that $E_Z(pz) \leq 1/2$.

(ii) Let A be a von Neumann subalgebra of M with contains strictly its center Z. Show that there exists a non-zero projection $q \in A$ such that $E_Z(q) \leq 1/2$ (use Exercise 9.3).

(iii) Let A be a von Neumann subalgebra of M and let $q_0 \in A$ be a projection with $q_0 Z \subsetneq q_0 A q_0$. Show that there exists a non-zero projection $q \in A$ such that $q \leq q_0$ and $E_Z(q) \leq E_Z(q_0)/2$.

Exercise 17.2. Let (M, τ) be a tracial von Neumann algebra and P a von Neumann subalgebra. Show that $P \prec_M \mathbb{C}1_M$ if and only if P contains a minimal projection.

Exercise 17.3. Let $G \curvearrowright (Q, \tau)$ be a trace preserving action of a countable group G. We set $M = Q \rtimes G$. Let P be a von Neumann subalgebra of M. Show that $P \not\prec_M Q$ if and only if there exists a net (v_i) of unitary elements in P such that $\lim_i \left\|E_Q(v_i u_g^*)\right\|_2 = 0$ for every $g \in G$, where the u_g's are the canonical unitaries in M (that is, for every $g \in G$, the net $\big((v_i)_g\big)$ of Fourier coefficients of index g of the v_i's goes to 0 in $\|\cdot\|_2$-norm).

Notes

The intertwining-by-bimodules technique is a major innovation introduced in [**Pop06a, Pop06d**] in the early 2000's. This new technology, combined with rigidity results, provides an exceptionally powerful tool allowing to solve a wealth of longstanding problems. The results of Section 17.1 come from [**Pop06a, Pop06d**]. For our presentation, we have also benefited from Vaes' survey [**Vae07**].

The first example of II_1 factor with two Cartan subalgebras which are not conjugate by an automorphism was given by Connes and Jones [**CJ82**]. In [**Pop86a**, Corollary 4.7.2], [**Pop90**], one finds an example of McDuff factor with uncountably many non conjugate Cartan subalgebras. More recently, many new classes of examples of II_1 factors with more than one Cartan subalgebra were found [**Pop08, OP10b, PV10b, KS13, KV17**]. There even exist II_1 factors with unclassifiably many Cartan subalgebras in the sense that the equivalence relation of being conjugate by an automorphism is not Borel [**SV12**]. In Section 17.3, we have followed the paper [**PV10b**] of Popa and Vaes.

As for Cartan subalgebras of the hyperfinite factor R, Proposition 17.4.1 had been proved by another method in [**Pac85**].

Some examples of II_1 factors without Cartan subalgebra are briefly presented in the comments at the end of Chapter 19. The hard problem concerning the uniqueness of Cartan subalgebras is also out of the scope of this monograph. It has been solved positively in an amazing variety of situations. Let us only mention the following striking result of Popa and Vaes [**PV14a, PV14b**], after a previous breakthrough of Ozawa and Popa [**OP10a, OP10b**]: any free ergodic p.m.p. action of a non-elementary hyperbolic group (*e.g.* a non-abelian free group) or of a lattice in a rank one simple Lie group, gives rise to a crossed product having a unique Cartan subalgebra, up to unitary conjugacy, thus extending Example 18.1.5. The countable groups with this uniqueness property for any free ergodic p.m.p. action are called *Cartan-rigid*. This class also contains all arbitrary (non trivial) free products [**Ioa15, Vae14**] and central quotients of braid groups [**CIK15**].

The class of Cartan-rigid groups has very powerful properties. For any free ergodic p.m.p. action of a Cartan-rigid group G on (X, μ) and any other free ergodic p.m.p. action $H \curvearrowright (Y, \nu)$ of any other group, whenever $L^\infty(X) \rtimes G$ and $L^\infty(Y) \rtimes H$ are isomorphic it follows that the actions are orbit equivalent. In fact, this property already holds if G is *group measure space Cartan-rigid* in the sense that for every free ergodic p.m.p. action

$G \curvearrowright (X,\mu)$, the crossed product $L^\infty(X) \rtimes G$ has a unique group measure space Cartan subalgebra, up to unitary conjugacy[4]. This class includes, in addition to Cartan-rigid groups, certain amalgamated free products [**PV10b, HPV13**], certain HNN-extensions [**FV12**] and many other groups [**Ioa12b, Ioa12a**], [**CP13**], [**Vae13**].

A notion stronger than orbit equivalence is conjugacy. Two p.m.p. actions $G \curvearrowright (X,\mu)$ and $H \curvearrowright (Y,\nu)$ are said to be *conjugate* if there exist an isomorphism $\theta : (X,\mu) \to (Y,\nu)$ and a group isomorphism $\delta : G \to H$ such that $\theta(gx) = \delta(g)\theta(x)$ for all $g \in G$ and almost every $x \in X$. In some case, it is possible to retrieve the conjugacy of the actions from their orbit equivalence. A free ergodic p.m.p. action $G \curvearrowright (X,\mu)$ is said to be *orbit equivalence superrigid* if it is conjugate to any other free ergodic p.m.p. action $H \curvearrowright (Y,\nu)$ as soon as the two actions are orbit equivalent. It is said to be W^*-*superrigid* if $L^\infty(X) \rtimes G$ remembers the group action in the sense that for any isomorphism $L^\infty(X) \rtimes G \simeq L^\infty(Y) \rtimes H$, the corresponding actions are conjugate. An action is W^*-superrigid if and only if it is orbit equivalence superrigid and $L^\infty(X)$ is the unique group measure space Cartan subalgebra up to conjugacy. Many examples of orbit equivalence superrigidity have been found:[**Fur99b**], [**MS06**], [**Pop06e, Pop07a, Pop08**], [**Kid06, Kid10, Kid11, Kid13**], [**PV11**], [**Ioa11a**], [**PS12**]. Various remarkable W^*-superrigidity theorems have been proved in [**Ioa11b**], [**Pet10**], [**PV10b**], [**HPV13**], [**CIK15**], [**CK15**], where the uniqueness of group measure space Cartan subalgebras could be combined with orbit equivalence superrigidity. More information on these developments, up to 2012, will be found in the surveys [**Pop07b**], [**Fur11**], [**Vae10**], [**Ioa13**].

[4] Indeed, conjugacy would already be enough to get the orbit equivalence.

CHAPTER 18

A II_1 factor with trivial fundamental group

In this chapter, we will show that factors of the form $L^\infty(\mathbb{T}^2)\rtimes\mathbb{F}_n$, $n\geq 2$, where $\mathbb{F}_n$ appears as a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$ with its obvious action on $\mathbb{T}^2$, have a fundamental group reduced to $\{1\}$. To that purpose, we develop in Section 18.1 a deformation/rigidity argument which implies that $L^\infty(\mathbb{T}^2)$ is the only rigidly embedded Cartan subalgebra of $L^\infty(\mathbb{T}^2)\rtimes\mathbb{F}_n$ up to unitary conjugacy. The deformation comes from the fact that $\mathbb{F}_n$ has the Haagerup property. From the uniqueness of this type of Cartan subalgebra we deduce that the fundamental group of the crossed product coincides with the fundamental group of the corresponding equivalence relation $\mathcal{R}_{\mathbb{F}_n\curvearrowright\mathbb{T}^2}$. This latter notion of fundamental group is introduced in Section 18.2 together with the notion of cost. We show in Section 18.3 that the equivalence relation $\mathcal{R}_{\mathbb{F}_n\curvearrowright\mathbb{T}^2}$ has a trivial fundamental group, thanks to the computation of its cost in Theorem 18.3.3.

18.1. A deformation/rigidity result

The following theorem is one of the many examples where a fight between a deformation property and rigidity concludes in an embedding.

Theorem 18.1.1. *Let (M,τ) be a tracial von Neumann algebra and P, Q two von Neumann subalgebras. We assume that P is relatively rigid in M and that M has the property* (H) *relative to Q. Then $P\prec_M Q$.*

Proof. We will establish that condition (i) of Theorem 17.1.1 holds. Since $P\subset M$ is rigid, given $\varepsilon=1/4$, there exist a finite subset F of M and $\delta>0$ such that whenever ϕ is a subtracial and subunital completely positive map from M to M satisfying $\max_{x\in F}\|\phi(x)-x\|_2\leq\delta$, then $\|\phi(y)-y\|_2\leq 1/4$ for every y in the unit ball of P. On the other hand, since M has Property (H) relative to Q, there exists a net (ϕ_i) of Q-bimodular, subtracial and subunital completely positive maps such that $T_{\phi_i}\in\mathcal{I}_0(\langle M,e_Q\rangle)$ and $\lim_i\|\phi_i(x)-x\|_2=0$ for every $x\in M$.[1] So, there exists a Q-bimodular, subtracial and subunital completely positive map ϕ such that $T_\phi\in\mathcal{I}_0(\langle M,e_Q\rangle)$ and, for y in the unit ball of P,

$$\|\phi(y)-y\|_2\leq 1/4.$$

[1] Recall that $\mathcal{I}_0(\langle M,e_Q\rangle)$ is the norm closure of the two-sided ideal of $\langle M,e_Q\rangle$ generated by e_Q and also of the linear span of the elements of the form $L_\xi L_\eta^*$ with $\xi,\eta\in\left(L^2(M)_Q\right)^0$ (Proposition 9.4.3).

Then for every $u \in \mathcal{U}(P)$ we have

$$\|\phi(u)\|_2 \geq 3/4.$$

The conclusion follows from the next lemma. $\square$

LEMMA 18.1.2. *Let (u_i) be a net of unitary elements in M such that $\lim_i \|E_Q(x^* u_i y)\|_2 = 0$ for every $x, y \in M$. Let $\phi : M \to M$ be a subtracial and subunital completely positive map such that $T_\phi \in \mathcal{I}_0(\langle M, e_Q \rangle)$. Then, $\lim_i \|\phi(u_i)\|_2 = 0$.*

PROOF. It suffices to show that we have $\lim_i \left\|L_\xi L_\eta^* u_i\right\|_2 = 0$ for every $\xi, \eta \in \left(L^2(M)_Q\right)^0$, and indeed, that $\lim_i \left\|L_\eta^* u_i\right\|_2 = 0$ for every $\eta \in \left(L^2(M)_Q\right)^0$. Recall that $L_\eta^* u_i = E_Q(\eta^* u_i)$ (see (9.5) in Section 9.4.1).

Given $\varepsilon > 0$, take $x \in M$ with $\|\eta - x\|_2 \leq \varepsilon/2$. Then we have

$$\begin{aligned}
\|E_Q(\eta^* u_i)\|_2 &\leq \|E_Q((\eta^* - x^*)u_i)\|_2 + \|E_Q(x^* u_i)\|_2 \\
&\leq \|\eta - x\|_2 + \|E_Q(x^* u_i)\|_2 \\
&\leq \varepsilon/2 + \|E_Q(x^* u_i)\|_2.
\end{aligned}$$

So, we conclude that $\left\|L_\eta^* u_i\right\|_2 \leq \varepsilon$ for i large enough. $\square$

THEOREM 18.1.3. *Let M be a II_1 factor, A, B two Cartan subalgebras of M. We assume that B is relatively rigid in M and that M has the property* (H) *relative to A. Then there exists $u \in \mathcal{U}(M)$ such that $uBu^* = A$. So A is also relatively rigid in M and there is only one relatively rigid Cartan subalgebra of M, up to unitary conjugacy.*

PROOF. Immediate consequence of Theorems 18.1.1 and 17.2.3. $\square$

COROLLARY 18.1.4. *Let $G_i \curvearrowright (X_i, \mu_i)$, $i = 1, 2$, be two free ergodic p.m.p. actions of countable groups. We assume that $L^\infty(X_1)$ is relatively rigid in $L^\infty(X_1) \rtimes G_1$ and that G_2 has the Haagerup property. Then the von Neumann algebras $L^\infty(X_1) \rtimes G_1$ and $L^\infty(X_2) \rtimes G_2$ are isomorphic if and only if the equivalence relations $\mathcal{R}_{G_1 \curvearrowright X_1}$ and $\mathcal{R}_{G_2 \curvearrowright X_2}$ are isomorphic.*

PROOF. Use the previous theorem together with Proposition 16.3.5 and Corollary 12.2.7. $\square$

EXAMPLE 18.1.5. Let G be any finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})$ and consider its natural action on $\mathbb{Z}^2$. The dual action of G on $\mathbb{T}^2$ is free and ergodic. We identify $L(\mathbb{Z}^2 \rtimes G)$ to the group measure space von Neumann algebra $M = L^\infty(\mathbb{T}^2) \rtimes G$, through the usual identification of $L(\mathbb{Z}^2)$ with $A = L^\infty(\mathbb{T}^2)$ by Fourier transform (see Example 14.2.8). Then, A is relatively rigid in M since $\mathbb{Z}^2$ is relatively rigid in $\mathbb{Z}^2 \rtimes G$ (see Section 14.1 and Proposition 14.2.7) and M has the relative property (H) relative to A by Proposition 16.3.5, since G has the Haagerup property. So, by the previous theorem, A is the unique rigidly embedded Cartan subalgebra of $L^\infty(\mathbb{T}^2) \rtimes G$ up to unitary conjugacy. For instance we may take G to be any non-abelian free subgroup of finite index in $\mathrm{SL}(2,\mathbb{Z})$ (*e.g.* consider the subgroup generated by the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, which is free of index 12 in $\mathrm{SL}(2,\mathbb{Z})$ and which contains every free group $\mathbb{F}_n$, $n \geq 2$, as a subgroup of finite index).

Let $\mathbb{F}_n$ be embedded as a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Our goal is to show that the fundamental group of $L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_n$ is reduced to $\{1\}$. To that purpose, we will compare it with the fundamental group of the equivalence relation $\mathcal{R}_{\mathbb{F}_n \curvearrowright \mathbb{T}^2}$. We need first to introduce two important invariants for equivalence relations.

18.2. Fundamental group and cost of an equivalence relation

18.2.1. Fundamental group of an ergodic equivalence relation. It is defined in analogy with that of a II_1 factor. Let $\mathcal{R}$ be an ergodic countable p.m.p. equivalence relation on the Lebesgue probability measure space (X,μ). We set $I_n = \{1,2,\dots,n\}$ and we define an equivalence relation $\mathcal{R}_n$ on $X_n = X \times I_n$ by saying that two elements (x,i) and (y,j) of X_n are equivalent if and only if $x \sim_{\mathcal{R}} y$. We equip X_n with the measure $\mu \times \lambda_n$, where λ_n is the counting measure on I_n. Then $\mathcal{R}_n$ is an ergodic measure preserving equivalence relation on X_n with $\mu_n(X_n) = n$. We note that $L(\mathcal{R}_n) = M_n(L(\mathcal{R}))$.

Let t be a positive real number and choose an integer n with $t \leq n$. Let $Y \subset X_n$ with $\mu_n(Y) = t$. The induced p.m.p. equivalence relation $\mathcal{R}_Y = \mathcal{R}_n \cap (Y \times Y)$ on Y equipped with the normalized measure $\mu_{n_{|_Y}}/\mu_n(Y)$ only depends on t, up to isomorphism (see Exercise 18.1). It is therefore denoted without ambiguity by $\mathcal{R}^t$. We say that $\mathcal{R}^t$ is an *amplification* of $\mathcal{R}$.

Observe that $Y = \cup_{i=1} Y_i \times \{i\}$, where $Y_1,\dots,Y_n$ are Borel subsets of X with $\sum_{i=1}^n \mu(Y_i) = t$. So $\mathcal{R}^t$ may be realized on the disjoint union $Y_1 \sqcup \cdots \sqcup Y_n$, where two elements $x \in Y_i$ and $y \in Y_j$ are $\mathcal{R}^t$-equivalent if and only if $x \sim_{\mathcal{R}} y$.

As in the case of factors (Lemma 4.2.3), one shows that the equivalence relations $(\mathcal{R}^s)^t$ and $\mathcal{R}^{st}$ are isomorphic.

Definition 18.2.1. Let $\mathcal{R}$ be an ergodic countable p.m.p. equivalence relation. We denote by $\mathfrak{F}(\mathcal{R})$ the subgroup of $\mathbb{R}_+^*$ formed of the positive real numbers t such that $\mathcal{R}^t$ is isomorphic to $\mathcal{R}$. It is called the *fundamental group* of $\mathcal{R}$.

For instance, since there is only one II_1 hyperfinite countable p.m.p. equivalence relation, up to isomorphism, we see that its fundamental group is the whole $\mathbb{R}_+^*$.

Let $\mathcal{R}$ be an ergodic countable p.m.p. equivalence relation on (X,μ). For every $t > 0$, the factors $L(\mathcal{R}^t)$ and $L(\mathcal{R})^t$ are isomorphic, and so we see that $\mathfrak{F}(\mathcal{R}) \subset \mathcal{F}(L(\mathcal{R}))$. However, an isomorphism between $L(\mathcal{R})$ and $L(\mathcal{R}^t)$

is not always induced by an isomorphism between $\mathcal{R}$ and $\mathcal{R}^t$, and we have to take Cartan subalgebras into account.

Recall that two Cartan subalgebra inclusions $A_i \subset M_i$, $i = 1, 2$, where M_1, M_2 are II_1 factors, are said to be isomorphic if there exists an isomorphism $\alpha : M_1 \simeq M_2$ such that $\alpha(A_1) = A_2$.

Let A be a Cartan subalgebra of a II_1 factor M. Let $0 < t \leq 1$ be given and choose a non-zero projection p in A with $\tau(p) = t$. It follows from Lemma 17.2.2 that the isomorphism class of the inclusion $(Ap \subset pMp)$ only depends on t. We denote it by $(A \subset M)^t = A^t \subset M^t$. Whenever $t > 1$, one proceeds in the same way, starting with the Cartan subalgebra inclusion $A \otimes D_n \subset M \otimes M_n(\mathbb{C})$, where D_n is the diagonal subalgebra of $M_n(\mathbb{C})$.

Whenever $M = L(\mathcal{R})$ and $A = L^\infty(X)$, the Cartan subalgebra inclusion defined by $\mathcal{R}^t$ is $(A \subset M)^t$. Obviously if t_1, t_2 are such that $\mathcal{R}^{t_1}$ and $\mathcal{R}^{t_2}$ are isomorphic, then $(A \subset M)^{t_1}$ and $(A \subset M)^{t_2}$ are isomorphic. The converse follows from Corollary 12.2.4. If it happens that every $t \in \mathfrak{F}(L(\mathcal{R}))$ is such that there exists an isomorphism from $L(\mathcal{R})$ onto $L(\mathcal{R}^t)$ sending A onto A^t then $\mathfrak{F}(L(\mathcal{R})) = \mathfrak{F}(\mathcal{R})$ and in this case the computation of $\mathfrak{F}(L(\mathcal{R}))$ is reduced to the computation of $\mathfrak{F}(\mathcal{R})$.

We will apply this strategy to $\mathcal{R}_{\mathbb{F}_n \curvearrowright \mathbb{T}^2}$ and therefore will have to compute the fundamental group of an equivalence relation induced by a free ergodic p.m.p. action of a free group. For this, we will use the notion of cost of an equivalence relation.

18.2.2. Cost of an equivalence relation. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X, μ). A *graphing* of $\mathcal{R}$ is a sequence (φ_n) in the pseudo group $[[\mathcal{R}]]$, which generates $\mathcal{R}$ in the sense that $\mathcal{R}$ is the smallest equivalence relation such that $x \in D(\varphi_n)$ implies $x \sim \varphi_n(x)$.

DEFINITION 18.2.2. The *cost* $\mathcal{C}((\varphi_n))$ of the graphing (φ_n) is defined as
$$\sum_n \mu(D(\varphi_n)).$$
The *cost* $\mathcal{C}(\mathcal{R})$ *of the equivalence relation* is defined as the infimum of the cost of all graphings of $\mathcal{R}$.

Note that $\mathcal{C}(\mathcal{R})$, as well as $\mathfrak{F}(\mathcal{R})$, are invariants of the isomorphism class of $\mathcal{R}$.

Recall that the *rank of a countable group* G if the minimal number $\mathrm{rank}(G)$ of elements that are needed to generate G. Assume that G acts on (X, μ) in a p.m.p. way. The equivalence relation $\mathcal{R}_{G \curvearrowright X}$ is generated by any family $\varphi_i : x \in X \mapsto g_i x$, where the g_i's range over a set of generators of G. Thus we have
$$\mathcal{C}(\mathcal{R}_{G \curvearrowright X}) \leq \mathrm{rank}(G).$$

THEOREM 18.2.3. *Let $\mathcal{R}$ be countable p.m.p. equivalence relation on (X, μ). For $t > 0$ we have*
$$\mathcal{C}(\mathcal{R}^t) - 1 = \big(\mathcal{C}(\mathcal{R}) - 1\big)/t.$$

In particular, if in addition $\mathcal{R}$ is ergodic and $1 < \mathcal{C}(\mathcal{R}) < +\infty$, then $\mathfrak{F}(\mathcal{R}) = \{1\}$.

PROOF. For simplicity, we will only prove the inequality $\mathcal{C}(\mathcal{R}^t) - 1 \leq \big(\mathcal{C}(\mathcal{R}) - 1\big)/t$ for $t > 1$. This is all that will be needed in the sequel. Let $Y_1, \dots, Y_n$ be Borel subsets of X such that $\sum_{i=1}^n \mu(Y_i) = t - 1$. We realize $\mathcal{R}^t$ on the disjoint union $Y = X \sqcup Y_1 \sqcup \cdots \sqcup Y_n$ as explained above. Let η be the normalized probability measure on Y. We denote by $\sigma_i \in [[\mathcal{R}^t]]$ the transformation that identifies Y_i viewed as a subset of X to Y_i in the disjoint union. Whenever (φ_k) is a graphing of $\mathcal{R}$, then together with $(\sigma_i)_{1\leq i\leq n}$ it gives a graphing of $\mathcal{R}^t$. It follows that

$$\sum_k \eta(D(\varphi_k)) + \sum_{i=1}^n \eta(D(\sigma_i)) = \frac{\mathcal{C}((\varphi_k))}{t} + \frac{t-1}{t}.$$

Since this holds for every choice of graphing of $\mathcal{R}$, the inequality $\mathcal{C}(\mathcal{R}^t) - 1 \leq \big(\mathcal{C}(\mathcal{R}) - 1\big)/t$ follows. $\square$

18.3. A II_1 factor with trivial fundamental group

THEOREM 18.3.1. *Let $\mathcal{R}$ be an ergodic countable p.m.p. equivalence relation on (X, μ). We assume that $A = L^\infty(X)$ is the unique rigidly embedded Cartan subalgebra of $M = L(\mathcal{R})$, up to isomorphism. Let $0 < t \leq 1$ be such that $M \simeq M^t$. Then $\mathcal{R}$ and $\mathcal{R}^t$ are isomorphic. Therefore, we have*

$$\mathfrak{F}(L(\mathcal{R})) = \mathfrak{F}(\mathcal{R}).$$

PROOF. Let $p \in M$ be a projection such that $\tau(p) = t$. By Lemma 17.2.1 we may take $p \in A$. The inclusion $A^t \subset M^t$ is rigid (see Proposition 14.2.11). Our uniqueness assumption implies that it is isomorphic to $A \subset M$ and therefore the equivalence relations $\mathcal{R}$ and $\mathcal{R}^t$ are isomorphic. $\square$

THEOREM 18.3.2. *Let $\mathbb{F}_n$, $2 \leq n < \infty$, be embedded as a finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})$. The fundamental group of $L(\mathbb{Z}^2 \rtimes \mathbb{F}_n) \simeq L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_n$ is equal to $\{1\}$.*

PROOF. We apply the previous theorem to the equivalence relation $\mathcal{R}$ on $\mathbb{T}^2$ which is induced by the natural action of $\mathbb{F}_n$. Indeed, $L^\infty(\mathbb{T}^2)$ is the unique rigidly embedded Cartan subalgebra of $L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_n$, up to unitary conjugacy (see Example 18.1.5). Then we conclude thanks to the next theorem. $\square$

THEOREM 18.3.3. *Let $\mathbb{F}_n \curvearrowright (X, \mu)$, $2 \leq n < \infty$, be a free p.m.p. action. Then the cost of $\mathcal{R}_{\mathbb{F}_n \curvearrowright X}$ is n. Therefore, if moreover the action is ergodic, the fundamental group of this equivalence relation is $\{1\}$.*

PROOF. We only give the proof for $n = 2$, to keep the notations simpler. We write $G = \mathbb{F}_2$, $\mathcal{R} = \mathcal{R}_{\mathbb{F}_2 \curvearrowright X}$ and $M = L^\infty(X) \rtimes G$. Recall that for $\varphi \in [[\mathcal{R}]]$, there exists a partition $E = \cup_{g\in G} E_g$ of the domain E of φ into Borel subsets such that $\varphi(x) = gx$ if $x \in E_g$ (see Exercise 1.17). Therefore, to compute the cost of $\mathcal{R}$, it suffices to consider graphings consisting of maps of the form $x \in E \mapsto gx$ form some Borel subset E of X and some $g \in G$.

Let $G = \{g_0, g_1, \dots\}$ be an enumeration of the elements of G. Let $(E_k)_{k\geq 0}$ be a sequence of Borel subsets of X and for k, consider $\varphi_k : x \in E_k \mapsto g_k x$. Whenever (φ_k) is a graphing of $\mathcal{R}$, we only need to prove that

$$\sum_{k\geq 0} \mu(E_k) \geq 2.$$

We denote by $Z^1(G, M)$ the set of maps $c : G \mapsto M$ such that

$$\forall g, h \in G, \quad c_{gh} = c_g + u_g c_h, \tag{18.1}$$

where the u_g's are the canonical unitaries in M. These maps c are called 1-cocycles and are of course determined by the values that they take on the generators a, b. We note that $Z^1(G, M)$ is a right M-module and, since a, b are free in G, we have a bijective right M-linear map $\Psi : M \oplus M \to Z^1(G, M)$ defined by

$$\Psi(m \oplus n) = c \quad \text{with} \quad c_a = m \text{ and } c_b = n.$$

Let p_k be the projection in $L^\infty(X)$ with support $g_k E_k$. Then we consider the M-linear map $\Theta : Z^1(G, M) \to \prod_{k=0}^{\infty} p_k M$ defined by

$$\Theta(c)_k = p_k c_{g_k} \quad \text{for all} \quad k \geq 0.$$

We claim that Θ is injective. Indeed, let $c \in Z^1(G, M)$ such that $\Theta(c) = 0$, that is, $p_k c_{g_k} = 0$ for all $k \geq 0$. Recall from Section 1.4.2 that M is canonically embedded into $L^2(X) \otimes \ell^2(G)$ by $m \mapsto m(1 \otimes \delta_e)$. Therefore, via the identification of $L^2(X) \otimes \ell^2(G)$ with $L^2(X, \ell^2(G))$, we may view every element $m = \sum_{g\in G} a_g u_g$ of M as the measurable function $x \mapsto m(x) \in \ell^2(G)$ where the component $m(x)_g$ of index $g \in G$ is $m(x)_g = a_g(x)$. Then the cocycle equation (18.1) becomes

$$c_{gh}(x) = c_g(x) + \lambda(g) c_h(g^{-1}x) \quad a.e., \tag{18.2}$$

where λ is the left regular representation of G.

Now, since the action $G \curvearrowright (X, \mu)$ is free, the map $(x, g) \mapsto (x, g^{-1}x)$ allows to identify without ambiguity $(g, x) \mapsto c_g(x) \in \ell^2(G)$ with

$$(x, y) = (x, g^{-1}x) \in \mathcal{R} \mapsto \omega(x, y) = c_g(x) \in \ell^2(G).$$

Then, (18.2) becomes

$$\omega(x, z) = \omega(x, y) + \lambda(x, y)\omega(y, z) \quad \text{for } a.e.\ (x, y, z) \in \mathcal{R}^{(2)}, \tag{18.3}$$

where we set $\lambda(x, g^{-1}x) = \lambda(g)$.

The fact that $p_k c_{g_k} = 0$ becomes $\omega(g_k y, y) = 0$ for $a.e.$ $y \in E_k$. By (18.3), the set of $(x, y) \in \mathcal{R}$ such that $\omega(x, y) = 0$ is a subequivalence relation of $\mathcal{R}$. Since (φ_k) is a graphing, we see that $\omega(x, y) = 0$ for $a.e.$ $(x, y) \in \mathcal{R}$. This means that $c_g = 0$ for all $g \in G$. So $c = 0$ and Θ is injective.

For every $n \geq 0$, we consider the right M-linear map

$$\theta_n : M \oplus M \to \bigoplus_{k=0}^{n} p_k M$$

defined by

$$\theta_n(m_1 \oplus m_2) = \bigoplus_{k=0}^{n} \Theta(\Psi(m_1 \oplus m_2))_k.$$

Since θ_n is right M-linear, there exists a unique element $V_n \in M_{n+1,2}(\mathbb{C}) \otimes M$ such that

$$\big(\theta_n(m_1 \oplus m_2)\big)_k = (V_n)_{k,1} m_1 + (V_n)_{k,2} m_2 \quad \text{for all } m_1, m_2 \in M.$$

We denote by $Q_n \in M_2(M)$ the right support of V_n and by $P_n \in M_{n+1}(M)$ its left support. Note that by construction, we have $P_n \leq \operatorname{diag}(p_0, \dots, p_n)$, where $\operatorname{diag}(p_0, \dots, p_n)$ is the diagonal matrix with entries $p_0, \dots, p_n$ on the diagonal. It follows that[2]

$$\sum_{k=0}^{n} \mu(E_k) = \sum_{k=0}^{n} \tau(p_k) \geq (\operatorname{Tr}_n \otimes \tau)(P_n) = (\operatorname{Tr}_2 \otimes \tau)(Q_n).$$

We claim that the sequence of projection Q_n increases to 1. Once we have proven this claim the theorem will follow since the above inequality yields

$$\sum_{k=0}^{\infty} \mu(E_k) \geq 2.$$

The sequence of projections Q_n is increasing since the kernels of the maps θ_n are decreasing. So (Q_n) converges strongly to some projection $Q \in M_2(M)$. Assume that $Q < 1$. Then either the first column or the second column of $1 - Q$ is non-zero and defines a non-zero element $m_1 \oplus m_2$ of $M \oplus M$ with the property that $\theta_n(m_1 \oplus m_2) = 0$ for all n. This means that $\Theta(\Psi(m_1 \oplus m_2)) = 0$ and contradicts the fact that Ψ and Θ are injective. $\square$

Remark 18.3.4. The equivalence relations $\mathcal{R}_{\mathbb{F}_n \curvearrowright \mathbb{T}^2}$ are mutually non-isomorphic since their costs are distinct. It follows from Corollary 18.1.4 that the II_1 factors $L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_n$ are mutually non-isomorphic.

Exercise

Exercise 18.1. Let $\mathcal{R}$ be an ergodic countable p.m.p. equivalence relation on (X, μ).

(i) Let Y_1, Y_2 be two Borel subsets of X. Show that $\mu(Y_1) = \mu(Y_2)$ if and only if there exists $\varphi \in [\mathcal{R}]$ such that $\varphi(Y_1) = \varphi(Y_2)$.

(ii) Show that the fundamental group $\mathfrak{F}(\mathcal{R})$ of $\mathcal{R}$ may be defined as

$$\Big\{ t_1 t_2^{-1} : t_1, t_2 \in]0, 1], \mathcal{R}^{t^1} \simeq \mathcal{R}^{t^2} \Big\}.$$

[2]below Tr_k is the usual trace in $M_k(\mathbb{C})$

(iii) Show that $\mathfrak{F}(\mathcal{R}) \subset \mathfrak{F}(L(\mathcal{R}))$.

Notes

The deformation/rigidity technique was discovered by Popa [**Pop06a, Pop06c, Pop06d, Pop06e**] between 2001-2004. Since then, more and more deformations were found [**Ioa07**], [**IPP08**], [**Pet09**], [**Sin11**], as well as more and more rigidity behaviours, not necessarily associated with property (T) but also with spectral gap properties [**Pop08**], [**Pop07d**], [**OP10a**], [**OP10b**], [**CS13**], [**CSU13**], these lists of references being not exhaustive. This is now an essential tool to detect the position of a somewhat rigid subalgebra in presence of an appropriate deformation property.

Due to the results of Section 18.1 obtained in [**Pop06a**], the computation of the fundamental group of $L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_n$ was reduced to that of the corresponding equivalence relation, in principle easier to achieve. Fundamental groups of countable p.m.p. equivalence relations were introduced by Gefter and Golodets [**GG88**]. They proved in particular that $\mathfrak{F}(\mathcal{R}_{G \curvearrowright X}) = \{1\}$ whenever $G \curvearrowright (X, \mu)$ is a free p.m.p. action of a lattice G in a connected simple Lie group with finite center and real rank ≥ 2. A major breakthrough is due to Gaboriau who could exhibit many actions $G \curvearrowright (X, \mu)$ whose equivalence relations have a trivial fundamental group, as a consequence of his remarkable study of the notions of cost and ℓ^2-Betti numbers for equivalence relations (see [**Gab00, Gab02**] and [**Gab10**] for a survey). Theorems 18.2.3 and 18.3.3 come from [**Gab02**]. In fact, as soon as G has at least one non-zero ℓ^2-Betti number, we have $\mathfrak{F}(\mathcal{R}_{G \curvearrowright X}) = \{1\}$ [**Gab02**]. The proof of Theorem 18.3.3 given in this chapter was communicated to us by Vaes. It is a version in the spirit of operator algebras of a previous proof by Gaboriau, expressed in a more geometric style. We thank them for allowing us to present it here.

The example of $L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_n$ described in this chapter was the first example of a II_1 factor with an explicitely computed fundamental group distinct of $\mathbb{R}_+^*$. More generally, if a group G having the Haagerup property and at least one non-zero Betti number (like $\mathrm{SL}(2, \mathbb{Z})$ or $\mathbb{F}_n$, $n \geq 2$) acts on (X, μ) in a free ergodic p.m.p. way and if $L^\infty(X)$ is rigidly embedded into $L^\infty(X) \rtimes G$, then the fundamental group of this factor is trivial. We remind the reader that Connes had established that the fundamental group of every $L(G)$, where G is an ICC group with Property (T), is countable but without an explicit description [**Con80a**]. The above example $L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_n$ answers positively a longstanding question of Kadison [**Kad67**], asking whether there exist II_1 factors M such that, for some $n \geq 1$, $M_n(\mathbb{C}) \otimes M$ is not isomorphic to M.

Since then, impressive advances have been achieved during the last decade. Thus, let $\mathfrak{F}$ be a subgroup of $\mathbb{R}_+^*$: if $\mathfrak{F}$ is countable, or uncountable of any Hausdorff dimension in $(0, 1)$, there exists a group measure space factor having $\mathfrak{F}$ as fundamental group (see [**PV10a, PV10c**], and [**Vae10**] for a survey). Other kinds of examples are given in [**IPP08**], [**Hou09**].

The example found by Connes and Jones [**CJ82**] of a free ergodic p.m.p. action $G \curvearrowright (X, \mu)$ such that $L^\infty(X) \rtimes G$ contains two non-conjugate group measure space Cartan subalgebras (see also Section 17.3) shows that the fundamental group of $\mathcal{R}_{G \curvearrowright X}$ can be strictly smaller than the fundamental group of $L^\infty(X) \rtimes G$. Other examples were given in [**Pop86a, Pop90**]. There even exist examples where $\mathfrak{F}(L^\infty(X) \rtimes G) = \mathbb{R}_+^*$ and $\mathfrak{F}(\mathcal{R}_{G \curvearrowright X}) = \{1\}$ (see [**Pop08**]). A wealth of results about fundamental groups is contained in the paper [**PV10c**].

Let us recall also that it is now known that $L^\infty(X) \rtimes \mathbb{F}_n$ has a unique Cartan subalgebra, up to unitary conjugacy, for any free ergodic p.m.p. action of $\mathbb{F}_n$ [**PV14a**]. In particular, the crossed products $\big(L^\infty([0,1]^{\mathbb{F}_n}\big) \rtimes \mathbb{F}_n$ and $\big(L^\infty([0,1]^{\mathbb{F}_m}\big) \rtimes \mathbb{F}_m$ arising from Bernoulli actions are isomorphic if and only if these actions are orbit equivalent and so, using the cost invariant, if and only if $n = m$. It follows that the factors $L(\mathbb{Z} \wr \mathbb{F}_n)$ and $L(\mathbb{Z} \wr \mathbb{F}_m)$ of the wreath products are isomorphic if and only if $n = m$, since $L(\mathbb{Z} \wr \mathbb{F}_k) \simeq \big(L^\infty([0,1]^{\mathbb{F}_k}\big) \rtimes \mathbb{F}_k$ for every k. However, this is still far from giving an answer to the major open problem asking whether $L(\mathbb{F}_n) \simeq L(\mathbb{F}_m)$, $n, m \geq 2$, implies $n = m$.

In [**IPV13, BV14, Ber15, CI17**], the reader will find examples of generalized wreath products G for which the group factor $L(G)$ remembers the group G in the sense that any isomorphism between $L(G)$ and an arbitrary group factor $L(H)$ is implemented by an isomorphism of the groups. An ICC group G with this property is called a *W^*-superrigid group*. Other examples of W^*-superrigid groups are given in [**CdSS16, CI17**].

CHAPTER 19

Free group factors are prime

In this chapter, we illustrate by another example the deformation/rigidity technique. The deformation is constructed in Lemma 19.1.1. It is a one parameter group $(\alpha_t)_{t\in\mathbb{R}}$ of automorphisms of the free product $M = N * N$ of two copies of the free group factor $N = L(\mathbb{F}_n)$. It moves $N * \mathbb{C}$ inside M in such a way that $M = N * \alpha_1(N * \mathbb{C})$. The rigidity comes from a spectral gap property of embeddings of non-amenable von Neumann algebras in $N * \mathbb{C} \subset M$ (see Lemma 19.1.3). Using intertwining techniques and the crucial fact that the deformation α carries a symmetry given by a period 2 automorphism of $N * N$ (one says that the deformation is *s-malleable*), we will prove the following result.

Theorem. *Let $\mathbb{F}_n$ be the free group on n generators, $2 \leq n \leq \infty$, and let $P \subset L(\mathbb{F}_n)$ be a von Neumann subalgebra such that $P' \cap L(\mathbb{F}_n)$ is diffuse. Then P is amenable.*

A II_1 factor satisfying the property of this theorem is called *solid* (see Exercice 19.2 for an equivalent formulation). One says that a II_1 factor M is *prime* if, whenever M is isomorphic to a tensor product $M_1 \overline{\otimes} M_2$, then either M_1 or M_2 is finite dimensional. The previous theorem has the following consequence, since $L(\mathbb{F}_n)$ cannot be written as a tensor product of the form $R \overline{\otimes} N$ where R is the hyperfinite II_1 factor.

Corollary. *The free group factors $L(\mathbb{F}_n)$, $n \geq 2$, are prime.*

19.1. Preliminaries

In this section we gather the lemmas that we will use in our proof of the above theorem.

19.1.1. Construction of the malleable deformation.

Lemma 19.1.1. *Let $N = (L(\mathbb{F}_n), \tau)$ be the factor of the free group on n generators, $2 \leq n \leq \infty$. There exist a continuous homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}\,(N * N)$ and a period two automorphism $\beta \in \mathrm{Aut}\,(N * N)$ such that*

(a) *$N_0 = N * \mathbb{C}$ and $N_1 = \alpha_1(N * \mathbb{C})$ are free with respect to τ and $M = N_0 * N_1$;*
(b) *$\beta\alpha_t\beta = \alpha_{-t}$ for all $t \in \mathbb{R}$;*
(c) *$\beta(x) = x$ for all $x \in N * \mathbb{C}$.*

Proof. Let $a_1, a_2, \cdots$ be the generators of $\mathbb{F}_n$, viewed as unitary elements in $N * \mathbb{C}$, and let $b_1, b_2, \cdots$ be the same generators but viewed as unitary elements in $\mathbb{C} * N$.

Fix k and let h_k be the self-adjoint operator with spectrum in $[-\pi, \pi]$, obtained from b_k by the Borel functional calculus, such that $b_k = \exp(ih_k)$. For $t \in \mathbb{R}$, we put $\alpha_t(a_k) = \exp(ith_k)a_k$ and $\alpha_t(b_k) = b_k$. Obviously, $\exp(ith_k)a_k$ and b_k generate the same von Neumann algebra as a_k, b_k. Moreover, since $\tau(a_k^n) = 0 = \tau(b_k^n)$ for $n \neq 0$, we see that

$$\tau(\alpha_t(a_k)^{i_1}\alpha_t(b_k)^{j_1} \cdots \alpha_t(a_k)^{i_l}\alpha_t(b_k)^{j_l}) = 0 = \tau(a_k^{i_1}b_k^{j_1} \cdots a_k^{i_l}b_k^{j_l})$$

for any sequence $i_1, \dots, i_l$, $j_1, \dots, j_l$ of elements in $\mathbb{Z}$ which are non-zero, except may be i_1 and j_l. It follows that α_t extends to an automorphim of the von Neumann algebra generated by a_k, b_k. Doing like this for every k and every $t \in \mathbb{R}$ we get a continuous one-parameter group $t \mapsto \alpha_t$ of automorphisms of the whole $M = N * N$.

Note that $\alpha_1(a_k) = b_k a_k$ and a_k are free with respect to τ and that $a_k, \alpha_1(a_k)$ generate the same von Neumann algebra as a_k, b_k. It follows that if we set $N_0 = N * \mathbb{C}$ and $N_1 = \alpha_1(N * \mathbb{C})$, then N_0, N_1 are free with respect to τ and generate M

Next, we define β by $\beta(a_k) = a_k$ and $\beta(b_k) = b_k^*$. Clearly, β is a period 2 automorphism of M which satisfies Condition (c). Moreover, we have

$$\beta \circ \alpha_t \circ \beta(a_k) = \beta(\exp(ith_k)a_k) = \exp(-ith_k)a_k = \alpha_{-t}(a_k),$$

and similarly

$$\beta \circ \alpha_t \circ \beta(b_k) = b_k = \alpha_{-t}(b_k).$$

Therefore, Condition (b) is also satisfied. □

The pair (α, β) is a *s-malleable deformation* of N is the following sense: there exists an embedding of N in a larger II_1 factor $\widetilde{N}$, together with a continuous path $t \in \mathbb{R} \mapsto \alpha_t$ of automorphisms of $\widetilde{N}$ and a period 2 automorphism β of $\widetilde{N}$ such that

(a) $\tau(x\alpha_1(y)) = 0$ for all $x, y \in N$ with $\tau(x) = 0 = \tau(y)$;
(b) $\beta\alpha_t\beta = \alpha_{-t}$ for all $t \in \mathbb{R}$;
(c) $\beta(x) = x$ for all $x \in N$.

19.1.2. A spectral gap property. We first give some clarification on the structure of $L^2(M_1 * M_2)$ as a M_1-M_1-bimodule.

Lemma 19.1.2. *Let $(M, \tau) = (M_1, \tau_1) * (M_2, \tau_2)$ be a free product of two tracial von Neumann algebras. Then, as a M_1-M_1-bimodule, the orthogonal $L^2(M) \ominus L^2(M_1)$ of $L^2(M_1)$ in $L^2(M)$ is isomorphic to an orthogonal direct sum of copies of the coarse M_1-M_1-bimodule.*

Proof. We keep the notation of Section 5.3.2. In particular, $\mathcal{H}_i = L^2(M_i)$. We have observed that $L^2(M)$ is isomorphic to $L^2(M_1) \otimes \mathcal{H}_l(1)$,

where
$$\mathcal{H}_l(1) = \mathbb{C}\xi \oplus \bigoplus_{n\geq 1}\Big(\bigoplus_{\substack{i_1\neq i_2\neq\cdots\neq i_n\\ i_1\neq 1}} \overset{o}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{i_n} \Big).$$
We identify $\mathcal{H}_1$ with $\mathcal{H}_1\otimes\xi$, $(\mathcal{H}_1\otimes \overset{o}{\mathcal{H}}_2)\oplus(\mathcal{H}_1\otimes \overset{o}{\mathcal{H}}_2 \otimes \overset{o}{\mathcal{H}}_1)$ with $\mathcal{H}_1\otimes \overset{o}{\mathcal{H}}_2 \otimes\mathcal{H}_1$, and so on.

Let us set
$$\mathcal{K}_n = \mathcal{H}_1 \otimes \overbrace{\overset{o}{\mathcal{H}}_2 \otimes \overset{o}{\mathcal{H}}_1 \cdots \otimes \overset{o}{\mathcal{H}}_1 \otimes \overset{o}{\mathcal{H}}_2}^{2n-1} \otimes\mathcal{H}_1.$$
Then, we have $L^2(M) = L^2(M_1) \oplus \bigoplus_{n\geq 1}\mathcal{K}_n$. A straightforward verification shows that the direct summands of this decomposition are M_1-M_1-subbimodules of ${}_{M_1}L^2(M)_{M_1}$. All of them are direct sums of the coarse M_1-M_1-bimodule, except for the first one which is the trivial M_1-M_1-bimodule. $\square$

LEMMA 19.1.3. *Let (M_1,τ_1), (M_2,τ_2) be two tracial von Neumann algebras and set $(M,\tau) = (M_1,\tau_1)*(M_2,\tau_2)$. Let P be a separable von Neumann subalgebra of M_1 which has no amenable direct summand. We identify P with $P*\mathbb{C}$ and M_1 with $M_1*\mathbb{C}$. Then, for any free ultrafilter ω, we have $P'\cap M^\omega \subset M_1^\omega$. In other terms, for every $\varepsilon > 0$, there exist a finite subset F of $\mathcal{U}(P)$ and $\delta > 0$ such that if x in the unit ball $(M)_1$ satisfies $\max_{u\in F}\|ux - xu\|_2 \leq \delta$, then $\|x - E_{M_1}(x)\|_2 \leq \varepsilon$.*

PROOF. We assume that P does not have any amenable direct summand but that $P'\cap M^\omega \not\subset M_1^\omega$. Take an element[1] $x = (x_n) \in P'\cap M^\omega$ which does not belong to M_1^ω. Observe that $E^{M^\omega}_{M_1^\omega}(x) = (E^M_{M_1}(x_n))$ commutes with P. By substracting $E^{M^\omega}_{M_1^\omega}(x)$ to x we may assume that $E^M_{M_1}(x_n) = 0$, so that $x_n \in L^2(M)\ominus L^2(M_1)$ for every n, with x still non-zero. Moreover, since P is separable, we may replace (x_n) by a subsequence satisfying $\lim_n \|[x_n, y]\|_2 = 0$ for every $y \in P$. Of course, we may assume that $\sup_n \|x_n\|_\infty \leq 1$.

We will apply Lemma 13.4.8 with $Q = \mathbb{C}$, M_1 instead of M, ${}_{\mathbb{C}}L^2(M_1)_{M_1} = \mathcal{K}$ and
$$\mathcal{H} = L^2(M)\ominus L^2(M_1) = \big(L^2(M_1)\otimes L^2(M_1)\big)\otimes \ell^2(I),$$
a direct sum of copies of the coarse M_1-M_1-bimodule. Let us check that the sequence (x_n) of the elements $x_n \in \mathcal{H}$ satisfies the conditions (a), (b) and (c) of this lemma (the net (ξ_i) of this lemma being replaced by (x_n)). It is immediate for (b) and (c) and Condition (a) holds since, for $x \in M_1$ we have
$$\|xx_n\|_2^2 = \tau(x_n^*x^*xx_n) = \tau(xx_nx_n^*x^*) \leq \|x_n\|_\infty^2\|x\|_2^2 \leq \|x\|_2^2.$$
It follows that there exists a non-zero projection $p' \in \mathcal{Z}(P'\cap M_1)$ such that Pp' is amenable relative to $\mathbb{C}$ inside M and so Pp' is amenable (see Exercise

[1] For simplicity, we denote in the same way a bounded sequence and its class in M^ω.

13.16). The greatest projection z in P such that $zp' = 0$ belongs to $\mathcal{Z}(P)$ and $P(1-z)$ is amenable since it is isomorphic to Pp'. This contradicts our assumption and therefore we have $P' \cap M^\omega \subset M_1^\omega$.

Let us show the last assertion of the lemma. If it does not hold, since P is separable, there exist $\varepsilon > 0$ and a sequence $x = (x_n)$ in the unit ball $(M)_1$ of M such that $\lim_n \|yx_n - x_n y\|_2 = 0$ for every $y \in P$ and $\|x_n - E_{M_1}(x_n)\|_2 \geq \varepsilon$ for every n. Then $x \in P' \cap M^\omega$ but $x \notin M_1^\omega$. The converse is also immediate. $\square$

19.1.3. Two more lemmas.

Lemma 19.1.4. *Let $(M, \tau) = (M_1, \tau_1) * (M_2, \tau_2)$ be the free product of two tracial von Neumann algebras. Let P be a separable, diffuse, w.o. closed self-adjoint subalgebra of M_1 and let $v \in M$ be a partial isometry such that $vPv^* \subset M_2$ and $v^*v = 1_P$. Then $v = 0$.*

Proof. The proof is similar to that of Lemma 5.3.6. Since P is diffuse, it contains a sequence (u_n) of unitary operators such that $\tau_1(u_n) = 0$ for every n and $\lim_n u_n = 0$ in the w.o. topology. We claim that for every $x, y \in M$ we have $\lim_n \|E_{M_2}(xu_n y)\|_2 = 0$. Using approximations by elements of the free algebraic product $\mathcal{M}$ of M_1 and M_2 defined in Remark 5.3.5, we may assume that x and y belong to $\mathcal{M}$. By linearity, it suffices to consider the case where $xu_n y = ax_1 u_n y_1 b$ with $x_1, y_1 \in M_1$, and a (resp. b) is an alternated product ending (resp. beginning) by some element in $\overset{o}{M}_2$ or is the identity. Then, we have

$$xu_n y = a\big((x_1 u_n y_1) - \tau_1(x_1 u_n y_1)1\big)b + \tau_1(x_1 u_n y_1)ab.$$

Since $a\big((x_1 u_n y_1) - \tau_1(x_1 u_n y_1)1\big)b$ is an alternated product not in M_2, its projection under E_{M_2} is 0 and so

$$E_{M_2}(xu_n y) = \tau_1(x_1 u_n y_1)E_{M_2}(ab).$$

But we have $\lim_n \tau_1(x_1 u_n y_1) = 0$ and therefore $\lim_n \|E_{M_2}(xu_n y)\|_2 = 0$.

In particular, we see that $\lim_n \|E_{M_2}(vu_n v^*)\|_2 = 0$. On the other hand, $E_{M_2}(vu_n v^*) = vu_n v^*$ and so $\|E_{M_2}(vu_n v^*)\|_2 = \tau(1_P)^{1/2}$. It follows that $1_P = 0 = v$. $\square$

Lemma 19.1.5. *Let (M, τ) be a tracial von Neumann algebra, P and Q two von Neumann subalgebras. Let $0 < \varepsilon < 2^{-1/2}$ be such that*

$$\|x - E_Q(x)\|_2 \leq \varepsilon$$

for x in the unit ball $(P)_1$ of P. Then a corner of P embeds into Q inside M.

Proof. A straightforward modification of the proof of Lemma 14.3.2 shows the existence of a non-zero element h in $P' \cap \langle M, e_Q\rangle$ such that $\widehat{\tau}(h) \leq \widehat{\tau}(e_Q) = 1$. It follows from Theorem 17.1.1 that $P \prec_M Q$. $\square$

19.2. Proof of the solidity of $\mathbb{F}_n$

We set $M = L(\mathbb{F}_n) * L(\mathbb{F}_n)$, $M_1 = L(\mathbb{F}_n) * \mathbb{C}$ and $M_2 = \mathbb{C} * L(\mathbb{F}_n)$. Assumet that there exists a non-amenable von Neumann subalgebra P of $L(\mathbb{F}_n)$ (identified with M_1) such that $P' \cap L(\mathbb{F}_n)$ is diffuse. There is a non-zero projection $e \in P$ such that ePe has no amenable corner (see Exercise 10.8). In particular, ePe is diffuse and so, replacing if necessary e by a smaller projection, we may assume that $\tau(e) = 1/k$ for some integer $k > 0$. Then, by Proposition 4.2.5 we have $M_1 = M_k(\mathbb{C}) \otimes (eM_1e)$. The von Neumann subalgebra $\widetilde{P} = M_k(\mathbb{C}) \otimes (ePe)$ has no amenable direct summand and $\widetilde{P}' \cap M_1$ is still diffuse. It follows that, replacing P by $\widetilde{P}$, we may assume that P has no amenable direct summand.

By Lemma 19.1.3, for $\varepsilon > 0$, there exist a finite subset F of $\mathcal{U}(P)$ and $\delta > 0$ such that if x in the unit ball $(M)_1$ satisfies $\max_{u \in F} \|ux - xu\|_2 \leq \delta$, then $\|x - E_{M_1}(x)\|_2 \leq \varepsilon$.

We set $P^0 = P' \cap M_1$. Let (α, β) be as in Lemma 19.1.1. Since $t \mapsto \alpha_t$ is continuous, there exists $t = 2^{-n}$ such that

$$\forall u \in F, \quad \|u - \alpha_{-t}(u)\|_2 \leq \delta/2$$

and therefore

$$\forall u \in F, \forall x \in (P^0)_1, \quad \|[\alpha_{-t}(x), u]\|_2 \leq 2\|u - \alpha_{-t}(u)\|_2 \leq \delta.$$

It follows that $\|\alpha_{-t}(x) - E_{M_1}(\alpha_{-t}(x))\|_2 \leq \varepsilon$ for all $x \in (P^0)_1$ or equivalently

$$\left\|x - E_{\alpha_t(M_1)}(x)\right\|_2 \leq \varepsilon$$

for all $x \in (P^0)_1$.

So, having chosen ε small enough, we have $P^0 \prec_M \alpha_t(M_1)$, by Lemma 19.1.5. Now, thanks to Theorem 17.1.2 we get non-zero projections $p \in P^0$, $q \in \alpha_t(M_1)$, a unital normal homomorphism $\theta : pP^0p \to q\alpha_t(M_1)q$ and a non-zero partial isometry $v \in M$ such that $vv^* = p' \leq p$, $v^*v = q' \leq q$, and $xv = v\theta(x)$ for all $x \in pP^0p$. Moreover, we have $p' \in (pP^0p)' \cap pMp$ and $q' \in \theta(pP^0p)' \cap qMq$.

Since pP^0p is diffuse, the remark 5.3.7 implies that $(pP^0p)' \cap pMp \subset M_1$ and so $p' \in M_1$. Similarly, since $\theta(pP^0p)$ is diffuse and since $M = \alpha_t(M_1) * \alpha_t(M_2)$, we get $q' \in \alpha_t(M_1)$. So, $P_0 = pP^0pp'$ lies in M_1 and v is a partial isometry such that $vv^* = 1_{P_0}$, $v^*P_0v \subset \alpha_t(M_1)$.

With n fixed as above, we now construct by induction over $k \geq 0$, partial isometries $v_k \in M$ and diffuse weakly closed self-adjoint subalgebras P_k of M_1 such that

$$\tau(v_k^*v_k) = \tau(v^*v), \quad v_kv_k^* = 1_{P_k}, \quad v_k^*P_kv_k \subset \alpha_{1/2^{n-k}}(M_1). \tag{19.1}$$

For $k = 0$, the above P_0 and $v_0 = v$ satisfy the required conditions. Assume that we have constructed v_j, P_j for $j = 0, 1, \cdots, k$. We have

$$\beta(v_k^*)P_k\beta(v_k) = \beta(v_k^*P_kv_k) \subset \alpha_{-1/2^{n-k}}(M_1).$$

So P_{k+1}, defined as $\alpha_{1/2^{n-k}}\big(\beta(v_k^*)P_k\beta(v_k)\big)$, lies in M_1. Note that

$$\alpha_{1/2^{n-k}}(\beta(v_k))P_{k+1}\alpha_{1/2^{n-k}}(\beta(v_k^*)) = \alpha_{1/2^{n-k}}(P_k). \tag{19.2}$$

On the other hand, by applying $\alpha_{1/2^{n-k}}$ in (19.1) we also have

$$\alpha_{1/2^{n-k}}(v_k^*)\alpha_{1/2^{n-k}}(P_k)\alpha_{1/2^{n-k}}(v_k) \subset \alpha_{1/2^{n-k-1}}(M_1). \tag{19.3}$$

Therefore, if we set $v_{k+1} = \alpha_{1/2^{n-k}}(\beta(v_k^*))\alpha_{1/2^{n-k}}(v_k)$, we deduce from (19.2) and (19.3) that

$$v_{k+1}^*P_{k+1}v_{k+1} \subset \alpha_{1/2^{n-k-1}}(M_1).$$

Moreover, since $\beta(v_kv_k^*) = v_kv_k^*$ we have $v_{k+1}^*v_{k+1} = \alpha_{1/2^{n-k}}(v_k^*v_k)$, so that $\tau(v_{k+1}^*v_{k+1}) = \tau(v_k^*v_k)$. This ends the induction argument.

Taking $k = n$, we see that

$$v_n^*P_nv_n \subset \alpha_1(M_1) = M_2.$$

Then, by Lemma 19.1.4, we get $v_n = 0$ and also $v = 0$ since $\tau(v^*v) = \tau(v_n^*v_n)$, a contradiction. □

Remark 19.2.1. Roughly speaking, the rigidity provided by the fact that P has no amenable direct summand is used to build intertwiners between subalgebras of $\alpha_t(M_1)$ and $\alpha_{t'}(M_1)$ for sufficiently small $t' - t$. The main difficulty is to glue together these intertwiners, which are partial isometries, in order to get a non-zero intertwiner from a subalgebra of M_1 into $\alpha_1(M_1)$. The role of β is to overcome this problem.

Exercises

Exercise 19.1. Let M be a tracial von Neumann algebra such that there exists an abelian diffuse von Neumann subalgebra in its center. Show that M is diffuse.

Exercise 19.2. Show that a II_1 factor M is solid if and only if it satisfies the following condition: for every diffuse von Neumann subalgebra Q of M the commutant $Q' \cap M$ is amenable.

Exercise 19.3. Let (M_i, τ_i), $i = 1, 2$, be two tracial von Neumann algebras. We assume that M_1 is a full II_1 factor. Show that the free product $(M_1, \tau_1) * (M_2, \tau_2)$ is a full II_1 factor.

Notes

The solidity of $L(\mathbb{F}_n)$, and more generally of any II_1 subfactor of $L(G)$, where G is a non-elementary ICC word-hyperbolic group G, was established by Ozawa in [**Oza04a**]. His approach uses sophisticated C^*-algebraic tools. With the purpose of giving a more elementary and self-contained proof, the Popa's s-malleable deformation/rigidity method of this chapter has been published in [**Pop07d**]. Note that a non-amenable II_1 factor having Property Gamma cannot be solid [**Oza04a**]. In [**Oza06**], Ozawa has produced examples of prime II_1 factors that are not solid (see also [**Fim11**] for other examples).

The first example of a II_1 factor that is not prime was given in [**Pop83**], namely the von Neumann algebra of the free group with uncountably many generators. Later, the primeness of $L(\mathbb{F}_n)$ was proved by Ge [**Ge98**], using Voiculescu's free entropy techniques. A new approach, based on the notion of L^2-rigidity for von Neumann algebras, was proposed by Peterson [**Pet09**]. He obtained in this way another proof of the primeness of every non-amenable II_1 subfactor of $L(\mathbb{F}_n)$, and more generally of any free products of diffuse finite von Neumann algebras. His examples have neither Property Gamma nor Property (T).

Malleability properties were discovered in [**Pop06c**], [**Pop06d**]. Since then, malleable deformations have been constructed in various contexts and are essential tools in studying group actions and II_1 factors (see the surveys [**Pop07b**] and [**Ioa13**] for informations and references).

In [**OP10a**], Ozawa and Popa found that the free group von Neumann algebras $L(\mathbb{F}_n)$ have an even more remarkable property than solidity, they named *strong solidity*. This means that the normalizer of any diffuse amenable von Neumann subalgebra of $L(\mathbb{F}_n)$ generates an amenable von Neumann algebra. This is stronger than solidity and implies that $L(\mathbb{F}_n)$ has no Cartan subalgebra, a fact initially proved by Voiculescu [**Voi96**] by free probability techniques. Recently, Chifan and Sinclair [**CS13**] have established the strong solidity of the von Neumann algebras of non-elementary ICC word-hyperbolic groups. For related results see also [**CH10**], [**Hou10**], [**HS11**], [**Sin11**], [**CSU13**]. All these results are obtained by using powerful deformation/rigidity strategies with various sources of deformations and rigidity.

APPENDIX

A. C^*-algebras

We collect below the results on C^*-algebras that we need in this monograph. For a concise reference we recommend [**Tak02**, Chapter I] and [**Mur90**] for an additional first course.

A.1. Definition an examples. A (*concrete*) *C^*-algebra* on a Hilbert space $\mathcal{H}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed with respect to the norm topology. For every operator $x \in \mathcal{B}(\mathcal{H})$, one has the crucial identity $\|xx^*\| = \|x\|^2$.

An *abstract C^*-algebra* A is a Banach $*$-algebra where this identity holds for every $x \in A$. An important consequence is that the spectral radius of every self-adjoint element is equal to its norm [**Tak02**, Proposition I.4.2], from which it follows that on any $*$-algebra there is at most one norm making it a C^*-algebra [**Mur90**, Corollary 2.1.2]. It also follows that every homomorphism[1] ϕ from a C^*-algebra into another one is norm decreasing [**Tak02**, Proposition I.5.2].

Apart from $\mathcal{B}(\mathcal{H})$, the most basic example of C^*-algebra is $C_0(X)$, the abelian $*$-algebra (endowed with the uniform norm) of complex-valued continuous functions that vanish at infinity on a locally compact (Hausdorff)[2] space X. Conversely, let A be an abelian C^*-algebra and $\widehat{A}$ its spectrum, that is, the space of non-zero homomorphisms χ from A to $\mathbb{C}$. We view $x \in A$ as the map $\widehat{x} : \chi \mapsto \chi(x)$. Equipped with the topology of pointwise convergence, $\widehat{A}$ is locally compact and the invaluable Gelfand theorem states that the *Gelfand map* $x \mapsto \widehat{x}$ is an isometric isomorphism from the C^*-algebra A onto the C^*-algebra $C_0(\widehat{A})$ [**Tak02**, Theorem I.4.4].

As a first consequence of this theorem, one gets the following result. Let A be a *unital* C^*-algebra, that is, having a unit element 1, and let $x \in A$ be *normal*, that is, $x^*x = xx^*$. Denote by $\mathrm{Sp}(x)$ its spectrum. Then there is a unique isomorphism ϕ from $C(\mathrm{Sp}(x))$ onto the C^*-subalgebra of A generated by x and 1 such that $\phi(1) = 1$ and $\phi(z) = x$, where $z : C(\mathrm{Sp}(x) \to \mathbb{C}$ is the identity function. If $f \in C(\mathrm{Sp}(x))$, then $\phi(f)$ is denoted $f(x)$. One says

[1] For us, a homomorphism from a C^*-algebra A into an other one B preserves the algebraic operations and the involution.

[2] If X is compact, we write $C(X)$ instead of $C_0(X)$. All topological spaces will be Hausdorff.

that $f(x)$ is obtained from x by *continuous functional calculus*. When A is not unital, there is an adapted version [**Tak02**, page 19] which allows to define similarly $f(x) \in A$ for every continuous complex-valued function f on $\mathbb{R}$ such that $f(0) = 0$.

From the Gelfand theorem, one deduces that every injective homomorphism $\phi : A \to B$ between two C^*-algebras is isometric [**Tak02**, Corollary I.5.4].

Every closed ideal of A is self-adjoint and the quotient is in a natural way a C^*-algebra [**Tak02**, Theorem I.8.1]. It follows that the range $\phi(A)$ of every homomorphism $\phi : A \to B$ is a C^*-subalgebra of B, *i.e.*, is automatically closed.

Particularly important is the study of the *representations* of A, that is, of the homomorphisms π from A into some $\mathcal{B}(\mathcal{H})$. In particular, $\pi(A)$ is a C^*-algebra on $\mathcal{H}$. The Gelfand-Naimark theorem states that every abstract C^*-algebra has an injective representation as a concrete C^*-subalgebra of some $\mathcal{B}(\mathcal{H})$ [**Tak02**, Theorem I.9.8]

The finite-dimensional C^*-algebras are well-understood. Indeed they are all isomorphic to some direct sum $\oplus_{k=1}^m M_{n_k}(\mathbb{C})$ where $M_n(\mathbb{C})$ is the C^*-algebra of $n \times n$ matrices with complex entries (see Exercise 2.2 or [**Tak02**, Section I.11]).

A.2. Positivity. Let A be a C^*-algebra and $A_{s.a}$ its subspace of self-adjoint elements. An essential feature of A is that $A_{s.a}$ carries a natural partial order. A self-adjoint element x is said to be *positive*, and one writes $x \geq 0$, if it is of the form $x = y^*y$ for some $y \in A$, or equivalently of the form h^2 with $h \in A_{s.a}$. For $A \subset \mathcal{B}(\mathcal{H})$, this means that x is a positive operator in the usual sense, that is, self-adjoint with it spectrum contained in $\mathbb{R}_+$ (or equivalently such that $\langle \xi, x\xi \rangle \geq 0$ for every $\xi \in \mathcal{H}$). We denote by A_+ the set of positive elements in A. It is a closed convex cone with $A_+ \cap (-A_+) = \{0\}$ [**Tak02**, Theorem I.6.1]. The partial order relation on $A_{s.a}$ is defined by $y \leq x$ if $x - y \in A_+$. This implies that $aya^* \leq axa^*$ for every $a \in A$. When A is unital, let us also observe that $x \leq \|x\|1$ for every $x \in A_{s.a}$. If $y \leq x$, but $y \neq x$, we write $y < x$.

We define the *absolute value* of $x \in A$ by $|x| = (x^*x)^{1/2}$. If $x \in A_{s.a}$, we set

$$x_+ = \frac{1}{2}(|x| + x), \quad x_- = \frac{1}{2}(|x| - x).$$

These elements x_+ and x_- are respectively the *positive* and *negative* part of x. Hence, x is the difference of two positive elements x_+ and x_- in A such that $x_+x_- = 0$.

It follows that A_+ generates linearly A. Indeed, let $x \in A$. Then first

$$x = \frac{1}{2}(x + x^*) + i\frac{1}{2i}(x - x^*),$$

where the self-adjoint operators $\Re(x) = (1/2)(x + x^*)$ and $\Im(x) = (1/2i)(x - x^*)$ are respectively called the *real* and *imaginary part* of x. Next we write these self-adjoint elements as the difference of their positive and negative part. In conclusion, $x = x_1 - x_2 + ix_3 - ix_4$, with x_i positive and $\|x_i\| \leq \|x\|$, $1 \leq i \leq 4$.

A linear functional φ on a C^*-algebra A is said to be *positive* if $\varphi(x) \geq 0$ for every $x \in A_+$. It is automatically continuous and satisfies $\varphi(x^*) = \overline{\varphi(x)}$ for every $x \in A$. Moreover, if A is unital, then $\|\varphi\| = \varphi(1)$ [**Tak02**, Section I.9]. Conversely, every bounded linear functional ψ on A such that $\|\psi\| = \psi(1)$ is positive [**Dix77**, Proposition 2.1.9].

A linear map $\phi : A \to B$ between C^*-algebra is said to be *positive* if $\phi(A_+) \subset B_+$. It is bounded, preserves the passage to the adjoint, *i.e.*, $\phi(x^*) = \phi(x)^*$ for $x \in A$, and moreover we have $\|\phi\| \leq 2\|\phi(1)\|$ whenever A is unital [**Pau02**, Proposition 2.1]. Homomorphisms are basic examples of positive maps. A linear positive map $\phi : A \to B$ is said to be *faithful* if whenever $x \in A_+$ is such that $\phi(x) = 0$, then $x = 0$. Note that a homomorphism is faithful if and only if it is injective.

For every integer $n \geq 1$, and every C^*-algebra C, we denote by $M_n(C)$ the set of $n \times n$ matrices with entries in C. There is a natural way to turn it in a C^*-algebra (see [**Tak02**, Section IV.3]). Given a linear map $\phi : A \to B$ between C^*-algebra, we define a map $\phi_n : M_n(A) \to M_n(B)$ by $\phi_n([a_{i,j}]) = [\phi(a_{i,j})]$. When ϕ is positive, we could expect that ϕ_n is still positive but this is not the case in general (see [**Pau02**, page 5]). Note however that ϕ_n is a homomorphism whenever ϕ is a homomorphism, and so positivity is preserved in this case.

A.3. Completely positive maps. They form a very important class of morphisms, intermediate between positive linear maps and homomorphisms. For a comprehensive study of these morphisms see [**Pau02**]

DEFINITION A.1. A linear map $\phi : A \to B$ between C^*-algebras is said to be *completely positive* if ϕ_n is positive for every integer $n \geq 1$.

The following result provides an easy way to check that a linear map is completely positive [**Tak02**, Corollary IV.3.4].

PROPOSITION A.2. *A linear map $\phi : A \to B$ is completely positive if and only if $\sum_{i,j=1}^n y_i^* \phi(x_i^* x_j) y_j \geq 0$ for every $n \geq 1$, $x_1, \dots, x_n \in A$ and $y_1, \dots, y_n \in B$.*

In particular, given $a \in A$, the map $x \mapsto a^* x a$ from A to A is completely positive. Every positive linear map from a C^*-algebra into an abelian C^*-algebra is completely positive [**Tak02**, Corollary IV.3.5]. The general completely positive maps from A into $\mathcal{B}(\mathcal{H})$ are described by the dilation theorem of Stinespring. We recall below its version in case A is unital (see [**BO08**, Theorem 1.5.3] for instance), which is the only case we need.

THEOREM A.3. *Let A be a unital C^*-algebra and $\phi : A \to \mathcal{B}(\mathcal{H})$ be a completely positive map. There exist a Hilbert space $\mathcal{K}$, a unital representation $\pi : A \to \mathcal{B}(\mathcal{K})$, i.e., $\pi(1) = 1$, and an operator $V : \mathcal{H} \to \mathcal{K}$ such* *that*

$$\phi(x) = V^*\pi(x)V$$

for every $x \in A$. *In particular, we have* $\|\phi\| = \|V^*V\| = \|\phi(1)\|$.

The *Schwarz inequality*

$$\forall x \in A, \quad \phi(x)^*\phi(x) \leq \|\phi\|\phi(x^*x)$$

follows immediately.

A.4. Norm-one projections and conditional expectations. Let A be a unital C^*-algebra and B a C^*-subalgebra with the same unit. The notion of *conditional expectation* from A onto B is defined in Section 9.1: it is a linear positive projection E which is B-bimodular, that is, $E(b_1xb_2) = b_1E(x)b_2$ for $b_1, b_2 \in B$ and $x \in A$. For $x, y \in A$, we have the generalized *Cauchy-Schwarz inequality*

$$E(y^*x)E(y^*x)^* \leq \|x\|^2 E(y^*y).$$

Assuming that $\|x\| = 1$, this is proved by developing $E((xb-y)^*(xb-y)) \geq 0$ with $b = E(x^*y)$.

Taking $y = 1$ in the Cauchy-Schwarz inequality, we see that $\|(E(x)\| \leq \|x\|$. Conversely, Tomyiama proved that every norm-one projection is a conditional expectation [**Tom57**]. More precisely we have the following result.

THEOREM A.4. *Let* A *be a unital* C^**-algebra,* $B \subset A$ *a* C^**-subalgebra with the same unit and* $E : A \to B$ *a linear map. The following conditions are equivalent:*

(i) E *is a conditional expectation;*
(ii) E *is a completely positive projection;*
(iii) E *is a norm-one projection.*

For a short proof, we refer to [**BO08**, Theorem 1.5.10].

A.5. Arveson's extension theorem. Let A be a unital C^*-algebra and B a C^*-subalgebra with the same unit. By the Hahn-Banach theorem, every positive linear functional φ on B extends to a bounded linear functional $\widetilde{\varphi}$ on A with the same norm. Thus we have $\|\widetilde{\varphi}\| = \widetilde{\varphi}(1)$ and therefore $\widetilde{\varphi}$ is positive. It is a remarkable result of Arveson [**Arv69**] that the same result still holds when $\mathbb{C}$ is replaced by any $\mathcal{B}(\mathcal{H})$, under an assumption of complete positivity (see [**BO08**, Theorem 1.6.1] for a concise proof).

THEOREM A.5. *Let* A *be a unital* C^**-algebra and* B *a* C^**-subalgebra with the same unit. Then, every completely positive map* ϕ *from* B *to* $\mathcal{B}(\mathcal{H})$ *extends to a completely positive map* $\widetilde{\phi} : A \to \mathcal{B}(\mathcal{H})$

B. Standard Borel and measure spaces

In this monograph we will only deal with these spaces. We gather now the few properties that we will use. For more information, we refer to [**Kec95**].

B.1. Standard Borel spaces.

Definition B.1. A *Polish space* is a separable topological space admitting a compatible complete metric.

Definition B.2. A *standard Borel space* is a Borel space isomorphic to some Borel space $(X, \mathcal{B})$, where $\mathcal{B}$ is the collection of Borel subsets of a Polish space X.

These Borel spaces satisfy the following important property (see [**Kec95**, Corollary 15.2]).

Proposition B.3. *Let $f : X \to Y$ be a Borel map between two standard Borel spaces. Let $A \subset X$ be a Borel subspace such that $f_{|A}$ is injective. Then $f(A)$ is a Borel subset of Y. In particular, if f is a Borel bijection, then it is a Borel isomorphism.*

Standard Borel spaces have a simple classification: they are either finite, or isomorphic to $\mathbb{Z}$, or to $[0, 1]$ (see for instance [**Kec95**, Chapter II, Theorem 15.6] or [**Tak02**, Corollary A.11]).

A Polish group is a topological group whose topology is Polish. A useful result is the automatic continuity property stated below [**Kec95**, Theorem 9.10].

Proposition B.4. *Let $f : G \to H$ be a continuous bijective homomorphism between Polish groups. Then f is a homeomorphism.*

For the next theorem, see [**Kec95**, Theorem 18.10].

Theorem B.5 (**Lusin-Novikov**). *Let X, Y be two standard Borel spaces and $E \subset X \times Y$ a Borel subset. We assume that for every $x \in X$, the fiber $\pi^{-1}(x) \cap E$ is countable, where $\pi : X \times Y \to X$ is the projection onto X. Then there is a countable partition $E = \cup_n E_n$ of $X \times Y$ into Borel subsets such that the restriction of π to each E_n is injective.*

B.2. Standard probability measure spaces. They have a nice behaviour and are sufficiently general for all our practical purposes. Their theory was started by von Neumann [**vN32a, vN32b**] and further studied in particular by Halmos and von Neumann [**HvN42**].

Let μ be a probability measure on a Borel space X. We say that μ is *continuous* (or *without atom*) if $\mu(\{t\}) = 0$ for all $t \in X$. We say that μ is *discrete* if $\mu = \sum_{t \in T} \mu(\{t\})\delta_t$, where T is a subset of X, which is necessarily countable. Every probability measure μ can be uniquely written as $\mu = \mu_c + \mu_d$ where μ_c is continuous and μ_d is discrete.

Definition B.6. A probability measure space (X, μ) is said to be *standard* if there exists a standard conull Borel subspace in X. If in addition μ is continuous, we say that (X, μ) is a *Lebesgue probability measure space*.

One of the most important facts about standard probability measure spaces is the Halmos-von Neumann theorem stating that they are isomorphic to $[0, 1]$ equipped with its natural Borel structure together with a convex combination of the Lebesgue probability measure on $[0, 1]$ and a discrete probability measure. For details, see [**Kec95**, Chapter II, §17] or [**Ram71**].

So there is only one Lebesgue probability measure space, up to isomorphism of probability measure spaces, and we speak of *the* Lebesgue probability measure space. For the reader's convenience, we give a proof the uniqueness of this space, based on the classification of standard Borel spaces.

Theorem B.7. *Let (X, μ) be a standard probability measure space, where μ is continuous. There is a Borel isomorphism $\theta : X \to [0, 1]$ such that $\theta_* \mu = \lambda$ where where $\theta_* \mu$ is the pushforward of μ under θ and where λ is the Lebesgue probability measure on* $[0, 1]$.

Proof. We follow [**Kec95**, Theorem 17.41]. Using the classification theorem of standard Borel spaces, we may assume that $X = [0, 1]$. Let g be the continuous function $t \mapsto \mu([0, t])$ defined on $[0, 1]$. It is non-decreasing, with $g(0) = 0$ and $g(1) = 1$. Furthermore, we have $g_* \mu = \lambda$ since for any $t \in [0, 1]$, if we choose s with $g(s) = t$, we get

$$\mu(g^{-1}([0, t]) = \mu([0, s]) = g(s) = t = \lambda([0, t]).$$

This function g is not necessarily injective, but for every $t \in [0, 1]$ we have $\mu(g^{-1}(\{t\})) = 0$. The subset T of $[0, 1]$ such that the interval $g^{-1}(t)$ is not reduced to a point is countable, and therefore $N = g^{-1}(T)$ is such that $\mu(N) = 0$. Note that g is a homeomorphism from $[0, 1] \setminus N$ onto $[0, 1] \setminus T$. Let $Q \subset [0, 1] \setminus T$ be an uncountable Borel set of Lebesgue measure 0 and set $P = g^{-1}(Q)$. Then $P \cup N$ and $Q \cup T$ are two uncountable standard Borel spaces, and therefore there is a Borel isomorphism h from $P \cup N$ onto $Q \cup T$. Now we define θ to be equal to h on $P \cup N$ and to g on the complement. Obviously, θ is a Borel isomorphism from $[0, 1]$ onto itself such that $\theta_* \mu = \lambda$. $\square$

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